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SUPERCLOSENESS AND SUPERCONVERGENCE OF STABILIZED LOW-ORDER FINITE ELEMENT DISCRETIZATIONS OF THE STOKES PROBLEM

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Abstract. The supercloseness and superconvergence properties of stabilized finite element methods applied to the Stokes problem are studied. We consider consistent residual based stabilization methods as well as inconsistent local projection type stabilizations. Moreover, we are able to show the supercloseness of the linear part of the MINI-element solution which has been previously observed in practical computations. The results on supercloseness hold on three-directional triangular, axiparallel rectangular, and brick-type meshes, respectively, but extensions to more general meshes are also discussed. Applying an appropriate postprocess to the computed solution, we establish superconvergence results. Numerical examples illustrate the theoretical predictions.

1. Introduction

In recent years, the superconvergence of finite element methods has been an active research field in numerical analysis. The main objective of the superconvergence research is to improve the existing approximation accuracy by applying certain postprocessing techniques which are cheap and easy to implement.

In this paper, we consider the supercloseness and the superconvergence properties of numerical solutions of the stationary Stokes problem

$$\left\{ \begin{array}{rcll} -\nu\Delta\mathbf{u}+\nabla p &=& \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &=& 0 & \text{in } \Omega, \\ \mathbf{u} &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1.1}$$

Here, $\nu > 0$ is the kinematic viscosity (we set $\nu = 1$ for simplicity) and the function $\mathbf{f}$ is sufficiently smooth. We study finite element approximations on uniform triangular (three-directional grids), axiparallel rectangular, and brick-type meshes, and increase the order of convergence of the original computed solution by postprocessing. The technique for the standard Galerkin finite element approach is well-understood; see e.g. [6, 12]. If the finite element spaces approximating velocity and pressure satisfy an inf-sup condition, stability and convergence of the discretization can be proven. So far, some superconvergence results have been obtained for the standard Galerkin method; see, for example, [17, 18, 20, 25, 36].

Here, we consider the superconvergence property of stabilized methods which has been developed in order to circumvent the inf-sup condition and to allow equal-order interpolations for velocity and pressure; see [7, 11, 14, 15]. The usual way of

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analyzing superconvergence properties of postprocessed computed solutions consists of two steps:

(1) Supercloseness property: an interpolation approximating the finite element solution of higher order. Often such interpolation does exist if the underlying mesh has a special structure.
(2) A postprocessing operator: an interpolation operator (in a higher-order finite element space) with certain stability, invariance and higher-order approximation properties. Applying this interpolation operator to the original finite element solution, we obtain the postprocessed solution, which has a superconvergence property.

There are many research contributions to these two steps. Currently two approaches are used to prove superconvergence. One is based on the analysis of the method: there are the integral identity, the integral expansion (which is based on the Bramble-Hilbert lemma) and others; see [8, 35]. The second concerns the mesh condition [2, 23, 36, 37, 38, 40, 44]. The supercloseness result has been extended from structured meshes to more general, practical and automatically generated meshes. In the case of the postprocessing operator, the first approach is manifested as higher-order finite element interpolation, while gradient recovery methods [39, 40, 43, 44] are used in the second approach.

The supercloseness phenomena have been already established for some kinds of mixed finite elements. For example, in [18, 19, 20, 23, 25], the supercloseness analysis for the Stokes problem and Navier-Stokes problems has been given.

For the error analysis, we introduce the standard notation for the Sobolev spaces $W^{k,p}(D)$, $H^k(D) = W^{k,2}(D)$, $H_0^k(D)$, $L^p(D) = W^{0,p}(D)$ with nonnegative integers k and $1 \leq p \leq \infty$. The corresponding vector-valued versions of these spaces will be indicated by boldface letters. The norm and seminorm corresponding to both the scalar and the vector-valued version of the space $W^{k,p}(D)$ are denoted by $\|\cdot\|_{k,p,D}$ and $|\cdot|_{k,p,D}$. For the inner product in $L^2(D)$, its vector-valued versions, and $L^2(\partial D)$, we write $(\cdot,\cdot)_D$ and $\langle\cdot,\cdot\rangle_{\partial D}$, respectively. We will drop the index D when $D = \Omega$. Throughout this paper C denotes a generic positive constant that is independent of the mesh size.

2. Discretizations of the Stokes problem

2.1. **Standard Galerkin.** Let $\Omega \subset \mathbb{R}^d$ be a polygonal ($d = 2$) or polyhedral ($d = 3$) domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$. Introducing the solution spaces $\mathbf{V} := (H_0^1(\Omega))^d$ and $Q := L_0^2(\Omega)$, a weak formulation of (1.1) is:

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that for all $(\mathbf{v}, q) \in \mathbf{V} \times Q$,

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) + (q, \operatorname{div} \mathbf{u}) = (\mathbf{f}, \mathbf{v}). \tag{2.1}$$

It is well known that the Babuška-Brezzi condition,

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(q, \operatorname{div} \mathbf{v})}{|\mathbf{v}|_1 \, \|q\|_0} > 0, \tag{2.2}$$

guarantees the existence and uniqueness of a solution of (2.1); cf. [6, 12].

We introduce a shape regular partition $\mathcal{T}_h$ of the computational domain Ω into cells K (triangles, quadrilaterals, tetrahedrons, hexahedrons) such that

$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} \bar{K}.$$

Here $h := \max_{K \in \mathcal{T}_h} \{h_K\}$ and $h_K = \operatorname{diam} K$ denote the global and local mesh size, respectively. Let $\mathbf{V}_h \subset \mathbf{V}$ and $Q_h \subset Q$ be finite element spaces approximating velocity and pressure. Then, the standard Galerkin discretization is:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$,

$$(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) + (q_h, \operatorname{div} \mathbf{u}_h) = (\mathbf{f}, \mathbf{v}_h). \tag{2.3}$$

The standard Galerkin approach for solving the Stokes problem (1.1) by finite element discretizations is well understood (see, e.g., [6, 12]). If the finite element spaces approximating velocity and pressure satisfy a discrete version of the Babuška-Brezzi condition (2.2) uniformly in h, stability and convergence of the discretization can be established. A large number of finite element pairs is known to satisfy this stability condition; however, there are several reasons for circumventing it. First, equal-order interpolations, in general, do not belong to this class of "stable methods", but they are simple to implement since the same finite element space is used for approximating the pressure and the velocity components. Second, and this is even more important, it is often not clear whether the stability property also holds on sequences of meshes with hanging nodes, which are popular in adaptive finite elements. In the case of the $Q_r - P_{r-1}^{\text{disc}}$ finite element pair, the validity of the Babuška-Brezzi condition on mesh families with hanging nodes has been shown in [29]. Alternative methods for solving the Stokes problem are based on consistent and inconsistent modifications of the discrete problem. These approaches do not require fulfilment of the Babuška-Brezzi condition and work also on families with hanging nodes.

2.2. **Local projection stabilization.** In this section, we consider equal-order interpolations stabilized by the local projection method in its one-level variant as developed in [11, 26]. For the two-level approach we refer to [3, 5, 27]. Let Y_h denote a scalar finite element space of continuous, piecewise polynomials over $\mathcal{T}_h$. The spaces for approximating velocity and pressure are given by $\mathbf{V}_h := Y_h^d \cap \mathbf{V}$ and $Q_h := Y_h \cap Q$. The discrete problem of our stabilized method is:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$,

$$(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) + (q_h, \operatorname{div} \mathbf{u}_h) + S_h(p_h, q_h) = (\mathbf{f}, \mathbf{v}_h), \tag{2.4}$$

where the stabilization term with user-chosen parameters α_K is given by

$$S_h(p, q) = \sum_{K \in \mathcal{T}_h} \alpha_K (\kappa_h \nabla p, \kappa_h \nabla q)_K. \tag{2.5}$$

Here, the fluctuation operator $\kappa_h : L^2(\Omega) \to L^2(\Omega)$ acting componentwise is defined as follows. Let $P_s(K)$ denote the set of all polynomials of degree less than or equal to s and let $D_h(K)$ be a finite-dimensional space on the cell $K \in \mathcal{T}_h$ with $P_s(K) \subset D_h(K)$. We extend the definition by allowing $P_{-1}(K) = D_h(K) = \{0\}$. We introduce the associated global space of discontinuous finite elements

$$D_h := \bigoplus_{K \in \mathcal{T}_h} D_h(K)$$

and the local $L^2(K)$-projection $\pi_K : L^2(K) \to D_h(K)$ generating the global projection $\pi_h : L^2(\Omega) \to D_h$ by

$$(\pi_h w)\big|_K := \pi_K(w|_K) \qquad \forall K \in \mathcal{T}_h,\ \forall w \in L^2(\Omega).$$

The fluctuation operator $\kappa_h : L^2(\Omega) \to L^2(\Omega)$ used in (2.5) is given by $\kappa_h := id - \pi_h$, where $id : L^2(\Omega) \to L^2(\Omega)$ is the identity on $L^2(\Omega)$.

In order to study the supercloseness and superconvergence properties of this method on structured meshes, we introduce the bilinear form

$$A_h((\mathbf{u}, p); (\mathbf{v}, q)) = (\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) + (q, \operatorname{div} \mathbf{u}) + S_h(p, q) \tag{2.6}$$

and the mesh-dependent norm

$$|||(\mathbf{v}, q)|||_A := \Big(|\mathbf{v}|_1^2 + \|q\|_0^2 + \sum_{K\in\mathcal{T}_h} \alpha_K \|\kappa_h \nabla q\|_{0,K}^2 \Big)^{1/2}. \tag{2.7}$$

From (2.1) and (2.4) follows the error equation

$$A_h((\mathbf{u} - \mathbf{u}_h, p - p_h), (\mathbf{v}_h, q_h)) = S_h(p, q_h) \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h, \tag{2.8}$$

which shows that in contrast to residual based stabilization methods [14, 15] the method is inconsistent.

The existence and uniqueness of discrete solutions of (2.4) have been studied in [11] for different pairs (Y_h, D_h) of approximation and projection spaces, respectively. Here, the supercloseness and superconvergence properties will be studied only for the lowest-order cases; i.e., we will consider

- on three-directional triangular meshes ($d = 2$) the cases
  $Y_h := \{v \in H^1(\Omega) : v|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\}$, $D_h := \{0\}$, $\alpha_K \sim h_K^2$,
  $Y_h := \{v \in H^1(\Omega) : v|_K \in P_1^+(K),\ \forall K \in \mathcal{T}_h\}$,
  $D_h := \{q \in L^2(\Omega) : q|_K \in P_0(K),\ \forall K \in \mathcal{T}_h\}$, $\alpha_K \sim h_K$ or $\alpha_K \sim h_K^2$,
- on rectangular or brick meshes ($d = 2$ or $d = 3$) the case
  $Y_h := \{v \in H^1(\Omega) : v|_K \in Q_1(K),\ \forall K \in \mathcal{T}_h\}$, $D_h := \{0\}$, $\alpha_K \sim h_K^2$,

where $Q_1(K)$ denotes the space of mapped bilinear and trilinear functions, respectively, and $P_1^+(K)$ is the space of linear functions enriched by cubic bubbles vanishing on the boundary of K. In the following, we will refer to these different cases shortly as the P_1, the P_1^+, and the Q_1 case, respectively.

All three cases fit to the theory developed in [11]; consequently we have the following stability and convergence result.

Lemma 2.1 ([11])**.** *Assume $h_K^2/\alpha_K \le C$. Then, there is a positive constant β_A independent of h such that*

$$\inf_{(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times Q_h} \sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h} \frac{A_h\big((\mathbf{v}_h, q_h); (\mathbf{w}_h, r_h)\big)}{|||(\mathbf{v}_h, q_h)|||_A \ |||(\mathbf{w}_h, r_h)|||_A} \ge \beta_A > 0 \tag{2.9}$$

holds.

Lemma 2.2 ([11])**.** *Let the solution $(\mathbf{u}, p)$ of (2.1) belong to $(\mathbf{V} \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$. Then, there exists a positive constant C independent of h such that the solution $(\mathbf{u}_h, p_h)$ of (2.4) satisfies*

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_A \le Ch\,(\|\mathbf{u}\|_2 + \|p\|_1). \tag{2.10}$$

Remark 2.3. In the two cases in which $D_h = \{0\}$, the fluctuation operator becomes the identity and (2.4) corresponds to the method studied in [7].

2.3. **Relationship to residual based stabilizations.** In the case of $Y_h = \{v \in H^1(\Omega) : v|_K \in P_1^+(K),\ \forall K \in \mathcal{T}_h\}$, the standard space of continuous, piecewise linear functions has been enriched by one cubic bubble function per cell. These additional degrees of freedom can be eliminated locally by static condensation [11, Section 4.3]. Based on the splitting of the approximation space $Y_h = Y_L \oplus Y_B$ into the piecewise linear part Y_L and the bubble part Y_B the solution $(\mathbf{u}_h, p_h)$ of (2.4) can be split into $\mathbf{u}_h = \mathbf{u}_L + \mathbf{u}_B$ and $p_h = \tilde{p}_L + p_B$ with $\mathbf{u}_L \in \mathbf{V}_L = Y_L^2 \cap \mathbf{V}$ and $\tilde{p}_L \in Y_L$. Let us define

$$p_L = \tilde{p}_L - \frac{1}{|\Omega|}\int_\Omega \tilde{p}_L\,dx \in Q_L := Y_L \cap Q.$$

Then, as shown in [11], the linear part $(\mathbf{u}_L, p_L) \in \mathbf{V}_L \times Q_L$ of $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ is a solution of the problem:

Find $(\mathbf{u}_L, p_L) \in \mathbf{V}_L \times Q_L$ such that for all $(\mathbf{v}_L, q_L) \in \mathbf{V}_L \times Q_L$,

$$B_h\big((\mathbf{u}_L, p_L); (\mathbf{v}_L, q_L)\big) = L_h(\mathbf{v}_L, q_L). \tag{2.11}$$

The bilinear form $B_h(\cdot;\cdot)$ and the linear form $L_h(\cdot)$ are defined by

$$\begin{aligned} B_h\big((\mathbf{u}, p); (\mathbf{v}, q)\big) &:= (\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \operatorname{div}\mathbf{v}) + (q, \operatorname{div}\mathbf{u}) \\ &\quad + \sum_{K\in\mathcal{T}_h} \gamma_K(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v})_K + \sum_{K\in\mathcal{T}_h}(-\Delta\mathbf{u} + \nabla p, \tau_K\nabla q)_K, \\ L_h(\mathbf{v}, q) &:= (\mathbf{f}, \mathbf{v}) + \sum_{K\in\mathcal{T}_h}(\mathbf{f}, \tau_K\nabla q)_K, \end{aligned}$$

with

$$\gamma_K = \frac{\|b_K\|_{0,1,K}^2}{\alpha_K |K|\, \|\kappa_h \nabla b_K\|_{0,K}^2}, \qquad \tau_K = \frac{\|b_K\|_{0,1,K}}{|b_K|_{1,K}^2}\, b_K, \qquad b_K = \lambda_1\lambda_2\lambda_3.$$

Here, λ_i, $i = 1, 2, 3$, denote the barycentric coordinates of K. Note that

$$\tau_K \sim h_K^2 b_K,$$

and that, depending on the choice of the stabilization parameter α_K (which is related to the approximation space D_h) in the LPS, we have

$$\alpha_K \sim h_K^2 \quad \Leftrightarrow \quad \gamma_K \sim 1, \qquad \alpha_K \sim h_K \quad \Leftrightarrow \quad \gamma_K \sim h_K.$$

We mention that the problem (2.11) corresponds to the Pressure Stabilized Petrov Galerkin (PSPG) method [14, 15, 31] in combination with the grad-div stabilization [10, 13, 32, 33]. The PSPG stabilization is consistent in the sense that for a smooth solution $(\mathbf{u}, p) \in (H_0^1(\Omega) \cap H^2(\Omega))^d \times (L_0^2(\Omega) \cap H^1(\Omega))$,

$$B_h\big((\mathbf{u}, p); (\mathbf{v}_L, q_L)\big) = L_h(\mathbf{v}_L, q_L) \quad \forall(\mathbf{v}_L, q_L) \in \mathbf{V}_L \times Q_L$$

holds. We have two options for analyzing the method (2.11): using the error estimate (2.10) and deriving estimates for the linear part of the solution, or studying directly the PSPG method (2.11). We follow the second option by proving the stability of the bilinear form $B_h : (\mathbf{V}_L \times Q_L) \times (\mathbf{V}_L \times Q_L) \to \mathbb{R}$ with respect to the mesh-dependent norm

$$|||(\mathbf{v}, q)|||_B := \Big(|\mathbf{v}|_1^2 + \|q\|_0^2 + \sum_{K\in\mathcal{T}_h}\gamma_K\|\operatorname{div}\mathbf{v}\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h}\|\tau_K^{1/2}\nabla q\|_{0,K}^2\Big)^{1/2}. \tag{2.12}$$

Lemma 2.4. *Assume $\gamma_K = \mathcal{O}(1)$ and $\tau_K \sim h_K^2 b_K$. Then, there is a positive constant β_B independent of h such that*

$$\inf_{(\mathbf{v}_L, q_L)\in \mathbf{V}_L\times Q_L} \sup_{(\mathbf{w}_L, r_L)\in \mathbf{V}_L\times Q_L} \frac{B_h\big((\mathbf{v}_L, q_L);(\mathbf{w}_L, r_L)\big)}{|||(\mathbf{v}_L, q_L)|||_B\ |||(\mathbf{w}_L, r_L)|||_B} \geq \beta_B > 0 \tag{2.13}$$

holds.

Proof. Let $(\mathbf{v}_L, q_L)$ be an arbitrary element of $\mathbf{V}_L \times Q_L$. We obtain

$$B_h\big((\mathbf{v}_L, q_L);(\mathbf{v}_L, q_L)\big) = |\mathbf{v}_L|_1^2 + \sum_{K\in\mathcal{T}_h} \gamma_K \|\text{div } \mathbf{v}_L\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h} \|\tau_K^{1/2}\nabla q_L\|_{0,K}^2. \tag{2.14}$$

Compared with (2.12), just the L^2 control over the pressure is missing. Due to the continuous inf-sup condition (2.2) there is for any $q_L \in Q_L$ an element $\mathbf{v}_{q_L} \in \mathbf{V}$ satisfying

$$-(q_L, \text{div } \mathbf{v}_{q_L}) = \|q_L\|_0^2, \qquad \|\mathbf{v}_{q_L}\|_1 \leq C\|q_L\|_0.$$

As a consequence, we have for the Scott-Zhang [30] interpolant $i_h : H_0^1(\Omega)^2 \to \mathbf{V}_L$,

$$\begin{aligned} B_h\big((\mathbf{v}_L, q_L);(i_h\mathbf{v}_{q_L}, 0)\big) &= -(q_L, \text{div } i_h\mathbf{v}_{q_L}) + (\nabla\mathbf{v}_L, \nabla i_h\mathbf{v}_{q_L}) \\ &\qquad + \sum_{K\in\mathcal{T}_h} \gamma_K(\text{div } \mathbf{v}_L, \text{div } i_h\mathbf{v}_{q_L})_K \\ &= \|q_L\|_0^2 + (q_L, \text{div } (\mathbf{v}_{q_L} - i_h\mathbf{v}_{q_L})) + (\nabla\mathbf{v}_L, \nabla i_h\mathbf{v}_{q_L}) \\ &\qquad + \sum_{K\in\mathcal{T}_h} \gamma_K(\text{div } \mathbf{v}_L, \text{div } i_h\mathbf{v}_{q_L})_K. \end{aligned} \tag{2.15}$$

Using integration by parts, the approximation property of the Scott-Zhang [30] interpolation

$$\|\mathbf{v} - i_h\mathbf{v}\|_{0,K} \leq Ch_K\, \|\mathbf{v}\|_{1,\omega(K)} \qquad \forall \mathbf{v} \in H^1(K),\ K \in \mathcal{T}_h,$$

the inequality

$$\|\tau_K^{1/2}\nabla q_L\|_{0,K} \geq Ch_K\|\nabla q_L\|_{0,K} \qquad \forall q_L \in P_1(K),$$

and the bound of $\|\mathbf{v}_{q_L}\|_1$, we estimate the second term in (2.15) as follows:

$$\begin{aligned} |(q_L, \text{div } (\mathbf{v}_{q_L} - i_h\mathbf{v}_{q_L}))| = |(\nabla q_L, \mathbf{v}_{q_L} - i_h\mathbf{v}_{q_L})| &\leq C \sum_{K\in\mathcal{T}_h} \|\nabla q_L\|_{0,K} h_K \|\mathbf{v}_{q_L}\|_{1,\omega(K)} \\ &\leq C \left(\sum_{K\in\mathcal{T}_h} \|\tau_K^{1/2}\nabla q_L\|_{0,K}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \|\mathbf{v}_{q_L}\|_{1,\omega(K)}^2\right)^{1/2} \\ &\leq \frac{1}{6}\|q_L\|_0^2 + C_1 \sum_{K\in\mathcal{T}_h} \|\tau_K^{1/2}\nabla q_L\|_{0,K}^2. \end{aligned}$$

The estimations of the third and fourth terms in (2.15) are standard:

$$|(\nabla\mathbf{v}_L, \nabla i_h\mathbf{v}_{q_L})| \leq C|\nabla\mathbf{v}_L|_1\, |\nabla\mathbf{v}_{q_L}|_1 \leq \frac{1}{6}\|q_L\|_0^2 + C_2|\nabla\mathbf{v}_L|_1^2,$$

$$\left|\sum_{K\in\mathcal{T}_h} \gamma_K(\text{div } \mathbf{v}_L, \text{div } i_h\mathbf{v}_{q_L})_K\right| \leq \frac{1}{6}\|q_L\|_0^2 + C_3 \sum_{K\in\mathcal{T}_h} \gamma_K\|\text{div } \mathbf{v}_L\|_{0,K}^2.$$

Summing up the last three inequalities, we obtain from (2.15),

$$B_h\big((\mathbf{v}_L, q_L); (i_h \mathbf{v}_{q_L}, 0)\big) \geq \frac{1}{2}\|q_L\|_0^2 - C_4 \left(|\nabla \mathbf{v}_L|_1^2 + \sum_{K\in\mathcal{T}_h} \Big[\|\tau_K^{1/2} \nabla q_L\|_{0,K}^2 + \gamma_K \|\text{div } \mathbf{v}_L\|_{0,K}^2 \Big] \right) \tag{2.16}$$

with $C_4 = \max(C_1, C_2, C_3)$. Multiplying this inequality by $2/(1+2C_4)$ and adding it to (2.14), we see that for any $(\mathbf{v}_L, q_L) \in \mathbf{V}_L \times Q_L$, there exists

$$(\mathbf{w}_L, r_L) := (\mathbf{v}_L, q_L) + (2/(1+2C_4))(i_h \mathbf{v}_{q_L}, 0)$$

such that

$$B_h\big((\mathbf{v}_L, q_L); (\mathbf{w}_L, r_L)\big) \geq \frac{1}{1+2C_4} |||(\mathbf{v}_L, q_L)|||_B^2.$$

Furthermore, the H^1-stability of the interpolation i_h, the upper bound of $\|\mathbf{v}_{q_L}\|_1$, and $\gamma_K \leq \gamma_0$ lead to

$$|||(\mathbf{w}_L, r_L)|||_B \leq |||(\mathbf{v}_L, q_L)|||_B + \frac{2}{1+2C_4} |||(i_h \mathbf{v}_L, 0)|||_B \leq (1 + C_5) |||(\mathbf{v}_L, q_L)|||_B.$$

Thus, the statement of the lemma holds true with $\beta = 1/((1+2C_4)(1+C_5))$. $\square$

Taking into consideration the approximation properties of the space $\mathbf{V}_L \times Q_L$ we get

Lemma 2.5. *Assume $\gamma_K = \mathcal{O}(1)$ and $\tau_K \sim h_K^2 b_K$. Let the solution $(\mathbf{u}, p)$ of (2.1) belong to $(\mathbf{V} \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$. Then, there exists a positive constant C independent of h such that the solution $(\mathbf{u}_L, p_L)$ of (2.11) satisfies*

$$|||(\mathbf{u} - \mathbf{u}_L, p - p_L)|||_B \leq Ch\,(\|\mathbf{u}\|_2 + \|p\|_1). \tag{2.17}$$

Finally in this section, we mention the relationship of the standard Galerkin discretization using the MINI-element to the residual based stabilization method (2.11). In this case, the velocity and pressure are approximated by elements from the spaces

$$\mathbf{V}_h = \mathbf{V}_L^+ = \{\mathbf{v} \in H_0^1(\Omega)^d : \mathbf{v}|_K \in P_1^+(K)^d,\ \forall K \in \mathcal{T}_h\},$$
$$Q_h = Q_L = \{q \in L_0^2(\Omega) \cap H^1(\Omega) : q|_K \in P_1(K)\, \forall K \in \mathcal{T}_h\}.$$

Note that the MINI-element satisfies the discrete version of (2.2); see [1]. Thus no stabilization term is needed. Again eliminating the bubble part in the velocity space we end up with the method (2.11) in $\mathbf{V}_L \times Q_L$ for $\gamma_K = 0$. This relationship between the MINI-element discretization and the residual based stabilization will allow us to prove supercloseness results for the linear part of the MINI-element discretization.

3. Supercloseness

In this section, we consider structured meshes, in particular three-directional triangular meshes in the two-dimensional case and uniform brick meshes in the three-dimensional case. All theorems for the brick meshes hold analogously also for rectangular meshes; however, we do not formulate them explicitly.

3.1. **Piecewise linear interpolations on three-directional meshes.** We start with some interpolation error estimates which are necessary for our supercloseness analysis. Let $i_h : H^2(\Omega)^2 \to \mathbb{R}^2$ and $j_h : H^2(\Omega) \to \mathbb{R}$ denote the standard piecewise linear nodal interpolation. In order to derive the supercloseness property of $(\mathbf{u}_h, p_h)$ to $(i_h\mathbf{u}, j_h p)$, we recall some estimates which can be found, e.g., in [22] and in the books [19, 24].

Lemma 3.1 ([19, 22, 24])**.** *Let $\mathbf{u} \in H^3(\Omega)^2$ and the mesh $\mathcal{T}_h$ be three-directional. Then, we have the estimate*

$$|(\nabla(\mathbf{u} - i_h\mathbf{u}), \nabla \mathbf{w}_h)| \quad \leq \quad Ch^2\|\mathbf{u}\|_3|\mathbf{w}_h|_1, \quad \forall \mathbf{w}_h \in \mathbf{V}_h. \tag{3.1}$$

Remark 3.2. The estimate (3.1) has been obtained by many researchers and may be the oldest supercloseness result ([28]). Nowadays, the proof of this estimate has been simplified and extended to more general meshes [2], which we consider in Section 5.

Let us define the notation illustrated in Figure 1. For an arbitrary triangle

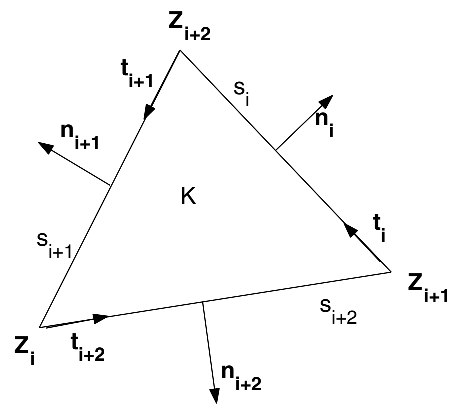

FIGURE 1. Some notation of a triangle $K \in \mathcal{T}_h$.

$K \in \mathcal{T}_h$, let $Z_i = (x_{i1}, x_{i2})$ $(1 \leq i \leq 3)$ be the counterclockwise oriented vertices, s_i $(1 \leq i \leq 3)$ denote the edge of length h_i $(1 \leq i \leq 3)$ opposite to Z_i; $\mathbf{n}_i$ $(1 \leq i \leq 3)$ is the unit outward normal vector on s_i, and $\mathbf{t}_i$ $(1 \leq i \leq 3)$ are the unit tangent vectors in the counterclockwise orientation. We use the periodic relation for the subscripts: $i + 3 = i$ and write for the derivative in the direction of $\mathbf{t}_i$ shortly $\partial_{\mathbf{t}_i} = \partial/\partial \mathbf{t}_i$. As usual, let the reference triangle $\widehat{K}$ have the vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

Lemma 3.3. *Let $\widehat{I}$ be the standard piecewise linear nodal interpolation on $\widehat{K}$. Then, there are positive constants C such that for all $\hat{\varphi} \in H^3(\widehat{K})$ and for all $\hat{\psi} \in P_0(\widehat{K})$,*

$$\left| \int_{\widehat{K}} (\hat{\varphi} - \widehat{I}\hat{\varphi})\, \hat{\psi}\, d\hat{x} + \frac{1}{12} \int_{\widehat{K}} (\hat{\varphi}_{\hat{x}_1\hat{x}_1} - \hat{\varphi}_{\hat{x}_1\hat{x}_2} + \hat{\varphi}_{\hat{x}_2\hat{x}_2})\, \hat{\psi}\, d\hat{x} \right| \leq C|\hat{\varphi}|_{3,\widehat{K}} \|\hat{\psi}\|_{0,\widehat{K}}. \tag{3.2}$$

Proof. We use a Bramble-Hilbert type argument [9, Theorem 4.1.3] in order to prove the expansion formulas on the reference element $\widehat{K}$. For fixed $\hat{\psi} \in P_0(\widehat{K})$, we consider the following continuous linear form $\Phi : H^3(\widehat{K}) \to \mathbb{R}$ given by

$$\hat{\varphi} \mapsto \Phi(\hat{\varphi}) = \int_{\widehat{K}} (\hat{\varphi} - \widehat{I}\hat{\varphi})\, \hat{\psi}\, d\hat{x} + \frac{1}{12} \int_{\widehat{K}} (\hat{\varphi}_{\hat{x}_1\hat{x}_1} - \hat{\varphi}_{\hat{x}_1\hat{x}_2} + \hat{\varphi}_{\hat{x}_2\hat{x}_2})\, \hat{\psi}\, d\hat{x}$$

for which

$$|\Phi(\hat{\varphi})| \le C\|\hat{\varphi}\|_{3,\widehat{K}}\|\hat{\psi}\|_{0,\widehat{K}}$$

holds true. When $\hat{\varphi}$ equals $\hat{x}_1^2$, $\hat{x}_1\hat{x}_2$, and $\hat{x}_2^2$, respectively, the corresponding interpolations $\widehat{I}\hat{u}$ become $\hat{x}_1$, 0, and $\hat{x}_2$. A direct computation shows that

$$\Phi(\hat{\varphi}) = 0, \qquad \forall\hat{\varphi} \in P_2(\widehat{K}).$$

Consequently, there is a positive constant C such that (3.2) holds true. $\square$

Lemma 3.4. *Let $\mathbf{u} \in H^3(\Omega)^2 \cap \mathbf{V}$ and the mesh $\mathcal{T}_h$ be three-directional. Then, we have the estimates*

$$|(r_h, \operatorname{div}(\mathbf{u} - i_h\mathbf{u}))| \le Ch^{3/2}\|\mathbf{u}\|_3\|r_h\|_0, \qquad \forall r_h \in Q_h, \tag{3.3}$$

$$|(\operatorname{div}(\mathbf{u} - i_h\mathbf{u}), \operatorname{div}\,\mathbf{v}_h)| \le Ch^2\|\mathbf{u}\|_3\,\|\mathbf{v}_h\|_1, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{3.4}$$

Proof. We use techniques similar to [41, 42]. We start with (3.3), integrate by parts

$$(r_h, \operatorname{div}\,(\mathbf{u} - i_h\mathbf{u})) = -(\mathbf{u} - i_h\mathbf{u}, \nabla r_h) = -\sum_{K\in\mathcal{T}_h}(\mathbf{u} - i_h\mathbf{u}, \nabla r_h)_K, \tag{3.5}$$

and define an affine mapping $F_K : \widehat{K} \to K$ by

$$x = F_K\hat{x} = (h_1\mathbf{t}_1, -h_3\mathbf{t}_3)\cdot\hat{x} + \mathbf{Z}_2 = B_K\hat{x} + \mathbf{Z}_2.$$

Then, for a function $\hat{w} : \widehat{K} \to \mathbb{R}$ and $w = \hat{w} \circ F_K^{-1}$ we have

$$\hat{w}_{\hat{x}_1} = h_1(\partial_{\mathbf{t}_1}w)\circ F_K, \qquad \hat{w}_{\hat{x}_2} = -h_3(\partial_{\mathbf{t}_3}w)\circ F_K,$$

$$\hat{w}_{\hat{x}_1\hat{x}_1} = h_1^2(\partial^2_{\mathbf{t}_1\mathbf{t}_1}w)\circ F_K, \quad \hat{w}_{\hat{x}_1\hat{x}_2} = -h_1h_3(\partial^2_{\mathbf{t}_1\mathbf{t}_3}w)\circ F_K, \quad \hat{w}_{\hat{x}_2\hat{x}_2} = h_3^2(\partial^2_{\mathbf{t}_3\mathbf{t}_3}w)\circ F_K.$$

Now, transforming onto the reference triangle $\widehat{K}$, using Lemma 3.3 componentwise, transforming back to the original element K, and integrating by parts, we get

$$\begin{aligned}
\int_K(\mathbf{u} - i_h\mathbf{u})\nabla r_h dx &= \det\,B_K\int_{\widehat{K}}(\hat{\mathbf{u}} - \widehat{I}\hat{\mathbf{u}})B_K^{-T}\hat{\nabla}\hat{r}_h d\hat{x}\\
&= -\frac{\det\,B_K}{12}\int_{\widehat{K}}\left(\hat{\mathbf{u}}_{\hat{x}_1\hat{x}_1} - \hat{\mathbf{u}}_{\hat{x}_1\hat{x}_2} + \hat{\mathbf{u}}_{\hat{x}_2\hat{x}_2}\right)B_K^{-T}\hat{\nabla}\hat{r}_h d\hat{x} + R\\
&= -\frac{1}{12}\int_K\left(h_1^2\partial^2_{\mathbf{t}_1\mathbf{t}_1}\mathbf{u} + h_1h_3\partial^2_{\mathbf{t}_1\mathbf{t}_3}\mathbf{u} + h_3^2\partial^2_{\mathbf{t}_3\mathbf{t}_3}\mathbf{u}\right)\nabla r_h dx + R\\
&= \frac{1}{12}\int_K r_h\operatorname{div}\left(h_1^2\partial^2_{\mathbf{t}_1\mathbf{t}_1}\mathbf{u} + h_1h_3\partial^2_{\mathbf{t}_1\mathbf{t}_3}\mathbf{u} + h_3^2\partial^2_{\mathbf{t}_3\mathbf{t}_3}\mathbf{u}\right)dx + R\\
&\quad - \frac{1}{12}\int_{\partial K} r_h\,\left(h_1^2\partial^2_{\mathbf{t}_1\mathbf{t}_1}\mathbf{u} + h_1h_3\partial^2_{\mathbf{t}_1\mathbf{t}_3}\mathbf{u} + h_3^2\partial^2_{\mathbf{t}_3\mathbf{t}_3}\mathbf{u}\right)\cdot\mathbf{n}_K\,ds.
\end{aligned}$$

The first term can be estimated by the Cauchy-Schwarz inequality, leading to

$$\left|\frac{1}{12}\int_K r_h\operatorname{div}\left(h_1^2\partial^2_{\mathbf{t}_1\mathbf{t}_1}\mathbf{u} + h_1h_3\partial^2_{\mathbf{t}_1\mathbf{t}_3}\mathbf{u} + h_3^2\partial^2_{\mathbf{t}_3\mathbf{t}_3}\mathbf{u}\right)dx\right| \le Ch_K^2|\mathbf{u}|_{3,K}\|r_h\|_{0,K},$$

and for the second we obtain from Lemma 3.3,

$$|R| \le C\det\,B_K|\hat{\mathbf{u}}|_{3,\widehat{K}}\|B_K^{-T}\hat{\nabla}\hat{r}_h\|_{0,\widehat{K}} \le C\|B_K\|^3|\mathbf{u}|_{3,K}|r_h|_{1,K} \le Ch_K^2|\mathbf{u}|_{3,K}\|r_h\|_{0,K},$$

where, in the last step, we used an inverse estimate and the standard estimates for affine-equivalent finite elements [9, Theorem 3.1.2]. Summing up over all $K \in \mathcal{T}_h$ we find that the integrals over all inner edges cancel out. Indeed, let K and K'

be two neighbouring cells with a common edge $E = \partial K \cap \partial K'$. Then, for the tangential and outer normal directions we have on E,

$$\mathbf{n}_K = -\mathbf{n}_{K'}, \qquad \mathbf{t}_i^K = -\mathbf{t}_i^{K'}, \quad i = 1, 2, 3.$$

Thus, apart from the sign of $\mathbf{n}_K$, we have the same traces of the second derivatives of $\mathbf{u}$ and of r_h on the common edge E. Thus, we have shown that

$$\begin{aligned}|(r_h, \operatorname{div}(\mathbf{u} - i_h\mathbf{u}))| &\leq Ch^2|\mathbf{u}|_3\|r_h\|_0 \\ &\quad + \frac{1}{12}\sum_{E\subset\partial\Omega}\int_E \left|r_h\left(h_1^2\partial^2_{\mathbf{t}_1\mathbf{t}_1}\mathbf{u} + h_1h_3\partial^2_{\mathbf{t}_1\mathbf{t}_3}\mathbf{u} + h_3^2\partial^2_{\mathbf{t}_3\mathbf{t}_3}\mathbf{u}\right)\cdot\mathbf{n}_E\right| ds,\end{aligned}$$

from which the statement of the lemma follows by using the discrete trace inequality

$$\|r_h\|_{0,E} \leq Ch_K^{-1/2}\|r_h\|_{0,K}, \qquad E \subset \partial K,\ \forall K \in \mathcal{T}_h \tag{3.6}$$

and the continuity of the trace operator $\varphi \mapsto \varphi|_{\partial\Omega}$ from $H^1(\Omega)$ in $L^2(\partial\Omega)$.

Consider now (3.4). Similar to [2, 4] we represent the contribution of one cell by an integral over the cell and line integrals over the edges including only the tangential derivatives of the test function $\mathbf{v}_h$. Integration by parts yields

$$\int_K \operatorname{div}\,(\mathbf{u} - i_h\mathbf{u})\,\operatorname{div}\,\mathbf{v}_h\,dx = \sum_{i=1}^{3}\int_{s_i}(\mathbf{u} - i_h\mathbf{u})\cdot\mathbf{n}_i\,\operatorname{div}\,\mathbf{v}_h\,ds.$$

Transformation to the reference cell, applying the Bramble-Hilbert lemma, and transforming back gives

$$\int_{s_i}(\mathbf{u} - i_h\mathbf{u})\,ds = -\frac{h_i^2}{12}\int_{s_i}\partial^2_{\mathbf{t}_i\mathbf{t}_i}\mathbf{u}\,ds + \mathcal{O}(h_K^3|\mathbf{u}|_{3,K}).$$

There are coefficients ω_{mn}^K, with $m, n \in \{1, 2\}$, such that

$$\operatorname{div}\,\mathbf{v}_h\big|_K = \omega_{11}^K\partial_{\mathbf{t}_i}v_{1,h} + \omega_{12}^K\partial_{\mathbf{t}_{i+1}}v_{1,h} + \omega_{21}^K\partial_{\mathbf{t}_i}v_{2,h} + \omega_{22}^K\partial_{\mathbf{t}_{i+1}}v_{2,h}.$$

The essential idea is to replace the resulting line integrals of $\partial_{\mathbf{t}_{i+1}}$ over s_i by line integrals over s_{i+1} taking into consideration that (Green's theorem)

$$h_{i+1}\int_{s_i}w\,ds - h_i\int_{s_{i+1}}w\,ds = \frac{h_1h_2h_3}{2|K|}\int_K\partial_{\mathbf{t}_{i+2}}w\,dx.$$

As a result, we obtain

$$\begin{aligned}\int_{s_i}(\mathbf{u} - i_h\mathbf{u})\cdot\mathbf{n}_i\,\operatorname{div}\,\mathbf{v}_h\,ds &= -\frac{h_i^2}{12}\int_{s_i}\partial^2_{\mathbf{t}_i\mathbf{t}_i}\mathbf{u}\cdot\mathbf{n}_i\,(\omega_{11}^K\partial_{\mathbf{t}_i}v_{1,h} + \omega_{21}^K\partial_{\mathbf{t}_i}v_{2,h})\,ds \\ &\quad - \frac{h_i^3}{12h_{i+1}}\int_{s_{i+1}}\partial^2_{\mathbf{t}_i\mathbf{t}_i}\mathbf{u}\cdot\mathbf{n}_i\,(\omega_{12}^K\partial_{\mathbf{t}_{i+1}}v_{1,h} + \omega_{22}^K\partial_{\mathbf{t}_{i+1}}v_{2,h})\,ds \\ &\quad - \frac{h_i^2h_1h_2h_3}{24h_{i+1}|K|}\int_K\partial^3_{\mathbf{t}_{i+2}\mathbf{t}_i\mathbf{t}_i}\mathbf{u}\cdot\mathbf{n}_i\,(\omega_{12}^K\partial_{\mathbf{t}_{i+1}}v_{1,h} + \omega_{22}^K\partial_{\mathbf{t}_{i+1}}v_{2,h})\,dx \\ &\quad + \mathcal{O}(h_K^3|\mathbf{u}|_{3,K}\left|\operatorname{div}\,\mathbf{v}_h|_K\right|).\end{aligned}$$

Now, summing over the edges s_i of the cell K and over all cells $K \in \mathcal{T}_h$, the line integrals cancel out, since for neighbouring cells K and K',

$$\mathbf{n}_i^K = -\mathbf{n}_i^{K'}, \quad \mathbf{t}_i^K = -\mathbf{t}_i^{K'}, \quad \omega_{mn}^K = -\omega_{mn}^{K'}$$

and on the boundary the tangential derivative of $\mathbf{v}_h$ vanishes. The sum over the integrals over K gives an $\mathcal{O}(h^2\|\mathbf{u}\|_3\,\|\mathbf{v}_h\|_1)$ term and

$$\sum_{K\in\mathcal{T}_h} h_K^3|\mathbf{u}|_{3,K}\big|\text{div } \mathbf{v}_h|_K\big| = \sum_{K\in\mathcal{T}_h} \frac{h_K^3}{|K|^{1/2}}|\mathbf{u}|_{3,K}\|\text{div } \mathbf{v}_h\|_{0,K} = \mathcal{O}(h^2\|\mathbf{u}\|_3\,\|\mathbf{v}_h\|_1).$$

Thus, (3.4) is proven. □

Moreover, we have the following estimate from interpolation theory.

Lemma 3.5. *Assume $\alpha_K \sim h_K^2$ and $p \in H^2(\Omega)$. Then, it follows that*

$$|(p - j_hp, \text{div } \mathbf{w}_h)| \le Ch^2\|p\|_2|\mathbf{w}_h|_1, \qquad \forall \mathbf{w}_h \in \mathbf{V}_h, \tag{3.7}$$

$$\Big|\sum_{K\in\mathcal{T}_h} \alpha_K(\nabla(p - j_hp), \nabla r_h)_K\Big| \le Ch^2\|p\|_2\|r_h\|_0, \qquad \forall r_h \in Q_h. \tag{3.8}$$

Finally, we need an estimate for the consistency error of the stabilized method.

Lemma 3.6. *Assume $\alpha_K \sim h_K^2$ and $p \in H^2(\Omega)$. Then, it follows that*

$$\Big|\sum_{K\in\mathcal{T}_h} \alpha_K(\nabla p, \nabla r_h)_K\Big| \quad\le\quad Ch^{3/2}\|p\|_2\|r_h\|_0, \quad \forall r_h \in Q_h. \tag{3.9}$$

Proof. Integration by parts, the continuity of the trace operator, and the discrete trace inequality (3.6) yield

$$\begin{aligned}\left|\sum_{K\in\mathcal{T}_h} \alpha_K(\nabla p, \nabla r_h)_K\right| &= \left|\int_\Omega \alpha_K \Delta p r_h dx\right| + \left|\int_{\partial\Omega} \alpha_K \frac{\partial p}{\partial \mathbf{n}} r_h ds\right| \\ &\le C(h^2 + h^{3/2})\|p\|_2\|r_h\|_0,\end{aligned}$$

from which the statement of the lemma follows. □

Now, we show the supercloseness of the finite element solution $(\mathbf{u}_h, p_h)$ of the stabilized scheme (2.4) to the piecewise linear interpolant $(i_h\mathbf{u}, j_hp) \in \mathbf{V}_h \times Q_h$.

Theorem 3.7. *Let $\alpha_K \sim h_K^2$, the mesh $\mathcal{T}_h$ be three-directional, and the solution $(\mathbf{u}, p)$ of (2.1) belong to $H^3(\Omega)^2\times H^2(\Omega)$. Then, we have the supercloseness estimate for the finite element approximation $(\mathbf{u}_h, p_h)$,*

$$|||(\mathbf{u}_h - i_h\mathbf{u}, p_h - j_hp)|||_A \quad\le\quad Ch^{3/2}(\|\mathbf{u}\|_3 + \|p\|_2). \tag{3.10}$$

Proof. From (2.1) and (2.4), we get

$$\begin{aligned} A_h((\mathbf{u}_h{-}i_h\mathbf{u}, p_h - j_hp); (\mathbf{w}_h, r_h)) &= A_h((\mathbf{u} - i_h\mathbf{u}, p - j_hp); (\mathbf{w}_h, r_h)) + S_h(p, r_h) \\ &= (\nabla\mathbf{u} - i_h\mathbf{u}, \nabla\mathbf{w}_h) - (p - j_hp, \text{div}\mathbf{w}_h) + (r_h, \text{div}(\mathbf{u} - i_h\mathbf{u})) \\ &\qquad + S_h(p - j_hp, r_h) + S_h(p, r_h).\end{aligned}$$

Using the stability of the bilinear form A_h with respect to the triple norm $|||\cdot|||_A$ (see Lemma 2.1), we obtain

$$\begin{aligned}|||(\mathbf{u}_h - i_h\mathbf{u}, p_h - j_hp)|||_A &\le \frac{1}{\beta_A} \sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h} \frac{A_h((\mathbf{u}_h - i_h\mathbf{u}, p_h - j_hp); (\mathbf{w}_h, r_h))}{|||(\mathbf{w}_h, r_h)|||_A} \\ &= \frac{1}{\beta_A} \sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h} \frac{A_h((\mathbf{u} - i_h\mathbf{u}, p - j_hp); (\mathbf{w}_h, r_h)) + S_h(p, r_h)}{|||(\mathbf{w}_h, r_h)|||_A} \\ &\le Ch^{3/2}(\|\mathbf{u}\|_3 + \|p\|_2),\end{aligned}$$

where the estimates in Lemmas 3.1, 3.4, 3.5, and 3.6 have been applied. □

In a similar way, we can show the supercloseness of the piecewise linear part $(\mathbf{u}_L, p_L)$ of the solution $(\mathbf{u}_h, p_h)$ of the LPS method for the pair of spaces

$$Y_h = \{v \in H^1(\Omega) : v|_K \in P_1^+(K), \forall K \in \mathcal{T}_h\},$$
$$D_h = \{q \in L^2(\Omega) : q|_K \in P_0(K), \forall K \in \mathcal{T}_h\}$$

and the choice $\alpha_K \sim h_K$ and $\alpha_K \sim h_K^2$, respectively. We remind the reader that the linear part is a solution of the PSPG stabilized method (2.11).

Theorem 3.8. *Let $\gamma_K = \mathcal{O}(1)$, $\tau_K \sim h_K^2 b_K$, the mesh $\mathcal{T}_h$ be three-directional, and the solution $(\mathbf{u}, p)$ of (2.1) belong to $H^3(\Omega)^2 \times H^2(\Omega)$. Then, we have the supercloseness estimate for the finite element solution $(\mathbf{u}_L, p_L)$ of (2.11) (or equivalently for the linear part of the LPS solution computed with $\alpha_K \sim h_K$ or $\alpha_K \sim h_K^2$)*

$$|||(\mathbf{u}_L - i_h\mathbf{u}, p_L - j_h p)|||_B \le Ch^{3/2}(\|\mathbf{u}\|_3 + \|p\|_2).$$

Proof. We start with the stability of the bilinear form B_h with respect to the triple norm $|||\cdot|||_B$ given in Lemma 2.4, i.e.

$$\begin{aligned}|||(\mathbf{u}_L - i_h\mathbf{u}, p_L - j_h p)|||_B &\le \frac{1}{\beta_B} \sup_{(\mathbf{v}_L, q_L)\in \mathbf{V}_L\times Q_L} \frac{B_h\big((\mathbf{u}_L - i_h\mathbf{u}, p_L - j_h p); (\mathbf{v}_L, q_L)\big)}{|||(\mathbf{v}_L, q_L)|||_B}\\ &= \frac{1}{\beta_B} \sup_{(\mathbf{v}_L, q_L)\in \mathbf{V}_L\times Y_L} \frac{B_h\big((\mathbf{u} - i_h\mathbf{u}, p - j_h p); (\mathbf{v}_L, q_L)\big)}{|||(\mathbf{v}_L, q_L)|||_B}.\end{aligned}$$

Next we use the following identity and estimate each term separately:

$$\begin{aligned}B_h\big((\mathbf{u} - i_h\mathbf{u}, p - j_h p);(\mathbf{v}_L, q_L)\big) &= (\nabla(\mathbf{u} - i_h\mathbf{u}), \nabla\mathbf{v}_L) - (p - j_h p, \operatorname{div} \mathbf{v}_L)\\ &\quad + \sum_{K\in\mathcal{T}_h} \gamma_K(\operatorname{div} (\mathbf{u} - i_h\mathbf{u}), \operatorname{div} \mathbf{v}_L)_K + (q_L, \operatorname{div} (\mathbf{u} - i_h\mathbf{u}))\\ &\quad + \sum_{K\in\mathcal{T}_h} (-\Delta(\mathbf{u} - i_h\mathbf{u}) + \nabla(p - j_h p), \tau_K\nabla q_L)_K.\end{aligned} \tag{3.11}$$

The first, second, and fourth terms on the right hand side can be considered as above. For the third term it follows from $\gamma_K = \mathcal{O}(1)$ and Lemma 3.4 that

$$\left|\sum_{K\in\mathcal{T}_h} \gamma_K(\operatorname{div} (\mathbf{u} - i_h\mathbf{u}), \operatorname{div} \mathbf{v}_L)_K\right| \le Ch^2\|\mathbf{u}\|_3\, |||(\mathbf{v}_L, q_L)|||_B.$$

We split the last term into

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h} (-\Delta(\mathbf{u} - i_h\mathbf{u}) + \nabla(p - j_h p), \tau_K\nabla q_L)_K\\ &\qquad = \sum_{K\in\mathcal{T}_h} (-\Delta(\mathbf{u} - i_h\mathbf{u}), \tau_K\nabla q_L)_K + \sum_{K\in\mathcal{T}_h} (\nabla(p - j_h p), \tau_K\nabla q_L)_K.\end{aligned}$$

Let $\Pi_K : L^2(K) \to P_0(K)$ denote the local L^2-projection onto $P_0(K)$. Since $\Delta i_h \mathbf{u} = 0$ on each cell $K \in \mathcal{T}_h$ we have

$$\left| \sum_{K\in\mathcal{T}_h} (-\Delta(\mathbf{u} - i_h\mathbf{u}), \tau_K \nabla q_L)_K \right| \le \sum_{K\in\mathcal{T}_h} |(\Delta\mathbf{u} - \Pi_K \Delta\mathbf{u}, \tau_K \nabla q_L)_K| + \left| \sum_{K\in\mathcal{T}_h} (\Pi_K \Delta\mathbf{u}, \tau_K \nabla q_L)_K \right|.$$

The estimation of the first term on the right hand side is standard:

$$\begin{aligned} \sum_{K\in\mathcal{T}_h} |(\Delta\mathbf{u} - \Pi_K \Delta\mathbf{u}, \tau_K \nabla q_L)_K| &\le C \sum_{K\in\mathcal{T}_h} h_K^2 \|\mathbf{u}\|_{3,K} \|\tau_K^{1/2} \nabla q_L\|_{0,K} \\ &\le Ch^2 \|\mathbf{u}\|_3 |||(\mathbf{v}_L, q_L)|||_B. \end{aligned}$$

For the second the relation

$$(\Pi_K \Delta\mathbf{u}, \tau_K \nabla q_L)_K = \frac{1}{|K|} \int_K \tau_K \, dx \ (\Delta\mathbf{u}, \nabla q_L)_K = \frac{1}{|K|} \frac{\|b_K\|_{0,1,K}^2}{|b_K|_{1,K}^2} \ (\Delta\mathbf{u}, \nabla q_L)_K$$

is taken into consideration where, on a three-directional mesh,

$$\frac{1}{|K|} \frac{\|b_K\|_{0,1,K}^2}{|b_K|_{1,K}^2} = C_0 h_K^2$$

with a fixed constant C_0. Integrating by parts, we get

$$\begin{aligned} \left| \sum_{K\in\mathcal{T}_h} (\Pi_K \Delta\mathbf{u}, \tau_K \nabla q_L)_K \right| &\le C_0 h^2 \left| \sum_{K\in\mathcal{T}_h} (\Delta\mathbf{u}, \nabla q_L)_K \right| \\ &\le C_0 h^2 \left\{ \langle \Delta\mathbf{u} \cdot \mathbf{n}, q_L \rangle_{\partial\Omega} - (\operatorname{div} \Delta\mathbf{u}, q_L)_\Omega \right\} \\ &\le Ch^{3/2} \|\mathbf{u}\|_3 \, \|q_L\|_0, \end{aligned}$$

where in the last step the discrete trace inequality (3.6) has been applied.

Finally, we get

$$\begin{aligned} \left| \sum_{K\in\mathcal{T}_h} (\nabla(p - j_h p), \tau_K \nabla q_L)_K \right| &\le \sum_{K\in\mathcal{T}_h} \|\tau_K^{1/2} \nabla(p - j_h p)\|_{0,K} \, \|\tau_K^{1/2} \nabla q_L\|_{0,K} \\ &\le C \sum_{K\in\mathcal{T}_h} h_K^2 \|p\|_{2,K} \, \|\tau_K^{1/2} \nabla q_L\|_{0,K} \\ &\le Ch^2 \|p\|_2 \, |||(\mathbf{v}_L, q_L)|||_B, \end{aligned}$$

which completes the arguments. □

From the relationship of the Standard Galerkin discretization using the MINI-element to the residual based stabilization method (2.11) with $\gamma_K = 0$ we get the following result:

Theorem 3.9. *Let $\gamma_K = 0$, $\tau_K \sim h_K^2 b_K$, the mesh $\mathcal{T}_h$ be three-directional, and the solution $(\mathbf{u}, p)$ of (2.1) belong to $H^3(\Omega)^2 \times H^2(\Omega)$. Then, we have the supercloseness estimate for the finite element solution $(\mathbf{u}_L, p_L)$ of (2.11) or equivalently for the linear part of the MINI-element Galerkin finite element solution*

$$|||(\mathbf{u}_L - i_h\mathbf{u}, p_L - j_h p)|||_B \le Ch^{3/2} (\|\mathbf{u}\|_3 + \|p\|_2).$$

Remark 3.10. This type of supercloseness has been observed experimentally in a number of papers starting in [34, page 312]. For example the numerical results in [16, Table 4] demonstrate clearly the 3/2 rate of $|\mathbf{u}_L - i_h\mathbf{u}|_1$.

3.2. **Piecewise trilinear interpolations on brick meshes.** Let us consider now the stabilized Q_1-Q_1 finite element on brick meshes in $\mathbb{R}^3$. All results are true analogously on rectangular meshes in $\mathbb{R}^2$. The edges of each cell K are parallel to the coordinate axes; their lengths are denoted by $2l_K, 2k_K$, and $2m_K$. We suppose that the family of meshes is shape regular; i.e., there is a constant C, such that

$$C\sqrt{l_K^2+k_K^2+m_K^2} \le \min\{l_K,k_K,m_K\}, \qquad \forall K\in\mathcal{T}_h.$$

Thus, h is defined by $h := \max_{K\in\mathcal{T}_h}\{2\sqrt{l_K^2+k_K^2+m_K^2}\}$. The reference cell is given by $\hat{K}=(-1,1)^3$. For simplicity of notation, we will write $(x,y,z)\in K$ instead of $(x_1,x_2,x_3)\in K$ and $(\xi,\eta,\zeta)\in\widehat{K}$ instead of $(\hat{x}_1,\hat{x}_2,\hat{x}_3)\in\widehat{K}$ in this section. We introduce the nodal interpolation operator $\widehat{I}: H^2(\hat{K})\to Q_1(\hat{K})$ with $\widehat{I}\hat{v}(a_i)=\hat{v}(a_i)$, where $a_i, i=1,\dots,8$ denote the vertices of $\hat{K}$. The interpolation $i_h(u)$ on an arbitrary cell K is given by $i_h(u)|_K := (\widehat{I}(u|_K\circ F_K))\circ F_K^{-1}$, with F_K being a bijective affine mapping from $\hat{K}$ to K. As usual, we apply the interpolation on vector-valued functions in a componentwise manner. The interpolation operator for the pressure p is denoted by j_h and uses the same degrees of freedom as i_h.

Lemma 3.11. *Let $\mathbf{u}\in H^3(\Omega)^3$ and $i_h\mathbf{u}$ be the piecewise trilinear interpolant. Then, on a family of brick meshes we have*

$$|(\nabla(\mathbf{u}-i_h\mathbf{u}),\nabla\mathbf{w}_h)| \le Ch^2|\mathbf{u}|_3|\mathbf{w}_h|_1 \qquad \forall\mathbf{w}_h\in\mathbf{V}_h.$$

Proof. The proof is similar to that of the supercloseness result in [25]. □

Lemma 3.12. *Let $\alpha_K\sim h_K^2$, $p\in H^2(\Omega)$, and j_h be the interpolant defined above. Then, the following estimation holds:*

$$|((p-j_hp),\operatorname{div}\mathbf{w}_h)| \le Ch^2\|p\|_2|\mathbf{w}_h|_1 \qquad \forall\mathbf{w}_h\in\mathbf{V}_h,$$
$$\Big|\sum_{K\in\mathcal{T}_h}\alpha_K(\nabla(p-j_hp),\nabla r_h)_K\Big| \le Ch^2\|p\|_2|||(\mathbf{w}_h,r_h)|||_A \qquad \forall r_h\in Q_h.$$

Proof. The estimation follows from Cauchy-Schwarz inequality and the approximation properties of the Q_1 interpolation operator. □

For the estimation of the term $|(r_h,\operatorname{div}(\mathbf{u}-i_h\mathbf{u}))|$ we need the following lemma:

Lemma 3.13. *Let $\hat{u}\in H^3(\widehat{K})$. Then for all $\hat{r}_h\in Q_1(\widehat{K})$ we have*

$$\int_{\widehat{K}}\hat{r}_h\partial_\xi(\hat{u}-\widehat{I}\hat{u})d\xi d\eta d\zeta = \frac{1}{3}\int_{\widehat{K}}\partial_\xi(\partial_{\xi\xi}\hat{u}\hat{r}_h)d\xi d\eta d\zeta + \mathcal{O}(|\hat{u}|_{3,\widehat{K}}\|\hat{r}_h\|_{0,\widehat{K}}).$$

Proof. Again, the Bramble-Hilbert lemma is used. For a fixed $\hat{r}_h\in Q_1(\widehat{K})$ we consider the mapping $\Psi: H^3(\widehat{K})\to\mathbb{R}$,

$$\hat{u}\mapsto\Psi(\hat{u}) := \int_{\widehat{K}}\hat{r}_h\partial_\xi(\hat{u}-\widehat{I}\hat{u})d\xi d\eta d\zeta - \frac{1}{3}\int_{\widehat{K}}\partial_\xi(\partial_{\xi\xi}\hat{u}\hat{r}_h)d\xi d\eta d\zeta,$$

which is obviously a linear and continuous mapping with

$$\begin{aligned}|\Psi(\hat{u})| &\le C(\|\hat{r}_h\|_{0,\widehat{K}}|\hat{u}-\widehat{I}\hat{u}|_{1,\widehat{K}} + |\hat{u}|_{2,\widehat{K}}|\hat{r}_h|_{1,\widehat{K}} + |\hat{u}|_{3,\widehat{K}}\|\hat{r}_h\|_{0,\widehat{K}})\\ &\le C\|\hat{r}_h\|_{0,\widehat{K}}\|\hat{u}\|_{3,\widehat{K}}.\end{aligned}$$

We need to show that $\Psi(\hat{u}) = 0$ for $\hat{u} \in P_2(\widehat{K})$. Since $\widehat{I}$ is the Q_1 interpolation operator, and $\partial_{\xi\xi}\hat{u} = 0$, for $\hat{u} \in Q_1(\widehat{K})$, it is sufficient to investigate $\hat{u} \in span\{\xi^2, \eta^2, \zeta^2\}$. Since the ξ-derivative appears in both integrals, only $\hat{u} = \xi^2$ remains to analyze. With $\widehat{I}\xi^2 = 1$ we get by direct computation,

$$\Psi(\xi^2) = \int_{\widehat{K}} \hat{r}_h(2\xi) d\xi d\eta d\zeta - \frac{1}{3}\int_{\widehat{K}} \partial_\xi(2\hat{r}_h) d\xi d\eta d\zeta = 0.$$

Applying the Bramble-Hilbert lemma, we finally have

$$|\Psi(\hat{u})| \le C\|\hat{r}_h\|_{0,\widehat{K}} |\hat{u}|_{3,\widehat{K}},$$

which is the statement of the lemma. □

Remark 3.14. Analogous estimates can be shown by replacing the ξ-derivatives by η-derivatives and ζ-derivatives, respectively.

Lemma 3.15. *Let $\mathbf{u} \in H^3(\Omega)^3$ and $\mathcal{T}_h$ be a decomposition into bricks of uniform size ($l_K = l$, $k_K = k$, and $m_K = m$). Then, we have*

$$|(r_h, \text{div } (\mathbf{u} - i_h\mathbf{u})| \le Ch^{3/2}\|\mathbf{u}\|_3 \|r_h\|_0 \quad \forall r_h \in Q_h.$$

Proof. Again, it is sufficient to consider $(\partial_x(u - i_h u), r_h)_K$. By mapping to the reference cell $\widehat{K}$ and using Lemma 3.13 we have

$$\begin{aligned}
\int_K \partial_x(u - i_h u) r_h dxdydz &= \frac{l_K k_K m_K}{l_K} \int_{\widehat{K}} \partial_\xi(\hat{u} - \widehat{I}\hat{u})\hat{r}_h d\xi d\eta d\zeta \\
&= k_K m_K \left\{ \frac{1}{3}\int_{\widehat{K}} \partial_\xi(\partial_{\xi\xi}\hat{u}\hat{r}_h) d\xi d\eta d\zeta + \mathcal{O}(|\hat{u}|_{3,\widehat{K}} \|\hat{r}_h\|_{0,\widehat{K}}) \right\} \\
&= k_K m_K \left\{ \frac{1}{3}\frac{l_K^3}{l_K k_K m_K} \int_K \partial_x(\partial_{xx} u r_h) dxdydz + \mathcal{O}(|u|_{3,K}\|r_h\|_{0,K}) \right\} \\
&= \frac{1}{3} l_K^2 \int_K \partial_x(\partial_{xx} u r_h) dxdydz + \mathcal{O}(h^2 |u|_{3,K}\|r_h\|_{0,K}).
\end{aligned}$$

Now the integrals over the cell K can be represented as the difference of integrals over opposite faces of K, e.g. if S_1 and S_2 are the opposite faces of K belonging to the planes $x = x_K \pm l_K$, we have for any smooth function Λ,

$$\int_K \partial_x \Lambda(x,y,z)\, dxdydz = \int_{S_1} \Lambda(x_K + l_K, y, z) dydz - \int_{S_2} \Lambda(x_K - l_K, y, z) dydz;$$

thus

$$\frac{1}{3} l_K^2 \int_K \partial_x(\partial_{xx} u r_h) dxdydz = \frac{1}{3} l_K^2 \left(\int_{S_1} - \int_{S_2} \right) \partial_{xx} u r_h dydz.$$

Summing over all cells K, the integrals cancel out inside Ω, since r_h is continuous over the inner element faces and $l_K = l_{K'}$ for neighbouring cells K and K'. Finally, we obtain

$$\begin{aligned}
\Big| \sum_{K \in \mathcal{T}_h} \int_K \partial_x(u - i_h u) r_h dxdydz \Big| &\le Ch^2 |u|_3 \|r_h\|_0 + Ch^2 \int_{\partial\Omega} |\partial_{xx} u r_h| d\gamma \\
&\le Ch^2 |u|_3 \|r_h\|_0 + Ch^2 \|u\|_3 \|r_h\|_{0,\partial\Omega}
\end{aligned}$$

by using both a global trace inequality and the discrete trace inequality (3.6). □

Lemma 3.16. *Assume $\alpha_K \sim h_K^2$ and $p \in H^2(\Omega)$. Then, the estimate*

$$|S_h(p, r_h)| \le Ch^{3/2}\|p\|_2\|r_h\|_0 \qquad \forall r_h \in Q_h$$

holds true for the stabilization term.

Proof. Analogous to the proof of Lemma 3.6. □

Theorem 3.17. *Let $(\mathbf{u}, p) \in H^3(\Omega)^3 \times H^2(\Omega)$ and let i_h and j_h be the piecewise trilinear interpolations. Then, on a familiy of uniform brick meshes we have for the LPS finite element solution,*

$$|||(\mathbf{u}_h - i_h\mathbf{u}, p_h - j_h p)|||_A \le Ch^{3/2}(\|\mathbf{u}\|_3 + \|p\|_2).$$

Proof. Using Lemmas 3.11, 3.12, 3.15 and 3.16 the proof follows the line of the proof of Theorem 3.7. □

4. Postprocessing and superconvergence

4.1. **Piecewise quadratic postprocessing.** In this section, we define an interpolation postprocessing operator I_{2h} allowing us to improve the original finite element approximations and to obtain a superconvergence result. In contrast to the standard approach of superconvergence for the Stokes problem [17, 18, 20, 23, 25, 38], we do not need any postprocessing for the pressure because the pressure approximation itself is superconvergent:

$$\|p_h - p\|_0 \le \|p_h - j_h p\|_0 + \|j_h p - p\|_0 \le C(h^{3/2} + h^2)(\|\mathbf{u}\|_3 + \|p\|_2). \tag{4.1}$$

Here, we assume that the mesh $\mathcal{T}_h$ is generated from a coarse mesh $\mathcal{T}_{2h}$ by a regular refinement (connecting the edge midpoints). Then, it is easy to see that each patch $\widetilde{K} \in \mathcal{T}_{2h}$ consists of 4 congruent child triangles $K_i \in \mathcal{T}_h, i = 1, 2, 3, 4$, indicated in Figure 2. The P_2 postprocessing interpolation operator I_{2h} will be

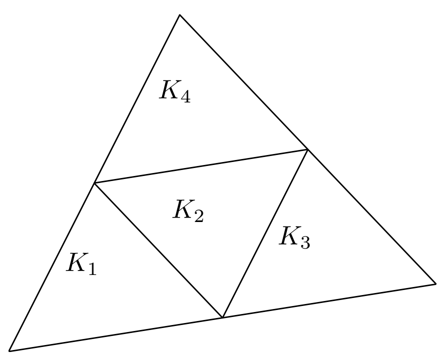

FIGURE 2. The patch $\widetilde{K} \in \mathcal{T}_{2h}$ and its 4 child triangles.

locally defined by

$$I_{2h}\mathbf{v}|_{\widetilde{K}} = I_{2h}(\mathbf{v}|_{\widetilde{K}}),$$

where on each patch, I_{2h} coincides with the standard quadratic Lagrange nodal interpolation in the six degrees of freedom, the function values in the three vertices and the three midpoints of edges. The postprocessing interpolation operator I_{2h} satisfies the following properties.

Lemma 4.1. *For the patchwise quadratic interpolation I_{2h} and the piecewise linear interpolations i_h and j_h the properties*

$$I_{2h} i_h \mathbf{w} = I_{2h}\mathbf{w} \qquad \forall \mathbf{w} \in C(\overline{\Omega})^2, \tag{4.2}$$

$$|||(I_{2h}\mathbf{v}_h, q_h)||| \le C\, |||(\mathbf{v}_h, q_h)||| \qquad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h, \tag{4.3}$$

$$|||(\mathbf{u} - I_{2h}\mathbf{u}, p - j_h p)||| \le Ch^2(\|\mathbf{u}\|_3 + \|p\|_2) \quad \forall (\mathbf{u}, p) \in H^3(\Omega)^2 \times H^2(\Omega) \tag{4.4}$$

hold true, where $|||\cdot|||$ denotes $|||\cdot|||_A$ (with $\alpha_K = \mathcal{O}(h_K^2)$) and $|||\cdot|||_B$ (with $\gamma_K = \mathcal{O}(1)$ and $\tau_K = \mathcal{O}(h_K^2)$), respectively.

Proof. Property (4.2) is simple to see and well known. The estimate (4.4) depends on the choices of the stabilization parameters; however, for the $|||\cdot|||_A$-norm we have $\alpha_K = \mathcal{O}(h_K^2)$ and for the $|||\cdot|||_B$-norm $\tau_K = \mathcal{O}(h_K^2)$, and $\gamma_K = \mathcal{O}(1)$, which is sufficient to get the second-order convergence.

For the stability (4.3) it is enough to show that

$$|I_{2h}\mathbf{v}_h|^2_{1,\widetilde{K}} \le C \sum_{K\subset\widetilde{K}} |\mathbf{v}_h|^2_{1,K} \qquad \forall \mathbf{v} \in \mathbf{V}_h.$$

This follows by transformation onto a reference patch and norm equivalence on finite-dimensional spaces. □

After constructing the postprocessing operator I_{2h}, we can state the following superconvergence result.

Theorem 4.2. *Assume that the postprocessing operator I_{2h} satisfies* (4.2)-(4.4). *Under the assumption of Theorem* 3.7*, we have the following superconvergence result for the case $Y_h = \{v \in H^1(\Omega) :\ v|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\}$ and $D_h = \{0\}$:*

$$|||(I_{2h}\mathbf{u}_h - \mathbf{u}, p_h - p)|||_A \quad \le \quad Ch^{3/2}(\|\mathbf{u}\|_3 + \|p\|_2) \tag{4.5}$$

and the following superconvergence result for the piecewise linear part in the approximation space $Y_h = \{v \in H^1(\Omega) : v|_K \in P_1^+(K), \forall K \in \mathcal{T}_h\}$ and the discontinuous projection space $D_h = \{q :\ q|_K \in P_0(K),\ \forall K \in \mathcal{T}_h\}$:

$$|||(I_{2h}\mathbf{u}_L - \mathbf{u}, p_L - p)|||_B \quad \le \quad Ch^{3/2}(\|\mathbf{u}\|_3 + \|p\|_2). \tag{4.6}$$

Proof. From (3.10) and (4.2)-(4.4), we have

$$\begin{aligned}
&|||(I_{2h}\mathbf{u}_h - \mathbf{u}, p_h - p)|||_A \\
&\qquad \le |||(I_{2h}(\mathbf{u}_h - i_h\mathbf{u}), p_h - j_h p)|||_A + |||(I_{2h} i_h \mathbf{u} - \mathbf{u}, j_h p - p)|||_A \\
&\qquad \le C|||(\mathbf{u}_h - i_h\mathbf{u}, p_h - j_h p)|||_A + |||(I_{2h}\mathbf{u} - \mathbf{u}, j_h p - p)|||_A \\
&\qquad \le Ch^{3/2}(\|\mathbf{u}\|_3 + \|p\|_2).
\end{aligned}$$

Thus, (4.5) is shown. The estimate (4.6) follows by the same arguments. □

4.2. **Piecewise d-quadratic postprocessing.** In this subsection, we construct a postprocessing interpolation operator for the rectangular or brick meshes. In the axiparallel rectangular, and brick case, no postprocessing for the pressure is needed. For the velocity, we assume similar to the triangular case that the mesh $\mathcal{T}_h$ was obtained from a coarse mesh $\mathcal{T}_{2h}$ by a regular refinement. Then, each patch $\widetilde{K} \in \mathcal{T}_{2h}$ consists of 2^d congruent child bricks $K_i \in \mathcal{T}_h, i = 1, 2, \ldots, 2^d$; see Figure 3.

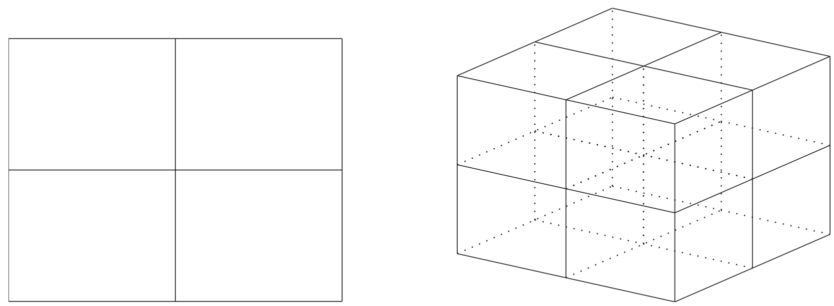

FIGURE 3. The patch $\widetilde{K} \in \mathcal{T}_{2h}$ and its child rectangles ($d = 2$) or bricks ($d = 3$).

Now, we define the d-quadratic interpolation operator I_{2h} locally by

$$I_{2h}\mathbf{v}|_{\widetilde{K}} = I_{2h}(\mathbf{v}|_{\widetilde{K}}),$$

and use on each patch $\widetilde{K}$ the Q_2 Lagrange interpolation defined by

$$I_{2h}\mathbf{v}(Z_i) \quad = \quad \mathbf{v}(Z_i), \tag{4.7}$$

where Z_i, $i = 1, 2, \ldots, 3^d$ are the vertices of the child bricks belonging to the patch.

The postprocessing interpolation I_{2h} satisfies the properties (4.2)-(4.4). Thus, we have a similar superconvergence result as Theorem 4.2 for the approximation $(I_{2h}\mathbf{u}_h, p_h)$.

Theorem 4.3. *Assume that $\mathcal{T}_h$ is a family of axiparallel uniform rectangular or uniform brick-type meshes. Let I_{2h} be the patchwise d-quadratic interpolation. Under the assumptions of Theorem* 3.17*, we have the superconvergence result:*

$$|||(I_{2h}\mathbf{u}_h - \mathbf{u}, p_h - p)|||_A \quad \leq \quad Ch^{3/2}(\|\mathbf{u}\|_3 + \|p\|_2). \tag{4.8}$$

Proof. The arguments are analogous to those of the proof of Theorem 4.2. $\square$

5. Extension to more general meshes

In realistic computations, we cannot always work with a three-directional mesh. Following the ideas of [2] for the Poisson equation, we extend the superconvergence result valid for three-directional meshes to more general meshes.

We state first the mesh conditions. Two adjacent triangles (sharing a common edge) are said to form an $\mathcal{O}(h^{1+\rho})$ ($\rho > 0$) approximate parallelogram if the lengths of any two opposite edges differ only by $\mathcal{O}(h^{1+\rho})$.

Definition 5.1. The triangulation $\mathcal{T}_h = \mathcal{T}_{1,h} \cup \mathcal{T}_{2,h}$ is said to satisfy *condition* (ρ, σ) if there exist positive constants ρ and σ such that every two adjacent triangles inside $\mathcal{T}_{1,h}$ form an $O(h^{1+\rho})$ parallelogram and

$$\overline{\Omega}_{1,h} \cup \overline{\Omega}_{2,h} = \overline{\Omega}, \quad |\Omega_{2,h}| = O(h^{\sigma}), \quad \overline{\Omega}_{i,h} = \bigcup_{K \in \mathcal{T}_{i,h}} \overline{K}, \quad i = 1, 2.$$

The *condition* (ρ, σ) is a reasonable condition in practice and can be satisfied by most automatically generated meshes. In fact, Ω_{2h} contains only exceptional elements, and these are relatively few in number. We set $\beta := \min\big(\rho, \frac{1}{2}, \frac{\sigma}{2}\big)$. Then, we have the following estimates for meshes satisfying the *condition* (ρ, σ).

Lemma 5.2 ([23, 37])**.** *Assume that* $\mathbf{u} \in (H^3(\Omega) \cap W^{2,\infty}(\Omega))^2$*. Then, we have the following estimate for meshes satisfying the condition* (ρ, σ)*:*

$$|(\nabla(\mathbf{u} - i_h\mathbf{u}), \nabla\mathbf{w}_h)| \quad \leq \quad Ch^{1+\beta}(\|\mathbf{u}\|_3 + \|\mathbf{u}\|_{2,\infty})|\mathbf{w}_h|_1 \qquad \forall \mathbf{w}_h \in \mathbf{V}_h. \tag{5.1}$$

Lemma 5.3. *Assume that* $\mathbf{u} \in (H^3(\Omega) \cap W^{2,\infty}(\Omega))^2$*. Then, we have the following estimate for meshes satisfying the condition* (ρ, σ)*:*

$$|(r_h, \operatorname{div}(\mathbf{u} - i_h\mathbf{u}))| \leq Ch^{1+\beta}(\|\mathbf{u}\|_3 + \|\mathbf{u}\|_{2,\infty})\|r_h\|_0 \qquad \forall r_h \in Q_h, \tag{5.2}$$

$$|(\operatorname{div}(\mathbf{u} - i_h\mathbf{u}), \operatorname{div}\mathbf{v}_h)| \leq Ch^{1+\beta}(\|\mathbf{u}\|_3 + \|\mathbf{u}\|_{2,\infty})|\mathbf{w}_h|_1 \qquad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{5.3}$$

Proof. As in the proof of Lemma 3.4, we obtain by integration by parts,

$$\begin{aligned}
&(r_h, \operatorname{div}\,(\mathbf{u} - i_h\mathbf{u}_h)) \\
&\quad = \frac{1}{12}\sum_{K\in\mathcal{T}_h}\int_K r_h \operatorname{div}\big(h_1^2\partial^2_{\mathbf{t}_1\mathbf{t}_1}\mathbf{u} + h_1h_3\partial^2_{\mathbf{t}_1\mathbf{t}_3}\mathbf{u} + h_3^2\partial^2_{\mathbf{t}_3\mathbf{t}_3}\mathbf{u}\big)\,dx + R \\
&\qquad - \frac{1}{12}\sum_{K\in\mathcal{T}_h}\int_{\partial K} r_h\big(h_1^2\partial^2_{\mathbf{t}_1\mathbf{t}_1}\mathbf{u} + h_1h_3\partial^2_{\mathbf{t}_1\mathbf{t}_3}\mathbf{u} + h_3^2\partial^2_{\mathbf{t}_3\mathbf{t}_3}\mathbf{u}\big)\cdot\mathbf{n}_K\,ds,
\end{aligned}$$

where again, the first and the second term can be bounded by $Ch^2\|\mathbf{u}\|_3\|r_h\|_0$. In the third term, we replace the derivatives in the tangential directions $\mathbf{t}_1$ and $\mathbf{t}_3$ by derivatives in the tangential direction $\mathbf{t}_2$ and the normal direction $\mathbf{n}_2$, respectively. With Θ_i the angle opposite to s_i, $i = 1, 2, 3$, we have

$$h_1^2\partial^2_{\mathbf{t}_1\mathbf{t}_1}\mathbf{u} + h_1h_3\partial^2_{\mathbf{t}_1\mathbf{t}_3}\mathbf{u} + h_3^2\partial^2_{\mathbf{t}_3\mathbf{t}_3}\mathbf{u} = F\partial^2_{\mathbf{t}_2\mathbf{t}_2}\mathbf{u} + G\partial^2_{\mathbf{t}_2\mathbf{n}_2}\mathbf{u} + H\partial^2_{\mathbf{n}_2\mathbf{n}_2}\mathbf{u},$$

where

$$\begin{aligned}
F &= h_1^2\cos^2\Theta_3 + h_1h_3\cos\Theta_1\cos\Theta_3 + h_3^2\cos^2\Theta_1, \\
G &= h_1\sin\Theta_3(h_3\cos\Theta_1 - h_1\cos\Theta_3), \\
H &= h_1^2\sin^2\Theta_3.
\end{aligned}$$

Let us split the set of all edges into three different classes. $\mathcal{E}_1$ is the set of inner edges E such that the two adjacent triangles sharing E form an $O(h^{1+\rho})$ approximate parallelogram, $\mathcal{E}_2$ is the set of remaining inner edges, and $\mathcal{E}_3$ is the set of all boundary edges. Consider now an edge $E = \partial K \cap \partial K' \in \mathcal{E}_1$. Then, we have $|h_1 - h_1'| \leq Ch_2^{1+\rho}$, $h_E = h_2 = h_2'$, and $|h_3 - h_3'| \leq Ch_2^{1+\rho}$, from which the estimates

$$|F - F'| \leq Ch_2^{2+\rho}, \quad |G - G'| \leq Ch_2^{2+\rho}, \quad |H - H'| \leq Ch_2^{2+\rho}$$

follow by geometric considerations. Since $\mathbf{n}_K = -\mathbf{n}_{K'}$, the sum of integrals over the common edge E can be estimated as

$$\begin{aligned}
&\left|\int_E r_h\left((F - F')\partial^2_{\mathbf{t}_2\mathbf{t}_2}\mathbf{u} + (G - G')\partial^2_{\mathbf{t}_2\mathbf{n}_2}\mathbf{u} + (H - H')\partial^2_{\mathbf{n}_2\mathbf{n}_2}\mathbf{u}\right)\cdot\mathbf{n}_K\,ds\right| \\
&\qquad \leq Ch_E^{2+\rho}\|r_h\|_{0,1,E}\,\|\mathbf{u}\|_{2,\infty,K\cup K'}, \qquad E \in \mathcal{E}_1.
\end{aligned} \tag{5.4}$$

Furthermore, we have the following estimates:

$$\left|\int_E r_h \left((F-F')\partial^2_{\mathbf{t}_2\mathbf{t}_2}\mathbf{u} + (G-G')\partial^2_{\mathbf{t}_2\mathbf{n}_2}\mathbf{u} + (H-H')\partial^2_{\mathbf{n}_2\mathbf{n}_2}\mathbf{u}\right)\cdot \mathbf{n}_K\, ds\right| \tag{5.5}$$
$$\leq Ch_E^2 \|r_h\|_{0,1,E}\, \|\mathbf{u}\|_{2,\infty,K\cup K'}, \qquad E \in \mathcal{E}_2,$$

$$\left|\int_E r_h \left(F\partial^2_{\mathbf{t}_2\mathbf{t}_2}\mathbf{u} + G\partial^2_{\mathbf{t}_2\mathbf{n}_2}\mathbf{u} + H\partial^2_{\mathbf{n}_2\mathbf{n}_2}\mathbf{u}\right)\cdot \mathbf{n}_K\, ds\right| \tag{5.6}$$
$$\leq Ch_E^2 \|r_h\|_{0,1,E}\, \|\mathbf{u}\|_{2,\infty,K}, \qquad E \in \mathcal{E}_3.$$

The estimate of the sums over the three types of edges is based on the discrete trace inequality

$$h_E\|r_h\|_{0,1,E} \leq C|K|^{1/2}\, \|r_h\|_{0,K} \qquad \forall r_h \in Q_h,\ E \subset \partial K, \tag{5.7}$$

from which we get by summation

$$\sum_{E\in\mathcal{E}_i} h_E\|r_h\|_{0,1,E} \leq C \left(\sum_{\substack{K\in\mathcal{T}_h \\ \exists E\in\mathcal{E}_i:\partial K\cap E\neq\emptyset}} |K| \right)^{1/2} \|r_h\|_0.$$

The discrete trace inequality (5.7) follows from scaling properties, the trace inequality on the reference cell, and the equivalence of norms in finite-dimensional spaces. Applying (5.7) to (5.4)-(5.6) we end up with

$$\left|\sum_{E\in\mathcal{E}_1}\int_E \cdots\, ds\right| \leq Ch^{1+\rho}\left(\sum_{K\in\mathcal{T}_{1,h}}|K|\right)^{1/2} \|\mathbf{u}\|_{2,\infty,\Omega}\|r_h\|_0 \leq Ch^{1+\rho}\|\mathbf{u}\|_{2,\infty,\Omega}\|r_h\|_0,$$

$$\left|\sum_{E\in\mathcal{E}_2}\int_E \cdots\, ds\right| \leq Ch\left(\sum_{K\in\mathcal{T}_{2,h}}|K|\right)^{1/2} \|\mathbf{u}\|_{2,\infty,\Omega}\|r_h\|_0 \leq Ch^{1+\sigma/2}\|\mathbf{u}\|_{2,\infty,\Omega}\|r_h\|_0,$$

$$\left|\sum_{E\in\mathcal{E}_3}\int_E \cdots\, ds\right| \leq Ch\left(\sum_{\overline{K}\cap\Gamma\neq\emptyset}|K|\right)^{1/2} \|\mathbf{u}\|_{2,\infty,\Omega}\|r_h\|_0 \leq Ch^{3/2}\|\mathbf{u}\|_{2,\infty,\Omega}\|r_h\|_0.$$

Thus, (5.2) is proven. Along the same lines, (5.3) can be shown. □

These meshes are not generated by a regular refinement. Therefore a reasonable postprocessing is given by recovery methods for linear finite elements as already used for second-order elliptic problems; see e.g. [39, 40, 43]. We define a recovery operator $G_h : \mathbf{V}_h \to (Y_h \times Y_h)^2$, where $Y_h = \{v \in H^1(\Omega) : v|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\}$. The piecewise linear function $G_h\mathbf{u}_h$ is an approximation to the gradient of the exact solution $\nabla\mathbf{u}$ constructed by the finite element solution $\mathbf{u}_h$. This operator has the following properties:

$$\|G_h\mathbf{v}_h\|_0 \leq C\,\|\mathbf{v}_h\|_1, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{5.8}$$

$$\|G_h\mathbf{u} - \nabla\mathbf{u}\|_0 \leq Ch^2\|\mathbf{u}\|_3, \qquad \forall \mathbf{u} \in H^3(\Omega)^2, \tag{5.9}$$

$$G_h\mathbf{u} = G_h(i_h\mathbf{u}), \qquad \forall \mathbf{u} \in H^3(\Omega)^2. \tag{5.10}$$

Theorem 5.4. *Let the solution* $(\mathbf{u}, p)$ *of* (2.1) *belong to* $(H^3(\Omega) \cap W^{2,\infty}(\Omega))^2 \times H^2(\Omega)$. *Then, we have the following supercloseness on meshes satisfying the condition* (ρ, σ)*:*

$$|||(\mathbf{u}_h - i_h\mathbf{u}, p_h - j_h p)|||_A \quad \leq \quad Ch^{1+\beta}(\|\mathbf{u}\|_3 + \|\mathbf{u}\|_{2,\infty} + \|p\|_2) \tag{5.11}$$

for the P_1 *case. Furthermore, for the* P_1^+ *case the linear part* $(\mathbf{u}_L, p_L)$ *of the discrete solution satisfies*

$$|||(\mathbf{u}_L - i_h\mathbf{u}, p_L - j_h p)|||_B \quad \leq \quad Ch^{1+\beta}(\|\mathbf{u}\|_3 + \|\mathbf{u}\|_{2,\infty} + \|p\|_2). \tag{5.12}$$

For the recovered gradients the superconvergence results

$$\|G_h\mathbf{u}_h - \nabla\mathbf{u}\|_0 + \|p_h - p\|_0 \leq Ch^{1+\beta}(\|\mathbf{u}\|_3 + \|\mathbf{u}\|_{2,\infty} + \|p\|_2), \tag{5.13}$$

$$\|G_h\mathbf{u}_L - \nabla\mathbf{u}\|_0 + \|p_L - p\|_0 \leq Ch^{1+\beta}(\|\mathbf{u}\|_3 + \|\mathbf{u}\|_{2,\infty} + \|p\|_2) \tag{5.14}$$

hold true.

Proof. The supercloseness results (5.11) and (5.12) follow from the stability of the underlying bilinear forms, the Lemmas 5.2, 5.3, 3.5, and 3.6, in the same way as shown in Section 3.1.

Combining (5.8)-(5.10) and (5.11), we have

$$\begin{aligned} \|G_h\mathbf{u}_h - \nabla\mathbf{u}\|_0 + \|p_h - p\|_0 \quad &\leq \quad \|G_h\mathbf{u}_h - G_h(i_h\mathbf{u})\|_0 + \|G_h(i_h\mathbf{u}) - G_h\mathbf{u}\|_0 \\ &\quad\quad +\|G_h\mathbf{u} - \nabla\mathbf{u}\|_0 + \|p_h - p\|_0 \\ &\leq \quad C\|\mathbf{u}_h - i_h\mathbf{u}\|_1 + Ch^2(\|\mathbf{u}\|_3 + \|p\|_2) \\ &\leq \quad Ch^{1+\beta}(\|\mathbf{u}\|_3 + \|\mathbf{u}\|_{2,\infty} + \|p\|_2). \end{aligned}$$

The estimate (5.14) can be proven similarly. □

In fact, the *condition* (ρ, σ) is very general. For a general domain, we start with a coarse mesh and use a regular refinement. Then, the resulting family of meshes satisfies the *condition* (ρ, σ).

6. Numerical tests

In this section we present numerical results for solving the Stokes problem by the stabilized method (2.4) for the P_1 case and the P_1^+ case, respectively.

Example 6.1. Let $\Omega = (0, 1)^2$. We consider the Stokes problem

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f}, \qquad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \qquad \mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{6.1}$$

where the right hand side $\mathbf{f}$ and the inhomogeneous Dirichlet boundary condition $\mathbf{g}$ are chosen such that

$$\mathbf{u} = \begin{pmatrix} \sin(x)\sin(y) \\ \cos(x)\cos(y) \end{pmatrix}, \quad p = 2\cos(x)\sin(y) - 2\sin(1)(1 - \cos(1)) \tag{6.2}$$

is the solution.

We compute the finite element solution on uniform triangular meshes of a regular pattern obtained by successive regular refinement of an initial coarse mesh. The mesh on level 1 is shown in Figure 4.

In the following, we evaluate the results of our computation by considering the H^1 seminorm of the velocity error and the L^2 norm of the pressure error. We also compute the H^1 seminorm error of the postprocessed velocity $I_{2h}\mathbf{u}_h$. The numerical results confirm the theoretically predicted convergence rates; see Figure 5 for the P_1 case and Figure 6 for the P_1^+ case, respectively.

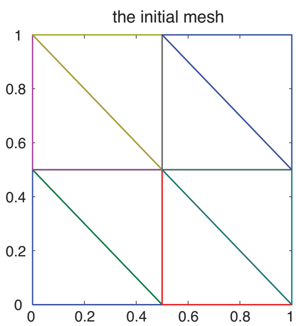

FIGURE 4. The initial uniform mesh.

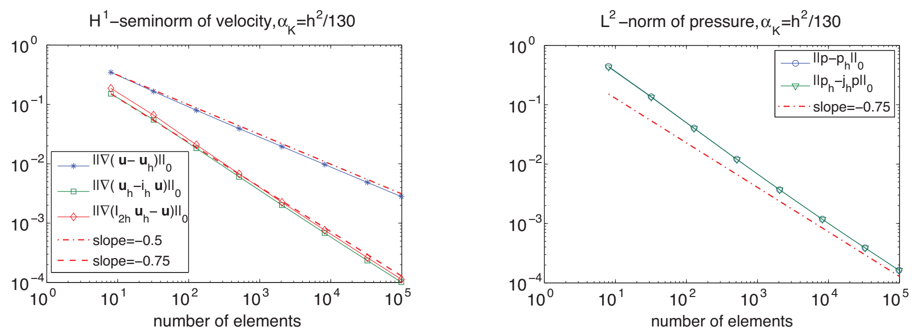

FIGURE 5. Velocity and pressure error for the P_1 case on three-directional meshes.

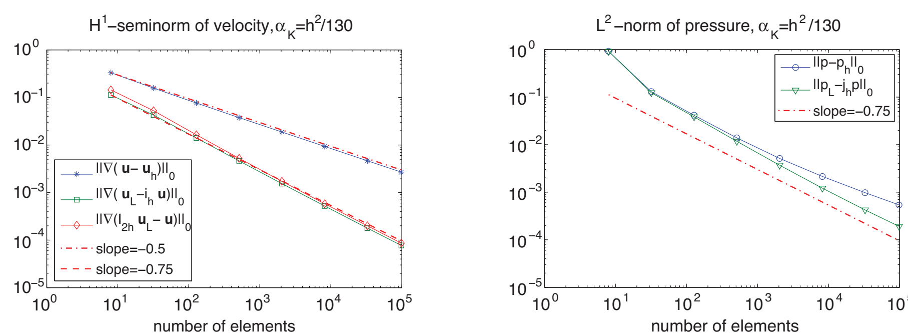

FIGURE 6. Velocity and pressure error for the P_1^+ case on three-directional meshes.

Example 6.2. In this example, we solve the Stokes problem (6.1) on a unit circle where the right hand side $\mathbf{f}$ and the inhomogeneous Dirichlet boundary condition $\mathbf{g}$ are chosen such that

$$\mathbf{u} = \begin{pmatrix} \sin(x)\sin(y) \\ \cos(x)\cos(y) \end{pmatrix}, \quad p = 2\cos(x)\sin(y) \tag{6.3}$$

is the exact solution.

In this example, the domain Ω cannot be triangulated by a family of three-directional meshes. Here, we used the Delaunay triangulation to produce an initial coarse mesh. Then, a family of meshes is generated by successive regular refinement. Figure 7 shows the initial mesh and the refined mesh on level 4. In Figures 8 and 9 the errors for the velocity and the pressure are shown for the P_1 case and the P_1^+ case, respectively. We observe superconvergence also over this type of successively refined meshes. Note that, the technique of [21] has been used to carry out the postprocessing.

We also considered an automatic generated family of meshes (Delaunay triangulation separately on each mesh level). Even in this case we observed some (less pronounced) superconvergence properties.

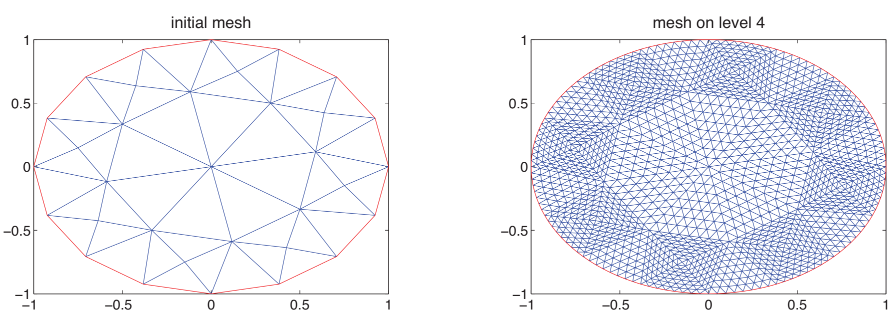

FIGURE 7. The meshes on level 0 and level 4 for a regular refinement.

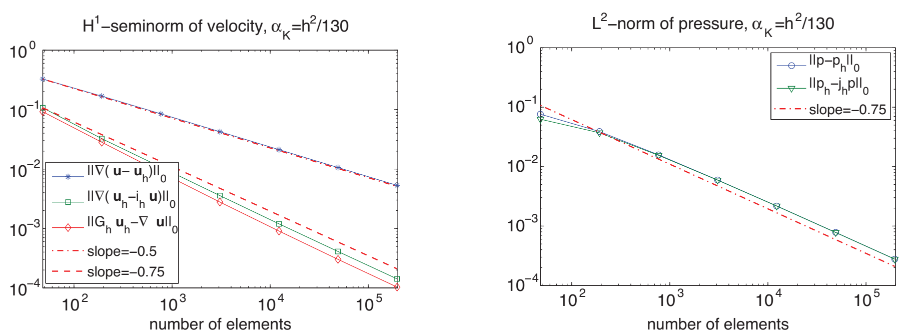

FIGURE 8. Velocity and pressure error for the regular refinement P_1 case.

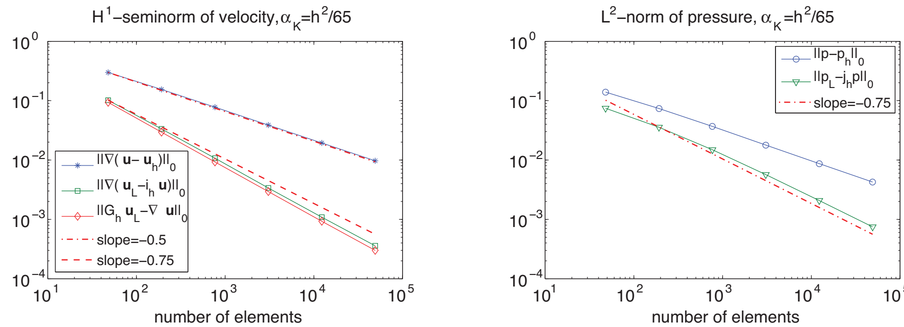

FIGURE 9. Velocity and pressure error for the regular refinement P_1^+ case.

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