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RATIONAL POINTS ON DIAGONAL QUARTIC SURFACES

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ABSTRACT. We searched up to height 10^7 for rational points on diagonal quartic surfaces. The computations fill several gaps in earlier lists computed by Pinch, Swinnerton-Dyer, and Bright.

1. Introduction

The set of rational points on a variety is one of the central objects in arithmetic geometry. For some classes of varieties, one has precise conjectures what it should look like.

In the case of Fano varieties, many rational points are expected. This expectation is described by the famous conjecture of Manin [FMT]. The case of a variety of general type is described by the Lang conjecture. It claims that the Zariski closure of the set of rational points has strictly smaller dimension than the variety considered. Both conjectures are proven only in a few special cases. However, in the intermediate case (i.e., varieties with are neither Fano nor of general type) it is not even clear what a general conjecture should look like.

In this note, we inspect diagonal quartic surfaces. These are special K3 surfaces and they form one of the most famous examples of varieties of intermediate type. More precisely, we focus on surfaces of the form

$$ax^4 + by^4 = cz^4 + dw^4$$

with coefficients $a, b, c, d \in \mathbb{Z}$ and $1 \leq a, b, c, d \leq 15$. We describe our methods to test local solvability and our search for rational points.

Remark 1.1. The only known technique to prove that there are no rational points on a K3 surface which has local points everywere is given by the Brauer-Manin obstruction. The algebraic part of the obstruction was intensively studied by Martin Bright in his PhD thesis [Br1]. As explained in [Br2], for many diagonal quartic surfaces the algebraic and the transcendental Brauer-Manin obstruction would not be able to explain the absence of rational points. See [ISZ] for details on the transcendental part.

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2. Point search algorithms

The main idea of our approach to searching for points of absolute height at most B on varieties of the form $f(x, y) = g(z, w)$ is as follows.

Compute the two sets $\{f(x, y) \mid |x|, |y| \leq B\}$ and $\{g(z, w) \mid |z|, |w| \leq B\}$. Each solution of the equation leads to an element in the intersection. On the other hand, one can find the solutions when one knows the intersection. If one can handle sets fast enough, an $O(B^2)$ algorithm results.

Since these sets tend to be very large, a more sophisticated approach has to be used. One way is given in [Be] for functions f and g being sums of two univariate functions. Then, one can enumerate the two sets above in sorted form. From this, the intersection can be formed easily.

A second approach is given in [EJ1] and [EJ2]. There, the sets are implemented using hash tables. To reduce the size of the sets, a page prime p_p is introduced. Then, one computes the intersection of

$$L_a := \left\{f(x,y) \,\middle|\, |x|, |y| \leq B \text{ and } f(x,y) \equiv a \pmod{p_p}\right\} \text{ and}$$
$$R_a := \left\{g(z,w) \,\middle|\, |z|, |w| \leq B \text{ and } g(z,w) \equiv a \pmod{p_p}\right\}$$

for each $a \in [0, p_p - 1]$. Assuming equidistribution, this reduces the size of the sets approximately by a factor of p_p.

As we focus on diagonal quartic surfaces, some additional optimizations can be done. First, one can restrict to non-negative values of x, y, z, w. In about one half of the cases, one can find at least one variable which must be divisible by 5. Further, in many cases, the parity of some or all variables in a primitive solution can be determined.

Usually, many other moduli lead to congruences which could be used for a speed-up if one knew how to handle them quickly on a computer. See [EJ3] for an analysis in a particular case.

Details of the program written. The point search was done using the hashing approach. In an initialization step, congruences modulo 5 and powers of 2 were checked to get congruence conditions for primitive solutions. The page prime 500083 was chosen and the hash table had 134217728 entries. To speed up the modular arithmetic, a table containing fourth roots and multiplicative inverse elements modulo the page prime was built up in the initialization part. Note that the page prime is congruent to 3 modulo 4 and, thus, the fourth root is unique up to sign.

To avoid multiprecision computations, the computations were done modulo 2^{64}. We found fewer than 100000 simultaneous coincidences modulo the page prime and modulo 2^{64}. Only these were checked by multiprecision computations.

The running time depends highly on the congruences found. Searching on one surface for points up to height 10^7 took between 12 and 86 days of CPU time on a 2.27GHz Xeon processor. In total, 13 years of CPU time were used.

3. Results

In total, there are 7194 quadruples (a, b, c, d) with $a, b, c, d \in \{1, \ldots, 15\}$, $a \leq b$, $a \leq c$, and $c \leq d$ and $\gcd(a, b, c, d) = 1$. Testing for local solvability excludes 3904 of the corresponding equations $ax^4 + by^4 = cz^4 + dw^4$.

A point search with height-bound 10 solves 3009 cases. Increasing the bound to 100 leads to solutions for 52 of the 281 remaining equations. Further, 31 equations have a first solution of height at most 1000. The remaining 198 equations (and all solutions found for them) are the entries of the list [E2].

In 21 cases, a solution of height between 10^3 and 10^4 was found by Martin Bright. In 18 cases a first solution of height between 10^4 and 10^5 was found. Further, in 14 cases, there is a first solution of height between 10^5 and 10^6. Finally, in 15 cases a first solution of height between 10^6 and 10^7 was detected. In 130 cases, still, no solution is known.

TABLE 1. Surfaces with smallest solutions of height above 10^6 found.

a	b	c	d	x	y	z	w	ρ
1	15	7	11	2903019	391311	1780640	549424	1
2	10	7	11	5742991	2277664	4262801	1865875	1
4	11	7	13	873483	1115876	1281143	448499	1
4	12	11	14	3902789	1356045	1015370	2875318	1
4	5	11	14	394427	1355547	1112545	308333	1
5	11	6	7	1545359	3316097	187414	3732530	1
7	9	11	13	3094925	7817089	6049224	6224852	1
2	9	12	15	3625719	1832215	1639331	2213957	1
2	11	7	9	2957980	1748992	468557	2308737	1
5	14	7	9	1943732	493862	984595	1643257	1
4	4	11	13	1668661	1272265	324881	1335627	1
2	3	8	11	1216988	924293	384555	873425	1
2	8	5	11	1315404	988742	1272177	470035	1
1	8	4	13	3730667	1735542	2189289	1913815	1
3	7	12	14	1116485	269121	345539	754095	2

As several equations have a first solution of height above 10^6, one cannot expect the unsolved examples to be unsolvable. However, $x^4 + y^4 = 6z^4 + 12w^4$ is known to be unsolvable. See [Br1] for details. Note that this surface is isomorphic over $\mathbb{Q}$ to $3x^4 + 6y^4 = 8z^4 + 8w^4$. This is the only pair of isomorphic surfaces in the list.

Remark 3.1. As all diagonal quartic surfaces are isomorphic over $\mathbb{C}$ this is not entirely obvious. First note that an isomorphism of surfaces with arithmetic Picard rank 1 is always given by a projective linear map $\mathbf{P}^3 \to \mathbf{P}^3$. In this situation one can try to map the 48 lines of one surface to the lines of another one with a projective linear map defined over $\mathbb{Q}$. In the case of higher arithmetic Picard rank one can distinguish surfaces by counting points modulo primes of good reduction. These arguments suffice in all cases.

It would be nice if one could make a complete list of solvable and unsolvable cases as done in [CKS] for diagonal cubic surfaces. A naive extrapolation suggests that this requires a search for points up to height 10^{15}. Further, one has to compute the transcendental Brauer-Manin obstruction in the case when [ISZ, Corollary 3.3] is not applicable.

Remark 3.2. Some people tend to believe that the arithmetic Picard rank ρ of a K3 surface has a strong influence on the set of rational points. We have no unsolved case with Picard rank greater than 2. The unsolved cases with rank equal to 2 are $[1, 1, 6, 12]$, $[2, 4, 9, 9]$, $[2, 4, 11, 11]$, $[2, 9, 6, 12]$, $[3, 6, 8, 8]$, $[3, 6, 11, 11]$, $[4, 9, 8, 8]$, and $[6, 12, 11, 11]$. On the other hand, the table above contains one equation with Picard rank 2 and a smallest solution of height 1116485.

Comparing the rank 1 and the rank 2 cases in the sample, one does not find a great difference for the proportion of unknown cases.

Appendix A. The list

List of the diagonal quartic surfaces $V\colon ax^4+by^4 = cz^4+dw^4$ and rational points found on them. Missing entries in the right columns mark unsolved equations. The surfaces are orderd by the Galois group that acts on the 48 lines of the surface. See [Br1] for details.

Galois group 1 (A222), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
1	13	6	11	2569	1570	2111	745
1	14	6	11				
1	15	10	11	33027	50285	55430	22246
1	15	11	13				
1	15	11	14				
1	15	7	11	2903019	391311	1780640	549424
1	7	5	12				
1	7	6	11	50699	7640	32419	4145
11	14	12	15				
2	10	7	11	5742991	2277664	4262801	1865875
2	11	12	15				
2	11	3	15	18201	7308	7171	11281
2	11	6	10				
2	12	7	11				
2	13	5	11				
2	13	6	10				
2	14	10	15				
2	14	12	15	1899	465	268	1154
2	14	5	12				
2	14	6	13				
2	15	7	11	814137	381238	641189	183115
2	3	7	11	479727	314300	210007	324245
2	6	5	11				
2	6	5	13				
2	6	7	11	887643	344188	643897	324877
2	6	7	15				
2	7	10	15	3615	18355	3253	15167
2	7	11	12				
3	13	10	15				
3	13	7	8				
3	13	8	11	2233	544	1355	1451
3	13	8	14				
3	15	8	11	2567	3892	339	4245

(Table continued)

a	b	c	d	x	y	z	w
3	4	5	14	3405	3838	3839	1907
				56825	15064	19469	38503
				683435	763534	276881	613601
3	7	4	15				
3	7	8	11				
3	8	13	14				
4	11	7	13	873483	1115876	1281143	448499
4	12	11	13				
4	12	11	14	3902789	1356045	1015370	2875318
4	12	5	14	142891	880825	176618	847432
4	12	7	10				
4	15	6	14				
4	5	11	14	394427	1355547	1112545	308333
4	5	12	13				
4	5	6	14	112525	220510	202663	113721
4	6	11	15				
4	7	6	10	33692	38898	9673	38125
5	11	6	7	1545359	3316097	187414	3732530
5	11	6	8				
5	11	7	13				
5	11	8	12	569	1995	2156	632
6	10	7	11	54113	64965	64930	55604
6	8	11	13	3596	7663	2899	6801
7	10	11	12	248911	22210	221045	84493
7	13	8	15				
7	8	11	13				
7	9	10	15	429335	116865	391868	125402
				490300	115345	444017	184637
7	9	11	13	3094925	7817089	6049224	6224852
7	9	12	13				
7	9	12	15				
8	11	13	14	48	1635	1103	1435
8	12	11	13				
8	15	11	13				
9	10	11	14				
9	10	12	13				
9	11	12	15				
9	11	13	14				
9	13	10	15				

Galois group 6 (A223), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
4	9	11	14				

Galois group 7 (A224), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
4	13	7	9				
4	13	9	11				
4	15	7	9				

Galois group 14 (A225), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
1	2	5	12				
1	2	6	11	5215	1874	2853	2403
1	2	7	10				
1	8	10	15				
1	8	11	14				
1	8	6	11				
2	14	3	6				
2	5	6	12				
2	7	6	12	41943	16890	31205	17281
				527799	362670	377845	336583
2	9	12	15	3625719	1832215	1639331	2213957
2	9	13	14	16747	10235	11847	253
3	6	11	14				
3	6	8	14				
4	8	11	13	23746	5375	17711	11083
				118358	4535	1109	88151
4	8	13	15				
4	8	7	13				
5	11	6	12	441	5637	4408	5210
6	12	7	15				
7	14	11	12	13467	10834	8039	13315
8	9	10	14				
8	9	11	13				
8	9	11	14				

Galois group 15 (A226), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
1	14	7	10				
1	15	2	11	16747	242669	12700	262234
				591731	113243	404020	284218
				136427	38287	44200	76208
1	15	8	11	109243	12833	3800	60028
				818273	174547	356770	417008
1	7	3	14	439019	332152	108087	305495
2	10	7	9				
2	11	4	12				
2	11	7	9	2957980	1748992	468557	2308737
2	14	3	7				
2	14	5	7				
2	14	6	7				
2	14	7	10				
2	14	7	12				
2	14	7	13				
2	14	9	10	959575	602221	165566	770822
2	14	9	13				
2	5	4	12	1304	4126	4075	2327
2	7	11	14				
3	13	4	6				
3	4	8	11				
4	12	6	7	93671	28107	84754	30230
4	14	7	13				
4	7	6	14				
5	14	7	9	1943732	493862	984595	1643257
6	10	12	13	27215	22985	26664	5998
				240525	27011	12314	198262
6	14	7	15				
6	7	10	12	1595	814	113	1367
				935	2858	1165	2479
6	7	8	14				
7	11	13	14				
7	13	14	15				
7	8	10	14				
7	9	10	14				
7	9	13	14				
7	9	8	11				

Galois group 17 (A227), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
2	2	7	11				
4	11	10	10	148879	53538	119731	14529
				148879	53538	14529	119731
4	4	11	13	224865	2639719	2049201	376219
				1668661	1272265	324881	1335627
4	4	7	13	24881	25128	18145	20661
				25128	24881	18145	20661
4	6	11	11				
7	7	11	12				
8	8	10	15				
8	8	11	13	6529	8580	155	8169
8	8	11	14				

Galois group 18 (A228), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
1	12	3	14				
1	14	4	13				
2	11	8	10	1767	862	1337	165
				2107	3126	11	3231
				11313	4822	4073	7773
2	14	8	13	3387	2291	164	2658
2	3	7	12	2718	3089	2715	635
2	3	8	11	1216988	924293	384555	873425
3	13	12	14				
3	14	4	12				
3	4	11	12				
3	7	11	12				

Galois group 25 (A229), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$:

a	b	c	d	x	y	z	w
1	4	3	14				
1	4	6	11				
2	8	11	13				
2	8	11	15	759389	135099	358775	424157
2	8	3	15				
2	8	5	11	4428912	3335126	1099539	3552505
				1315404	988742	1272177	470035
				1311878	1614416	1415335	1382675
2	8	5	13				
2	8	7	11	32445	20922	13597	23767

Galois group 37 (A234), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
2	14	4	9				

Galois group 38 (A235), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
1	14	8	9				
2	9	4	14				
4	8	7	9				
4	8	9	11				
4	8	9	15				

Galois group 41 (A242), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
1	9	4	13	3955	3086	687	3005

Galois group 42 (A238), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
4	4	7	9				
4	4	9	11				

Galois group 56 (A251), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/4\mathbb{Z}$:

a	b	c	d	x	y	z	w
1	2	8	11	3213	587	1857	1015
				24167	2763	14231	5875
				64307	28629	38887	10825
1	8	4	13	3730667	1735542	2189289	1913815
2	14	8	9				
2	3	6	12	735	4342	2397	2917
				160887	451082	377235	135455

Galois group 57 (A248), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
2	2	7	9				
2	2	9	11				
7	12	14	14				
7	9	14	14	89855	66941	82130	7552
				89855	66941	7552	82130
8	8	9	11				
8	8	9	13				

Galois group 61 (A240), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$:

a	b	c	d	x	y	z	w
1	4	8	13	7957	41635	9527	30969
2	8	7	9				
3	12	6	10				
3	12	6	13				

Galois group 64 (A241), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
4	12	9	10	99071	356615	88962	373160

Galois group 72 (A249), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/4\mathbb{Z}$:

a	b	c	d	x	y	z	w
4	6	9	13				

Galois group 135 (A123), Picard rank 2, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
2	4	9	9				

Galois group 142 (A118), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$:

a	b	c	d	x	y	z	w
1	4	8	9				

Galois group 172 (A112), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
4	14	7	9				
4	7	9	14				

Galois group 228 (A18), Picard rank 2, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
4	9	8	8				

Galois group 241 (A8), Picard rank 2, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
7	8	9	14	5145	18832	11843	15623

Galois group 260 (A121), Picard rank 2, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
1	1	6	12				
2	4	11	11				
3	6	11	11				
3	6	8	8				
6	12	11	11				

Galois group 263 (A127), Picard rank 2, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = 0$:

a	b	c	d	x	y	z	w
2	14	7	8	3367	1275	954	2450
				9137	3555	5058	6170
				27607	53755	2962	61980
3	14	7	12	14333	132	10041	8245
				51001	13458	18333	35915
				150715	37776	99567	92761
3	7	12	14	695827	2215287	1896995	998025
				1116485	269121	345539	754095

Galois group 330 (A257), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
4	9	10	15				

Galois group 336 (A109), Picard rank 1, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
4	12	10	15				

Galois group 373 (A51), Picard rank 3, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

a	b	c	d	x	y	z	w
2	8	13	13	995	1227	1115	71
				277575	326921	117017	296755
				336725	198409	232205	70143

Galois group 646 (A76), Picard rank 2, $\mathrm{Br}_1(V)/\mathrm{Br}_0(V) = \mathbb{Z}/8\mathbb{Z}$:

a	b	c	d	x	y	z	w
2	9	6	12				

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