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LOWER BOUNDS FOR THE ARTIN CONDUCTOR

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Abstract. In this paper we improve on Odlyzko's lower bounds for the Artin conductor.

1. Introduction

Let K be an algebraic number field such that $K/\mathbb{Q}$ is Galois and let χ be the character of an n-dimensional linear representation of $\mathcal{G} = \mathrm{Gal}(K/\mathbb{Q})$. Assuming the analyticity of the Artin L-function $L(s,\chi)$, Odlyzko ([4], p. 382) proved that the Artin conductor f_χ ([3], p. 540) satisfies the lower bound

$$f_\chi \geq (3.70)^{a_\chi}(2.38)^{b_\chi},$$

where a_χ and b_χ are nonnegative integers giving the Γ-factors of the completed Artin L-function. Namely, $a_\chi + b_\chi = \chi(1) = n$, $a_\chi - b_\chi = \chi(g_0)$, with $g_0 \in \mathcal{G}$ a complex conjugation. In §3 we use Weil's explicit formulas, as simplified by Mestre [1], to improve these bounds to

$$f_\chi \geq (4.90)^{a_\chi}(2.91)^{b_\chi}.$$

These bounds are nearly best possible. Indeed, the quadratic field $\mathbb{Q}(\sqrt{5})$ has a character χ_5 with $\chi_5(1) = 1$, $a_\chi = 1$, $b_\chi = 0$ and $f_{\chi_5} = 5$. The quadratic field $\mathbb{Q}(\sqrt{-3})$ has a character χ_3 with $\chi_3(1) = 1$, $a_\chi = 0$, $b_\chi = 1$ and $f_{\chi_3} = 3$. Thus, if χ is the (in general, reducible) character $\chi := a\chi_5 + b\chi_3$ (for arbitrary non-negative integers a and b), then

$$f_\chi = 5^a\, 3^b,$$

with $a = a_\chi$ and $b = b_\chi$.

For irreducible characters, and assuming the Artin conjecture for $\chi\overline{\chi}$, Odlyzko ([4], p. 385) was able to improve his bounds to

$$f_\chi^{1/n} \geq 4.71(1.645)^{\frac{(a_\chi - b_\chi)^2}{n^2}} + O(1/n^2), \quad \text{as } n \to \infty.$$

Odlyzko also gave lower bounds for small degrees of n. Using the explicit formulas we are able to improve Odlyzko's bounds only slightly for large n, namely

$$f_\chi^{1/n} \geq 4.73(1.648)^{\frac{(a_\chi - b_\chi)^2}{n^2}} e^{-(13.34/n)^2},$$

but we make considerable improvements on the lower bounds for small n (see Table 1 below).

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The above improvements on Odlyzko's bounds are directly inspired by the story of lower bounds for the discriminant D_K of a number field K. Odlyzko's complicated analytical method can be greatly simplified, as Serre [7] noted, by the use of explicit formulas. This was developed by Odlyzko himself and by Poitou and collaborators [5] and [6]. In SS2–4 we adapt these methods to Artin L-functions.

In §5 we introduce a new idea into Odlyzko's method, rather than just cleaning up his techniques using the explicit formulas. The main observation is that the first method (valid for all characters) does not yield good lower bounds only because the primes may contribute negative terms. When we pass to irreducible characters and consider $\chi\overline{\chi}$, the primes always contribute positively. However, without further information on the primes, we have to drop these terms. Our idea is to consider simultaneously both inequalities and remark that we need not take the worst possible case in both methods. If the primes hurt us (that is, amount to a negative term) in the first method, then they exist and will help us in the second one. It turns out that this simple idea yields substantial improvements whenever $a_\chi \neq 0$ (see Table 4 for small degrees). In particular, in Corollaries 5.2 and 5.3 we obtain,

$$f_\chi^{1/n} \geq 9.482\, e^{-10.359/n},$$

for $a_\chi = n$, and

$$f_\chi^{1/n} \geq 5.542\, e^{-16.859/n},$$

for $a_\chi = b_\chi$. This improves on the lower bounds

$$f_\chi^{1/n} \geq 7.797\, e^{-(13.34/n)^2}$$

and

$$f_\chi^{1/n} \geq 4.73\, e^{-(13.34/n)^2},$$

respectively, from §4.

2. Explicit formulas and Odlyzko's method

2.1. **Mestre's explicit formulas.** Let K be an algebraic number field. Suppose that $K/\mathbb{Q}$ is Galois, χ is a linear character of $\mathcal{G} = \mathrm{Gal}(K/\mathbb{Q})$ and f_χ is its Artin conductor ([3], p. 527). Let us define the completed Artin L-function by

$$\Lambda(s,\chi) = \Big(\frac{f_\chi}{\pi^{\chi(1)}}\Big)^{s/2}\Gamma\Big(\frac{s}{2}\Big)^{a_\chi}\Gamma\Big(\frac{s+1}{2}\Big)^{b_\chi}L(s,\chi), \tag{2.1}$$

where $L(s,\chi)$ is the Artin L-function associated to χ with base field $\mathbb{Q}$, a_χ and b_χ are integers such that

$$a_\chi + b_\chi = \chi(1), \qquad a_\chi - b_\chi = \chi(g_0), \tag{2.2}$$

with 1 the identity element of $\mathcal{G}$ and $g_0 \in \mathcal{G}$ a complex conjugation ([3], pp. 522 and 540). This function verifies the functional equation ([3], p. 540)

$$\Lambda(1-s,\overline{\chi}) = W(\chi)\Lambda(s,\chi), \tag{2.3}$$

where $W(\chi) \in \mathbb{C}$ is such that $|W(\chi)| = 1$ and $\overline{\chi}$ is the character of the dual (or contragredient) representation of χ.

We will need Mestre's form ([1], pp. 212–213) of Weil's explicit formulas for rather general L-functions. We assume our L-functions L_i have Euler products of the type

$$L_1(s) = \prod_p \prod_{i=1}^{M'} \left(1 - \alpha_{i,p}\, p^{-s}\right)^{-1},$$
$$L_2(s) = \prod_p \prod_{i=1}^{M'} \left(1 - \beta_{i,p}\, p^{-s}\right)^{-1},$$

where p runs over the prime numbers and $\alpha_{i,p}, \beta_{i,p}$ are complex numbers such that

$$|\alpha_{i,p}|, |\beta_{i,p}| \le p^c. \tag{2.4}$$

For positive real numbers A, B, a_i and a'_i $(1 \le i \le M)$ such that $\sum_{i=1}^M a_i = \sum_{i=1}^M a'_i$ and complex numbers b_i and b'_i, with $\mathrm{Re}(b_i) \ge 0$ and $\mathrm{Re}(b'_i) \ge 0$, we consider meromorphic functions

$$\Lambda_1(s) = A^s L_1(s) \prod_{i=1}^M \Gamma(a_i s + b_i),$$
$$\Lambda_2(s) = B^s L_2(s) \prod_{i=1}^M \Gamma(a'_i s + b'_i),$$

verifying by assumption that

$$\Lambda_1(1-s) = \omega \Lambda_2(s),$$

for some $\omega \in \mathbb{C}^*$.

Let $F : \mathbb{R} \to \mathbb{R}$ be a function such that:

(1) There exists $\varepsilon > 0$ such that $F(x)e^{(\frac{1}{2}+c+\varepsilon)x}$ is integrable over $\mathbb{R}$, with $c \ge 0$ satisfying (2.4) above.

(2) There exists $\varepsilon > 0$ such that $F(x)e^{(\frac{1}{2}+c+\varepsilon)x}$ is of bounded variation, the value at each point being the average of the right- and left-hand limits.

(3) The function $\frac{F(x)-F(0)}{x}$ is of bounded variation.

We define the Mellin transform of F by

$$\phi(s) = \int_{-\infty}^{\infty} F(x) e^{(s-\frac{1}{2})x} dx, \qquad (-\epsilon < \mathrm{Re}(s) < 1+\epsilon). \tag{2.5}$$

Define[1]

$$I(a,b) = a \int_0^\infty \Big(\frac{F(ax)e^{-(\frac{a}{2}+b)x}}{1-e^{-x}} - \frac{F(0)e^{-x}}{x}\Big) dx$$

and

$$J(a,b) = a \int_0^\infty \Big(\frac{F(-ax)e^{-(\frac{a}{2}+b)x}}{1-e^{-x}} - \frac{F(0)e^{-x}}{x}\Big) dx.$$

[1] There is a slight misprint in the definition of $I(a,b)$ in ([1], p. 212), where $f(ax)$ appears instead of $F(ax)$.

For a function F verifying the conditions (1)–(3), Mestre obtained the following explicit formula ([1], pp. 212–213):

(2.6)
$$\sum_{\rho}\phi(\rho)-\sum_{\mu}\phi(\mu)+\sum_{i=1}^{M}I(a_i,b_i)+\sum_{i=1}^{M}J(a_i',b_i')$$
$$=F(0)\log(AB)-\sum_{i=1}^{M'}\sum_{p\ \text{prime}}\sum_{m=1}^{\infty}\big(\alpha_{i,p}^{m}F(m\log p)+\beta_{i,p}^{m}F(-m\log p)\big)\frac{\log p}{p^{m/2}},$$

where ρ and μ run respectively over all zeros and poles of Λ_1 (counted according to their multiplicity) in the vertical strip $\big\{s\in\mathbb{C}\big|-c\le \mathrm{Re}(s)\le 1+c\big\}$.

We will apply Mestre's formula as follows. Let χ be any character of $\mathcal{G}$. If $\rho:\mathcal{G}\longrightarrow \mathrm{GL}(V)$ with V a $\mathbb{C}$-vector space, is the representation associated to χ, β is any prime ideal of K over p and φ_β is a corresponding Frobenius automorphism, we can write the Artin L-function as a product of Euler factors for each prime as

$$L(s,\chi)=\prod_{p\ \text{prime}}\big(\det(\mathrm{Id}-p^{-s}\rho(\varphi_\beta);V^{I_\beta})\big)^{-1},$$

where V^{I_β} is the subspace of invariants in V under the inertia group I_β. If $\lambda_{1,p},\dots,\lambda_{m_p,p}$ are the eigenvalues of $\rho(\varphi_\beta)$ acting on V^{I_β}, then $m_p\le n=\chi(1)$ and

$$\det(\mathrm{Id}-p^{-s}\rho(\varphi_\beta);V^{I_\beta})=\prod_{i=1}^{m_p}\big(1-p^{-s}\lambda_{i,p}\big)=\prod_{i=1}^{n}\big(1-p^{-s}\lambda_{i,p}\big),$$

where we have put $\lambda_{i,p}=0$ if $n\ge i>m_p$. Thus,

$$L(s,\chi)=\prod_{p}\prod_{i=1}^{n}\big(1-p^{-s}\lambda_{i,p}\big)^{-1}. \tag{2.7}$$

In Mestre's formula take

$$\begin{aligned} L_1(s)&=L(s,\chi), \qquad & L_2(s)&=L(s,\overline{\chi}),\\ \Lambda_1(s)&=\Lambda(s,\chi), \qquad & \Lambda_2(s)&=\Lambda(s,\overline{\chi}), \end{aligned} \tag{2.8}$$

with Λ the completed Artin L-function in (2.1). Note that $|\lambda_{i,p}|\le 1$, because $\mathcal{G}$ is a finite group. Take $\alpha_{i,p}=\lambda_{i,p}$, so that $\alpha_{i,p}=\overline{\beta_{i,p}}$ and $c=0$ in (2.4). As $\lambda_{i,p}^{m}$ is an eigenvalue of $\rho(\varphi_\beta^m)$, if we denote by $\chi(p^m)$ the character χ evaluated on φ_β^m acting on V^{I_β}, we have

$$\chi(p^m)=\sum_{i=1}^{n}\lambda_{i,p}^{m}, \qquad 2\mathrm{Re}\big(\chi(p^m)\big)=\sum_{i=1}^{n}\big(\alpha_{i,p}^{m}+\beta_{i,p}^{m}\big). \tag{2.9}$$

We also take

$$\begin{aligned} &M'=M=n=\chi(1)=a_\chi+b_\chi, \qquad a_i'=a_i=1/2\quad\text{for } 1\le i\le n,\\ &b_i'=b_i=0\quad \text{for } 1\le i\le a_\chi, \qquad b_i'=b_i=1/2\quad\text{for }\ a_\chi+1\le i\le n, \end{aligned}$$

and

$$A=\Big(\frac{f_\chi}{\pi^{\chi(1)}}\Big)^{1/2}, \qquad B=\Big(\frac{f_{\overline{\chi}}}{\pi^{\overline{\chi}(1)}}\Big)^{1/2}.$$

Actually,

$$A=B, \tag{2.10}$$

because $f_\chi = f_{\overline{\chi}}$ and $\chi(1) = n = \overline{\chi}(1)$. Here is an analytic proof of $f_\chi = f_{\overline{\chi}}$. Take absolute values of both sides of the functional (2.3) for $s = \frac{1}{2} + it$ and $t \in \mathbb{R}$ such that $L(s,\chi) \neq 0$, to get

$$\left|\frac{f_\chi}{f_{\overline{\chi}}}\right|^{1/4} = \left|\frac{\overline{L(\frac{1}{2}+it,\chi)}}{L(\frac{1}{2}+it,\chi)}\right| = 1.$$

Here we used $\overline{L(s,\chi)} = L(\overline{s},\overline{\chi})$ and $|W(\chi)| = 1$. As the conductor is a positive integer, we conclude that $f_\chi = f_{\overline{\chi}}$.

A more satisfactory arithmetic proof of this same fact can be carried out as follows. If ψ is any representation of $\mathcal{G}$ let $f_\psi = \prod_{p\nmid\infty} p^{f_p(\psi)}$. If G_j is the j-th ramification group at p in the lower numbering, we have for each prime p,

$$f_p(\psi) = \frac{1}{|G_0|}\sum_{j\geq 0} |G_j|\psi(1) - \psi(G_j),$$

where $\psi(G_j) = \sum_{g\in G_j}\psi(g)$ (see [3], pp. 528–530). Since $f_p(\psi)$ is a real number, we have $f_p(\psi) = \overline{f_p(\psi)} = f_p(\overline{\psi})$ for each prime p.

Let us now assume that the function F in Mestre's formula (2.6) verifies $F(-x) = F(x)$ and $F(0) = 1$. Thus $I(a,b) = J(a,b)$, since F is even, and

$$\sum_{i=1}^{n} I(a_i,b_i) + \sum_{j=1}^{n} J(a'_i,b'_i) = 2\sum_{i=1}^{n} I(a_i,b_i) = 2a_\chi I(\tfrac{1}{2},0) + 2b_\chi I(\tfrac{1}{2},\tfrac{1}{2}),$$

where

$$\begin{aligned} I(\tfrac{1}{2},0) &= \frac{1}{2}\int_0^\infty \Big(\frac{e^{-x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx\,, \\ I(\tfrac{1}{2},\tfrac{1}{2}) &= \frac{1}{2}\int_0^\infty \Big(\frac{e^{-3x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx\,. \end{aligned} \tag{2.11}$$

Finally, if $(s-1)^r L(s,\chi)$ is entire, with r being exactly the order of the pole at $s = 1$, from (2.6) to (2.11), we obtain the explicit formula

$$\begin{aligned} \log f_\chi &= \sum_\rho \phi(\rho) - r\big(\phi(0)+\phi(1)\big) + \chi(1)\log(\pi) \\ &+ a_\chi\int_0^\infty \Big(\frac{e^{-x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx + b_\chi\int_0^\infty \Big(\frac{e^{-3x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx \\ &+ 2\sum_{p\text{ prime}}\sum_{m=1}^{\infty}\frac{\log(p)}{p^{m/2}}\mathrm{Re}\big(\chi(p^m)\big)F(m\log p), \end{aligned} \tag{2.12}$$

where $\phi(s)$ is like (2.5) and ρ runs over all the zeros of $\Lambda(s,\chi)$ in the critical strip $0 < \mathrm{Re}(\rho) < 1$.[2]

Remark 1. We shall obtain lower bounds for conductors by controlling the signs of various terms appearing in the explicit formula. For this we will have to impose sign conditions on F and its Mellin transform. On the other hand, since we only want an inequality, we may weaken slightly some of the analytic conditions.

[2] $L(s,\chi)$ has neither zeroes nor poles on the lines $\mathrm{Re}(s) = 0$ or $\mathrm{Re}(s) = 1$, except possibly at $s = 1$, where there may only be a pole. Its order is exactly the multiplicity of the trivial representation in χ (see [2], p. 6). This will be important when we consider $L(s,\chi\overline{\chi})$ in the last chapter.

Poitou-Mestre Hypothesis. Let $F:\mathbb{R}\to\mathbb{R}$ be a function such that:

(A) There exists $\varepsilon \geq 0$ such that $F(x)e^{(\frac{1}{2}+c+\varepsilon)x}$ is integrable over $\mathbb{R}$, with $c \geq 0$ satisfying (2.4). If $\varepsilon = 0$, assume in addition that

$$\sum_{p \text{ prime}} \sum_{m=1}^{\infty} \log(p) \frac{F\big(m\log(p)\big)}{p^{m/2}} < \infty.$$

(B) There exists $\varepsilon \geq 0$ such that $F(x)e^{(\frac{1}{2}+c+\varepsilon)x}$ is of bounded variation, the value at each point being the average of the right- and left-hand limits.

(C) The function $\frac{F(x)-F(0)}{x}$ is of bounded variation.

(D) F is even, $F(0)=1$, $F(x)\geq 0$ for $x \in \mathbb{R}$, and $\mathrm{Re}\big(\phi(s)\big)\geq 0$ for $0< \mathrm{Re}(s)< 1$.

The purpose of the last condition is to ensure that the contributions from the zeroes ρ are all nonnegative. In (A) and (B) we have weakened the conditions (1)–(3) by allowing $\varepsilon = 0$ (cf. Proposition 5 in [6]).

Under the Poitou-Mestre Hypothesis we then have

$$\begin{aligned}\log f_\chi \geq \chi(1)\log(\pi) &+ a_\chi \int_0^\infty \Big(\frac{e^{-x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx \\ &+ b_\chi \int_0^\infty \Big(\frac{e^{-3x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx - 4r\int_0^\infty F(x)\cosh(x/2)\,dx. \\ &+ 2\sum_{p \text{ prime}}\sum_{m=1}^{\infty} \frac{\log(p)}{p^{m/2}} \mathrm{Re}\big(\chi(p^m)\big)F(m\log p)\,.\end{aligned} \tag{2.13}$$

As Odlyzko pointed out (cf. [6]), the conditions of nonnegativity on $F(x)$, and on $\mathrm{Re}\big(\phi(s)\big)$ on the critical strip, are equivalent to the requirement that

$$F(x) = \frac{f(x)}{\cosh(x/2)}, \tag{2.14}$$

where $f(x)\geq 0$ and $f(x)$ has a nonnegative Fourier transform.

If we assume the Riemann Hypothesis for $L(s,\chi)$, we only have to ensure that $\mathrm{Re}\big(\phi(\frac{1}{2}+it)\big) \geq 0$ for all real t. In this case we will only need to assume that $F(x)\geq 0$ and that F has a nonnegative Fourier transform.

3. Bounds for arbitrary characters

A preliminary result is the following:

Theorem 3.1. *Suppose that χ is a character of $\mathcal{G}$ such that $Re\big(\chi(g)\big) \geq 0$ for all $g \in \mathcal{G}$ and that for some integer r, $(s-1)^r L(s,\chi)$ is entire. Then*

$$f_\chi \geq (6.5735)^{a_\chi}(3.9046)^{b_\chi}(0.1134)^r. \tag{3.1}$$

Proof. Consider the family of functions (introduced by L. Tartar [6])

$$F_y(x) = \frac{f(x\sqrt{y})}{\cosh(x/2)}, \tag{3.2}$$

where

$$f(x) = \frac{9\big(\sin(x) - x\cos(x)\big)^2}{x^6} \tag{3.3}$$

and $y > 0$ is a positive parameter. F_y satisfies the Poitou-Mestre Hypothesis (see [6]). In ([6], p. 13) it is shown that f has a nonnegative Fourier transform.[3] Since we have assumed $\mathrm{Re}\big(\chi(g)\big) \geq 0$ we may drop from inequality (2.13) the sum over the primes. Putting $a_\chi + b_\chi = \chi(1)$, $F = F_y$ and $y = 12$ in (2.13), yields numerically,

$$\log f_\chi \geq 1.88305\, a_\chi + 1.36216\, b_\chi - 2.17656\, r,$$

and this is equivalent to (3.1). □

In general $\mathrm{Re}(\chi)$ is not positive, so Theorem 3.1 does not apply. Nevertheless, following Odlyzko we can prove the following.

Theorem 3.2. *Let χ be a character of $\mathcal{G}$ such that its Artin L-function $L(s,\chi)$ is entire. Then its conductor f_χ satisfies*

$$f_\chi \geq (4.90)^{a_\chi}(2.91)^{b_\chi},$$

where a_χ (resp. b_χ) is the number of $\Gamma(\frac{s}{2})$ (resp. $\Gamma(\frac{1+s}{2})$) factors in the completed Artin L-function.

Proof. Consider the character

$$\widetilde{\chi} = \chi + \chi(1)\chi_0,$$

where χ_0 is the one-dimensional identity character. Since $L(s,\chi)$ is assumed to be entire, we see that $(s-1)^{\chi(1)}L(s,\widetilde{\chi})$ is entire. Indeed,

$$\begin{aligned} L(s,\chi+\chi(1)\chi_0) &= L(s,\chi)L(s,\chi(1)\chi_0) \\ &= L(s,\chi)L(s,\chi_0)^{\chi(1)} \\ &= L(s,\chi)\zeta(s)^{\chi(1)}, \end{aligned}$$

where $\zeta(s)$ is the Riemann-zeta function. Since $|\chi(g)| \leq \chi(1)$ for all $g \in \mathcal{G}$, we have

$$\mathrm{Re}\big(\widetilde{\chi}(g)\big) = \chi(1) + \mathrm{Re}\big(\chi(g)\big) \geq 0.$$

From the properties of the conductor ([3], p. 533),

$$f_{\widetilde{\chi}} = f_{\chi+\chi(1)\chi_0} = f_\chi f_{\chi(1)\chi_0} = f_\chi. \tag{3.4}$$

Also (see (2.1)),

$$a_{\widetilde{\chi}} = a_\chi + \chi(1) = 2a_\chi + b_\chi, \qquad b_{\widetilde{\chi}} = b_\chi, \qquad \widetilde{\chi}(1) = 2\chi(1).$$

Applying Theorem 3.1 to the character $\widetilde{\chi}$ we obtain

$$\begin{aligned} f_\chi &\geq (6.5735)^{(2a_\chi+b_\chi)}(3.9046)^{b_\chi}(0.1134)^{\chi(1)} \\ &= (6.5735)^{(2a_\chi+b_\chi)}(3.9046)^{b_\chi}(0.1134)^{(a_\chi+b_\chi)} \\ &> (4.90)^{a_\chi}(2.91)^{b_\chi}. \end{aligned}$$ □

[3] We note that there is an error in ([6], p. 13) concerning the normalization constant required to ensure $F_y(0) = 1$. There the 9 in (3.3) is incorrectly replaced by $4/\pi^2$.

3.1. **Contribution of zeros.** So far, we have not considered the positive contribution from the zeros in the explicit formulas. In general, we know almost nothing about the location of zeros of $L(s,\chi)$, but in the proof of Theorem 3.2 we introduced the Riemann zeta function and dropped the contribution from its zeros. If we restore the contribution from the lowest zeros $\rho_0 = \frac{1}{2} \pm i14.134725142$ of the Riemann-zeta function we gain $2\mathrm{Re}\,\phi_y(\rho_0)$, where ϕ_y is the Mellin transform of F_y. In this way we obtain, with $y = 10.35$,

$$f_\chi \geq (4.947)^{a_\chi}(2.833)^{b_\chi},$$

which is slightly better than Theorem 3.2 if a_χ is much larger than b_χ.

Another possibility is to take $y = 13.5$ to obtain, likewise,

$$f_\chi \geq (4.832)^{a_\chi}(2.95)^{b_\chi}.$$

With $y = 12$ we obtain a (minor) improvement for all a_χ and b_χ. Namely, under the hypotheses of Theorem 3.2,

$$f_\chi \geq (4.905)^{a_\chi}(2.913)^{b_\chi}.$$

4. Bounds for irreducible characters

We have seen that our results above are nearly optimal for arbitrary (i.e., possibly reducible) characters. In this section we again follow Odlyzko to obtain better lower bounds for irreducible characters. We will need the following lemma, valid for any character χ.

Lemma 4.1 (Odlyzko). *$f_{\chi\overline{\chi}}$ divides $f_\chi^{2(\chi(1)-1)}$.*

Proof (Odlyzko). Since the conductor f_χ is a product of local conductors $p^{f_p(\chi)}$ ([3], p. 532), we need to prove that

$$f_p(\chi\overline{\chi}) \leq 2(\chi(1)-1)f_p(\chi). \tag{4.1}$$

For this, we will show that for every subgroup H of $\mathcal{G}$ we have

$$|H|\chi(1)^2 - \chi\overline{\chi}(H) \leq 2(\chi(1)-1)\big(|H|\chi(1) - \chi(H)\big), \tag{4.2}$$

where $f(H) = \sum_{h\in H} f(h)$ and $|H|$ denotes the cardinality of H. We decompose

$$\chi|_H = r\phi_0 + \sum_{i\geq 1} r_i\phi_i, \tag{4.3}$$

where ϕ_0 is the trivial character of H, the ϕ_i are distinct, irreducible, nontrivial characters of H, and $r_i \geq 0$, $r \geq 0$. We have that

$$\chi(H) = r\sum_{h\in H}\phi_0(h) + \sum_{i\geq 1} r_i \sum_{h\in H}\phi_i(h)$$

and $\sum_{h\in H}\phi_0(h) = |H|$, and that $\sum_{h\in H}\phi_i(h) = 0$ (see [8], p. 17). Hence $\chi(H) = r|H|$. Also,

$$\begin{aligned}\chi\overline{\chi}|_H &= r^2\phi_0 + r\sum_{i\geq 1} r_i\overline{\phi_i} + r_1|\phi_1|^2 + r_1\sum_{i\neq 1} r_i\phi_1\overline{\phi}_i + r_2|\phi_2|^2 + r_2\sum_{i\neq 2} r_i\phi_2\overline{\phi}_i + \dots \\ &+ r_k^2|\phi_k|^2 + r_k\sum_{i\neq k} r_i\phi_k\overline{\phi}_i.\end{aligned}$$

Thus,

$$\chi\overline{\chi}(H) = r^2|H| + r\sum_{i\geq 1} r_i \sum_{h\in H} \overline{\phi}_i(h) + r_1^2 \sum_{h\in H} |\phi_1(h)|^2 + \cdots + r_k^2 \sum_{h\in H} |\phi_k(h)|^2,$$

and so $\chi\overline{\chi}(H) = \big(r^2 + \sum_{i\geq 1} r_i^2\big)|H|$. But (4.2) is equivalent to

$$\chi(1)^2 - r^2 - \sum_{i\geq 1} r_i^2 \leq 2(\chi(1)-1)(\chi(1)-r),$$

and to

$$-\sum_{i\geq 1} r_i^2 \leq (\chi(1)-r)(\chi(1)-r-2).$$

From (4.3), $\chi(1) = r + \sum_{i\geq 1} r_i\phi_i(1)$, so the last inequality is equivalent to

$$-\sum_{i\geq 1} r_i^2 \leq \big(\sum_{i\geq 1} r_i\phi_i(1)\big)\big(\sum_{i\geq 1} r_i\phi_i(1) - 2\big). \tag{4.4}$$

The right side is negative only if $\sum_{i\geq 1} r_i\phi_i(1) < 2$, and this can happen only if $r_j = \phi_j(1) = 1$, for some j and $r_i = 0$ if $i \neq j$. In this case we obtain equality in (4.4), so (4.2) is true.

Returning to the proof of the lemma, let G_j be the j-th ramification group in the lower numbering ([3], p. 528) associated to a prime of K above p. Then, from (4.1)and (4.2) we obtain

$$\begin{aligned} f_p(\chi\overline{\chi}) &= \frac{1}{|G_0|}\sum_{j\geq 0}\big(|G_j|\chi(1)^2 - \chi\overline{\chi}(G_j)\big) \\ &\leq \frac{1}{|G_0|}\sum_{j\geq 0} 2(\chi(1)-1)(|G_j|\chi(1) - \chi(G_j)) \\ &= 2(\chi(1)-1)\frac{1}{|G_0|}\sum_{j\geq 0}(|G_j|\chi(1) - \chi(G_j)) \\ &= 2(\chi(1)-1)f_p(\chi). \end{aligned}$$

□

If we take χ as an irreducible character and assume the Artin Conjecture for the (reducible) character $\chi\overline{\chi}$ of $\mathcal{G}$, then $(s-1)L(s,\chi\overline{\chi})$ is entire. Lemma 4.1 implies

$$f_{\chi\overline{\chi}} \leq f_\chi^{2(\chi(1)-1)}, \tag{4.5}$$

and therefore

$$f_{\chi\overline{\chi}}^{1/2\chi(1)} \leq f_\chi. \tag{4.6}$$

Now, applying (2.13) to the character $\chi\overline{\chi}$ with $r = 1$, we get

$$\log f_{\chi\overline{\chi}} \geq a_{\chi\overline{\chi}}(I_F(y) + \log(\pi)) + b_{\chi\overline{\chi}}(J_F(y) + \log(\pi)) - 4R_F(y) \tag{4.7}$$

where

$$I_F(y) := \int_0^\infty \Big(\frac{e^{-x/4}F_y(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx, \tag{4.8}$$

$$J_F(y) := \int_0^\infty \Big(\frac{e^{-3x/4}F_y(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx, \tag{4.9}$$

and

$$R_F(y) := \int_0^\infty F_y(x)\cosh(x/2)\,dx. \tag{4.10}$$

Observe that (4.7) is equivalent to

$$\begin{aligned}\log f_{\chi\overline{\chi}} \geq\ & (a_{\chi\overline{\chi}} - b_{\chi\overline{\chi}})(I_F(y) + \log(\pi)) \\ & + 2b_{\chi\overline{\chi}}\Big(\frac{I_F(y) + J_F(y) + 2\log(\pi)}{2}\Big) - 4R_F(y).\end{aligned} \tag{4.11}$$

Hence, from (4.6) with $n = \chi(1)$, we get

$$\begin{aligned}&\frac{1}{n}\log f_\chi \geq \frac{1}{2n^2}\log f_{\chi\overline{\chi}} \\ &\quad \geq \frac{(a_{\chi\overline{\chi}} - b_{\chi\overline{\chi}})}{n^2}\Big(\frac{I_F(y) + \log(\pi)}{2}\Big) + \frac{2b_{\chi\overline{\chi}}}{n^2}\Big(\frac{I_F(y) + J_F(y) + 2\log(\pi)}{4}\Big) - \frac{2R_F(y)}{n^2}.\end{aligned}$$

Using the definition (2.2) of $a_{\chi\overline{\chi}}$ and $b_{\chi\overline{\chi}}$, we get

$$a_{\chi\overline{\chi}} - b_{\chi\overline{\chi}} = \chi\overline{\chi}(g_0) = \chi(g_0)^2 = (a_\chi - b_\chi)^2$$

and

$$b_{\chi\overline{\chi}} = \frac{\chi\overline{\chi}(1) - \chi\overline{\chi}(g_0)}{2} = \frac{\chi(1)^2 - \chi(g_0)^2}{2} = 2a_\chi b_\chi;$$

we have

$$\begin{aligned}\frac{\log f_\chi}{n} \ \geq\ & \frac{(a_\chi - b_\chi)^2}{n^2}\Big(\frac{I_F(y) + \log(\pi)}{2}\Big) + \frac{4a_\chi b_\chi}{n^2}\Big(\frac{I_F(y) + J_F(y) + 2\log(\pi)}{4}\Big) \\ & - \frac{2R_F(y)}{n^2}.\end{aligned} \tag{4.12}$$

From here, we obtain a lower bound that is useful for large n.

Theorem 4.1. *Let χ be an irreducible character of degree n with conductor f_χ such that $L(s, \chi\overline{\chi})$ satisfies the Artin conjecture. Then*

$$f_\chi^{1/n} \geq 4.73(1.648)^{\frac{(a_\chi - b_\chi)^2}{n^2}} e^{-(13.34/n)^2}. \tag{4.13}$$

Proof. Evaluate (4.12) with $y = 0.0045$ to obtain

$$f_\chi^{1/n} \geq (7.797)^{\frac{(a_\chi - b_\chi)^2}{n^2}}(4.73)^{\frac{4a_\chi b_\chi}{n^2}} e^{-(13.34/n)^2},$$

which is equivalent to (4.13) since $a_\chi + b_\chi = n$ and $\frac{7.797}{4.73} > 1.648$. □

If we assume the Generalized Riemann Hypothesis (see the end of Remark 1), we can improve the lower bounds.

Theorem 4.2. *Let χ be an irreducible character of degree n with conductor f_χ such that $L(s, \chi\overline{\chi})$ satisfies the Artin conjecture and the Riemann hypothesis. Then*

$$f_\chi^{1/n} \geq 6.59(2.163)^{\frac{(a_\chi - b_\chi)^2}{n^2}} e^{-(13278.42/n)^2}. \tag{4.14}$$

Proof. Consider the even function[4] $F = F_{(y)} : \mathbb{R} \to \mathbb{R}$ which vanishes for $x > y^{-1/2}$ and fpr which $x \in [0, y^{-1/2}]$ is given by

$$F_{(y)}(x) = (1 - x\sqrt{y})\cos(\pi x\sqrt{y}) + \frac{\sin(\pi x\sqrt{y})}{\pi}. \tag{4.15}$$

Setting $y = 0.0004$ and using (4.12) we obtain (4.14). □

Odlyzko ([4], p. 385) obtained that

$$f_\chi^{1/n} \geq 4.71(1.645)^{\frac{(a_\chi - b_\chi)^2}{n^2}} + O(1/n^2), \quad \text{as } n \to \infty,$$

and, assuming the Riemann hypothesis, that

$$f_\chi^{1/n} \geq 6.44(2.13)^{\frac{(a_\chi - b_\chi)^2}{n^2}} + O(1/n^2), \quad \text{as } n \to \infty.$$

Taking $y = .001$ in the above proof, we can get (still under the Riemann hypothesis for $L(s, \chi\overline{\chi})$)

$$f_\chi^{1/n} \geq 6.458(2.094)^{\frac{(a_\chi - b_\chi)^2}{n^2}} e^{-(260.81/n)^2}.$$

For large n our bounds are only marginally better than Odlyzko's. In the next subsection we shall substantially improve on his bounds for small degrees.

4.1. **Tables for small degrees.** In the previous section we used inequality (4.6), since we were interested only in large n. In this section we are interested in small n, so we use the stronger original inequality (4.5). The net effect is to replace every n^2 on the right-hand side of (4.12) by $n(n-1)$. From (4.11) we therefore obtain

$$\begin{aligned} \frac{\log f_\chi}{n} \geq \; & \frac{(a_\chi - b_\chi)^2}{n(n-1)}\Big(\frac{I_F(y) + \log(\pi)}{2}\Big) + \frac{4a_\chi b_\chi}{n(n-1)}\Big(\frac{I_F(y) + J_F(y) + 2\log(\pi)}{4}\Big) \\ & - \frac{2R_F(y)}{n(n-1)}. \end{aligned} \tag{4.16}$$

As before, we obtain bounds by evaluating (4.16) with Tartar's F_y as in (3.2) and y as given in Table 1.

From (4.8) and (4.9) we find $J_F(y) < I_F(y)$. Hence, from (4.16) we have the lower bound, valid for any nonnegative a_χ, b_χ with $a_\chi + b_\chi = n > 1$,

$$\frac{\log f_\chi}{n} \geq \frac{n}{n-1} \cdot \frac{I_F(y) + J_F(y) + 2\log(\pi)}{4} - \frac{2R_F(y)}{n(n-1)}, \quad \text{for } n \text{ even} \tag{4.17}$$

and

$$\begin{aligned} \frac{\log f_\chi}{n} \geq \; & \frac{n}{n-1} \cdot \frac{I_F(y) + J_F(y) + 2\log(\pi)}{4} + \frac{1}{n(n-1)} \cdot \frac{I_F(y) - J_F(y)}{4} \\ & - \frac{2R_F(y)}{n(n-1)}, \quad \text{for } n \text{ odd}. \end{aligned} \tag{4.18}$$

These bounds are given in the third column of the table below for $2 \leq n \leq 20$. We also give lower bounds for the extreme cases in which $a_\chi = 0$ or $b_\chi = 0$, this time using (4.16). Finally, for the bounds under GRH we use Odlyzko's function (4.15) with y as shown.

[4] Introduced by Odlyzko; cf. [6]. The crucial property of f_y is that it and its Fourier transform are nonnegative.

TABLE 1. Lower bounds for irreducible characters.[5]

Assuming Artin's Conjecture							Artin's Conjecture and G.R.H.			
Any a_χ, b_χ				$a_\chi b_\chi = 0$			Any a_χ, b_χ		$a_\chi b_\chi = 0$	
n	y	$f_\chi^{1/n} \geq$	Odl	y	$f_\chi^{1/n} \geq$	Odl	y	$f_\chi^{1/n} \geq$	y	$f_\chi^{1/n} \geq$
2	4.71	3.255	2.83	2.65	5.067	4.21	0.2353	3.266	0.14	5.127
3	1.5	4.103	–	0.84	6.370	–	0.084	3.953	0.053	6.615
4	0.8	4.245	3.74	0.460	7.059	5.86	0.052	4.347	0.033	7.544
5	0.48	4.528	–	0.300	7.432	–	0.034	4.606	0.024	8.169
6	0.4	4.553	4.07	0.220	7.649	6.47	0.0269	4.785	0.019	8.619
7	0.25	4.681	–	0.169	7.782	–	0.022	4.918	0.016	8.962
8	0.2	4.684	4.22	0.140	7.867	6.74	0.0192	5.022	0.014	9.235
9	0.19	4.748	–	0.11	7.922	–	0.017	5.106	0.013	9.460
10	0.18	4.738	4.30	0.096	7.960	6.88	0.0153	5.175	0.012	9.647
11	0.120	4.782	–	0.084	7.984	–	0.014	5.233	0.011	9.810
12	0.11	4.776	4.35	0.074	8.002	7.06	0.0129	5.283	0.01	9.952
13	0.1	4.799	–	0.065	8.013	–	0.0120	5.327	0.0094	10.076
14	0.116	4.776	4.39	0.059	8.020	7.38	0.0113	5.365	0.0091	10.185
15	0.08	4.808	–	0.064	8.025	–	0.011	5.399	0.0085	10.287
16	0.09	4.798	4.47	0.049	8.027	7.57	0.0101	5.431	0.0081	10.377
17	0.067	4.812	–	0.045	8.028	–	0.01	5.457	0.0077	10.46
18	0.06	4.806	4.55	0.042	8.028	7.69	0.00925	5.484	0.0074	10.536
19	0.05	4.813	–	0.039	8.026	–	0.009	5.507	0.0071	10.606
20	0.036	4.809	4.61	0.036	8.025	7.77	0.00855	5.529	0.0069	10.671

5. Beyond Odlyzko's method

In the previous section we obtained lower bounds for the conductor f_χ of the irreducible character χ by two different methods. In the first one (where irreducibility was irrelevant) we had to compensate for the possible negativity of $\mathrm{Re}(\chi)$. In the second method the primes entered positively, but we dropped them. In this section we improve on these bounds by noting that if the first method requires primes to be compensated for, then they must make a substantial contribution to the second method. If primes do not require compensation, then the first method can be substantially improved. Thus we are able to obtain an improvement regardless of the behavior of the primes.

We shall need a lemma which will allow us to balance gains against losses in the two methods.

Lemma 5.1. *Let j run over a finite set of indices and let τ, δ_j and β_j be real numbers, with $\tau > 0$ and $\delta_j > 0$ for all j. If*

$$\sum_j x_j \beta_j \leq -\tau, \tag{5.1}$$

then

$$\sum_j x_j^2 \delta_j \geq \frac{\tau^2}{\Gamma}, \qquad \textit{where} \qquad \Gamma = \sum_j \frac{\beta_j^2}{\delta_j}. \tag{5.2}$$

[5] The columns labeled Odl show the lower bounds obtained by Odlyzko ([4], p. 404). Cases not covered by Odlyzko's tables have a $-$ in the Odl column. All bounds are rounded down so that the inequality is rigorous. We assume χ is irreducible and that $L(s, \chi\overline{\chi})$ is analytic for $s \neq 1$. The last four columns on the right apply when we also assume GRH, i.e., that all zeroes ρ of $L(s, \chi\overline{\chi})$ satisfy $\mathrm{Re}(\rho) = \frac{1}{2}$. The first lower bound in each case (columns labeled "Any a_χ, b_χ") apply for any value of a_χ or b_χ with $a_\chi + b_\chi = n$. The lower bounds in the columns labeled "$a_\chi b_\chi = 0$" only apply when $a_\chi = n$ or $b_\chi = n$.

We note that for large n our non-GRH bounds will *drop* toward 4.78 because the term $-\frac{2R_F(y)}{n(n-1)}$ in (4.17) becomes irrelevant (it approaches 0) and the decrease in the factor $\frac{n}{n-1}$ takes over.

Proof. Since the δ_j are assumed to be positive, there is a minimum value m of the positive quadratic form $\sum_j x_j^2\delta_j$ as the x_j range over the region defined by $\sum_j x_j\beta_j \le -\tau$. First, we show that m can only be a assumed on the boundary. Indeed, suppose that there exist $\widetilde{x}_j$ such that $\sum_j \widetilde{x}_j\beta_j < -\tau$ and m is assumed at $\widetilde{x} = (\widetilde{x}_j)$. Then $\widetilde{x}$ is a critical point of the quadratic form. Taking partial derivatives we find $2\widetilde{x}_j\delta_j = 0$ for all j. Hence $\widetilde{x} = 0$, contradicting $\sum_j \widetilde{x}_j\beta_j < -\tau$, since τ is assumed to be positive.

Thus we seek to minimize the expression (5.2) using the condition (5.1) with equality. We will use Lagrange multipliers. Note that the minimum is known to exist, and hence will be given as a critical point of the auxiliary function $F(\mathbf{x},\lambda)$ used with Lagrange multipliers. We shall see that there is a unique critical point, and hence this yields the minimum m. Consider the function

$$F(\mathbf{x},\lambda) = g(\mathbf{x}) - \lambda h(\mathbf{x}),$$

where

$$g(\mathbf{x}) = \sum_j x_j^2\delta_j$$

and

$$h(\mathbf{x}) = \tau + \sum_j x_j\beta_j.$$

Now, we will find a critical point for F. This is equivalent to solving the system

$$\frac{\partial F}{\partial x_j} = 0, \qquad \frac{\partial F}{\partial \lambda} = 0,$$

which is equivalent to

$$\begin{aligned} \frac{\partial F}{\partial x_j} &= 2x_j\delta_j - \lambda\beta_j = 0, \\ \frac{\partial F}{\partial \lambda} &= \tau + \sum_j x_j\beta_j = 0. \end{aligned}$$

Thus,

$$x_j = \lambda\frac{\beta_j}{2\delta_j},$$

and so

$$-\tau = \sum_j x_j\beta_j = \frac{\lambda}{2}\sum_j \frac{\beta_j^2}{\delta_j} = \frac{\lambda\Gamma}{2}.$$

Hence, $x_j = -\frac{\tau\beta_j}{\Gamma\delta_j}$. Moreover,

$$\sum_j x_j^2\delta_j = \frac{\tau^2}{\Gamma^2}\sum_j \frac{\beta_j^2}{\delta_j^2}\delta_j = \frac{\tau^2}{\Gamma^2}\sum_j \frac{\beta_j^2}{\delta_j} = \frac{\tau^2}{\Gamma}.$$

Therefore,

$$\sum_j x_j^2\delta_j \ge \frac{\tau^2}{\Gamma},$$

as claimed in the lemma. ∎

To describe our main inequality we need some notation. Fix nonnegative integers a and b, and set $n = a + b$. For $F : \mathbb{R} \to \mathbb{R}$ an even function satisfying the Poitou-Mestre Hypothesis, set

$$I_F := \int_0^\infty \Big(\frac{e^{-x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\, dx, \tag{5.3}$$

$$J_F := \int_0^\infty \Big(\frac{e^{-3x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\, dx, \tag{5.4}$$

$$R_F := \int_0^\infty F(x)\cosh(x/2)\, dx, \tag{5.5}$$

$$G_F := \log(\pi) + \frac{a}{n} I_F + \frac{b}{n} J_F, \tag{5.6}$$

$$\begin{aligned} H_F &:= \frac{(a-b)^2}{n(n-1)}\Big(\frac{I_F + \log(\pi)}{2}\Big) + \frac{4ab}{n(n-1)}\Big(\frac{I_F + J_F + 2\log(\pi)}{4}\Big) \\ &- \frac{2}{n(n-1)} R_F, \end{aligned} \tag{5.7}$$

$$\alpha_{F,p,m} := \frac{F(m\log(p))\log(p)}{p^{m/2}}. \tag{5.8}$$

We note H_F is exactly the right-hand side of inequality (4.16) for $\frac{1}{n}\log f_\chi$ for irreducible characters, while G_F would also be a lower bound for $\frac{1}{n}\log f_\chi$ (cf. (2.13) with $r = 0$) if the primes had not forced us to replace χ by $\tilde{\chi}$ to ensure $\mathrm{Re}\big(\tilde{\chi}(p^m)\big) \geq 0$. Terms like $\alpha_{F,p,m}$ had not appeared in our inequalities as we had arranged to drop all terms coming from the primes in the explicit formulas.

Theorem 5.1. *Let χ be an irreducible character of $\mathcal{G}$ of dimension $n \geq 2$ and assume the Artin conjecture for $L(s,\chi)$ and $L(s,\chi\overline{\chi})$. Suppose further that F and $\widetilde{F}$ satisfy the Poitou-Mestre Hypothesis and $G_F > H_{\widetilde{F}}$, with F compactly supported and $\widetilde{F} > 0$ on the support of F. Then*

$$\frac{1}{n}\log f_\chi \geq H_{\widetilde{F}} + \frac{(n-1)\Gamma}{n} \cdot \left(\sqrt{1 + \frac{n}{n-1}\cdot\frac{G_F - H_{\widetilde{F}}}{\Gamma}}\; - \;1\right)^2, \tag{5.9}$$

where $\Gamma = \sum_{p,m} \frac{\alpha^2_{F,p,m}}{\alpha_{\widetilde{F},p,m}}$, the sum ranging over all primes p and positive integers m such that $m\log p$ is contained in the support of F.

The way to interpret the messy expression (5.9) is to think of $H_{\widetilde{F}}$ as the lower bound we had from the previous chapter, with the rest of the expression as the gain from the primes. In calculating (5.6) and (5.7) we take $a = a_\chi,\ b = b_\chi$ and $n = a + b$.

Proof. From the basic inequality (2.13) with $r = 0$, we obtain

$$\frac{1}{n}\log f_\chi \geq G_F + \frac{2}{n}\sum_{p,m} \alpha_{F,p,m}\cdot c_{p,m}, \qquad \text{where} \qquad c_{p,m} := \mathrm{Re}\big(\chi(p^m)\big). \tag{5.10}$$

Now consider the character $\chi\overline{\chi}$, for which we have proved in Lemma 4.1 that

$$\frac{1}{n}\log f_\chi \geq \frac{1}{2n(n-1)}\log f_{\chi\overline{\chi}} = \frac{n}{(n-1)}\frac{1}{2n^2}\log f_{\chi\overline{\chi}}.$$

We now apply (2.13) to $\chi\overline{\chi}$ (which corresponds to a representation of dimension n^2), the basic inequality (2.13) with $r = 1$ and $F = \widetilde{F}$ to obtain

$$\begin{aligned} \frac{1}{n}\log f_\chi &\geq H_{\widetilde{F}} + \frac{1}{n(n-1)}\sum_{p,m}\alpha_{\widetilde{F},p,m}\cdot|\chi(p^m)|^2 \\ &\geq H_{\widetilde{F}} + \frac{1}{n(n-1)}\sum_{p,m}\alpha_{\widetilde{F},p,m}\cdot c_{p,m}^2\,. \end{aligned} \tag{5.11}$$

In the last sum over p and m we may (and do) drop all p and m for which $F(m\log(p)) = 0$.[6] Dropping these terms ensures that sums over p and m are finite, which will be required when we apply Lemma 5.1 below. From the hypotheses in the theorem we have the strict inequality $\alpha_{\widetilde{F},p,m} > 0$ for terms p and m remaining in the sum.[7]

Let

$$T := H_{\widetilde{F}} + \frac{(n-1)\Gamma}{n}\cdot\left(\sqrt{1+\frac{n}{n-1}\cdot\frac{G_F - H_{\widetilde{F}}}{\Gamma}}\; - \;1\right)^2. \tag{5.12}$$

We claim,

$$G_F > T > H_{\widetilde{F}}.$$

Indeed, the second inequality is trivial and the first one is equivalent to (on letting $\Gamma' = \frac{(n-1)\Gamma}{n}$)

$$G_F - H_{\widetilde{F}} > \Gamma'\left(\sqrt{1+\frac{G_F - H_{\widetilde{F}}}{\Gamma'}}\; - \;1\right)^2,$$

which, on expanding the square, is equivalent to

$$G_F - H_{\widetilde{F}} > G_F - H_{\widetilde{F}} + 2\Gamma'\left(1 - \sqrt{1+\frac{G_F - H_{\widetilde{F}}}{\Gamma'}}\right),$$

which is clearly true since $G_F - H_{\widetilde{F}} > 0$ by assumption. Let us write (5.10) as

$$\frac{1}{n}\log f_\chi \geq \; T + t + \frac{2}{n}\sum_{p,m} c_{p,m}\cdot\alpha_{F,p,m},$$

where

$$t := G_F - T > 0. \tag{5.13}$$

If we had

$$t + \frac{2}{n}\sum_{p,m} c_{p,m}\cdot\alpha_{F,p,m} \geq 0, \tag{5.14}$$

we would have

$$\frac{1}{n}\log f_\chi \geq T,$$

proving the theorem in this case. Hence, we may suppose that (5.14) is false, i.e.,

$$\sum_{p,m} c_{p,m}\cdot\alpha_{F,p,m} < -\frac{nt}{2}.$$

---

[6] This is permissible since condition (D) in the Poitou-Mestre Hypothesis ensures $\alpha_{\widetilde{F},p,m} \geq 0$.

[7] In (4.16) previously we had simply dropped all of the sums over the primes using $\chi\overline{\chi}(p^m) \geq 0$. We wish to exploit in the explicit formula for $\chi\overline{\chi}$ the finitely many primes appearing in the explicit formula for χ with nonzero coefficients.

As a consequence of Lemma 5.1, with j indexed by p and m as in the lemma, $x_j = c_{p,m},\ \beta_j = \alpha_{F,p,m},\ \delta_j = \alpha_{\widetilde{F},p,m},\ \tau = nt/2$ we have

$$\sum_{p,m} c_{p,m}^2 \alpha_{\widetilde{F},p,m} \geq \frac{t^2}{4\Gamma} n^2.$$

Therefore, in (5.11) using (5.13), we have

$$\frac{1}{n} \log f_\chi \geq H_{\widetilde{F}} + \frac{t^2}{4\Gamma} \frac{n}{(n-1)} = H_{\widetilde{F}} + \frac{(G_F - T)^2}{4\Gamma} \frac{n}{(n-1)} = T, \tag{5.15}$$

where at the end we used definition (5.12) and some algebraic manipulations. Our last inequality proves the theorem. □

We now apply the above theorem to obtain improved lower bounds for large degrees.[8] In this case, we can replace every occurrence of $n-1$ in (5.7) and (5.9) by n.[9] Then (5.9) simplifies to

$$\frac{1}{n} \log f_\chi \geq H_{\widetilde{F}} + \Gamma \cdot \left(\sqrt{1 + \frac{G_F - H_{\widetilde{F}}}{\Gamma}} \ - \ 1 \right)^2. \tag{5.16}$$

In (5.16) we will take F to be Bernardette Perrin-Riou's function, introduced in [6, p. 13],[10]

$$F(x) := \frac{fr(x\sqrt{y_G})}{\cosh(x/2)},$$

where y_G is a positive parameter to be specified later, $fr(x)$ is even, vanishes for $x > 2\pi$ and for $x \in [0, 2\pi]$ is given by

$$fr(x) = \frac{1}{3\pi} \left(2\pi - x + \frac{3\sin(x) + \pi \cos(x) - (x - \pi)\cos(x)}{2} \right). \tag{5.17}$$

Now since F in Theorem 5.1 depends on an extra parameter, we add it everywhere to the notation, writing, for example, $G_F(y_G)$ for G_F in (5.6).

For $\widetilde{F}$ in (5.16) we will take Tartar's function

$$\widetilde{F}(x) = \frac{f(x\sqrt{y_H})}{\cosh(x/2)}, \tag{5.18}$$

where

$$f(x) = \frac{9\big(\sin(x) - x\cos(x)\big)^2}{x^6} \tag{5.19}$$

and $y_H > 0$ is another positive parameter to be specified below. Since Tartar's (nonnegative) function is positive for $0 \leq x \leq 4.49/\sqrt{y_H}$, one finds that $\widetilde{F}$ is positive on the support of Perrin-Riou's F if $y_H < y_G/2$.

We have the following numerical corollaries of Theorem 5.1.[11]

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[8] In the next section we will tabulate such bounds for small degrees.

[9] To see this, note that the $n-1$ comes from inequality (4.16), which becomes strictly weaker if we replace every occurrence of $n-1$ by n.

[10] There the function is described as a convolution square, but not explicitly calculated. The formula we give in (5.17) is the result of carrying out the calculation of this convolution square.

[11] The case $a = 0$ is not treated below, as Theorem 5.1 gives no significant improvement in this case.

Corollary 5.1. *Let χ be an irreducible character of $\mathcal{G}$ and assume the Artin conjecture for $L(s,\chi)$ and $L(s,\chi\overline{\chi})$. Then*

$$\begin{aligned}\frac{1}{n}\log f_\chi &\geq 1.378+\frac{(a-b)^2}{n^2}0.374-\frac{4.743}{n^2}\\ &+ 0.609\Big(\sqrt{1.1+\frac{a}{n}1.677+\frac{4a(n-a)}{n^2}0.374+\frac{7.782}{n^2}}-1\Big)^2.\end{aligned}$$

Proof. Take $y_H=0.632$ and $y_G=2.968$, which satisfy $y_H<y_G/2$. Also, we must verify $G_F(y_G)>H_{\widetilde{F}}(y_H)$ for all a,b. In fact, we can consider $G_F(y_G)-H_{\widetilde{F}}(y_H)$ as a function of a letting $b=n-a$, then with $G_F(y_G)=g(a)$ and $H_{\widetilde{F}}(y_H)=h(a)$ we have (replacing n by $n-1$)

$$\begin{aligned}g(a)-h(a) &= \log(\pi)+J_F(y_G)-\widetilde{I}_{\widetilde{F}}(y_H)+\frac{a}{n}(I_F(y_G)-J_F(y_G))\\ &+ \frac{4a(n-a)}{n^2}\big(\widetilde{I}_{\widetilde{F}}(y_H)-\widetilde{J}_{\widetilde{F}}(y_H)\big)+\frac{2R_{\widetilde{F}}(y_H)}{n^2},\end{aligned}\tag{5.20}$$

where $I_F(y), J_F(y)$ and $R_F(y)$ were defined in (5.3), (5.4) and (5.5), respectively (but we have now put the dependence on y into the notation), and

$$\begin{aligned}\widetilde{I}_F(y) &= \frac{I_F(y)+\log(\pi)}{2},\\ \widetilde{J}_F(y) &= \frac{I_F(y)+J_F(y)+2\log(\pi)}{4}.\end{aligned}$$

Since $I_F(y)-J_F(y)>0$, the expression (5.20) is quadratic in a with negative leading coefficient, its minimal value in any interval is attained at one of the interval extremes. One thus checks that on the interval $[0,n]$ the minimum is attained at $a=0$. This implies

$$\Gamma\left(\sqrt{1+\frac{g(a)-h(a)}{\Gamma}}-1\right)^2\geq\Gamma\left(\sqrt{1+\frac{g(0)-h(0)}{\Gamma}}-1\right)^2.$$

We have that $g(0)-h(0)=0.0615$, then $G_F(y_G)>H_{\widetilde{F}}(y_H)$ for all a. After this, one simply evaluates (5.16). □

Corollary 5.2. *Let χ be an irreducible character of $\mathcal{G}$ with $a=n$ and assume the Artin conjecture for $L(s,\chi)$ and $L(s,\chi\overline{\chi})$. Then,*

$$f_\chi^{1/n}\geq 9.482\,e^{-10.359/n}.\tag{5.21}$$

This improves on the lower bound $f_\chi^{1/n}\geq 7.797\,e^{-(13.34/n)^2}$ from Theorem 4.1 in the previous chapter.

Proof. Evaluating (5.16) with $y_G=1.2$ and $y_H=0.033$, we obtain

$$\begin{aligned}\frac{1}{n}\log f_\chi &\geq 2.0302-\frac{20.7526}{n^2}+1.2925\Big(\sqrt{1.9934+\frac{16.0556}{n^2}}-1\Big)^2\\ &\geq 5.8933-2.585\sqrt{1.9934+\frac{16.0556}{n^2}}\\ &\geq 2.2494-\frac{10.359}{n},\end{aligned}$$

where in the last step we used $\sqrt{A+B}\leq\sqrt{A}+\sqrt{B}$ for A and B positive. □

Corollary 5.3. *Let χ be an irreducible character of $\mathcal{G}$ with $a = b = n/2$ and assume the Artin conjecture for $L(s,\chi)$ and $L(s,\chi\overline{\chi})$. Then,*

$$f_\chi^{1/n} \geq 5.542\, e^{-16.859/n}. \tag{5.22}$$

This improves on the lower bound $f_\chi^{1/n} \geq 4.73\, e^{-(13.34/n)^2}$ from the previous chapter.

Proof. We evaluate (5.16) with $y_G = 2.069$ and $y_H = 0.05$ and use the same procedure as in the previous corollary. □

If we assume the Generalized Riemann Hypothesis as in the previous chapter, we can improve the lower bounds. Now, instead of Tartar's function we will take $\widetilde{F} : \mathbb{R} \to \mathbb{R}$ to be Odlyzko's function,

$$\widetilde{F}(x) = (1 - x\sqrt{y_H})\cos(\pi x\sqrt{y_H}) + \frac{\sin(\pi x\sqrt{y_H})}{\pi} \qquad \Big(x \in [0, y_H^{-1/2}]\Big), \tag{5.23}$$

which is even and vanishes for $x > y_H^{-1/2}$. We will take $F = fr$ to be Perrin- Riou's function (5.17). Note that $\widetilde{F}$ is positive on the support of F if $y_H < y_G/40$.

Then we have the following conditional improvements on Corollaries 5.1, 5.2, and 5.3.

Corollary 5.4. *Let χ be an irreducible character of $\mathcal{G}$ and assume the Artin conjecture and the Riemann Hypothesis for $L(s,\chi)$ and $L(s,\chi\overline{\chi})$. Then*

$$\begin{aligned} \frac{1}{n}\log f_\chi \quad &\geq \quad 1.425 + \frac{(a-b)^2}{n^2}0.402 - \frac{5.216}{n^2} \\ &+ \quad 0.6567\Big(\sqrt{1.02 + \frac{a}{n}1.618 + \frac{4a(n-a)}{n^2}0.612 + \frac{7.942}{n^2}} - 1\Big)^2. \end{aligned}$$

Proof. Proceeding as in Corollary 5.1, we take $y_G = 3.268$ and $y_H = 0.042$ (which satisfy $y_H < y_G/40$ and $G_F(y_G) > H_{\widetilde{F}}(y_H)$) and evaluate (5.16). □

Corollary 5.5. *Let χ be an irreducible character of $\mathcal{G}$ with $a = n$ and assume the Artin conjecture and the Riemann Hypothesis for $L(s,\chi)$ and $L(s,\chi\overline{\chi})$. Then,*

$$f_\chi^{1/n} \geq 11.438\, e^{-5.538/n}. \tag{5.24}$$

Proof. Evaluate (5.16) with $y_G = 1.046$ and $y_H = 0.0059$ and use the same procedure as in Corollary 5.2. □

Corollary 5.6. *Let χ be an irreducible character of $\mathcal{G}$ with $a = b = n/2$ and assume the Artin conjecture and the Riemann Hypothesis for $L(s,\chi)$ and $L(s,\chi\overline{\chi})$. Then,*

$$f_\chi^{1/n} \geq 6.294\, e^{-10.555/n}. \tag{5.25}$$

Proof. Evaluate (5.16) with $y_G = 2.062$ and $y_H = 0.009$. □

6. Tables

Table 2. Lower bounds for irreducible characters[12]

$a_\chi = n$				$a_\chi = b_\chi$		
n	y_G	y_H	$f_\chi^{1/n} \geq$	y_G	y_H	$f_\chi^{1/n} \geq$
2	3.147	1.570	7.467	5.997	2.696	4.599
3	1.890	0.675	8.636	–	–	–
4	1.517	0.378	9.207	2.968	0.632	5.336
5	1.35	0.265	9.509	–	–	–
6	1.263	0.190	9.68	2.433	0.301	5.559
7	1.21	0.153	9.781	–	–	–
8	1.016	0.121	9.834	2.253	0.197	5.645
9	1.153	0.105	9.882	–	–	–
10	1.115	0.088	9.906	2.169	0.135	5.684
11	1.102	0.079	9.920	–	–	–
12	1.1	0.069	9.929	2.134	0.116	5.7
13	1.1	0.059	9.934	–	–	–
14	1.142	0.052	9.935	2.099	0.08	5.71
15	1.138	0.047	9.935	–	–	–
16	1.136	0.043	9.933	2.085	0.072	5.714
17	1.160	0.041	9.930	–	–	–
18	1.1	0.038	9.928	2.081	0.060	5.715
19	1.13	0.035	9.924	–	–	–
20	1.2	0.033	9.917	2.069	0.050	5.714

[12] We assume the Artin conjecture for $L(s,\chi)$ and $L(s,\chi\overline{\chi})$ and use Theorem 5.1 with y_H and y_G as given. The auxiliary functions are Tartar's and Perrin-Rious's as in the previous section. We have omitted the case $a = 0$ as the gains over Table 1 in Chapter 3 are minor in this case.

TABLE 3. Lower bounds for irreducible characters with GRH[13]

$a_\chi = n$				$a_\chi = b_\chi$		
n	y_G	y_H	$f_\chi^{1/n} \geq$	y_G	y_H	$f_\chi^{1/n} \geq$
2	3.601	0.09	7.608	6.424	0.148	4.631
3	2.143	0.044	9.002	–	–	–
4	1.738	0.029	9.843	3.268	0.042	5.463
5	1.550	0.020	10.373	–	–	–
6	1.451	0.018	10.812	2.773	0.025	5.799
7	1.309	0.015	11.127	–	–	–
8	1.277	0.013	11.366	2.449	0.018	5.986
9	1.230	0.012	11.578	–	–	–
10	1.200	0.010	11.700	2.168	0.015	6.11
11	1.185	0.010	11.895	–	–	–
12	1.147	0.01	12.029	2.163	0.012	6.195
13	1.099	0.009	12.147	–	–	–
14	1.076	0.008	12.227	2.193	0.01	6.253
15	1.078	0.008	12.339	–	–	–
16	1.070	0.008	12.425	2.147	0.01	6.315
17	1.068	0.008	12.491	–	–	–
18	1.06	0.008	12.540	2.107	0.01	6.350
19	1.042	0.006	12.595	–	–	–
20	1.040	0.006	12.640	2.062	0.009	6.392

[13] We apply Theorem 5.1, assuming the Generalized Riemann Hypothesis, using Odlyzko's and Perrin-Riou's functions defined in (5.23) and (5.17).

TABLE 4. Lower bounds for $f_\chi^{1/n}$ for irreducible characters with and without GRH.[14]

$a \setminus b$	0	1	2	3	4	5	6	7	8	9	10
0	– – –	2.91 2.31 2.95	5.067 4.21 5.127	6.370 – 6.615	7.059 5.86 7.544	7.432 – 8.169	7.649 6.47 8.619	7.782 – 8.962	7.867 6.74 9.235	7.922 – 9.460	7.960 6.88 9.647
1	4.90 3.70 4.94	4.599 2.83 4.631	4.888 – 4.960	5.269 3.74 5.428	5.621 – 5.907	5.908 4.07 6.349	6.163 – 6.743	6.361 4.22 7.116	6.525 – 7.394	6.660 4.30 7.681	
2	7.467 4.21 7.608	5.652 – 5.730	5.336 3.74 5.463	5.328 – 5.516	5.426 4.07 5.691	5.553 – 5.916	5.689 4.22 6.141	5.825 – 6.405	5.949 4.30 6.634		
3	8.636 – 9.002	6.420 3.74 6.635	5.773 – 5.989	5.559 4.07 5.799	5.496 – 5.788	5.508 4.22 5.858	5.556 – 5.981	5.623 4.30 6.126	– – –		
4	9.207 5.86 9.843	6.985 – 7.353	6.175 4.07 6.485	5.814 – 6.125	5.645 4.22 5.986	5.573 – 5.958	5.554 4.30 5.993	– – –	– – –		
5	9.509 – 10.384	7.400 4.07 7.940	6.512 – 6.937	6.060 4.22 6.455	5.818 – 6.217	5.684 4.30 6.111	– – –	– – –	– – –		
6	9.68 5.86 10.812	7.728 – 8.433	6.798 4.22 7.348	6.291 – 6.772	5.993 4.30 6.464	– – –	– – –	– – –	– – –		
7	9.781 – 11.123	7.979 4.22 8.849	7.044 – 7.721	6.501 4.30 7.072	– – –	– – –	– – –	– – –	– – –		
8	9.834 6.74 11.374	8.179 – 9.212	7.254 4.30 8.052	– – –	– – –	– – –	– – –	– – –	– – –		
9	9.882 – 11.580	8.339 4.30 9.515	– – –	– – –	– – –	– – –	– – –	– – –	– – –		
10	9.906 6.88 11.700	– – –	– – –	– – –	– – –	– – –	– – –	– – –	– – –		

[14] We assume the Artin conjecture for χ and $\chi\overline{\chi}$. For each entry a, b the top number is a lower bound for $f_\chi^{1/n}$. The bottom number is a lower bound for $f_\chi^{1/n}$ if one also assumes the Generalized Riemann Hypothesis for χ and $\chi\overline{\chi}$. The middle number, when given, is Odlyzko's lower bound, which does not assume GRH. Table 5 below gives the values of the parameters y_G and y_H used in Theorem 5.1 to obtain Table 4. The auxiliary functions are as in Tables 2 and 3, except for $a = 0$ where we only use (4.16) and Tartar's F_y.

TABLE 5. Parameters used in Table 4.[15]

$a \setminus b$	0	1	2	3	4	5	6	7	8	9
0	–	12	2.65	0.84	0.460	0.300	0.220	0.169	0.140	0.11
	–	12	0.14	0.053	0.033	0.024	0.019	0.016	0.014	0.013
1	12	12	3.473	2.370	1.918	1.930	1.102	1.059	0.885	0.796
			1.131	0.608	0.380	0.268	0.196	0.155	0.130	0.112
	10.35	10.35	1.125	2.584	2.017	1.784	1.726	1.068	1.068	1.922
			0.028	0.042	0.030	0.023	0.017	0.016	0.016	0.018
2	3.147	3.373	5.997	2.530	2.156	2.092	1.892	1.452	1.300	–
	1.570	1.003	2.696	0.457	0.301	0.224	0.181	0.143	0.120	–
	3.601	2.165	6.424	2.688	2.338	1.805	1.698	1.543	1.663	–
	0.09	0.054	0.148	0.032	0.0259	0.021	0.015	0.016	0.012	–
3	1.890	2.409	2.427	–	2.281	2.092	1.935	1.767	–	–
	0.675	0.529	0.476	–	0.230	0.191	0.156	0.130	–	–
	2.143	2.632	2.681	–	2.343	1.970	1.757	1.727	–	–
	0.044	0.035	0.032	–	0.019	0.016	0.017	0.012	–	–
4	1.517	1.981	2.218	2.286	2.968	2.172	2.065	–	–	–
	0.378	0.338	0.280	0.229	0.632	0.152	0.135	–	–	–
	1.738	2.088	2.302	2.374	3.268	2.073	2.010	–	–	–
	0.029	0.025	0.025	0.019	0.042	0.016	0.015	–	–	–
5	1.350	2.003	2.006	1.972	2.185	–	–	–	–	–
	0.265	0.241	0.213	0.179	0.156	–	–	–	–	–
	1.550	1.804	2.066	2.214	2.175	–	–	–	–	–
	0.020	0.019	0.020	0.019	0.013	–	–	–	–	–
6	1.263	1.685	1.752	1.976	2.102	–	–	–	–	–
	0.190	0.178	0.162	0.147	0.130	–	–	–	–	–
	1.451	1.662	1.699	1.988	2.076	–	–	–	–	–
	0.018	0.016	0.015	0.017	0.015	–	–	–	–	–
7	1.210	1.370	1.719	1.913	–	–	–	–	–	–
	0.153	0.141	0.132	0.120	–	–	–	–	–	–
	1.309	1.544	1.738	1.896	–	–	–	–	–	–
	0.015	0.014	0.013	0.012	–	–	–	–	–	–
8	1.016	1.441	1.629	–	–	–	–	–	–	–
	0.121	0.117	0.110	–	–	–	–	–	–	–
	1.277	1.469	1.678	–	–	–	–	–	–	–
	0.013	0.012	0.413	–	–	–	–	–	–	–
9	1.153	1.281	–	–	–	–	–	–	–	–
	0.015	0.098	–	–	–	–	–	–	–	–
	1.230	1.404	–	–	–	–	–	–	–	–
	0.012	0.011	–	–	–	–	–	–	–	–
10	1.115	–	–	–	–	–	–	–	–	–
	0.088	–	–	–	–	–	–	–	–	–
	1.200	–	–	–	–	–	–	–	–	–
	0.010	–	–	–	–	–	–	–	–	–

[15] For each entry a, b the top number is the value of y_G used to obtain the non-GRH bounds in Table 4. This is followed by the corresponding y_H. The third number is the value of y_G used to obtain the conditional bounds. The last number is the corresponding y_H. For $a = 0$ we only give y used in Tartar's F_y.

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