

A HIGH-ORDER, ANALYTICALLY DIVERGENCE-FREE DISCRETIZATION METHOD FOR DARCY'S PROBLEM

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ABSTRACT. We develop and analyze a meshfree discretization method for Darcy's problem. Our approximation scheme is based upon optimal recovery which leads to a collocation scheme using divergence-free positive definite kernels. Besides producing analytically incompressible flow fields, our method can be of arbitrary order, works in arbitrary space dimension and for arbitrary geometries. After deriving the scheme, we investigate the approximation error for smooth target functions and derive optimal approximation orders. Finally, we illustrate the method by giving numerical examples.

1. INTRODUCTION

Darcy's law is often used to describe the creeping flow of Newtonian fluids in porous media [3]. It is given by

$$(1.1) \quad \mathbf{u} = -\frac{K}{\mu} \nabla p,$$

where the viscosity $\mu > 0$ and the permeability tensor K are given and the velocity \mathbf{u} and the pressure p have to be determined.

We will incorporate the viscosity into the permeability tensor. Furthermore, we will assume that the velocity field is incompressible. Then, appropriate boundary conditions are Neumann-boundary conditions, such that Darcy's problem can be stated in the following way:

$$(1.2) \quad \mathbf{u} + K \nabla p = \mathbf{f} \quad \text{in } \Omega,$$

$$(1.3) \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega,$$

$$(1.4) \quad \mathbf{u} \cdot \mathbf{n} = \mathbf{g} \cdot \mathbf{n} \quad \text{on } \partial\Omega.$$

Here, \mathbf{n} denotes the outer unit normal vector of the boundary $\partial\Omega \subseteq \mathbb{R}^d$. The right-hand sides \mathbf{f} and $\mathbf{g} \cdot \mathbf{n}$ and the tensor K are given. The boundary function \mathbf{g} must satisfy the compatibility condition

$$(1.5) \quad \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, dS = 0.$$

The tensor K is supposed to be symmetric, $K = K^T$, and strongly elliptic in the sense that there is a constant $\alpha > 0$ such that

$$(1.6) \quad \boldsymbol{\xi}^T K(\mathbf{x}) \boldsymbol{\xi} \geq \alpha \|\boldsymbol{\xi}\|_2^2, \quad \boldsymbol{\xi} \in \mathbb{R}^d, \mathbf{x} \in \Omega.$$

The velocity $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ and the pressure $p : \Omega \rightarrow \mathbb{R}$ are sought.

In the particular form of $K = \text{const} \cdot I$, which refers to isotropic and homogeneous material, Darcy's law also plays an important role in projection methods for discretizing the Navier-Stokes equations for incompressible Newtonian fluids, where it is usually solved by converting it into a Poisson problem with Neumann boundary conditions for the pressure and then defining the velocity by (1.2). However, there are also good reasons for dealing with Darcy's problem directly.

The goal of this paper is to derive a high-order method for solving Darcy's problem efficiently. We will use methods based on analytically divergence-free kernels, as they have been developed in [15] and further studied in [13, 12, 7, 6, 5]. Our method will follow ideas from [22].

For this purpose the paper is organized as follows. In the rest of this section we collect the necessary notation on vector-valued Sobolev spaces. The next section is devoted to our discretization scheme, hence covering matrix-valued kernels, their reproducing kernel Hilbert spaces and optimal recovery. In the third section, we analyze the discretization scheme and derive error estimates. In the final section, we give numerical examples to corroborate our theoretical estimates.

1.1. Sobolev spaces. We will work with the usual scalar-valued Sobolev spaces. For a domain $\Omega \subseteq \mathbb{R}^d$, a real number $r \geq 1$ or $r = \infty$ and an integer $k \in \mathbb{N}_0$, we denote by $W_r^k(\Omega)$ the space of all functions $f \in L_r(\Omega)$ having weak derivatives $D^\alpha f \in L_r(\Omega)$ for every multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = \alpha_1 + \dots + \alpha_d \leq k$. We will also work with fractional order Sobolev spaces $W_r^\tau(\Omega)$, particularly with $\tau > d/2$ so that we have continuous functions. For the introduction of such fractional order Sobolev spaces we refer, for example, to [1, 4, 21].

Since the pressure p in the solution of (1.2)–(1.4) is determined only up to a constant we will work with the quotient spaces $W_r^\tau(\Omega)/\mathbb{R}$ equipped with the norm

$$\|p\|_{W_r^\tau(\Omega)/\mathbb{R}} := \inf_{c \in \mathbb{R}} \|p + c\|_{W_r^\tau(\Omega)}.$$

We define the vector-valued Sobolev space $\mathbf{W}_r^\tau(\Omega)$ to consist of all vector-valued functions $\mathbf{u} = (u_1, \dots, u_n)^T : \Omega \rightarrow \mathbb{R}^n$, where each component u_j belongs to $W_r^\tau(\Omega)$. A norm on $\mathbf{W}_r^\tau(\Omega)$ can be defined by taking the discrete ℓ_r norm of the $W_r^\tau(\Omega)$ norms of the components, i.e. by

$$\|\mathbf{u}\|_{\mathbf{W}_r^\tau(\Omega)} = \begin{cases} \left(\sum_{j=1}^n \|u_j\|_{W_r^\tau(\Omega)}^r \right)^{1/r} & \text{for } 1 \leq r < \infty, \\ \max_{1 \leq j \leq n} \|u_j\|_{W_\infty^\tau(\Omega)} & \text{for } r = \infty. \end{cases}$$

Note that we do not use an index to indicate the dimension n since it will become clear from the context. We only distinguish between scalar-valued function spaces and vector-valued function spaces by using boldface for the latter. Finally, in the case $r = 2$, we will also use the notation $\mathbf{H}^\tau(\Omega) = \mathbf{W}_2^\tau(\Omega)$.

2. THE DISCRETIZATION SCHEME

In this section, we will review the necessary material on matrix-valued kernels and the way we will use them for discretizing Darcy's problem.

First, we will discuss the kernels, their reproducing Hilbert spaces and optimal recovery in general form and then adapt them to our problem. For this, we will mainly rely on material from [7, 6, 5, 22]. In the last part of this section, we will modify the general theory such that it fits to our problem.

2.1. Positive definite matrix-valued kernels.

Definition 2.1. A function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is positive definite if for all $N \in \mathbb{N}$, all pairwise distinct $\mathbf{x}_1, \dots, \mathbf{x}_N \in \mathbb{R}^d$ and all $\boldsymbol{\alpha} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$, the quadratic form

$$\sum_{j,k=1}^N \alpha_j \alpha_k \phi(\mathbf{x}_j - \mathbf{x}_k)$$

is positive. More generally, a matrix-valued function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^{n \times n}$ is said to be positive definite if it is even $\Phi(-\mathbf{x}) = \Phi(\mathbf{x})$, symmetric $\Phi(\mathbf{x}) = \Phi(\mathbf{x})^T$ and satisfies

$$\sum_{j,k=1}^N \boldsymbol{\alpha}_j^T \Phi(\mathbf{x}_j - \mathbf{x}_k) \boldsymbol{\alpha}_k > 0$$

for all pairwise distinct $\mathbf{x}_j \in \mathbb{R}^d$ and all $\boldsymbol{\alpha}_j \in \mathbb{R}^n$ such that not all $\boldsymbol{\alpha}_j$ are vanishing.

The theory of the associated function spaces can be formulated for positive definite matrix-valued functions as it can also be done for scalar-valued functions. First, we introduce the function space

$$\mathbf{F}_\Phi(\Omega) := \left\{ \sum_{j=1}^N \Phi(\cdot - \mathbf{x}_j) \boldsymbol{\alpha}_j : \mathbf{x}_j \in \Omega, \boldsymbol{\alpha}_j \in \mathbb{R}^n \right\}$$

and equip it with the natural inner product

$$\left(\sum_{j=1}^N \Phi(\cdot - \mathbf{x}_j) \boldsymbol{\alpha}_j, \sum_{k=1}^M \Phi(\cdot - \mathbf{y}_k) \boldsymbol{\beta}_k \right)_\Phi := \sum_{j=1}^N \sum_{k=1}^M \boldsymbol{\alpha}_j^T \Phi(\mathbf{x}_j - \mathbf{y}_k) \boldsymbol{\beta}_k.$$

Then, the closure of $\mathbf{F}_\Phi(\Omega)$ with respect to the norm stemming from this inner product is the associated *native Hilbert* space.

Definition 2.2. The reproducing kernel Hilbert space (RKHS), or native space, of a positive definite, matrix-valued kernel Φ is defined to be the closure of $\mathbf{F}_\Phi(\Omega)$ with respect to $\|\cdot\|_{\mathcal{N}_\Phi(\Omega)} := \|\cdot\|_\Phi$ and it will be denoted by $\mathcal{N}_\Phi(\Omega)$.

For every vector-valued function $\mathbf{f} \in \mathcal{N}_\Phi(\Omega)$ and every $\boldsymbol{\alpha} \in \mathbb{R}^n$ and every $\mathbf{x} \in \Omega$, we have the relations

$$(2.1) \quad \begin{aligned} \Phi(\cdot - \mathbf{x}) \boldsymbol{\alpha} &\in \mathcal{N}_\Phi(\Omega), \\ (\mathbf{f}, \Phi(\cdot - \mathbf{x}) \boldsymbol{\alpha})_\Phi &= \mathbf{f}(\mathbf{x})^T \boldsymbol{\alpha}, \end{aligned}$$

which generalize the reproducing kernel properties of the scalar-valued case.

For us, it will be important that for specific kernels these spaces can be identified to be norm equivalent to classical Sobolev spaces. The kernels we are interested in are defined as follows. Let $\phi, \psi : \mathbb{R}^d \rightarrow \mathbb{R}$ be positive definite functions, where ϕ is at least twice continuously differentiable. Then, we define

$$(2.2) \quad \tilde{\Phi} : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}, \quad \tilde{\Phi} := (-\Delta I + \nabla \nabla^T) \phi,$$

$$(2.3) \quad \Phi : \mathbb{R}^d \rightarrow \mathbb{R}^{(d+1) \times (d+1)}, \quad \Phi := \begin{pmatrix} \tilde{\Phi} & \mathbf{0} \\ \mathbf{0} & \psi \end{pmatrix} =: \tilde{\Phi} \otimes \psi,$$

where Δ is the usual Laplace operator, ∇ denotes the gradient and I the identity matrix. It is well known (see [15, 6, 5, 22]) that both matrix-valued kernels

are positive definite. Furthermore, their reproducing kernel Hilbert spaces can be characterized using Fourier transform; see [6, 5, 22].

Theorem 2.3. (1) *Suppose $\phi \in W_1^2(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)$ is positive definite. Define $\tilde{\Phi} = (-\Delta I + \nabla \nabla^T)\phi$. Then, $\mathcal{N}_{\tilde{\Phi}}(\mathbb{R}^d)$ consists of all divergence-free functions $\mathbf{f} \in \mathbf{L}_2(\mathbb{R}^d)$ with*

$$\|\mathbf{f}\|_{\mathcal{N}_{\tilde{\Phi}}(\mathbb{R}^d)}^2 = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{f}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2 \widehat{\phi}(\boldsymbol{\omega})} d\boldsymbol{\omega} < \infty.$$

(2) *Let $\Phi = \tilde{\Phi} \otimes \psi$ with a positive definite function ψ . Then,*

$$\mathcal{N}_{\Phi}(\mathbb{R}^d) = \mathcal{N}_{\tilde{\Phi}}(\mathbb{R}^d) \times \mathcal{N}_{\psi}(\mathbb{R}^d)$$

with the norm for $\mathbf{f} = (\mathbf{f}_u, f_p)$ given by

$$\begin{aligned} \|\mathbf{f}\|_{\mathcal{N}_{\Phi}(\mathbb{R}^d)}^2 &= \|\mathbf{f}_u\|_{\mathcal{N}_{\tilde{\Phi}}(\mathbb{R}^d)}^2 + \|f_p\|_{\mathcal{N}_{\psi}(\mathbb{R}^d)}^2 \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \left[\frac{\|\widehat{\mathbf{f}}_u(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2 \widehat{\phi}(\boldsymbol{\omega})} + \frac{|\widehat{f}_p(\boldsymbol{\omega})|^2}{\widehat{\psi}(\boldsymbol{\omega})} \right] d\boldsymbol{\omega}. \end{aligned}$$

We are particularly interested in reproducing kernel Hilbert spaces that are norm equivalent to Sobolev spaces. A scalar-valued RKHS $\mathcal{N}_{\phi}(\mathbb{R}^d)$ is norm equivalent to the Sobolev space $H^\tau(\mathbb{R}^d)$ if the kernel function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ has a Fourier transform $\widehat{\phi}$ satisfying

$$c_1(1 + \|\boldsymbol{\omega}\|_2^2)^{-\tau} \leq \widehat{\phi}(\boldsymbol{\omega}) \leq c_2(1 + \|\boldsymbol{\omega}\|_2^2)^{-\tau}, \quad \boldsymbol{\omega} \in \mathbb{R}^d,$$

with two constants $0 < c_1 \leq c_2$.

Corollary 2.4. *Assume ϕ generates $H^{\tau+1}(\mathbb{R}^d)$ and ψ generates $H^\sigma(\mathbb{R}^d)$, i.e., $\mathcal{N}_{\phi}(\mathbb{R}^d) = H^{\tau+1}(\mathbb{R}^d)$ and $\mathcal{N}_{\psi}(\mathbb{R}^d) = H^\sigma(\mathbb{R}^d)$ with equivalent norms. The associated reproducing kernel Hilbert space of the combined kernel is given by*

$$\mathcal{N}_{\Phi}(\mathbb{R}^d) = \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^\sigma(\mathbb{R}^d).$$

Here,

$$\begin{aligned} \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) &= \left\{ \mathbf{f} \in \mathbf{H}^\tau(\mathbb{R}^d; \text{div}) : \int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{f}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\boldsymbol{\omega}\|_2^2)^{\tau+1} d\boldsymbol{\omega} < \infty \right\}, \\ \mathbf{H}^\tau(\mathbb{R}^d; \text{div}) &= \{ \mathbf{f} \in \mathbf{H}^\tau(\mathbb{R}^d) : \nabla \cdot \mathbf{f} = 0 \}. \end{aligned}$$

Finally, since we mainly work on bounded domains, we need, for technical reasons, to extend our locally defined Sobolev functions to globally defined functions. We will use the following result from [22].

Proposition 2.5. *Let $d = 2, 3$. Let $\tau, \sigma \geq 0$ and let $\Omega \subseteq \mathbb{R}^d$ be a simply-connected domain with $C^{k,1}$ boundary, where $k \geq \tau$ is an integer. Then there exists a continuous operator $\mathbf{E} = (\tilde{\mathbf{E}}_{\text{div}}, E_S) : \mathbf{H}^\tau(\Omega; \text{div}) \times H^\sigma(\Omega) \rightarrow \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^\sigma(\mathbb{R}^d)$ such that $\mathbf{E}\mathbf{v}|_\Omega = \mathbf{v}|_\Omega$ for all $\mathbf{v} = (\mathbf{u}, p) \in \mathbf{H}^\tau(\Omega; \text{div}) \times H^\sigma(\Omega)$ and*

$$\|\tilde{\mathbf{E}}_{\text{div}} \mathbf{u}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div})} + \|E_S p\|_{H^\sigma(\mathbb{R}^d)} \leq C (\|\mathbf{u}\|_{\mathbf{H}^\tau(\Omega)} + \|p\|_{H^\sigma(\Omega)}).$$

The extension operator for the pressure part is the standard Stein extension operator E_S ; see [19].

2.2. Optimal recovery to build the approximant. We will use functionals from the dual space $\mathcal{N}_{\Phi}(\Omega)^* = \{\lambda : \mathcal{N}_{\Phi}(\Omega) \rightarrow \mathbb{R} : \lambda \text{ is linear and continuous}\}$ to describe our discretization scheme.

Since $\mathbf{f} = \Phi(\cdot - \mathbf{y})\mathbf{e}_j$ belongs to $\mathcal{N}_{\Phi}(\Omega)$, where \mathbf{e}_j is the j th unit vector, we see that the columns of Φ and, since Φ is symmetric, its rows also belong to $\mathcal{N}_{\Phi}(\Omega)$. Thus, we can define $\lambda^{\mathbf{y}}\Phi(\mathbf{x} - \mathbf{y})$ as the vector-valued function, which is generated by applying λ with respect to \mathbf{y} to every column of Φ . The resulting vector-valued function is the Riesz representer of λ in $\mathcal{N}_{\Phi}(\Omega)$ in the sense of

$$\lambda(\mathbf{f}) = (\mathbf{f}, \lambda^{\mathbf{y}}\Phi(\cdot - \mathbf{y}))_{\Phi}.$$

Thus the following result, which is well known in the context of scalar-valued kernels, remains true for matrix-valued kernels.

Proposition 2.6. *Suppose $\Phi : \Omega \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^{n \times n}$ is a positive definite, matrix-valued kernel. Suppose further that $\lambda_1, \dots, \lambda_N \in \mathcal{N}_{\Phi}(\Omega)^*$ are linearly independent and $f_1, \dots, f_N \in \mathbb{R}$ are given. Then, the problem of finding the solution of*

$$(2.4) \quad \min\{\|\mathbf{s}\|_{\mathcal{N}_{\Phi}(\Omega)} : \lambda_j(\mathbf{s}) = f_j, 1 \leq j \leq N\}$$

has a unique solution, which has the representation

$$\mathbf{s}_f = \sum_{j=1}^N \alpha_j \lambda_j^{\mathbf{y}_j} \Phi(\cdot - \mathbf{y}_j).$$

The coefficients α_j are determined via the interpolation conditions $\lambda_i(\mathbf{s}_f) = f_i$, $1 \leq i \leq N$.

To apply this result to Darcy's problem, we pick discretization points $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subseteq \Omega$ in the interior and $Y = \{\mathbf{y}_1, \dots, \mathbf{y}_M\} \subseteq \partial\Omega$ on the boundary. For $\mathbf{v} = (\mathbf{u}, p)$ we define the functionals

$$(2.5) \quad \lambda_j^{(i)}(\mathbf{v}) = u_i(\mathbf{x}_j) + (K\nabla p)_i(\mathbf{x}_j)$$

$$(2.6) \quad = u_i(\mathbf{x}_j) + \sum_{k=1}^d K_{ik}(\mathbf{x}_j) \partial_k p(\mathbf{x}_j), \quad 1 \leq i \leq d, 1 \leq j \leq N =: N_i$$

$$(2.7) \quad \lambda_j^{(d+1)}(\mathbf{v}) = \sum_{k=1}^d u_k(\mathbf{y}_j) n_k(\mathbf{y}_j), \quad 1 \leq j \leq M =: N_{d+1},$$

such that the approximant according to Proposition 2.6 becomes

$$(2.8) \quad \mathbf{s}_{\mathbf{v}}(\mathbf{x}) := \sum_{k=1}^{d+1} \sum_{j=1}^{N_k} \alpha_j^{(k)} (\lambda_j^{(k)})^{\mathbf{y}} \Phi(\mathbf{x} - \mathbf{y}).$$

The following result ensures that the so-defined functionals are linearly independent.

Theorem 2.7. *Let $\Omega \subseteq \mathbb{R}^d$ with a $C^{k,1}$ boundary, where $k \geq 1$. Assume that the building functions $\phi, \psi : \mathbb{R}^d \rightarrow \mathbb{R}$ are positive definite and chosen such that $\mathcal{N}_{\Phi}(\mathbb{R}^d) = \widetilde{\mathbf{H}}^{\tau}(\mathbb{R}^d; \text{div}) \times H^{\tau+1}(\mathbb{R}^d)$ with $\tau > d/2$. Then, the interpolation function $\mathbf{s}_{\mathbf{v}} = (\mathbf{s}_{\mathbf{u}}, s_p)^T$ from (2.8) is well defined and uniquely determined by the interpolation conditions (2.6) and (2.7). It satisfies $L\mathbf{s}_{\mathbf{v}}(\mathbf{x}_j) = \mathbf{f}(\mathbf{x}_j)$ with $L\mathbf{v} := \mathbf{u} + K\nabla p$ and $\mathbf{s}_{\mathbf{u}}(\mathbf{y}_j) \cdot \mathbf{n}(\mathbf{y}_j) = \mathbf{g}(\mathbf{y}_j) \cdot \mathbf{n}(\mathbf{y}_j)$. Furthermore, we have $\nabla \cdot \mathbf{s}_{\mathbf{u}} = 0$ in \mathbb{R}^d .*

Proof. Since $\tau + 1 > d/2 + 1$, we have $\phi, \psi \in C^2(\mathbb{R}^d)$ and thus $\tilde{\Phi} \in C(\mathbb{R}^d; \mathbb{R}^{d \times d})$. Hence, the kernel is sufficiently smooth and the functionals indeed belong to the dual of the native space. Thus, we only have to show that the functionals are linearly independent over $\mathcal{N}_{\Phi}(\mathbb{R}^d) = \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^{\tau+1}(\mathbb{R}^d)$.

Let us assume that there are coefficients $\alpha_j^{(k)} \in \mathbb{R}$ such that

$$(2.9) \quad \sum_{k=1}^{d+1} \sum_{j=1}^{N_k} \alpha_j^{(k)} \lambda_j^{(k)}(\gamma) = 0$$

for all $\gamma \in \mathcal{N}_{\Phi}(\mathbb{R}^d)$. We will now pick a specific test function γ for every index pair (i, ℓ) . First, we choose γ to have compact support such that the only data site contained in the support of this specific γ is \mathbf{x}_i (for $1 \leq \ell \leq d$) or \mathbf{y}_i (for $\ell = d+1$). Hence, in the first case, (2.9) reduces to

$$0 = \sum_{k=1}^d \alpha_i^{(k)} \lambda_i^{(k)}(\gamma) = \sum_{k=1}^d \alpha_i^{(k)} (L\gamma)_k(\mathbf{x}_i).$$

Since we have not yet exploited the second index ℓ , we can now modify γ such that $(L\gamma)_k(\mathbf{x}_i) = \delta_{k,\ell}$, which gives $\alpha_i^{(\ell)} = 0$. Since we can do the same in the case $\ell = d+1$, we see that all coefficients have to be zero, showing that the functionals are linearly independent. \square

3. ERROR ANALYSIS

Our error analysis is mainly based on a “shift” type theorem for the analytical solution of Darcy’s problem, which can be easily derived from a corresponding result for elliptic problems with Neumann boundary conditions. This immediately follows by taking the divergence of (1.2) and incorporating (1.3). The boundary conditions follow by taking the inner product of (1.2) with the unit outer normal vector \mathbf{n} of $\partial\Omega$ and to also use (1.4). In doing so, we see that (1.2)–(1.4) is equivalent to solving

$$(3.1) \quad \nabla \cdot (K\nabla p) = \tilde{f} := \nabla \cdot \mathbf{f} \quad \text{in } \Omega,$$

$$(3.2) \quad K \frac{\partial p}{\partial \mathbf{n}} = \tilde{g} := (\mathbf{f} - \mathbf{g}) \cdot \mathbf{n} \quad \text{on } \partial\Omega$$

and defining

$$(3.3) \quad \mathbf{u} := (\mathbf{f} - K\nabla p).$$

The compatibility conditions for the Neumann problem are satisfied since we have with the divergence theorem and (1.5) that

$$\int_{\partial\Omega} \tilde{\mathbf{g}} \, dS - \int_{\Omega} \tilde{\mathbf{f}} \, d\mathbf{x} = \int_{\partial\Omega} (\mathbf{f} - \mathbf{g}) \cdot \mathbf{n} \, dS - \int_{\Omega} \nabla \cdot \mathbf{f} \, d\mathbf{x} = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, dS = 0.$$

For the elliptic Neumann problem (3.1) and (3.2) the following existence and smoothness result is well known. For integer order τ and $K = I$, its proof can be found in [10, Theorem 1.10], the integer case for general tensor K follows from [11] and the general fractional order case follows by interpolation theory in Sobolev spaces. Though the result was originally derived for weak solutions, the higher regularity assumption on the given data implies that it also holds for classical solutions.

Proposition 3.1. *Let Ω be a bounded open subset of \mathbb{R}^d with a $C^{[\tau]+1,1}$ boundary $\partial\Omega$. Assume that the permeability tensor $K = (K_{ij})$ satisfies (1.6), $K = K^T$ and $K_{ij} \in W_r^{\tau+1}(\overline{\Omega})$. Assume for the data that $\tilde{f} \in W_r^\tau(\Omega)$ and $\tilde{g} \in W_r^{\tau+1-1/r}(\partial\Omega)$ for $1 < r < \infty$ and $\int_{\partial\Omega} \tilde{g} dS = \int_{\Omega} \tilde{f} d\mathbf{x}$. Then there exists a function $p \in W_r^{\tau+2}(\Omega)/\mathbb{R}$ solving (3.1) and (3.2), which satisfies*

$$\|p\|_{W_r^{\tau+2}(\Omega)/\mathbb{R}} \leq C \left\{ \|\tilde{f}\|_{W_r^\tau(\Omega)} + \|\tilde{g}\|_{W_r^{\tau+1-1/r}(\partial\Omega)} \right\}$$

with a constant $C = C(\tau, r, \Omega)$.

Applying this to our special situation, the existence and smoothness of the solutions of Darcy's problem follow.

Theorem 3.2. *Let Ω be a bounded open subset of \mathbb{R}^d with a $C^{[\tau]+1,1}$ boundary $\partial\Omega$. Assume that $\mathbf{f} \in \mathbf{W}_r^{\tau+1}(\Omega)$ and that $\mathbf{g} \in \mathbf{W}_r^{\tau+1-1/r}(\partial\Omega)$ for $1 < r < \infty$ satisfies $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} dS = 0$. Assume that the permeability tensor $K = (K_{ij})$ satisfies (1.6), $K = K^T$ and $K_{ij} \in W_r^{\tau+1}(\overline{\Omega})$. Then there exist a velocity $\mathbf{u} \in \mathbf{W}_r^{\tau+1}(\Omega)$ and a pressure $p \in W_r^{\tau+2}(\Omega)/\mathbb{R}$, solutions to (1.2)–(1.4), which satisfy*

$$\|\mathbf{u}\|_{\mathbf{W}_r^{\tau+1}(\Omega)} + \|p\|_{W_r^{\tau+2}(\Omega)/\mathbb{R}} \leq C \left(\|\mathbf{f}\|_{\mathbf{W}_r^{\tau+1}(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{W_r^{\tau+1-1/r}(\partial\Omega)} \right).$$

Proof. Our assumptions on the given data immediately yield $\tilde{f} = \nabla \cdot \mathbf{f} \in W_r^\tau(\Omega)$. Since the boundary is also assumed to be smooth enough, we have $\tilde{g} = (\mathbf{f} - \mathbf{g}) \cdot \mathbf{n} \in W_r^{\tau+1-1/r}(\partial\Omega)$. Furthermore, we have the obvious estimates $\|\tilde{f}\|_{W_r^\tau(\Omega)} \leq \|\mathbf{f}\|_{\mathbf{W}_r^{\tau+1}(\Omega)}$ and, since the boundary is sufficiently smooth,

$$\begin{aligned} \|\tilde{g}\|_{W_r^{\tau+1-1/r}(\partial\Omega)} &= \|(\mathbf{f} - \mathbf{g}) \cdot \mathbf{n}\|_{W_r^{\tau+1-1/r}(\partial\Omega)} \\ &\leq \|\mathbf{f}\|_{\mathbf{W}_r^{\tau+1-1/r}(\partial\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{W_r^{\tau+1-1/r}(\partial\Omega)}, \\ &\leq C \left(\|\mathbf{f}\|_{\mathbf{W}_r^{\tau+1}(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{W_r^{\tau+1-1/r}(\partial\Omega)} \right), \end{aligned}$$

where we have used the standard trace theorem for Sobolev spaces. From (3.3), we see that

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}_r^{\tau+1}(\Omega)} &\leq \|\mathbf{f}\|_{\mathbf{W}_r^{\tau+1}(\Omega)} + \|K\nabla p\|_{\mathbf{W}_r^{\tau+1}(\Omega)} \\ &\leq \|\mathbf{f}\|_{\mathbf{W}_r^{\tau+1}(\Omega)} + C\|p\|_{W_r^{\tau+2}(\Omega)/\mathbb{R}}. \end{aligned}$$

All of this, together with Proposition 3.1, gives

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}_r^{\tau+1}(\Omega)} + \|p\|_{W_r^{\tau+2}(\Omega)/\mathbb{R}} &\leq \|\mathbf{f}\|_{\mathbf{W}_r^{\tau+1}(\Omega)} + C\|p\|_{W_r^{\tau+2}(\Omega)/\mathbb{R}} \\ &\leq C \left(\|\mathbf{f}\|_{\mathbf{W}_r^{\tau+1}(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{W_r^{\tau+1-1/r}(\partial\Omega)} \right), \end{aligned}$$

which is the desired estimate. \square

Using again the notation $\mathbf{v} = (\mathbf{u}, p)$ and $L\mathbf{v} = \mathbf{u} + K\nabla p$, the estimate from Theorem 3.2 can be rewritten in the form

$$\|\mathbf{u}\|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \|p\|_{W_r^{\eta+2}(\Omega)/\mathbb{R}} \leq C \left(\|L\mathbf{v}\|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \|\mathbf{u} \cdot \mathbf{n}\|_{W_r^{\eta+1-1/r}(\partial\Omega)} \right)$$

for any $0 \leq \eta \leq \tau$. We will use this for $\mathbf{v} - \mathbf{s}_\mathbf{v}$ instead of \mathbf{v} , i.e.,

$$(3.4) \quad \begin{aligned} & \|\mathbf{u} - \mathbf{s}_{\mathbf{u}}\|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \|p - s_p\|_{W_r^{\eta+2}(\Omega)/\mathbb{R}} \\ & \leq C \left(\|L(\mathbf{v} - \mathbf{s}_{\mathbf{v}})\|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \|(\mathbf{u} - \mathbf{s}_{\mathbf{u}}) \cdot \mathbf{n}\|_{W_r^{\eta+1-1/r}(\partial\Omega)} \right). \end{aligned}$$

To estimate the two terms on the right-hand side of the last equation, we first observe that we have

$$\begin{aligned} (L\mathbf{v} - L\mathbf{s}_{\mathbf{v}})(\mathbf{x}_j) &= \mathbf{0}, & 1 \leq j \leq N, \\ (\mathbf{u} - \mathbf{s}_{\mathbf{u}}) \cdot \mathbf{n}(\mathbf{y}_j) &= 0, & 1 \leq j \leq M. \end{aligned}$$

Hence, we are dealing with smooth functions that have a large number of zeros. In the first case we have functions defined on a bounded region of \mathbb{R}^d , while in the second case we are dealing with functions on a manifold. For such functions, we can apply the following ‘‘sampling inequalities’’.

To state them, we have to introduce a measure for the data density on Ω and $\partial\Omega$. In the first case we introduce the ‘‘fill distance’’

$$h_{X,\Omega} := \sup_{\mathbf{x} \in \Omega} \min_{\mathbf{x}_j \in X} \|\mathbf{x} - \mathbf{x}_j\|_2.$$

The following result comes from [2, 16, 17] in its vector-valued form from [22].

Lemma 3.3. *Let $1 < r < \infty$, and $\tau, \eta \in \mathbb{R}$ with $\tau > d/2$ and $0 \leq \eta \leq \tau - d(1/2 - 1/r)_+$. Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded domain having a Lipschitz boundary. Let $X \subseteq \Omega$ be a discrete set with fill distance $h_{X,\Omega}$ sufficiently small. Assume that $\mathbf{u} \in \mathbf{H}^\tau(\Omega)$ satisfies $\mathbf{u}|_X = \mathbf{0}$. Then we also have*

$$\|\mathbf{u}\|_{\mathbf{W}_r^\eta(\Omega)} \leq ch_{X,\Omega}^{\tau-\eta-d(1/2-1/r)_+} \|\mathbf{u}\|_{\mathbf{H}^\tau(\Omega)}.$$

Proposition 3.4. *Let Ω be a bounded, simply connected, open subset of \mathbb{R}^d with a $C^{\lceil\tau\rceil+1,1}$ boundary $\partial\Omega$ where $d = 2, 3$. Let the permeability tensor $K = K_{ij}$ satisfy (1.6), $K = K^T$ and $K_{ij} \in H^{\tau+1}(\bar{\Omega})$. Assume that the data satisfy $\mathbf{f} \in \mathbf{H}^{\tau+1}(\Omega)$, $\mathbf{g} \in \mathbf{H}^{\tau+1/2}(\partial\Omega)$ and $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} dS = 0$. Suppose that the kernel Φ is chosen such that $\mathcal{N}_\Phi(\mathbb{R}^d) = \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^{\tau+1}(\mathbb{R}^d)$ with $\tau > d/2$. Then, for $0 \leq \eta \leq \tau - d(1/2 - 1/r)_+ - 1$ and for $1 < r < \infty$ we have*

$$\|L\mathbf{v} - L\mathbf{s}_{\mathbf{v}}\|_{\mathbf{W}_r^{\eta+1}(\Omega)} \leq ch_{X,\Omega}^{\tau-\eta-1-d(1/2-1/r)_+} (\|\mathbf{f}\|_{\mathbf{H}^\tau(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{\mathbf{H}^{\tau-1/2}(\partial\Omega)}).$$

Proof. First, we have

$$\|L\mathbf{v} - L\mathbf{s}_{\mathbf{v}}\|_{\mathbf{W}_r^{\eta+1}(\Omega)} \leq Ch_{X,\Omega}^{\tau-\eta-1-d(1/2-1/r)_+} \|L\mathbf{v} - L\mathbf{s}_{\mathbf{v}}\|_{\mathbf{H}^\tau(\Omega)}.$$

To bound the latter norm we first extend the function \mathbf{v} to $\mathbf{E}\mathbf{v} = (\tilde{\mathbf{E}}_{\text{div}}\mathbf{u}, E_S p) \in \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^{\tau+1}(\mathbb{R}^d)$ and note that the generalized interpolant $\mathbf{s}_{\mathbf{v}}$ coincides with $\mathbf{s}_{\mathbf{E}\mathbf{v}}$ on Ω . Furthermore, if we pick the representer p for the pressure such that

$\|p\|_{H^{\tau+1}(\Omega)} = \|p\|_{H^{\tau+1}(\Omega)/\mathbb{R}}$ then we have

$$\begin{aligned}
\|L\mathbf{v} - L\mathbf{s}_{\mathbf{v}}\|_{\mathbf{H}^{\tau}(\Omega)} &= \|L\mathbf{E}_{\mathbf{v}} - L\mathbf{s}_{\mathbf{E}_{\mathbf{v}}}\|_{\mathbf{H}^{\tau}(\Omega)} \\
&\leq \|\tilde{\mathbf{E}}_{\text{div}}\mathbf{u} - \mathbf{s}_{\tilde{\mathbf{E}}_{\text{div}}\mathbf{u}}\|_{\mathbf{H}^{\tau}(\Omega)} + \|K(\nabla E_{SP} - \nabla s_{E_{SP}})\|_{\mathbf{H}^{\tau}(\Omega)} \\
&\leq \|\tilde{\mathbf{E}}_{\text{div}}\mathbf{u} - \mathbf{s}_{\tilde{\mathbf{E}}_{\text{div}}\mathbf{u}}\|_{\mathbf{H}^{\tau}(\Omega)} + C\|E_{SP} - s_{E_{SP}}\|_{H^{\tau+1}(\Omega)} \\
&\leq \|\tilde{\mathbf{E}}_{\text{div}}\mathbf{u} - \mathbf{s}_{\tilde{\mathbf{E}}_{\text{div}}\mathbf{u}}\|_{\tilde{\mathbf{H}}^{\tau}(\mathbb{R}^d; \text{div})} + C\|E_{SP} - s_{E_{SP}}\|_{H^{\tau+1}(\mathbb{R}^d)} \\
&\leq C\|\mathbf{E}_{\mathbf{v}} - \mathbf{s}_{\mathbf{E}_{\mathbf{v}}}\|_{\mathcal{N}_{\Phi}(\mathbb{R}^d)} \\
&\leq C\|\mathbf{E}_{\mathbf{v}}\|_{\mathcal{N}_{\Phi}(\mathbb{R}^d)} \\
&\leq C\left(\|\tilde{\mathbf{E}}_{\text{div}}\mathbf{u}\|_{\tilde{\mathbf{H}}^{\tau}(\mathbb{R}^d; \text{div})} + \|E_{SP}\|_{H^{\tau+1}(\mathbb{R}^d)}\right) \\
&\leq C(\|\mathbf{u}\|_{\mathbf{H}^{\tau}(\Omega)} + \|p\|_{H^{\tau+1}(\Omega)}) \\
&\leq C(\|\mathbf{f}\|_{\mathbf{H}^{\tau}(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{\mathbf{H}^{\tau-1/2}(\partial\Omega)}),
\end{aligned}$$

where we have used Theorem 3.2 in the last step. \square

To bound the boundary part $\|(\mathbf{u} - \mathbf{s}_{\mathbf{u}}) \cdot \mathbf{n}\|_{W_r^{\eta+1-1/\tau}(\partial\Omega)}$ in the estimate (3.4) we have to carry the concept of Lemma 3.3 to the manifold $\partial\Omega$. This has been done in [8] for the special case of $\partial\Omega$ being the sphere in \mathbb{R}^d and in a more general context in [9].

We will represent the boundary $\partial\Omega$ by a finite atlas consisting of smooth diffeomorphisms with a slight abuse of terminology. To be more precise, we assume that $\partial\Omega = \bigcup_{j=1}^J V_j$, where $V_j \subseteq \partial\Omega$ are relatively open sets. Moreover, the sets V_j are images of $C^{k,s}$ -diffeomorphism

$$\varphi_j : B \rightarrow V_j,$$

where $B = B(\mathbf{0}, 1)$ denotes the unit ball in \mathbb{R}^{d-1} and k and s are to be specified. Finally, suppose $\{w_j\}$ is a partition of unity with respect to $\{V_j\}$. Then, the Sobolev norms on $\partial\Omega$ can equivalently be defined via

$$\|u\|_{W_p^{\mu}(\partial\Omega)}^p = \sum_{j=1}^J \|(uw_j) \circ \varphi_j\|_{W_p^{\mu}(B)}^p.$$

It is well known that this norm is independent of the chosen atlas $\{V_j, \varphi_j\}$ but this is of less importance here, since we will assume that the atlas is fixed.

We will measure the density of the points Y on $\partial\Omega$ by introducing the mesh norm

$$h_{Y, \partial\Omega} := \max_{1 \leq j \leq J} h_{T_j, B}$$

with $T_j = \varphi_j^{-1}(Y \cap V_j) \subseteq B$. As mentioned before, we will assume the atlas is fixed and hence we do not have to worry about the dependence of $h_{Y, \partial\Omega}$ on the atlas.

If $u \in W_2^{\tau}(\Omega)$, then $u \in W_2^{\tau-1/2}(\partial\Omega)$ and if $\tau > d/2$, then this together with the Sobolev embedding theorem guarantees that u is continuous on the boundary $\partial\Omega$.

The proof of [9, Theorem 3.10] implies the following result. We give an extended version which also deals with non-integer orders η . Its proof can be found in [22].

Lemma 3.5. *Let $1 < r < \infty$ and $\tau = k + s > d/2$, where $k \in \mathbb{N}_0$ and $0 < s \leq 1$. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain having a $C^{k,s}$ smooth boundary. Assume that*

$Y \subseteq \partial\Omega$ with $h_{Y,\partial\Omega}$ sufficiently small. Then there is a constant $c > 0$ such that for all $\mathbf{u} \in \mathbf{H}^\tau(\Omega)$ with $\mathbf{u}|_Y = 0$ we have for $0 \leq \eta \leq \tau - 1/2 - (d-1)(1/2 - 1/r)_+$ that

$$\|\mathbf{u}\|_{\mathbf{W}_r^\eta(\partial\Omega)} \leq ch_{Y,\partial\Omega}^{\tau-1/2-\eta-(d-1)(1/2-1/r)_+} \|\mathbf{u}\|_{\mathbf{H}^\tau(\Omega)}.$$

Now, the same procedure as the one employed in the proof of Proposition 3.4 leads to the following result.

Proposition 3.6. *Let $d = 2, 3$. Assume that Ω , K and \mathbf{f}, \mathbf{g} satisfy the smoothness assumptions of Proposition 3.4. Suppose that the kernel Φ is chosen such that $\mathcal{N}_\Phi(\mathbb{R}^d) = \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^{\tau+1}(\mathbb{R}^d)$ with $\tau > d/2$. Then*

$$\begin{aligned} & \|(\mathbf{u} - \mathbf{s}_u) \cdot \mathbf{n}\|_{W_r^{\eta+1-1/r}(\partial\Omega)} \\ & \leq Ch_{Y,\partial\Omega}^{\tau-\eta-1-1/2+1/r-(d-1)(1/2-1/r)_+} (\|\mathbf{f}\|_{\mathbf{H}^\tau(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{H^{\tau+1/2}(\partial\Omega)}) \end{aligned}$$

with $C > 0$ independent of \mathbf{u} and \mathbf{s}_u , where $1 < r < \infty$ and $0 \leq \eta \leq \tau - 1/2 - (d-1)(1/2 - 1/r)_+ - 1 + 1/r$.

Proof. First, since the boundary of Ω is $C^{[\tau]+1,1}$ it is also $C^{k,s}$ with $\tau + 1 = k + s$, $k \in \mathbb{N}_0$ and $0 < s \leq 1$ by the embedding theorem for Hölder spaces.

The domain Ω has a $C^{[\tau]+1,1}$ boundary, therefore the normals $\mathbf{n} \in \mathbf{C}^{[\tau],1}(\partial\Omega)$ exist almost everywhere and can be extended to a vector field $\tilde{\mathbf{n}} \in \mathbf{C}^{[\tau],1}(\bar{\Omega})$ with $\tilde{\mathbf{n}}|_{\partial\Omega} = \mathbf{n}$. This means that $\mathbf{n} \in \mathbf{H}^{[\tau]}(\partial\Omega)$ and $\tilde{\mathbf{n}} \in \mathbf{H}^{[\tau]+1/2}(\Omega)$.

This enables us to apply Lemma 3.5 to derive

$$\|(\mathbf{u} - \mathbf{s}_u) \cdot \mathbf{n}\|_{W_r^{\eta+1-1/r}(\partial\Omega)} \leq Ch_{Y,\partial\Omega}^{\tau-\eta-1-1/2+1/r-(d-1)(1/2-1/r)_+} \|(\mathbf{u} - \mathbf{s}_u) \cdot \tilde{\mathbf{n}}\|_{H^\tau(\Omega)}.$$

Then

$$\|(\mathbf{u} - \mathbf{s}_u) \cdot \tilde{\mathbf{n}}\|_{H^\tau(\Omega)} \leq \|\tilde{\mathbf{n}}\|_{\mathbf{H}^\tau(\Omega)} \|\mathbf{u} - \mathbf{s}_u\|_{\mathbf{H}^\tau(\Omega)} \leq C \|\mathbf{u} - \mathbf{s}_u\|_{\mathbf{H}^\tau(\Omega)}$$

and, according to the proof of Proposition 3.4, we also have

$$\|\mathbf{u} - \mathbf{s}_u\|_{\mathbf{H}^\tau(\Omega)} \leq (\|\mathbf{f}\|_{\mathbf{H}^\tau(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{H^{\tau-1/2}(\partial\Omega)});$$

our proof is complete. \square

After estimating both parts in (3.4) we have proven our main result.

Theorem 3.7. *Let Ω be a bounded, simply connected, open subset of \mathbb{R}^d , $d = 2, 3$, with a $C^{[\tau]+1,1}$ boundary $\partial\Omega$. Suppose that Φ is chosen such that its native space is $\mathcal{N}_\Phi(\mathbb{R}^d) = \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^{\tau+1}(\mathbb{R}^d)$ and the permeability tensor $K = K_{ij}$ satisfies (1.6), $K = K^T$ and $K_{ij} \in H^{\tau+1}(\bar{\Omega})$. Furthermore, assume that the data satisfy $\mathbf{f} \in \mathbf{H}^{\tau+1}(\Omega)$, $\mathbf{g} \in \mathbf{H}^{\tau+1/2}(\partial\Omega)$ and $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} dS = 0$, where $\tau > d/2$. Then, the error between the true solution and the collocation approximation can be bounded by*

$$\begin{aligned} & \|\mathbf{u} - \mathbf{s}_u\|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \|p - s_p\|_{W_r^{\eta+2}(\Omega)/\mathbb{R}} \\ & \leq C \left(h_{X,\Omega}^{\tau-\eta-1-d(1/2-1/r)_+} + h_{Y,\partial\Omega}^{\tau-\eta-1-1/2+1/r-(d-1)(1/2-1/r)_+} \right) \\ & \quad \times (\|\mathbf{f}\|_{\mathbf{H}^\tau(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{H^{\tau-1/2}(\partial\Omega)}) \end{aligned}$$

for $1 < r < \infty$ and $0 \leq \eta \leq \tau - d(1/2 - 1/r)_+ - 1$. If $r \geq 2$ and $h = h_{X,\Omega} \approx h_{Y,\partial\Omega}$, this reduces to

$$\begin{aligned} & \|\mathbf{u} - \mathbf{s}_u\|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \|p - s_p\|_{W_r^{\eta+2}(\Omega)/\mathbb{R}} \\ & \leq Ch^{\tau-\eta-1-d(1/2-1/r)} (\|\mathbf{f}\|_{\mathbf{H}^\tau(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{H^{\tau-1/2}(\partial\Omega)}). \end{aligned}$$

4. NUMERICAL EXAMPLES

We will test the theoretical error estimates for two numerical examples. Our first example will deal with homogeneous and isotropic permeability, where K reduces to a constant times the identity matrix. Our second example will deal with inhomogeneous but still isotropic material.

We will employ Wendland functions $\phi_{2,\ell} \in C^{2\ell}(\mathbb{R}^2)$ for both ϕ and ψ , which generate Sobolev spaces $H^{\ell+3/2}(\mathbb{R}^2)$. Thus by choosing $\phi = \psi = \phi_{2,\ell}$, we have

$$\mathcal{N}_{\Phi}(\mathbb{R}^d) = \mathbf{H}^{\ell+1/2}(\mathbb{R}^d; \text{div}) \times H^{\ell+3/2}(\mathbb{R}^d),$$

which means $\tau = \ell + 1/2$. To ensure that a sufficient amount of collocation points is in the support of the basis functions, we scale them with $\delta := 10$. Moreover, since the error estimates only exist for the case $\phi = \psi$, we choose $\phi = \psi = \phi_{2,\ell}(\frac{\cdot}{\delta})$ for all numerical examples. We will concentrate on the L_∞ and L_2 error only. Hence, we want to verify the estimates

$$\begin{aligned} \|\mathbf{u} - \mathbf{s}_{\mathbf{u}}\|_{\mathbf{H}^k(\Omega)} + \|p - s_p\|_{H^{k+1}(\Omega)/\mathbb{R}} &\leq C_{\mathbf{f},\mathbf{g}} h^{\tau-k} = C_{\mathbf{f},\mathbf{g}} h^{\ell+\frac{1}{2}-k}, \\ \|\mathbf{u} - \mathbf{s}_{\mathbf{u}}\|_{\mathbf{W}_\infty^k(\Omega)} + \|p - s_p\|_{W_\infty^{k+1}(\Omega)/\mathbb{R}} &\leq C_{\mathbf{f},\mathbf{g}} h^{\tau-k-d/2} = C_{\mathbf{f},\mathbf{g}} h^{\ell-\frac{1}{2}-k}. \end{aligned}$$

Note that the first estimate was only shown for $k \geq 1$ in Theorem 3.7. Moreover, the second estimate is actually not justified by our theoretical analysis.

In all cases the notation $\mathbf{e}_{\mathbf{u}} = \mathbf{u} - \mathbf{s}_{\mathbf{u}}$ and $e_p = p - s_p$ is used. The numerical tests were run on a sequence of equidistant grids $X_n = (\frac{1}{n}\mathbb{Z})^2 \cap \Omega$. The computational approximation orders are derived by

$$\frac{\log(e_n/e_{n+1})}{\log(n/(n+1))},$$

where e_n is the error on an $n \times n$ input grid.

4.1. Homogeneous permeability. In our first example, we choose $\Omega = [0, 1]^2$ and $K = I$ and \mathbf{f} and \mathbf{g} such that the true solution is given by

$$\mathbf{u}(x, y) = (-2x^3y, 3x^2y^2)^T, \quad p(x, y) = x^3y^2.$$

We tested this for a variety of basis functions as explained above. The error has been computed using discretized versions of the various norms on a fine 300×300 grid. From the classical stability analysis for radial basis function interpolation (see [14, 18, 6]) we have to expect ill-conditioned matrices. Hence, to avoid any negative influence from such ill-conditioning, as a precaution, all computations were done using quad double precision.

The results are presented in Tables 1 to 4 and in Figure 1. They indicate that the numerical approximation orders more than match the theoretical ones.

TABLE 1. Approximation errors with $\phi = \psi = \phi_{2,2}$.

n	$\ \mathbf{e}_{\mathbf{u}}\ _{\mathbf{L}_2}$	$\ \mathbf{e}_{\mathbf{u}}\ _{\mathbf{L}_\infty}$	$\ \mathbf{e}_{\mathbf{u}}\ _{\mathbf{H}^1}$	$\ \mathbf{e}_{\mathbf{u}}\ _{\mathbf{H}^2}$	$\ \nabla e_p\ _{\mathbf{L}_2}$	$\ \nabla e_p\ _{\mathbf{L}_\infty}$
4	1.6335e-01	7.7583e-01	1.1289e+00	6.8905e+00	2.9920e-01	2.1715e+00
8	2.9333e-02	2.1230e-01	4.6740e-01	5.3876e+00	4.9407e-02	6.9474e-01
16	4.7724e-03	5.5458e-02	1.6321e-01	3.7585e+00	6.9639e-03	1.8473e-01
32	6.5486e-04	1.3832e-02	4.7138e-02	2.2668e+00	8.8743e-04	4.4247e-02
64	7.9729e-05	3.0498e-03	1.2110e-02	1.2395e+00	1.0140e-04	9.2248e-03

TABLE 2. Approximation orders with $\phi = \psi = \phi_{2,2}$.

	$\ \mathbf{e}_u\ _{L_2}$	$\ \mathbf{e}_u\ _{L_\infty}$	$\ \mathbf{e}_u\ _{H^1}$	$\ \mathbf{e}_u\ _{H^2}$	$\ \nabla e_p\ _{L_2}$	$\ \nabla e_p\ _{L_\infty}$
computed	2.4774	1.8697	1.2722	0.3550	2.5983	1.6442
	2.6197	1.9366	1.5179	0.5195	2.8267	1.9111
	2.8655	2.0033	1.7918	0.7295	2.9722	2.0618
	3.0380	2.1813	1.9606	0.8710	3.1296	2.2620
estimated	2.5	1.5	1.5	0.5	2.5	1.5

TABLE 3. Approximation errors with $\phi = \psi = \phi_{2,3}$.

n	$\ \mathbf{e}_u\ _{L_2}$	$\ \mathbf{e}_u\ _{L_\infty}$	$\ \mathbf{e}_u\ _{H^1}$	$\ \mathbf{e}_u\ _{H^2}$	$\ \nabla e_p\ _{L_2}$	$\ \nabla e_p\ _{L_\infty}$
4	1.0127e-01	3.6764e-01	7.0026e-01	4.1830e+00	2.1525e-01	1.4904e+00
8	8.6886e-03	4.3353e-02	1.4323e-01	1.7673e+00	1.2082e-02	1.8352e-01
16	6.6247e-04	5.3868e-03	2.4002e-02	6.3930e-01	7.6629e-04	2.0568e-02
32	4.0582e-05	5.8268e-04	3.0337e-03	1.7604e-01	4.1610e-05	1.7658e-03
64	2.2916e-06	5.6109e-05	3.3587e-04	4.0901e-02	2.1316e-06	8.3619e-05

TABLE 4. Approximation orders $\phi = \psi = \phi_{2,3}$.

	$\ \mathbf{e}_u\ _{L_2}$	$\ \mathbf{e}_u\ _{L_\infty}$	$\ \mathbf{e}_u\ _{H^1}$	$\ \mathbf{e}_u\ _{H^2}$	$\ \nabla e_p\ _{L_2}$	$\ \nabla e_p\ _{L_\infty}$
computed	3.5430	3.0841	2.2896	1.2430	4.1550	3.0217
	3.7132	3.0086	2.5771	1.4670	3.9788	3.1575
	4.0289	3.2087	2.9840	1.8606	4.2029	3.5420
	4.1464	3.3764	3.1751	2.1057	4.2869	4.4003
estimated	3.5	2.5	2.5	1.5	3.5	2.5

TABLE 5. Approximation errors with $\phi = \psi = \phi_{2,4}$.

n	$\ \mathbf{e}_u\ _{L_2}$	$\ \mathbf{e}_u\ _{L_\infty}$	$\ \mathbf{e}_u\ _{H^1}$	$\ \mathbf{e}_u\ _{H^2}$	$\ \nabla e_p\ _{L_2}$	$\ \nabla e_p\ _{L_\infty}$
4	6.1153e-02	2.0845e-01	4.3615e-01	2.8112e+00	2.0304e-01	1.2941e+00
8	2.3251e-03	1.1990e-02	4.1046e-02	5.5070e-01	3.2751e-03	5.0862e-02
16	7.9533e-05	7.1673e-04	3.0418e-03	9.1047e-02	9.5225e-05	2.7176e-03
32	2.3199e-06	3.8217e-05	1.7927e-04	1.1659e-02	2.4788e-06	1.0642e-04
64	7.4599e-08	1.1181e-06	1.2205e-05	1.6011e-03	1.0738e-07	3.4691e-06

TABLE 6. Approximation orders with $\phi = \psi = \phi_{2,4}$.

	$\ \mathbf{e}_u\ _{L_2}$	$\ \mathbf{e}_u\ _{L_\infty}$	$\ \mathbf{e}_u\ _{H^1}$	$\ \mathbf{e}_u\ _{H^2}$	$\ \nabla e_p\ _{L_2}$	$\ \nabla e_p\ _{L_\infty}$
computed	4.7171	4.1198	3.4095	2.3518	5.9541	4.6693
	4.8696	4.0643	3.7543	2.5966	5.1040	4.2262
	5.0994	4.2291	4.0847	2.9652	5.2636	4.6745
	4.9588	5.0951	3.8765	2.8643	4.5289	4.9390
estimated	4.5	3.5	3.5	2.5	4.5	3.5

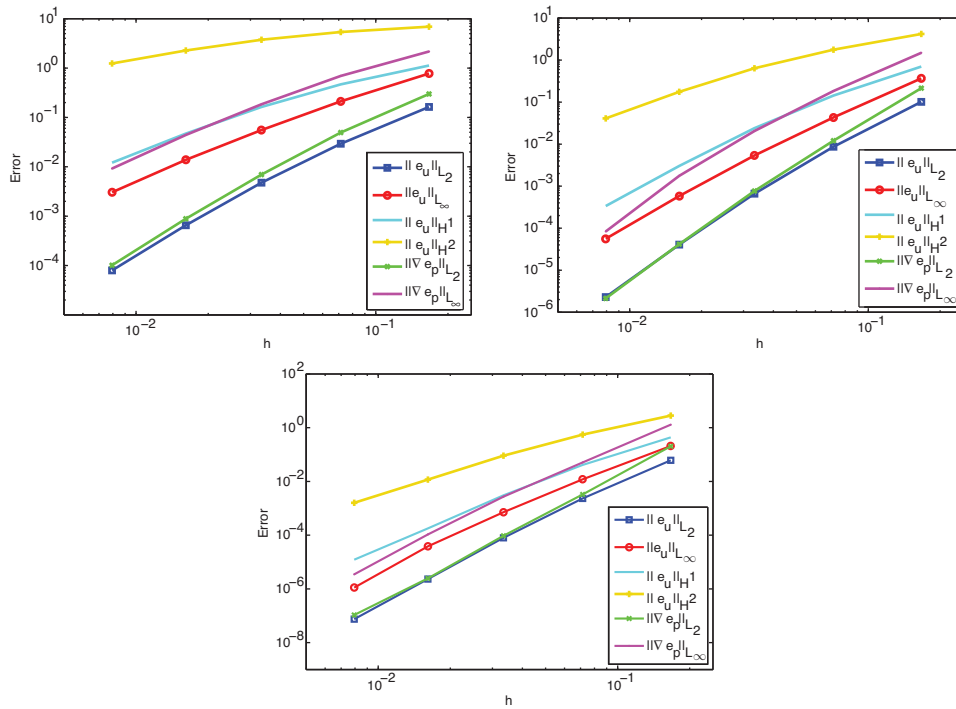


FIGURE 1. Approximation orders for $\phi = \psi = \phi_{2,2}$ (upper left), $\phi = \psi = \phi_{2,3}$ (upper right) and $\phi = \psi = \phi_{2,4}$.

4.2. Inhomogeneous permeability. Our second example deals with inhomogeneous and isotropic material, meaning that $K = \kappa I$ with a non-constant function κ . Our example is motivated by a similar example from [20] and describes the flow through a two-dimensional cylinder with varying permeability. To be more precise, pressure, velocity and permeability are given by

$$\begin{aligned}
 p(x, y) &= \frac{p_1 - p_0}{L}x + p_0, \\
 \mathbf{u}(x, y) &= \left(\frac{p_0 - p_1}{L\mu}(y - y_a)(y - y_b), 0 \right)^T, \\
 \kappa(x, y) &= (y - y_a)(y - y_b),
 \end{aligned}$$

where μ is the viscosity and L the length of the cylinder. Obviously, these quantities satisfy (1.1) and $\nabla \cdot \mathbf{u} = 0$. The permeability is constant along horizontal lines and is zero at the top and bottom boundary of the cylinder. The flow is also horizontal; see Figure 2.

For our computations, we set $L = 1$, $y_a = 0$, $y_b = 1$, $\mu = 1$, $p_1 = 2$ and $p_0 = 1$. For $\phi = \psi$ we have chosen the C^6 compactly supported function.

The results are represented in Tables 7 and 8 and in Figure 3.

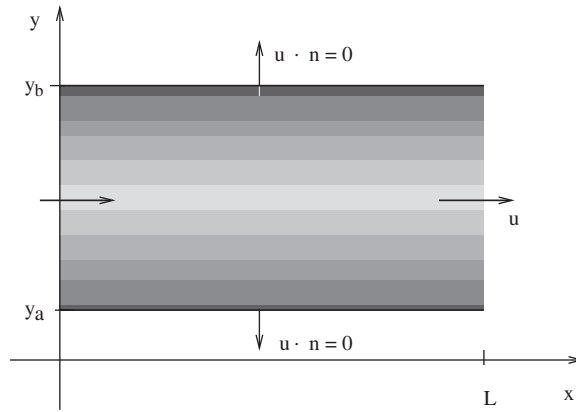


FIGURE 2. The schematic set up for the second example.

TABLE 7. Approximation errors for the inhomogeneous problem

n	$\ e_u\ _{L_2}$	$\ e_u\ _{L_\infty}$	$\ e_u\ _{H^1}$	$\ e_u\ _{H^2}$	$\ \nabla e_p\ _{L_2}$	$\ \nabla e_p\ _{L_\infty}$
4	6.2920e-03	2.3987e-02	5.2479e-02	5.1676e-01	4.9587e-02	1.2463e-01
8	5.3211e-04	4.6569e-03	8.8350e-03	1.3797e-01	2.2983e-03	9.9190e-03
16	4.0233e-05	5.5241e-04	1.4920e-03	4.5095e-02	1.8430e-04	1.4879e-03
32	2.9049e-06	5.3613e-05	2.3020e-04	1.3925e-02	1.7299e-05	2.4952e-04
64	1.8260e-07	4.1769e-06	3.0837e-05	3.8451e-03	1.5696e-06	3.8521e-05

TABLE 8. Approximation orders for the inhomogeneous problem.

	$\ e_u\ _{L_2}$	$\ e_u\ _{L_\infty}$	$\ e_u\ _{H^1}$	$\ e_u\ _{H^2}$	$\ \nabla e_p\ _{L_2}$	$\ \nabla e_p\ _{L_\infty}$
computed	3.5637	2.3648	2.5705	1.9052	4.4313	3.6513
	3.7253	3.0756	2.5660	1.6133	3.6404	2.7370
	3.7918	3.3651	2.6963	1.6953	3.4133	2.5760
	3.9917	3.6821	2.9001	1.8566	3.4623	2.6954
estimated	3.5	2.5	2.5	1.5	3.5	2.5

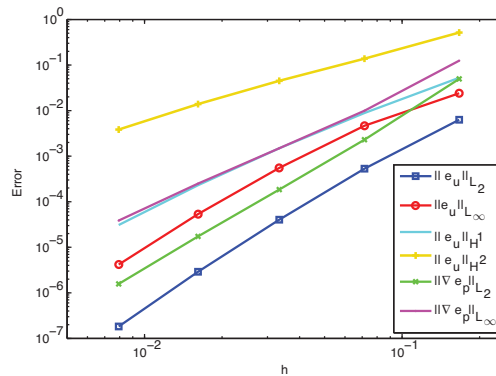


FIGURE 3. Convergence for the inhomogeneous example.

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