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NUMERICAL METHODS FOR SOLVING A TWO-DIMENSIONAL VARIABLE-ORDER ANOMALOUS SUBDIFFUSION EQUATION

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ABSTRACT. Anomalous subdiffusion equations have in recent years received much attention. In this paper, we consider a two-dimensional variable-order anomalous subdiffusion equation. Two numerical methods (the implicit and explicit methods) are developed to solve the equation. Their stability, convergence and solvability are investigated by the Fourier method. Moreover, the effectiveness of our theoretical analysis is demonstrated by some numerical examples.

1. INTRODUCTION

Because of their practical applications, anomalous subdiffusion equations have received much attention in recent years. For example, Schwille et al. [31] studied anomalous subdiffusion of proteins and lipids in membranes observed by fluorescence correlation spectroscopy, Saxton [30] researched anomalous subdiffusion in fluorescence photobleaching recovery, Ratto et al. [27] considered anomalous subdiffusion in heterogeneous lipid bilayers, Weiss et al. [34] analyzed anomalous subdiffusion as a measure for cytoplasmic crowding in living cells, Marseguerra et al. [22] investigated the Monte Carlo and fractional kinetics approaches to the underground anomalous subdiffusion of contaminants, Tan et al. [33] discussed an anomalous subdiffusion model for calcium spark in cardiac myocytes, and Langlands et al. [12] researched anomalous subdiffusion with multispecies linear reaction dynamics. At the same time, a number of authors have developed numerical methods for solving fractional diffusion equations (e.g. [1], [2]-[6], [13], [16]-[19], [35]-[37]). However, work on fractional diffusion equations in higher dimensions is still at an early stage. Meerschaert et al. [23] analyzed finite difference methods, while Tadjeran et al. [24] discussed a numerical method of second-order accuracy for the two-dimensional fractional diffusion equation, Zhuang et al. [38] proposed an implicit difference approximation for the two-dimensional space-time fractional diffusion equation, Chen et al. [7] investigated the following two-dimensional anomalous subdiffusion equation

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$$\frac{\partial u(x,y,t)}{\partial t} = {}_0D_t^{1-\gamma}\left(\kappa_1 \frac{\partial^2 u(x,y,t)}{\partial x^2} + \kappa_2 \frac{\partial^2 u(x,y,t)}{\partial y^2}\right) + f(x,y,t), \tag{1.1}$$

proposed two numerical schemes with first order temporal accuracy and second order spatial accuracy, and discussed the stability, convergence and solvability of these numerical schemes by the Fourier method.

In order to more accurately characterize the evolution of a system, the so-called variable-order operator calculus has been developed [8], [9], [10], [11], [14], [20], [21], [26]-[32]. However, only a few authors have studied numerical methods and numerical analysis of variable-order fractional differential equations. Lin et al. [15] studied the stability and convergence of an explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation. Zhuang et al. [39] presented some numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term.

In this paper, we will study a two-dimensional variable-order anomalous subdiffusion equation of the form

$$\frac{\partial u(x,y,t)}{\partial t} = {}_0D_t^{1-\gamma(x,y,t)}\left(\kappa_1 \frac{\partial^2 u(x,y,t)}{\partial x^2} + \kappa_2 \frac{\partial^2 u(x,y,t)}{\partial y^2}\right) + f(x,y,t) \tag{1.2}$$

with initial and boundary conditions:

$$u(x,y,0) = \phi(x,y),\ 0 \le x,y \le L, \tag{1.3}$$

$$u(x,0,t) = \varphi_1(x,t),\ u(x,L,t) = \varphi_2(x,t),\ 0 \le x \le L,\ 0 < t \le T, \tag{1.4}$$

$$u(0,y,t) = \psi_1(y,t),\ u(L,y,t) = \psi_2(y,t),\ 0 \le y \le L,\ 0 < t \le T, \tag{1.5}$$

where the constants $\kappa_1, \kappa_2 > 0$, $\phi(x,y)$, $\varphi_1(x,t)$, $\varphi_2(x,t)$, $\psi_1(y,t)$, $\psi_2(y,t)$ are sufficiently smooth functions, $0 < \gamma_{\min} \le \gamma(x,y,t) \le \gamma_{\max} < 1$ and ${}_0D_t^{1-\gamma(x,y,t)} g(x,y,t)$ is the variable-order Riemann-Liouville fractional partial derivative of order $1 - \gamma(x,y,t)$ for $g(x,y,t)$ defined by [15], [39]

$${}_0D_t^{1-\gamma(x,y,t)} g(x,y,t) = \frac{1}{\Gamma(\gamma(x,y,t))}\left[\frac{\partial}{\partial \xi}\int_0^{\xi} \frac{g(x,y,\eta)}{(\xi-\eta)^{1-\gamma(x,y,t)}} d\eta\right]_{\xi=t}.$$

Implicit and explicit numerical methods will be developed for this problem. Their stability and convergence will be described in detail. Some numerical examples will be presented to demonstrate the effectiveness of the methods.

2. An implicit numerical method

2.1. Derivation of the implicit numerical method. In this paper, we let

$$x_i = i\Delta_x,\ i = 0,1,\ldots,M_1;\ y_j = j\Delta_y,\ j = 0,1,\ldots,M_2;\ t_k = k\Delta_t, k = 0,1,\ldots,N,$$

respectively, where $\Delta_x = L/M_1$, $\Delta_y = L/M_2$ and $\Delta_t = T/N$ are the spatial and temporal steps, respectively. Also, we denote

$$\Omega = \{(x,y,t) |\ 0 \le x,y \le L,\ 0 \le t \le T\},$$

$$U(\Omega) = \left\{u(x,y,t) | \frac{\partial^4 u(x,y,t)}{\partial x^4}, \frac{\partial^4 u(x,y,t)}{\partial y^4}, \frac{\partial^3 u(x,y,t)}{\partial x^2 \partial t}, \frac{\partial^3 u(x,y,t)}{\partial y^2 \partial t}, \frac{\partial^2 u(x,y,t)}{\partial t^2} \in C(\Omega)\right\}. \tag{2.1}$$

In this paper, we always assume $u(x,y,t) \in U(\Omega)$. At the grid point (x_i, y_j, t_k), (1.2) becomes

$$\frac{\partial u(x_i,y_j,t_k)}{\partial t} = {}_0D_t^{1-\gamma_{i,j}^k}\left(\kappa_1 \frac{\partial^2 u(x_i,y_j,t_k)}{\partial x^2} + \kappa_2 \frac{\partial^2 u(x_i,y_j,t_k)}{\partial y^2}\right) + f_{i,j}^k, \tag{2.2}$$

where $\gamma_{i,j}^k \equiv \gamma(x_i,y_j,t_k)$, $f_{i,j}^k \equiv f(x_i,y_j,t_k)$.

The Grünwald-Letnikov fractional partial derivative of order $1-\gamma_{i,j}^k$ for $g(x,y,t)$, defined by

$$\lim_{\Delta_t \to 0} \Delta_t^{\gamma_{i,j}^k - 1} \sum_{l=0}^{[t/\Delta_t]} (-1)^l \binom{1-\gamma_{i,j}^k}{l} g(x,y,t-l\Delta_t),$$

provides an approximation of the Riemann-Liouville fractional partial derivative. From [25], we have

$$\begin{aligned} &\lim_{\Delta_t \to 0} \Delta_t^{\gamma_{i,j}^k - 1} \sum_{l=0}^{[t/\Delta_t]} (-1)^l \binom{1-\gamma_{i,j}^k}{l} g(x,y,t-l\Delta_t) \\ =&\Delta_t^{\gamma_{i,j}^k - 1} \sum_{l=0}^{[t/\Delta_t]} (-1)^l \binom{1-\gamma_{i,j}^k}{l} g(x,y,t-l\Delta_t) + O(\Delta_t). \end{aligned} \tag{2.3}$$

Under the condition that the function $g(x,y,t)$ has continuous partial derivative $\frac{\partial g(x,y,t)}{\partial t}$ for $t > 0$, the Riemann-Liouville and Grünwald-Letnikov fractional partial derivatives of order $1-\gamma_{i,j}^k$ for $g(x,y,t)$ have the following relation:

$$\begin{aligned} &{}_0D_t^{1-\gamma_{i,j}^k} g(x,y,t) \\ =&\lim_{\Delta_t \to 0} \Delta_t^{\gamma_{i,j}^k - 1} \sum_{l=0}^{[t/\Delta_t]} (-1)^l \binom{1-\gamma_{i,j}^k}{l} g(x,y,t-l\Delta_t). \end{aligned} \tag{2.4}$$

By (2.3) and (2.4) we get

$$\begin{aligned} &{}_0D_t^{1-\gamma_{i,j}^k} g(x,y,t) \\ =&\Delta_t^{\gamma_{i,j}^k - 1} \sum_{l=0}^{[t/\Delta_t]} (-1)^l \binom{1-\gamma_{i,j}^k}{l} g(x,y,t-l\Delta_t) + O(\Delta_t), \end{aligned} \tag{2.5}$$

which yields

$$\left[{}_0D_t^{1-\gamma_{i,j}^k} g(x,y,t)\right]_{t=t_k} = \Delta_t^{\gamma_{i,j}^k - 1} \sum_{l=0}^{k} \lambda_{i,j}^{k,l} g(x,y,t_{k-l}) + O(\Delta_t) \tag{2.6}$$

and

$${}_0D_t^{1-\gamma_{i,j}^k} g(x_i,y_j,t_k) = \Delta_t^{\gamma_{i,j}^k - 1} \sum_{l=0}^{k} \lambda_{i,j}^{k,l} g(x_i,y_j,t_{k-l}) + O(\Delta_t), \tag{2.7}$$

where $\lambda_{i,j}^{k,l} = (-1)^l \binom{1-\gamma_{i,j}^k}{l} = (-1)^l \frac{(1-\gamma_{i,j}^k)(-\gamma_{i,j}^k)\dots(1-\gamma_{i,j}^k-l+1)}{l!}$. From (2.7) and $u(x,y,t) \in U(\Omega)$ we get, respectively,

$$\begin{aligned} &{}_0D_t^{1-\gamma_{i,j}^k}\left(\frac{\partial^2 u(x_i,y_j,t_k)}{\partial x^2}\right) \\ =&\Delta_t^{\gamma_{i,j}^k-1}\sum_{l=0}^{k}\lambda_{i,j}^{k,l}\frac{\partial^2 u(x_i,y_j,t_{k-l})}{\partial x^2}+O(\Delta_t) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &{}_0D_t^{1-\gamma_{i,j}^k}\left(\frac{\partial^2 u(x_i,y_j,t_k)}{\partial y^2}\right) \\ =&\Delta_t^{\gamma_{i,j}^k-1}\sum_{l=0}^{k}\lambda_{i,j}^{k,l}\frac{\partial^2 u(x_i,y_j,t_{k-l})}{\partial y^2}+O(\Delta_t). \end{aligned} \tag{2.9}$$

Since $u(x,y,t) \in U(\Omega)$, the following formulas are obtained:

$$\frac{\partial u(x_i,y_j,t_k)}{\partial t} = \frac{u(x_i,y_j,t_k)-u(x_i,y_j,t_{k-1})}{\Delta_t}+O(\Delta_t), \tag{2.10}$$

$$\frac{\partial^2 u(x_i,y_j,t_k)}{\partial x^2} = \frac{\delta_x^2 u(x_i,y_j,t_k)}{\Delta_x^2}+O(\Delta_x^2), \tag{2.11}$$

$$\frac{\partial^2 u(x_i,y_j,t_k)}{\partial y^2} = \frac{\delta_y^2 u(x_i,y_j,t_k)}{\Delta_y^2}+O(\Delta_y^2), \tag{2.12}$$

where

$$\begin{aligned} \delta_x^2 u(x_i,y_j,t_k) &= u(x_{i-1},y_j,t_k)-2u(x_i,y_j,t_k)+u(x_{i+1},y_j,t_k), \\ \delta_y^2 u(x_i,y_j,t_k) &= u(x_i,y_{j-1},t_k)-2u(x_i,y_j,t_k)+u(x_i,y_{j+1},t_k). \end{aligned}$$

Now, by (2.8)-(2.12) we get

$$\begin{aligned} u(x_i,y_j,t_k) = u(x_i,y_j,t_{k-1}) &+ \overline{\mu}_{i,j}^k\sum_{l=0}^{k}\lambda_{i,j}^{k,l}\delta_x^2 u(x_i,y_j,t_{k-l}) \\ +\widetilde{\mu}_{i,j}^k\sum_{l=0}^{k}&\lambda_{i,j}^{k,l}\delta_y^2 u(x_i,y_j,t_{k-l})+\Delta_t f_{i,j}^k+R_{i,j}^k, \\ i=1,2,\dots,M_1-1;\ j&=1,2,\dots,M_2-1;\ k=1,2,\dots,N, \end{aligned} \tag{2.13}$$

where

$$\overline{\mu}_{i,j}^k = \kappa_1\frac{\Delta_t^{\gamma_{i,j}^k}}{\Delta_x^2},\ \widetilde{\mu}_{i,j}^k = \kappa_2\frac{\Delta_t^{\gamma_{i,j}^k}}{\Delta_y^2},$$

and

$$R_{i,j}^k = O(\Delta_x^2+\Delta_y^2)\Delta_t^{\gamma_{i,j}^k}\sum_{l=0}^{k}\lambda_{i,j}^{k,l}+O(\Delta_t^2). \tag{2.14}$$

In view of the above analysis, we present the following implicit numerical scheme for solving (1.2) with the initial and boundary conditions (1.3)-(1.5):

$$u_{i,j}^k = u_{i,j}^{k-1} + \overline{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l} \delta_x^2 u_{i,j}^{k-l} + \widetilde{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l} \delta_y^2 u_{i,j}^{k-l} + \Delta_t f_{i,j}^k, \tag{2.15}$$
$$i = 1, 2, \ldots, M_1 - 1,\ j = 1, 2, \ldots, M_2 - 1,\ k = 1, 2, \ldots, N,$$

$$u_{i,j}^0 = \phi(x_i, y_j),\ i = 0, 1, \ldots, M_1,\ j = 0, 1, \ldots, M_2, \tag{2.16}$$

$$u_{i,0}^k = \varphi_1(x_i, t_k), u_{i,M_2}^k = \varphi_2(x_i, t_k), \tag{2.17}$$
$$i = 1, 2, \ldots, M_1 - 1,\ k = 1, 2, \ldots, N,$$

$$u_{0,j}^k = \psi_1(y_j, t_k),\ u_{M_1,j}^k = \psi_2(y_j, t_k), \tag{2.18}$$
$$j = 1, 2, \ldots, M_2 - 1,\ k = 1, 2, \ldots, N,$$

2.2. **Some lemmas.** In this subsection, we establish two lemmas for use in later analysis.

Lemma 2.1. *If $0 < \gamma_{\min} \leq \gamma(x, y, t) \leq \gamma_{\max} < 1$, for $i = 1, 2, \ldots, M_1$, $j = 1, 2, \ldots, M_2$, $k = 1, 2, \ldots, N$, $l = 0, 1, \ldots,$ the coefficients $\lambda_{i,j}^{k,l}$ satisfy:*

(1) $\lambda_{i,j}^{k,0} = 1;\ \lambda_{i,j}^{k,1} = \gamma_{i,j}^k - 1 < 0;\ \lambda_{i,j}^{k,l} < 0,\ l = 2, 3, \ldots;$

(2) $\sum\limits_{l=0}^{\infty} \lambda_{i,j}^{k,l} = 0;$

(3) *for* $n = 1, 2, \ldots,\ -\sum\limits_{l=1}^{n} \lambda_{i,j}^{k,l} < 1.$

Proof. Obviously,

$$\lambda_{i,j}^{k,0} = (-1)^0 \binom{1 - \gamma_{i,j}^k}{0} = 1,$$
$$\lambda_{i,j}^{k,1} = (-1)^1 \binom{1 - \gamma_{i,j}^k}{1} = \gamma_{i,j}^k - 1 < 0.$$

Using the condition $0 < \gamma_{\min} \leq \gamma(x, y, t) \leq \gamma_{\max} < 1$, it holds for $i = 1, 2, \ldots, M_1$, $j = 1, 2, \ldots, M_2$, $k = 1, 2, \ldots, N$ that

$$0 < \gamma_{i,j}^k < 1. \tag{2.19}$$

So, for $l = 2, 3, \ldots$, we have

$$\begin{aligned} \lambda_{i,j}^{k,l} =& (-1)^l \binom{1 - \gamma_{i,j}^k}{l} \\ =& (-1)^l \frac{(1 - \gamma_{i,j}^k)(-\gamma_{i,j}^k)(-\gamma_{i,j}^k - 1) \ldots (1 - \gamma_{i,j}^k - l + 1)}{l!} \\ =& -\frac{(1 - \gamma_{i,j}^k)\gamma_{i,j}^k(\gamma_j^k + 1) \ldots (\gamma_{i,j}^k + l - 2)}{l!} < 0. \end{aligned} \tag{2.20}$$

Second, taking $t = 1$ in the formula

$$\sum_{l=0}^{\infty} \lambda_{i,j}^{k,l} t^l = (1 - t)^{1 - \gamma_{i,j}^k},$$

we have

$$\sum_{l=0}^{\infty} \lambda_{i,j}^{k,l} = 0.$$

Finally, from the conclusions (1) and (2), for $n = 1, 2, \ldots,$ it holds that

$$-\sum_{l=1}^{n} \lambda_{i,j}^{k,l} = \lambda_{i,j}^{k,0} + \sum_{l=n+1}^{\infty} \lambda_{i,j}^{k,l} = 1 + \sum_{l=n+1}^{\infty} \lambda_{i,j}^{k,l} < 1.$$

The proof of Lemma 2.1 is completed. □

Lemma 2.2. *For* $i = 1, 2, \ldots, M_1,\ j = 1, 2, \ldots, M_2,\ k = 1, 2, \ldots, N,$ *it holds that*

$$\Delta_t^{\gamma_{i,j}^k - 1} \sum_{l=0}^{k} \lambda_{i,j}^{k,l} = \frac{1}{\Gamma(\gamma_{i,j}^k)} + O(\Delta_t).$$

Proof. Taking $g(x, y, t) = 1$ and $t_k = 1$ in (2.6) gives

$$\left[{}_0D_t^{1-\gamma_{i,j}^k}(1) \right]_{t=1} = \Delta_t^{\gamma_{i,j}^k - 1} \sum_{l=0}^{k} \lambda_{i,j}^{k,l} + O(\Delta_t).$$

On the other hand,

$$\left[{}_0D_t^{1-\gamma_{i,j}^k}(1) \right]_{t=1} = \left[\frac{t^{\gamma_{i,j}^k - 1}}{\Gamma(\gamma_{i,j}^k)} \right]_{t=1} = \frac{1}{\Gamma(\gamma_{i,j}^k)}.$$

Therefore,

$$\Delta_t^{\gamma_{i,j}^k - 1} \sum_{l=0}^{k} \lambda_{i,j}^{k,l} = \frac{1}{\Gamma(\gamma_{i,j}^k)} + O(\Delta_t).$$

Lemma 2.2 is then proved. □

2.3. **Stability analysis of the implicit numerical method.** In this subsection, we will analyze the stability of the implicit numerical method (2.15)-(2.18). For this we let $U_{j,l}^k$ be the approximation of the solution for the implicit numerical method (2.15)-(2.18), and define

$$\rho_{i,j}^k = u_{i,j}^k - U_{i,j}^k,$$
$$i = 1, 2, \ldots, M_1 - 1,\ j = 1, 2, \ldots, M_2 - 1,\ k = 1, 2, \ldots, N$$

and

$$\rho^k = \left[\rho_{1,1}^k, \rho_{1,2}^k, \ldots, \rho_{1,M_2-1}^k, \ldots, \rho_{M_1-1,1}^k, \rho_{M_1-1,2}^k, \ldots, \rho_{M_1-1,M_2-1}^k \right]^T,$$

respectively. We then obtain the following roundoff error equation:

$$\rho_{i,j}^k = \rho_{i,j}^{k-1} + \overline{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l} \delta_x^2 \rho_{i,j}^{k-l} + \widetilde{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l} \delta_y^2 \rho_{i,j}^{k-l}, \tag{2.21}$$
$$i = 1, 2, \ldots, M_1 - 1,\ j = 1, 2, \ldots, M_2 - 1,\ k = 1, 2, \ldots, N.$$

For $k = 0, 1, \ldots, N$, we define the following grid function:

$$\rho^k(x, y) = \begin{cases} \rho_{i,j}^k, & \text{when} \quad (x, y) \in \Omega_1, \\ 0, & \text{when} \quad (x, y) \in \Omega_2, \end{cases}$$

where

$$\begin{aligned}\Omega_1 = \Big\{(x,y)|\ & x_{i-\frac{1}{2}} < x \le x_{i+\frac{1}{2}},\ i=1,2,\dots,M_1-1, \\ & y_{j-\frac{1}{2}} < y \le y_{j+\frac{1}{2}},\ j=1,2,\dots,M_2-1\Big\},\end{aligned} \tag{2.22}$$

$$\begin{aligned}\Omega_2 = \Big\{(x,y)|\ & 0 \le x \le \frac{\Delta_x}{2},\ \text{or}\ L-\frac{\Delta_x}{2} < x \le L, \\ & \text{or}\ 0 \le y \le \frac{\Delta_y}{2},\ \text{or}\ L-\frac{\Delta_y}{2} < y \le L\Big\}.\end{aligned} \tag{2.23}$$

Then $\rho^k(x,y)$ has the Fourier series expansion

$$\rho^k(x,y) = \sum_{l_1,l_2=-\infty}^{\infty} \zeta_k(l_1,l_2)e^{I2\pi(l_1x/L+l_2y/L)},\ k=0,1,\dots,N,$$

where

$$I=\sqrt{-1},\ \zeta_k(l_1,l_2) = \frac{1}{L^2}\int\limits_{0\le x,y\le L} \rho^k(x,y)e^{-I2\pi(l_1x/L+l_2y/L)}dxdy.$$

Using the Parseval equalities

$$\int\limits_{0\le x,y\le L} \left|\rho^k(x,y)\right|^2 dxdy = \sum_{l_1,l_2=-\infty}^{\infty} |\zeta_k(l_1,l_2)|^2,\ k=0,1,\dots,N$$

and

$$\int\limits_{0\le x,y\le L} \left|\rho^k(x,y)\right|^2 dxdy = \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} \Delta_x\Delta_y|\rho_{i,j}^k|^2,\ k=0,1,\dots,N,$$

we have

$$\begin{aligned}\|\rho^k\|_2 &\equiv \left(\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} \Delta_x\Delta_y|\rho_{i,j}^k|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{l_1,l_2=-\infty}^{\infty} |\zeta_k(l_1,l_2)|^2\right)^{\frac{1}{2}}, \quad k=0,1,\dots,N.\end{aligned} \tag{2.24}$$

Assume that the solution of the difference equation (2.21) has the form

$$\rho_{i,j}^k = \zeta_k e^{I(\sigma_1 i\Delta_x+\sigma_2 j\Delta_y)}, \tag{2.25}$$

where $\sigma_1 = 2\pi l_1/L,\ \sigma_2 = 2\pi l_2/L$. Substituting (2.25) into (2.21) gives

$$\begin{aligned}\zeta_k =& \zeta_{k-1} - 4\overline{\mu}_{i,j}^k \sin^2\frac{\sigma_1\Delta_x}{2}\sum_{l=0}^{k}\lambda_{i,j}^{k,l}\zeta_{k-l} \\ & - 4\widetilde{\mu}_{i,j}^k \sin^2\frac{\sigma_2\Delta_y}{2}\sum_{l=0}^{k}\lambda_{i,j}^{k,l}\zeta_{k-l},\ k=1,2,\dots,N.\end{aligned} \tag{2.26}$$

Using Lemma 2.1, (2.26) can be rewritten as

$$\zeta_k = \frac{1+(1-\gamma_{i,j}^k)\mu_{i,j}^k}{1+\mu_{i,j}^k}\zeta_{k-1} - \frac{\mu_{i,j}^k}{1+\mu_{i,j}^k}\sum_{l=2}^{k}\lambda_{i,j}^{k,l}\zeta_{k-l},\ k=1,2,\ldots,N, \tag{2.27}$$

where

$$\mu_{i,j}^k = 4\overline{\mu}_{i,j}^k \sin^2\frac{\sigma_1\Delta_x}{2} + 4\widetilde{\mu}_{i,j}^k \sin^2\frac{\sigma_2\Delta_y}{2} \geq 0. \tag{2.28}$$

Theorem 2.3. *The implicit numerical scheme* (2.15)-(2.18) *is unconditionally stable.*

Proof. Applying mathematical induction, we can demonstrate that

$$|\zeta_k| \leq |\zeta_0|,\ k=1,2,\ldots,N, \tag{2.29}$$

where ζ_k $(k=1,2,\ldots,N)$ is the solution of (2.27). When $k=1$, by (2.27) we get

$$\zeta_1 = \frac{1+(1-\gamma_{i,j}^1)\mu_{i,j}^1}{1+\mu_{i,j}^1}\zeta_0.$$

From (2.19) and (2.28), it follows that

$$|\zeta_1| = \frac{1+(1-\gamma_{i,j}^1)\mu_{i,j}^1}{1+\mu_{i,j}^1}|\zeta_0| \leq |\zeta_0|.$$

Assume that

$$|\zeta_n| \leq |\zeta_0|,\ n=1,2,\ldots,k-1,$$

then, according to (2.19), (2.28) and Lemma 2.1, from (2.27) we get

$$\begin{aligned}
|\zeta_k| &= \left|\frac{1+(1-\gamma_{i,j}^k)\mu_{i,j}^k}{1+\mu_{i,j}^k}\zeta_{k-1} + \frac{\mu_{i,j}^k}{1+\mu_{i,j}^k}\sum_{l=2}^{k}\lambda_{i,j}^{k,l}\zeta_{k-l}\right| \\
&\leq \frac{1+(1-\gamma_{i,j}^k)\mu_{i,j}^k}{1+\mu_{i,j}^k}|\zeta_{k-1}| + \frac{\mu_{i,j}^k}{1+\mu_{i,j}^k}\sum_{l=2}^{k}|\lambda_{i,j}^{k,l}||\zeta_{k-l}| \\
&\leq \left[\frac{1+(1-\gamma_{i,j}^k)\mu_{i,j}^k}{1+\mu_{i,j}^k} + \frac{\mu_{i,j}^k}{1+\mu_{i,j}^k}\sum_{l=2}^{k}|\lambda_{i,j}^{k,l}|\right]|\zeta_0| \\
&= \left\{\frac{1+(1-\gamma_{i,j}^k)\mu_{i,j}^k}{1+\mu_{i,j}^k} + \frac{\mu_{i,j}^k}{1+\mu_{i,j}^k}\left[\sum_{l=1}^{k}|\lambda_{i,j}^{k,l}| - |\lambda_{i,j}^{k,1}|\right]\right\}|\zeta_0| \\
&= \left\{\frac{1+(1-\gamma_{i,j}^k)\mu_{i,j}^k}{1+\mu_{i,j}^k} + \frac{\mu_{i,j}^k}{1+\mu_{i,j}^k}\left[-\sum_{l=1}^{k}\lambda_{i,j}^{k,l} - \left(1-\gamma_{i,j}^k\right)\right]\right\}|\zeta_0| \\
&\leq \left\{\frac{1+(1-\gamma_{i,j}^k)\mu_{i,j}^k}{1+\mu_{i,j}^k} + \frac{\mu_{i,j}^k}{1+\mu_{i,j}^k}\left[1-\left(1-\gamma_{i,j}^k\right)\right]\right\}|\zeta_0| \\
&= |\zeta_0|.
\end{aligned} \tag{2.30}$$

The conclusion (2.29) then follows. From (2.24) and the conclusion (2.29), the solution of the difference equation (2.21) satisfies

$$\|\rho^k\|_2 \leq \|\rho^0\|_2,\ k=1,2,\ldots,N.$$

This completes the proof of Theorem 2.3. □

2.4. **Solvability analysis of the implicit numerical method.** It is clear that the corresponding homogeneous linear algebraic equations for the implicit numerical scheme (2.15)-(2.18) are

$$u_{i,j}^k = u_{i,j}^{k-1} + \overline{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l} \delta_x^2 u_{i,j}^{k-l} + \widetilde{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l} \delta_y^2 u_{i,j}^{k-l}, \tag{2.31}$$

$$i = 1, 2, \ldots, M_1 - 1,\ j = 1, 2, \ldots, M_2 - 1,\ k = 1, 2, \ldots, N,$$

$$u_{i,j}^0 = 0,\ i = 0, 1, \ldots, M_1,\ j = 0, 1, \ldots, M_2, \tag{2.32}$$

$$u_{i,0}^k = u_{i,M_2}^k = 0,\ i = 1, 2, \ldots, M_1 - 1,\ k = 1, 2, \ldots, N, \tag{2.33}$$

$$u_{0,j}^k = u_{M_1,j}^k = 0,\ j = 1, 2, \ldots, M_2 - 1,\ k = 1, 2, \ldots, N. \tag{2.34}$$

Similarly to the proof of Theorem 2.3, we can also verify that the solution of the homogeneous linear algebraic equations (2.31)-(2.34) satisfy

$$\|u^k\|_2 \le \|u^0\|_2,\ k = 1, 2, \ldots, N.$$

Because $u^0 = 0$, we get

$$u^k = 0,\ k = 1, 2, \ldots, N.$$

This indicates that the homogeneous linear algebraic equations (2.31)-(2.34) have only zero solution. Then, we have the following result:

Theorem 2.4. *The implicit numerical scheme* (2.15)-(2.18) *is uniquely solvable.*

2.5. **Convergence analysis of the implicit numerical method.** In this subsection, we analyze the convergence of the implicit numerical scheme (2.15)-(2.18). Subtracting (2.15) from (2.13), we obtain the following error equation:

$$E_{i,j}^k = E_{i,j}^{k-1} + \overline{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l} \delta_x^2 E_{i,j}^{k-l} + \widetilde{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l} \delta_y^2 E_{i,j}^{k-l} + R_{i,j}^k, \tag{2.35}$$

$$i = 1, 2, \ldots, M_1 - 1,\ j = 1, 2, \ldots, M_2 - 1,\ k = 1, 2, \ldots, N,$$

where $E_{i,j}^k = u(x_i, y_j, t_k) - u_{i,j}^k$. For $k = 0, 1, \ldots, N$, we define the following grid functions, respectively:

$$E^k(x, y) = \begin{cases} E_{i,j}^k, & \text{when} \quad (x, y) \in \Omega_1, \\ 0, & \text{when} \quad (x, y) \in \Omega_2, \end{cases}$$

and

$$R^k(x, y) = \begin{cases} R_{i,j}^k, & \text{when} \quad (x, y) \in \Omega_1, \\ 0, & \text{when} \quad (x, y) \in \Omega_2, \end{cases}$$

where Ω_1 and Ω_2 are defined in subsection 2.3. Then, $E^k(x, y)$ and $R^k(x, y)$ have the Fourier series expansions

$$E^k(x, y) = \sum_{l_1, l_2 = -\infty}^{\infty} \eta_k(l_1, l_2) e^{I 2\pi (l_1 x/L + l_2 y/L)},\ k = 0, 1, \ldots, N$$

and

$$R^k(x, y) = \sum_{l_1, l_2 = -\infty}^{\infty} \xi_k(l_1, l_2) e^{I 2\pi (l_1 x/L + l_2 y/L)},\ k = 0, 1, \ldots, N,$$

where

$$\eta_k(l_1,l_2) = \frac{1}{L^2}\int\limits_{0\le x,y\le L} E^k(x,y)e^{-I2\pi(l_1x/L+l_2y/L)}dxdy,$$
$$\xi_k(l_1,l_2) = \frac{1}{L^2}\int\limits_{0\le x,y\le L} R^k(x,y)e^{-I2\pi(l_1x/L+l_2y/L)}dxdy.$$

Letting

$$E^k = \left[E^k_{1,1}, E^k_{1,2}, \ldots, E^k_{1,M_2-1}, \ldots, E^k_{M_1-1,1}, E^k_{M_1-1,2}, \ldots, E^k_{M_1-1,M_2-1}\right]^T,$$
$$R^k = \left[R^k_{1,1}, R^k_{1,2}, \ldots, R^k_{1,M_2-1}, \ldots, R^k_{M_1-1,1}, R^k_{M_1-1,2}, \ldots, R^k_{M_1-1,M_2-1}\right]^T,$$

and applying the Parseval equalities

$$\int\limits_{0\le x,y\le L} \left|E^k(x,y)\right|^2 dxdy = \sum_{l_1,l_2=-\infty}^{\infty} |\eta_k(l_1,l_2)|^2,\ k=0,1,\ldots,N,$$
$$\int\limits_{0\le x,y\le L} \left|R^k(x,y)\right|^2 dxdy = \sum_{l_1,l_2=-\infty}^{\infty} |\xi_k(l_1,l_2)|^2,\ k=0,1,\ldots,N$$

and

$$\int\limits_{0\le x,y\le L} \left|E^k(x,y)\right|^2 dxdy = \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} \Delta_x\Delta_y |E^k_{i,j}|^2,\ k=0,1,\ldots,N,$$
$$\int\limits_{0\le x,y\le L} \left|R^k(x,y)\right|^2 dxdy = \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} \Delta_x\Delta_y |R^k_{i,j}|^2,\ k=0,1,\ldots,N,$$

we have, respectively,

$$(2.36)\qquad \|E^k\|_2 \equiv \left(\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} \Delta_x\Delta_y |E^k_{i,j}|^2\right)^{\frac{1}{2}} = \left(\sum_{l_1,l_2=-\infty}^{\infty} |\eta_k(l_1,l_2)|^2\right)^{\frac{1}{2}}$$
$$k=0,1,\ldots,N,$$

and

$$(2.37)\qquad \|R^k\|_2 \equiv \left(\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} \Delta_x\Delta_y |R^k_{i,j}|^2\right)^{\frac{1}{2}} = \left(\sum_{l_1,l_2=-\infty}^{\infty} |\xi_k(l_1,l_2)|^2\right)^{\frac{1}{2}},$$
$$k=0,1,\ldots,N.$$

We now assume $E_{i,j}^k$ and $R_{i,j}^k$ have the forms

$$E_{i,j}^k = \eta_k e^{I(\sigma_1 i\Delta_x + \sigma_2 j\Delta_y)},\ R_{i,j}^k = \xi_k e^{I(\sigma_1 i\Delta_x + \sigma_2 j\Delta_y)}, \tag{2.38}$$

where σ_1 and σ_2 are as defined in subsection 2.3. By substituting (2.38) into (2.35) we get

$$\begin{aligned}\eta_k =&\eta_{k-1} - 4\overline{\mu}_{i,j}^k \sin^2 \frac{\sigma_1\Delta_x}{2} \sum_{l=0}^{k} \lambda_{i,j}^{k,l}\eta_{k-l}\\ &- 4\widetilde{\mu}_{i,j}^k \sin^2 \frac{\sigma_2\Delta_y}{2} \sum_{l=0}^{k} \lambda_{i,j}^{k,l}\eta_{k-l} + \xi_k,\ k = 1, 2, \ldots, N.\end{aligned} \tag{2.39}$$

Using Lemma 2.1, (2.39) can be rewritten as

$$\begin{gathered}\eta_k = \frac{1 + (1-\gamma_{i,j}^k)\mu_{i,j}^k}{1+\mu_{i,j}^k}\eta_{k-1} - \frac{\mu_{i,j}^k}{1+\mu_{i,j}^k}\sum_{l=2}^{k}\lambda_{i,j}^{k,l}\eta_{k-l} + \frac{\xi_k}{1+\mu_{i,j}^k},\\ k = 1, 2, \ldots, N.\end{gathered} \tag{2.40}$$

where $\mu_{i,j}^k$ is defined by (2.28). Applying (2.14) and Lemma 2.2, we have

$$\begin{aligned}R_{i,j}^k =&O(\Delta_x^2 + \Delta_y^2)\Delta_t^{\gamma_{i,j}^k} \sum_{l=0}^{k} \lambda_{i,j}^{k,l} + O(\Delta_t^2)\\ =&O(\Delta_x^2 + \Delta_y^2)\Delta_t\Delta_t^{\gamma_{i,j}^k-1} \sum_{l=0}^{k} \lambda_{i,j}^{k,l} + O(\Delta_t^2)\\ =&O(\Delta_x^2 + \Delta_y^2)\Delta_t\left(\frac{1}{\Gamma(\gamma_j^k)} + O(\Delta_t)\right) + O(\Delta_t^2)\\ =&O\left(\Delta_t^2 + \Delta_t\Delta_x^2 + \Delta_t\Delta_y^2\right).\end{aligned} \tag{2.41}$$

Then there is a positive constant C_1 such that

$$\begin{gathered}|R_{i,j}^k| \le C_1\left(\Delta_t^2 + \Delta_t\Delta_x^2 + \Delta_t\Delta_y^2\right),\\ i = 1, 2, \ldots, M_1-1,\ j = 1, 2, \ldots, M_2-1,\ k = 1, 2, \ldots, N.\end{gathered} \tag{2.42}$$

Now, by the first equality of (2.37) we get

$$\begin{aligned}\|R^k\|_2 <&\sqrt{M_1\Delta_x}\sqrt{M_2\Delta_y}C_1\left(\Delta_t^2 + \Delta_t\Delta_x^2 + \Delta_t\Delta_y^2\right)\\ =&C_1L\left(\Delta_t^2 + \Delta_t\Delta_x^2 + \Delta_t\Delta_y^2\right),\ k = 1, 2, \ldots, N.\end{aligned} \tag{2.43}$$

According to the convergence of the series in the second equality of (2.37), there is a constant $C_2 \ge 1$ such that

$$|\xi_k| \equiv |\xi_k(l_1, l_2)| \le C_2|\xi_1(l_1, l_2)| \equiv C_2|\xi_1|,\ k = 1, 2, \ldots, N. \tag{2.44}$$

Theorem 2.5. *The implicit numerical scheme* (2.15)-(2.18) *is convergent with the order* $O(\Delta_t + \Delta_x^2 + \Delta_y^2)$.

Proof. Using mathematical induction, we can verify that

$$|\eta_k| \le C_2k|\xi_1|,\ k = 1, 2, \ldots, N, \tag{2.45}$$

where $\eta_k (k = 1, 2, \ldots, N)$ is the solution of (2.40), and the constant $C_2 \geq 1$ is given by (2.44). When $k = 1$, it follows from (2.40) that

$$\eta_1 = \frac{1 + (1 - \gamma_{i,j}^1)\mu_{i,j}^1}{1 + \mu_{i,j}^1}\eta_0 + \frac{\xi_1}{1 + \mu_{i,j}^1}.$$

From $E^0 = 0$ and (2.36) we get

$$\eta_0 \equiv \eta_0(l_1, l_2) = 0. \tag{2.46}$$

From (2.28) and (2.44) it follows that

$$|\eta_1| = \frac{|\xi_1|}{1 + \mu_{i,j}^1} \leq |\xi_1| \leq C_2|\xi_1|.$$

Assume

$$|\eta_n| \leq C_2 n|\xi_1|, \ \ n = 1, 2, \ldots, k - 1.$$

In view of (2.19), (2.28), (2.44) and Lemma 2.1, (2.40) leads to

$$\begin{aligned}
|\eta_k| \leq & \frac{1 + (1 - \gamma_{i,j}^k)\mu_{i,j}^k}{1 + \mu_{i,j}^k}|\eta_{k-1}| + \frac{\mu_{i,j}^k}{1 + \mu_{i,j}^k}\sum_{l=2}^{k}|\lambda_{i,j}^{k,l}||\eta_{k-l}| + \frac{|\xi_k|}{1 + \mu_{i,j}^k} \\
\leq & \left[\frac{1 + (1 - \gamma_{i,j}^k)\mu_{i,j}^k}{1 + \mu_{i,j}^k}(k-1) + \frac{\mu_{i,j}^k}{1 + \mu_{i,j}^k}\sum_{l=2}^{k}|\lambda_{i,j}^{k,l}|(k-l) + \frac{1}{1 + \mu_{i,j}^k}\right]C_2|\xi_1| \\
\leq & \left[\frac{1 + (1 - \gamma_{i,j}^k)\mu_{i,j}^k}{1 + \mu_{i,j}^k}(k-1) + \frac{\mu_{i,j}^k}{1 + \mu_{i,j}^k}(k-1)\sum_{l=2}^{k}|\lambda_{i,j}^{k,l}| + \frac{1}{1 + \mu_{i,j}^k}\right]C_2|\xi_1| \\
\leq & \left[\frac{1 + (1 - \gamma_{i,j}^k)\mu_{i,j}^k}{1 + \mu_{i,j}^k}(k-1) + \frac{\mu_{i,j}^k}{1 + \mu_{i,j}^k}(k-1)\left(\sum_{l=1}^{k}|\lambda_{i,j}^{k,l}| - |\lambda_{i,j}^{k,1}|\right)\right. \\
& \left. + \frac{1}{1 + \mu_{i,j}^k}\right]C_2|\xi_1| \\
\leq & \left\{\frac{1 + (1 - \gamma_{i,j}^k)\mu_{i,j}^k}{1 + \mu_{i,j}^k}(k-1) + \frac{\mu_{i,j}^k}{1 + \mu_{i,j}^k}(k-1)\left[-\sum_{l=1}^{k}\lambda_{i,j}^{k,l} - (1 - \gamma_{i,j}^k)\right]\right. \\
& \left. + \frac{1}{1 + \mu_{i,j}^k}\right\}C_2|\xi_1| \\
\leq & \left\{\frac{1 + (1 - \gamma_{i,j}^k)\mu_{i,j}^k}{1 + \mu_{i,j}^k}(k-1) + \frac{\mu_{i,j}^k}{1 + \mu_{i,j}^k}(k-1)\left[1 - (1 - \gamma_{i,j}^k)\right]\right. \\
& \left. + \frac{1}{1 + \mu_{i,j}^k}\right\}C_2|\xi_1| \\
= & \left(k - 1 + \frac{1}{1 + \mu_{i,j}^k}\right)C_2|\xi_1| \\
\leq & C_2 k|\xi_1|.
\end{aligned} \tag{2.47}$$

This verifies the conclusion (2.45). Based on (2.36), (2.37), (2.43), the conclusion (2.45), and $k\Delta_t \leq T$ we obtain

$$\|E^k\|_2 \leq C_2 k\|R^1\|_2 \leq C_1C_2kL\left(\Delta_t^2 + \Delta_t\Delta_x^2 + \Delta_t\Delta_y^2\right) \leq C(\Delta_t + \Delta_x^2 + \Delta_y^2),$$

where $C = C_1C_2TL$. The proof of Theorem 2.5 is therefore completed. □

3. An explicit numerical method

3.1. **Derivation of the explicit numerical method.** Because $u(x,y,t) \in U(\Omega)$, it holds that

$$\frac{\partial u(x_i,y_j,t_k)}{\partial t} = \frac{u(x_i,y_j,t_{k+1}) - u(x_i,y_j,t_k)}{\Delta_t} + O(\Delta_t). \tag{3.1}$$

From (2.8), (2.9), (2.11), (2.12) and (3.1), we have

$$\begin{aligned} u(x_i,y_j,t_{k+1}) =& u(x_i,y_j,t_k) + \overline{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l}\delta_x^2 u(x_i,y_j,t_{k-l}) \\ &+ \widetilde{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l}\delta_y^2 u(x_i,y_j,t_{k-l}) + \Delta_t f_{i,j}^k + R_{i,j}^{k+1}, \end{aligned} \tag{3.2}$$

$$i = 1,2,\ldots,M_1-1,\ j = 1,2,\ldots,M_2-1,\ k = 0,1,\ldots,N-1,$$

where $\overline{\mu}_{i,j}^k$ and $\widetilde{\mu}_{i,j}^k$ are as defined in subsection 2.1, whereas

$$R_{i,j}^{k+1} = O(\Delta_x^2 + \Delta_y^2)\Delta_t^{\gamma_{i,j}^k} \sum_{l=0}^{k} \lambda_{i,j}^{k,l} + O(\Delta_t^2). \tag{3.3}$$

According to the above analysis, we now present an explicit numerical scheme for solving the two-dimensional variable-order anomalous subdiffusion equation (1.2) with the initial and boundary conditions (1.3)-(1.5) as follows:

$$u_{i,j}^{k+1} = u_{i,j}^k + \overline{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l}\delta_x^2 u_{i,j}^{k-l} + \widetilde{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l}\delta_y^2 u_{i,j}^{k-l} + \Delta_t f_{i,j}^k, \tag{3.4}$$

$$i = 1,2,\ldots,M_1-1,\ j = 1,2,\ldots,M_2-1,\ k = 0,1,\ldots,N-1,$$

$$u_{i,j}^0 = \phi(x_i,y_j),\ i = 0,1,\ldots,M_1,\ j = 0,1,\ldots,M_2, \tag{3.5}$$

$$\begin{aligned} &u_{i,0}^k = \varphi_1(x_i,t_k), u_{i,M_2}^k = \varphi_2(x_i,t_k), \\ &i = 1,2,\ldots,M_1-1,\ k = 1,2,\ldots,N, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &u_{0,j}^k = \psi_1(y_j,t_k),\ u_{M_1,j}^k = \psi_2(y_j,t_k), \\ &j = 1,2,\ldots,M_2-1,\ k = 1,2,\ldots,N. \end{aligned} \tag{3.7}$$

3.2. **Stability analysis of the explicit numerical method.** Obviously, the roundoff error equation of the explicit numerical scheme (3.4)-(3.7) is

$$\rho_{i,j}^{k+1} = \rho_{i,j}^k + \overline{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l}\delta_x^2 \rho_{i,j}^{k-l} + \widetilde{\mu}_{i,j}^k \sum_{l=0}^{k} \lambda_{i,j}^{k,l}\delta_y^2 \rho_{i,j}^{k-l}, \tag{3.8}$$

$$i = 1,2,\ldots,M_1-1,\ j = 1,2,\ldots,M_2-1,\ k = 0,1,\ldots,N-1,$$

where $\rho_{i,j}^k$ is as defined in subsection 2.3. We now also assume that the solution of the roundoff error (3.8) has the form defined by (2.25). Substituting (2.25) into (3.8) gives

$$\begin{aligned}\zeta_{k+1} &= \zeta_k - 4\overline{\mu}_{i,j}^k \sin^2\frac{\sigma_1\Delta_x}{2}\sum_{l=0}^{k}\lambda_{i,j}^{k,l}\zeta_{k-l} - 4\widetilde{\mu}_{i,j}^k \sin^2\frac{\sigma_2\Delta_y}{2}\sum_{l=0}^{k}\lambda_{i,j}^{k,l}\zeta_{k-l}\\ &= \zeta_k - \mu_{i,j}^k\sum_{l=0}^{k}\lambda_{i,j}^{k,l}\zeta_{k-l},\ k=0,1,\ldots,N-1,\end{aligned}\tag{3.9}$$

where $\mu_{i,j}^k$ is defined by (2.28).

Theorem 3.1. *If* $\mu_{i,j}^k \le 1$, *then the explicit numerical scheme* (3.4)-(3.7) *is stable.*

Proof. We prove the result by mathematical induction. If $\mu_{i,j}^k \le 1$, then

$$|\zeta_{k+1}| \le |\zeta_0|,\ k=0,1,\ldots,N-1,\tag{3.10}$$

where $\zeta_{k+1}(k=0,1,\ldots,N-1)$ is the solution of (3.9). When $k=0$, it follows from (3.9) that

$$\zeta_1 = (1-\mu_{i,j}^0)\zeta_0.$$

Using $0 \le \mu_{i,j}^k \le 1$ we arrive at

$$|\zeta_1| \le (1-\mu_{i,j}^0)|\zeta_0| \le |\zeta_0|.$$

Suppose that

$$|\zeta_n| \le |\zeta_0|,\ n=1,2,\ldots,k.$$

Based on $0 \le \mu_{i,j}^k \le 1$ and Lemma 2.1, we obtain from (3.9) that

$$\begin{aligned}|\zeta_{k+1}| &\le (1-\mu_{i,j}^k)|\zeta_k| + \mu_{i,j}^k\sum_{l=1}^{k}|\lambda_{i,j}^{k,l}||\zeta_{k-l}|\\ &\le \left(1-\mu_{i,j}^k + \mu_{i,j}^k\sum_{l=1}^{k}|\lambda_{i,j}^{k,l}|\right)|\zeta_0|\\ &= \left(1-\mu_{i,j}^k - \mu_{i,j}^k\sum_{l=1}^{k}\lambda_{i,j}^{k,l}\right)|\zeta_0|\\ &\le (1-\mu_{i,j}^k+\mu_{i,j}^k)|\zeta_0|\\ &= |\zeta_0|.\end{aligned}\tag{3.11}$$

The conclusion (3.10) is then proved. From this we have

$$\|\rho^{k+1}\|_2 \le \|\rho^0\|_2,\ k=0,1,\ldots,N-1.$$

The proof of Theorem 3.1 is therefore completed. □

3.3. **Convergence analysis of the explicit numerical method.** In this subsection, we investigate the convergence of the explicit numerical scheme (3.4)-(3.7). Subtracting (3.4) from (3.2), we obtain the error equation

$$E_{i,j}^{k+1} = E_{i,j}^k + \overline{\mu}_{i,j}^k\sum_{l=0}^{k}\lambda_{i,j}^{k,l}\delta_x^2 E_{i,j}^{k-l} + \widetilde{\mu}_{i,j}^k\sum_{l=0}^{k}\lambda_{i,j}^{k,l}\delta_y^2 E_{i,j}^{k-l} + R_{i,j}^{k+1},\tag{3.12}$$

$$i=1,2,\ldots,M_1-1,\ j=1,2,\ldots,M_2-1,\ k=0,1,\ldots,N-1,$$

where $E_{i,j}^k$ is as defined in subsection 2.5. We also assume that the solution of the difference equation (3.12) has the form defined by (2.38). Substituting (2.38) into (3.12) gives

$$\begin{aligned}\eta_{k+1} =& \eta_k - 4\overline{\mu}_{i,j}^k \sin^2 \frac{\sigma_1 \Delta_x}{2} \sum_{l=0}^{k} \lambda_{i,j}^{k,l} \eta_{k-l} \\ &- 4\widetilde{\mu}_{i,j}^k \sin^2 \frac{\sigma_2 \Delta_y}{2} \sum_{l=0}^{k} \lambda_{i,j}^{k,l} \eta_{k-l} + \xi_{k+1},\ k = 0, 1, \ldots, N-1.\end{aligned} \tag{3.13}$$

Using Lemma 2.1, (3.13) can be rewritten as

$$\eta_{k+1} = (1 - \mu_{i,j}^k)\eta_k - \mu_{i,j}^k \sum_{l=1}^{k} \lambda_{i,j}^{k,l} \eta_{k-l} + \xi_{k+1},\ k = 0, 1, \ldots, N-1, \tag{3.14}$$

where $\mu_{i,j}^k$ is defined by (2.28). Applying (3.3) and Lemma 2.2 gives

$$\begin{aligned}R_{i,j}^{k+1} =& O(\Delta_x^2 + \Delta_y^2)\Delta_t^{\gamma_{i,j}^k} \sum_{l=0}^{k} \lambda_{i,j}^{k,l} + O(\Delta_t^2) \\ =& O(\Delta_x^2 + \Delta_y^2)\Delta_t \Delta_t^{\gamma_{i,j}^k - 1} \sum_{l=0}^{k} \lambda_{i,j}^{k,l} + O(\Delta_t^2) \\ =& O(\Delta_x^2 + \Delta_y^2)\Delta_t \left(\frac{1}{\Gamma(\gamma_j^k)} + O(\Delta_t) \right) + O(\Delta_t^2) \\ =& O\left(\Delta_t^2 + \Delta_t \Delta_x^2 + \Delta_t \Delta_y^2\right).\end{aligned} \tag{3.15}$$

Then there is a positive constant C_3 such that

$$\begin{gathered}|R_{i,j}^{k+1}| \leq C_3 \left(\Delta_t^2 + \Delta_t \Delta_x^2 + \Delta_t \Delta_y^2\right), \\ i = 1, 2, \ldots, M_1 - 1,\ j = 1, 2, \ldots, M_2 - 1,\ k = 0, 1, \ldots, N-1.\end{gathered} \tag{3.16}$$

It is clear that for the explicit numerical scheme (3.4)-(3.7), the results (2.36), (2.37), (2.44) and (2.46) are valid. From (3.16) and the first equality of (2.37), we have

$$\begin{aligned}\|R^{k+1}\|_2 <& \sqrt{M_1 \Delta_x}\sqrt{M_2 \Delta_y} C_3 \left(\Delta_t^2 + \Delta_t \Delta_x^2 + \Delta_t \Delta_y^2\right) \\ =& C_3 L \left(\Delta_t^2 + \Delta_t \Delta_x^2 + \Delta_t \Delta_y^2\right),\ k = 0, 1, \ldots, N-1.\end{aligned} \tag{3.17}$$

Theorem 3.2. *If $\mu_{i,j}^k \leq 1$, then the explicit numerical scheme* (3.4)-(3.7) *is convergent with order $O(\Delta_t + \Delta_x^2 + \Delta_y^2)$.*

Proof. We prove the result by mathematical induction: if $\mu_{i,j}^k \leq 1$, then there is a positive constant C_2 such that

$$|\eta_{k+1}| \leq C_2(k+1)|\xi_1|, \;\; k = 0, 1, \ldots, N-1, \tag{3.18}$$

where $\eta_{k+1}(k = 0, 1, \ldots, N-1)$ is the solution of (3.14), and the constant $C_2 \geq 1$ is given by (2.44). When $k = 0$, it follows from (3.14) and (2.46) that

$$\eta_1 = \xi_1.$$

By (2.44), we have

$$|\eta_1| = |\xi_1| \leq C_2|\xi_1|.$$

Suppose that

$$|\eta_n| \leq C_2 n|\xi_1|, \;\; n = 1, 2, \ldots, k.$$

According to $0 \leq \mu_{i,j}^k \leq 1$ and Lemma 2.1, (3.14) gives

$$\begin{aligned}
|\eta_{k+1}| &\leq \left(1 - \mu_{i,j}^k\right)|\eta_k| + \mu_{i,j}^k \sum_{l=1}^{k} \left|\lambda_{i,j}^{k,l}\right| |\eta_{k-l}| + |\xi_{k+1}| \\
&\leq \left[\left(1 - \mu_{i,j}^k\right) k + \mu_{i,j}^k \sum_{l=1}^{k} \left|\lambda_{i,j}^{k,l}\right| (k-l) + 1\right] C_2|\xi_1| \\
&\leq \left[\left(1 - \mu_{i,j}^k\right) k + \mu_{i,j}^k k \sum_{l=1}^{k} \left|\lambda_{i,j}^{k,l}\right| + 1\right] C_2|\xi_1| \\
&= \left[\left(1 - \mu_{i,j}^k\right) k + \mu_{i,j}^k k \left(-\sum_{l=1}^{k} \lambda_{i,j}^{k,l}\right) + 1\right] C_2|\xi_1| \\
&\leq [\left(1 - \mu_{i,j}^k\right) k + \mu_{i,j}^k k + 1] C_2|\xi_1| \\
&= C_2(k+1)|\xi_1|.
\end{aligned} \tag{3.19}$$

The conclusion (3.18) then follows. From this we obtain

$$\begin{aligned}
\|E^{k+1}\|_2 \leq& C_2(k+1)\|R^1\|_2 \leq C_2C_3(k+1)L\left(\Delta_t^2 + \Delta_t\Delta_x^2 + \Delta_t\Delta_y^2\right) \\
\leq& \widehat{C}(\Delta_t + \Delta_x^2 + \Delta_y^2),
\end{aligned} \tag{3.20}$$

where $\widehat{C} = C_2C_3TL$. This completes the proof of Theorem 3.2. $\square$

4. Numerical examples

In this section, in order to demonstrate the accuracy of our theoretical results, we apply the implicit numerical method (2.15)-(2.18) and the explicit numerical method (3.4)-(3.7) to solve the following two-dimensional variable-order anomalous subdiffusion equation:

$$\begin{aligned}\frac{\partial u(x,y,t)}{\partial t} &= {}_0D_t^{1-\gamma(x,y,t)}\left(\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}\right)\\ &\quad + 2e^{x+y}\left(t-2\frac{t^{1+\gamma(x,y,t)}}{\Gamma(2+\gamma(x,y,t))}\right),\ 0<t\le 1,\ 0<x,y<1,\end{aligned}\tag{4.1}$$

with the initial and boundary conditions:

$$u(x,y,0)=0,\tag{4.2}$$

$$u(x,0,t)=e^x t^2,\ u(x,1,t)=e^{1+x}t^2,\tag{4.3}$$

$$u(0,y,t)=e^y t^2,\ u(1,y,t)=e^{1+y}t^2.\tag{4.4}$$

The exact solution of the problem (4.1)-(4.4) is

$$u(x,y,t)=e^{x+y}t^2.$$

We let

$$E_{\max}=\max_{0\le k\le N}\left\{\|E^k\|_2\right\}.$$

Table 1 provides the maximum errors of the numerical solutions for the problem (4.1)-(4.4) using the implicit numerical method (2.15)-(2.18) for various $\Delta_t=\Delta_x^2=\Delta_y^2$ and $\gamma(x,y,t)$, where on the finite domain $0\le x,y,t\le 1$, the function $\gamma(x,y,t)$ satisfies $0<\gamma(x,y,t)<1$.

TABLE 1. The maximum error $E_{\max}$ of the implicit numerical method (2.15)-(2.18) for $0<\gamma(x,y,t)<1$.

$\gamma(x,y,t)$	$\Delta_t=\Delta_x^2=\frac{1}{4}$	$\Delta_t=\Delta_x^2=\frac{1}{16}$	$\Delta_t=\Delta_x^2=\frac{1}{64}$	$\Delta_t=\Delta_x^2=\frac{1}{256}$
$\sin(xyt+\frac{2\pi}{5})$	2.4286×10^{-2}	7.8620×10^{-3}	2.0681×10^{-3}	5.3845×10^{-4}
$\cos(xyt+\frac{1}{100})$	2.3408×10^{-2}	7.4865×10^{-3}	1.9660×10^{-3}	5.0486×10^{-4}
$\frac{e^{xyt}-\sin(xyt)}{10}$	1.2208×10^{-2}	3.9143×10^{-3}	1.0309×10^{-3}	3.4173×10^{-4}
$\frac{e^{xyt}+\cos(xyt)}{20}$	1.1952×10^{-2}	3.8370×10^{-3}	1.0112×10^{-3}	3.3466×10^{-4}
$\frac{e^{xyt}-xyt}{8}$	1.1091×10^{-2}	3.5259×10^{-3}	9.2618×10^{-4}	3.0723×10^{-4}
$\frac{e^{xyt}-(xyt)^3}{12}$	1.2447×10^{-2}	4.0124×10^{-3}	1.0597×10^{-3}	3.4946×10^{-4}
$e^{xyt-2.5}$	1.2492×10^{-2}	4.0195×10^{-3}	1.0606×10^{-3}	3.5039×10^{-4}
$e^{-xyt-1.8}$	1.0361×10^{-2}	3.2696×10^{-3}	8.5603×10^{-4}	2.8827×10^{-4}
$\frac{\sqrt{xyt}+1}{15}$	1.2664×10^{-2}	4.1117×10^{-3}	1.0883×10^{-3}	3.6031×10^{-4}
$\frac{1+(xyt)^5}{9}$	1.1781×10^{-2}	3.7736×10^{-3}	9.9398×10^{-4}	3.2941×10^{-4}

Table 2 provides the maximum errors of the numerical solutions for the problem (4.1)-(4.4) using the implicit numerical method (2.15)-(2.18) for various $\Delta_t=\Delta_x^2=\Delta_y^2$ and $\gamma(x,y,t)$, where on the finite domain $0\le x,y,t\le 1$, the function $\gamma(x,y,t)$ satisfies $0\le\gamma(x,y,t)\le 1$. In other words, the order of the fractional derivative is now allowed to reach 0 or 1 on the given domain. The results also strongly support the numerical method.

TABLE 2. The maximum error $E_{\max}$ of the implicit numerical method (2.15)-(2.18) for $0 \leq \gamma(x,y,t) \leq 1$.

$\gamma(x,y,t)$	$\Delta_t = \Delta_x^2 = \frac{1}{4}$	$\Delta_t = \Delta_x^2 = \frac{1}{16}$	$\Delta_t = \Delta_x^2 = \frac{1}{64}$	$\Delta_t = \Delta_x^2 = \frac{1}{256}$
xyt	9.8590×10^{-3}	3.0285×10^{-3}	7.9403×10^{-4}	2.5184×10^{-4}
xy^2t^3	1.4792×10^{-2}	4.6940×10^{-3}	1.2337×10^{-3}	3.9508×10^{-4}
$\frac{x+y+t}{3}$	9.0507×10^{-3}	3.3864×10^{-3}	9.1541×10^{-4}	2.2891×10^{-4}
$\frac{x+y^2+t^3}{3}$	2.5695×10^{-3}	1.4821×10^{-3}	4.2358×10^{-4}	1.0179×10^{-4}
$\frac{xy+xt+yt}{3}$	3.5841×10^{-3}	1.1529×10^{-3}	3.0776×10^{-4}	9.7060×10^{-5}
$\frac{x+xy+xyt}{3}$	3.2323×10^{-3}	1.0642×10^{-3}	2.9591×10^{-4}	9.2645×10^{-5}
$\sin\frac{xyt\pi}{2}$	5.7740×10^{-3}	1.8470×10^{-3}	4.9167×10^{-4}	1.5160×10^{-4}
$\cos\frac{xyt\pi}{2}$	2.2202×10^{-2}	6.9722×10^{-3}	1.8306×10^{-3}	4.6569×10^{-4}
$\sin\frac{(x+y+t)\pi}{6}$	1.8163×10^{-2}	6.1365×10^{-3}	1.6269×10^{-3}	4.1510×10^{-4}
$cos\frac{(x+y+t)\pi}{6}$	1.2978×10^{-2}	4.4917×10^{-3}	1.2051×10^{-3}	3.0607×10^{-4}

Table 3 provides the maximum errors of the numerical solutions for the problem (4.1)-(4.4) using the explicit numerical method (3.4)-(3.7) for various $\Delta_x = \Delta_y$, Δ_t and $\gamma(x,y,t)$.

TABLE 3. The maximum error $E_{\max}$ of the explicit numerical method (3.4)-(3.7).

$\gamma(x,y,t)$	$\sin(xyt + \frac{2\pi}{5})$	$\cos(xyt + \frac{1}{100})$
$\Delta_x = \Delta_y = \frac{1}{2}, \Delta_t = \frac{1}{8}$	3.1089×10^{-2}	3.7502×10^{-2}
$\Delta_x = \Delta_y = \frac{1}{3}, \Delta_t = \frac{1}{26}$	1.4076×10^{-2}	3.6094×10^{-3}
$\Delta_x = \Delta_y = \frac{1}{4}, \Delta_t = \frac{1}{65}$	1.6009×10^{-3}	1.5754×10^{-3}
$\Delta_x = \Delta_y = \frac{1}{5}, \Delta_t = \frac{1}{122}$	8.6809×10^{-4}	8.5657×10^{-4}
$\Delta_x = \Delta_y = \frac{1}{6}, \Delta_t = \frac{1}{266}$	3.9543×10^{-4}	3.9336×10^{-4}

From Tables 1-3, it is seen that our theoretical analysis results have been verified by the numerical results.

A comparison of the numerical solution using the implicit numerical method (2.15)-(2.18) and exact solution of the problem (4.1)-(4.4) at $x = 0.25$, $t = 1$, for $\gamma(x,y,t) = \frac{e^{xyt} + \cos(xyt)}{20}$ and $\Delta_t = \Delta_x^2 = \Delta_y^2 = \frac{1}{256}$ is shown in Figure 1.

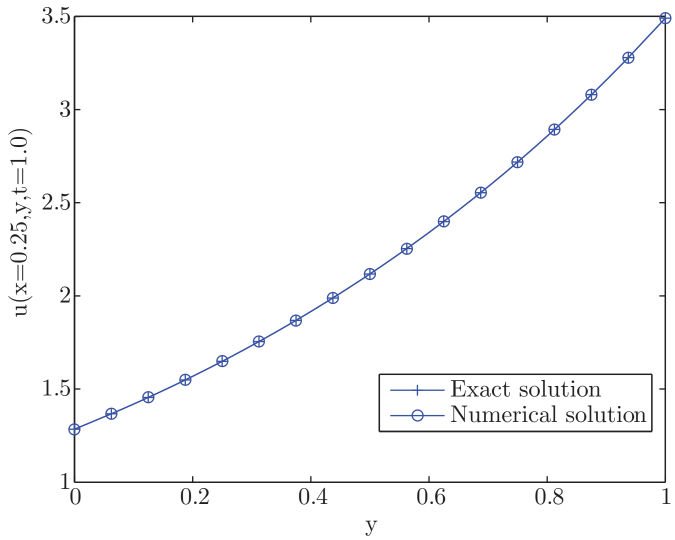

FIGURE 1. A comparison of the numerical solutions using the implicit numerical method (2.15)-(2.18) and exact solutions of the problem (4.1)-(4.4) at $x = 0.25$, $t = 1$, for $\gamma(x, y, t) = \frac{e^{xyt}+\cos(xyt)}{20}$.

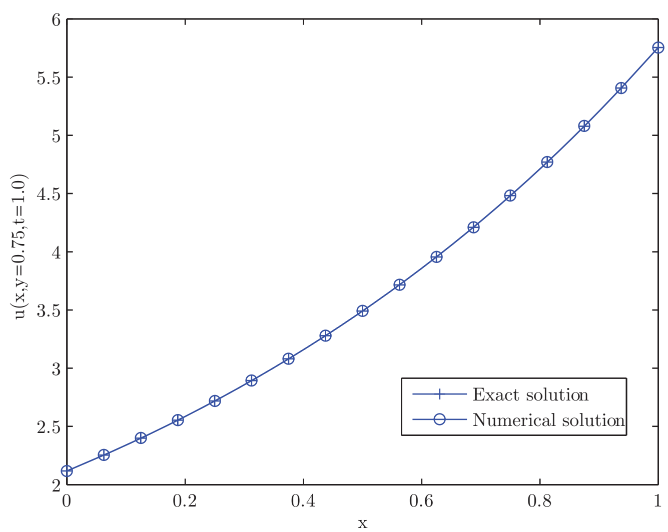

FIGURE 2. A comparison of the numerical solutions using the implicit numerical method (2.15)-(2.18) and exact solutions of the problem (4.1)-(4.4) at $y = 0.75$, $t = 1$, for $\gamma(x, y, t) = \frac{1+(xyt)^5}{9}$.

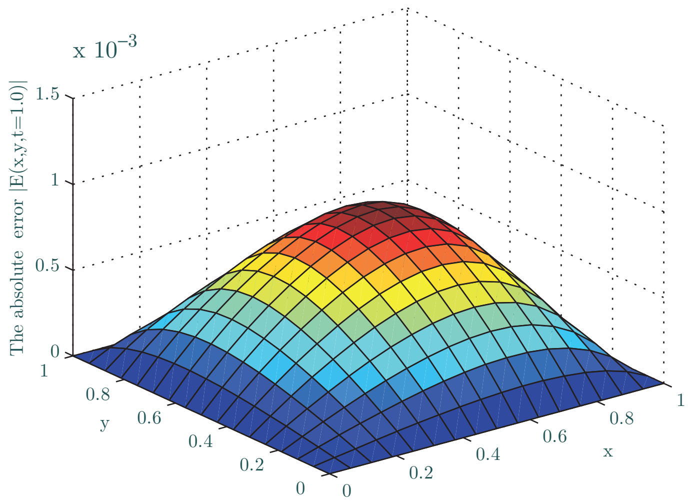

FIGURE 3. The absolute error of the numerical solution of the problem (4.1)-(4.4) using the implicit numerical method (2.15)-(2.18) at $t = 1.0$ for $\gamma(x, y, t) = \sin\frac{(x+y+t)\pi}{6}$.

A comparison of the numerical solution using the implicit numerical method (2.15)-(2.18) and exact solution of the problem (4.1)-(4.4) at $y = 0.75$, $t = 1$, for $\gamma(x, y, t) = \frac{1+(xyt)^5}{9}$ and $\Delta_t = \Delta_x^2 = \Delta_y^2 = \frac{1}{256}$ is shown in Figure 2.

The absolute error of the numerical solution of the problem (4.1)-(4.4) using the implicit numerical method (2.15)-(2.18) at $t = 1.0$ for $\gamma(x, y, t) = \sin\frac{(x+y+t)\pi}{6}$ and $\Delta_t = \Delta_x^2 = \Delta_y^2 = \frac{1}{256}$ is shown in Figure 3, where

$$E(x, y, t = 1.0) = u(x, y, 1.0) - \widetilde{u}(x, y, 1.0),$$

$\widetilde{u}(x, y, 1.0)$ being the numerical approximation for $u(x, y, 1.0)$.

From Figures 1-3, it can be seen that the numerical solution is consistent with the exact solution.

5. Conclusion

In this paper, two numerical methods for solving a two-dimensional variable-order anomalous subdiffusion equation have been developed. Their stability, convergence and solvability have been discussed using Fourier analysis. Moreover, the effectiveness of our theoretical analysis has been demonstrated by some numerical examples.

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