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ERGODIC SCALES IN FRACTAL MEASURES

PALLE E. T. JORGENSEN

Abstract. We will consider a family of fractal measures on the real line $\mathbb{R}$ which are fixed, in the sense of Hutchinson, under a finite family of contractive affine mappings. The maps are chosen such as to leave gaps on $\mathbb{R}$. Hence they have fractal dimension strictly less than 1. The middle-third Cantor construction is one example. Depending on the gaps and the scaling factor, it is known that the corresponding Hilbert space $L^2(\mu)$ exhibits strikingly different properties. In this paper we show that when μ is fixed in a certain class, there are positive integers p such that multiplication by p modulo 1 induces an ergodic automorphism on the measure space (support(μ), μ).

1. Introduction

We consider a family of fractal measures on the real line $\mathbb{R}$ which are fixed, in the sense of Hutchinson [14], under a finite family of contractive affine mappings, called affine iterated function systems (IFSs). The maps are chosen such as to leave gaps on $\mathbb{R}$. Hence they have fractal dimension strictly less than 1. The middle-third Cantor construction is one example.

Our main focus is on a family of harmonic measures considered first in a paper by Jorgensen and Pedersen [17], and since then by many authors; see e.g., Dutkay and Jorgensen [10]. Depending on the gaps and the scaling factor, we show that the harmonic analysis of the Hilbert space $L^2(\mu)$ exhibits strikingly different properties. We will focus on the case when the scaling factor for μ is an even integer at least 4. The principal result in this work (Theorem 5.1) is that when μ is fixed in a certain class, there are positive integers p such that multiplication by p modulo 1 induces an ergodic automorphism on the measure space (support(μ), μ).

Assuming that the scaling factor for μ is an even integer, a main tool in our proofs is a novel interplay between, on the one hand, a family of orthonormal bases in $L^2(\mu)$, and on the other, certain isometric operators in $L^2(\mu)$ induced by multiplication by p modulo 1. See Corollaries 4.2 and 4.5.

The study of special affine iterated function system (IFS)-measures was motivated by the paper [14] on the general class of IFS-measures, and by the result in [17], as well as more recent papers by Dutkay and the present author [8, 9, 10] (see also [16]), to the effect that a subclass of affine IFS-measures μ have an orthonormal basis (ONB) of Fourier frequencies (complex exponentials) for their $L^2(\mu)$ spaces.

Applications are many, see for example [19] and the papers cited there.

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Our focus is on *harmonic analysis* and *symmetries* of fractal measures possessing self-similarity intrinsic to a fixed system of affine transformations. Other papers in the general area include [1, 2, 3, 4, 6, 7, 12, 13, 14, 15, 18, 21, 22, 23, 24], and we refer to these for additional background, and for fundamentals in the theory.

Past work has identified measures μ which have a "sufficient" supply of a particular kind of basis functions, a family of functions consisting of complex exponentials e_λ for a suitable collection of points λ; $e_\lambda(x) = e^{i2\pi\lambda\cdot x}$, i.e., measures having a Fourier basis.

The ideal possibility for some given measure μ is that the corresponding Hilbert space $L^2(\mu)$ possesses an orthonormal basis (ONB) of complex exponentials. For this to make sense we must restrict consideration to measures μ with support in one or several real dimensions, i.e., in $\mathbb{R}^d$ for some d. It is interesting enough to study compact support, and to restrict further to the case $d = 1$. We make these restrictions here. For reasons described below, we further restrict our consideration to measures μ arising as infinite convolutions, so-called Bernoulli measures. Each μ has its support contained in a compact interval which we choose to be centered at $x = 0$. The measures are indexed by a single parameter s, and each value of the parameter offers a distinct measure class μ_s. If $s = 1/2$, then μ_s is Lebesgue measure restricted to a compact interval. If $s > 1/2$, then no Fourier basis is possible. We identify values of $s < 1/2$ for which $L^2(\mu_s)$ has a sufficient supply of orthogonal Fourier basis functions. (For details, see equations (3.1) and (3.2) below!)

All the measures μ_s for $s < 1/2$ are singular. For fixed s, the support X_s of μ_s is a fractal, for example for $s = 1/3$, it is the middle-third Cantor set.

As a result, for fixed s, the set X_s is identified inside a compact interval J by its probability law. While the ambient interval J carries its normalized Lebesgue measure, the compact subset X_s carries a different measure, here the probability measure μ_s. Since μ_s is singular, the Lebesgue measure of X_s is zero. Our representation of the measure μ_s is consistent with a *Monte Carlo limit* consideration (see e.g., [24]), or random sampling, but with a different probability law for the set X_s to be determined inside J. This entails new difficulties as compared to the more familiar cases where one may be determining the measure of a planar region A inside say a square Q of area one. Monte Carlo applied to this case in the simplest instance consists of a procedure for random generation of say n points in Q, and then counting the number r of those falling in the set A. The fraction r/n is an approximation to the area of A. But in this example for both the ambient set Q and the subset A, we are applying the restriction of Lebesgue measure. In the present fractal case there is a sequence of renormalizations involved in reaching the measure μ with its fractal support, and Hausdorff dimension equal to a corresponding fraction.

2. L^2-spaces of fractal measures

The paper is organized as follows. In section 2 below, we introduce the fundamental concepts needed both in the statement of our results and their proofs. The applications section (section 3) will also serve to motivate. Our aim is to combine symmetry and harmonic analysis in the study of a family of selfsimilar measures μ with compact support in $\mathbb{R}^d$. While our results apply more generally than stated, we feel that the main idea is more transparent in one dimension, i.e., $d = 1$, and for the special classes of measures μ that result from infinite convolutions of the Bernoulli type.

We find a family of unitary operators U in $L^2(\mu)$ implementing certain spectral dilations.

Theorem 2.1. *Let μ be a Borel probability measure on $\mathbb{R}$ and let $p \in \mathbb{R}_+$ be given. Suppose there is an orthonormal basis (ONB) of exponentials $e_\lambda(x) = e^{i2\pi\lambda x}$, for λ in some set $\Lambda \subset \mathbb{Z}$ such that $p\Lambda = \{p\lambda : \lambda \in \Lambda\}$ is orthonormal in $L^2(\mu)$.*

(a) If $f \in L^2(\mu)$, *then there is a natural extension of f to $f \in L^2(0,1)$ obtained using the embedding of $L^2(\mu)$ into $L^2(0,1)$. Setting $\sigma_p(x) := px \bmod \mathbb{Z}$ this extension has the property that*

$$\int f(\sigma_p(x))\, d\mu(x) = \int f(x)\, d\mu = \int f(x)\, d\mu(x). \tag{2.1}$$

In particular, this holds for all $f \in L^1(\mu)$ because μ is a probability measure. If, in addition, it happens that σ_p maps the support of μ to itself, then this implies that σ_p leaves μ invariant, i.e., we have

$$\mu \circ \sigma_p^{-1} = \mu. \tag{2.2}$$

(b) *If we set*

$$(V_p f)(x) = f(\sigma_p(x)) \text{ for } f \in L^2(\mu), \tag{2.3}$$

then V_p is unitary *on $L^2(\mu)$ if and only if*

$$E(p\Lambda) = \{e_\xi : \xi \in p\Lambda\} \tag{2.4}$$

is an ONB in $L^2(\mu)$.

Proof. The measure μ is fixed and we consider the Hilbert space $L^2(\mu)$. The assumption placed on the set Λ is that

$$E(\Lambda) = \{e_\lambda \mid \lambda \in \Lambda\} \tag{2.5}$$

forms an ONB in $L^2(\mu)$. The inner product in $L^2(\mu)$ will be denoted

$$\langle f_1, f_2 \rangle = \int \overline{f_1(x)} f_2(x)\; d\mu(x).$$

Orthogonality of the set Λ means that

$$\langle e_\lambda, e_{\lambda'} \rangle_\mu = \hat{\mu}(\lambda' - \lambda) = 0, \quad \lambda \neq \lambda',$$

where $\hat{\mu}$ is the Fourier transform of the measure μ. For $f \in L^2(\mu)$, setting

$$\begin{aligned} c(\lambda) = c_\mu(\lambda) &= \langle e_\lambda, f \rangle_\mu \\ &= \int \overline{e_\lambda(x)} f(x)\; d\mu(x), \end{aligned} \tag{2.6}$$

we get the Parseval representation

$$f(x) = \sum_{\lambda \in \Lambda} c(\lambda) e_\lambda(x) \tag{2.7}$$

and

$$\|f\|_\mu^2 = \int |f(x)|^2\; d\mu(x) = \sum_{\lambda \in \Lambda} |c(\lambda)|^2 < \infty. \tag{2.8}$$

Since $\Lambda \subset \mathbb{Z}$, it follows from (2.7) that every $f \in L^2(\mu)$ is periodic with period 1; i.e., that

$$f(x+1) = f(x) \text{ for } \mu \text{ a.e. } x \in \mathbb{R}. \tag{2.9}$$

□

The preceding allows us to define the extension $f \mapsto F$ by

$$f = \sum_{\lambda \in \Lambda} c(\lambda) e_\lambda \mapsto \sum_{\lambda \in \Lambda} c(\lambda) e^{i2\pi\lambda x} = F(x) \tag{2.10}$$

and see that this is an isometric embedding of $L^2(\mu)$ onto a closed subspace of $L^2(0,1)$.

Lemma 2.2. *Under the assumptions of the theorem in* (*a*)*, the operator in* V_p *is well defined in* (3) *and isometric in* $L^2(\mu)$*.*

Proof. By assumption the function system $E(p\Lambda)$ is orthonormal in $L^2(\mu)$, so using the representation (2.7), we get

$$\sum_{\lambda \in \Lambda} |c_f(p\lambda)|^2 \leq \|f\|_\mu^2. \tag{2.11}$$

Let $f \in L^2(\mu)$, and consider the representation (2.7), thus

$$\begin{aligned}
\|V_p f\|_\mu^2 &= \int \left| \sum_{\lambda \in \Lambda} c(\lambda) e_\lambda(px) \right|^2 d\mu(x) \\
&= \int \left| \sum_{\lambda \in \Lambda} c(\lambda) e_{p\lambda}(x) \right|^2 d\mu(x) \\
&= \sum_{\lambda, \lambda' \in \Lambda} \sum \overline{c(\lambda)} c(\lambda') \langle e_{p\lambda}, e_{p\lambda'} \rangle_\mu \, d\mu(x) \\
&=_{(\text{by orthogonality of } p\Lambda)} \sum_{\lambda \in \Lambda} |c(\lambda)|^2 \\
&=_{(\text{by } (2.8))} \|f\|_\mu^2.
\end{aligned}$$

□

Proof of the theorem continued. Let p and Λ be as above, and let $E(\Lambda)$ and $E(p\Lambda)$ be the corresponding families of exponentials; see (2.4) above. Then it follows from the lemma that

$$V_p E(\Lambda) = E(p\Lambda). \tag{2.12}$$

In part (2.6) of Theorem 2.1, we are assuming that both sets in (2.12) are ONBs.

It follows that V_p thus maps an ONB onto an ONB, subject to the condition in (b). But this property characterizes when V_p is a unitary operator. Note V_p is isometric by (a). Hence it is unitary if and only if

$$V_p L^2(\mu) = L^2(\mu). \tag{2.13}$$

We now turn to property (2.1), i.e., the invariance of μ under the transformation σ_p in the measure space $(\mathbb{R}, \mathcal{B}, \mu)$ where $\mathcal{B}$ denotes the Borel sets $\subseteq \mathbb{R}$. Note that by (2.9) functions f in $L^2(\mu)$ pass to the quotient $\mathbb{R}/\mathbb{Z}$, and hence are determined by a period-interval of length 1.

If $p = 3$, the transformation σ_p may thus be represented as in Figure 1.

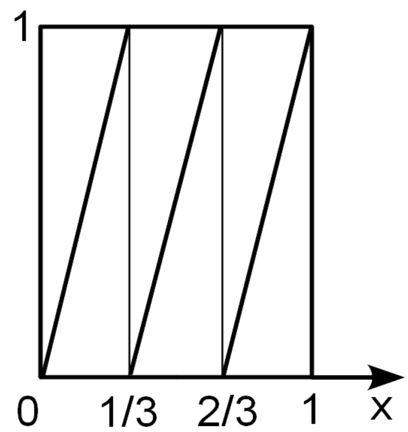

Figure 1

Let $E \in \mathcal{B}$, and set $f = \chi_E$. Substitution into

$$\|V_p f\|_\mu^2 = \|f\|_\mu^2 \tag{2.14}$$

from Lemma 2.2 then yields

$$\int \chi_E (px) \; d\mu (x) = \mu (E) . \tag{2.15}$$

Since

$$\chi_E \circ \sigma_p = \chi_{\sigma_p^{-1}(E)}, \tag{2.16}$$

we get

$$\mu \left(\sigma_p^{-1} (E)\right) = \mu (E) , \tag{2.17}$$

which is the desired conclusion (2.1). An application of standard measure theory further shows that the converse holds; indeed (2.1) implies (2.14), i.e., the isometric property of the operator V_p. □

Remark 2.3. Let $\mu = \mathcal{H}^s$ be Hausdorff measure of dimension s where $s \in (0, 1)$, i.e., for $\mathcal{H}^s$-measurable sets E, we have

$$\mathcal{H}^s (E) := \lim_{\delta \to 0} \inf \left\{ \sum_i |U_i|^s ; \; \bigcup_i U_i \supseteq E, |U_i| < \delta \right\}, \tag{2.18}$$

where $|U_i| =$ diameter of U_i.

Then

$$\mathcal{H}^s (cE) = c^s \mathcal{H}^s (E) . \tag{2.19}$$

For example, if $s = \frac{\ln 2}{\ln 3} = \log_3 2$, then $\mathcal{H}^s$ extends the middle-third Cantor measure, and

$$\mathcal{H}^s (3E) = 2\mathcal{H}^s (E) \tag{2.20}$$

or, equivalently,

$$2\int f(x)\ d\mathcal{H}^s(x) = \int f\left(\frac{x}{3}\right)\ d\mathcal{H}^s(x) \tag{2.21}$$

for all $\mathcal{H}^s$-measurable functions f.

We will show likewise that if μ is the Cantor measure of scale 3, then

$$\int f(x)\ d\mu(x) = \int f(3x)\ d\mu(x) \tag{2.22}$$

which may appear surprising in view of (2.21).

3. Applications

Below we introduce the measures μ that result from infinite convolutions of the Bernoulli type. While our results apply more generally, the Bernoulli case is of independent interest. In section 5 below, the framework will be extended to the context of the Hilbert spaces $L^2(\mu)$ where μ is an affine selfsimilar measure with compact support in $\mathbb{R}^d$.

Let $c \in \mathbb{R}_+$ be given; set

$$\tau_{\pm}^{(c)}(x) := c(x \pm 1); \tag{3.1}$$

i.e., two distinct transformations $\mathbb{R} \to \mathbb{R}$.

If $c < 1$, there is a unique probability Borel measure $\mu = \mu^{(c)}$ such that

$$\widehat{\mu^{(c)}}(t) = \prod_{n=1}^{\infty} \cos(2\pi c^n t). \tag{3.2}$$

Equivalently, $\mu^{(c)}$ is the distribution of the random power series $\sum_{k=1}^{\infty}(\pm 1)c^k$, i.e., random assignment of $\pm$ coefficients. See section 4 below. The case $c = 1/3$ is the middle-Cantor measure.

It is known that if $c = 1/2m$, i.e., c^{-1} = an even integer, then $L^2(\mu_{1/2m})$ has an ONB of the form $E(\Lambda_m)$ where

$$\Lambda_m := \left\{ \sum_{k=0}^{\text{finite}} a_k (2m)^k \mid a_k \in \left\{0, \frac{m}{2}\right\} \right\}. \tag{3.3}$$

In particular, $L^2(\mu_{1/8})$ has an ONB

$$\Lambda_4 = \{0, 2, 16, 18, 128, 130, 144, \ldots\}. \tag{3.4}$$

Similarly,

$$\Lambda_2 = \Lambda\left(\frac{1}{4}\right) = \{0, 1, 4, 5, 16, 17, 20, 21, \ldots\} \tag{3.5}$$

is an ONB in $L^2(\mu_{1/4})$. For the support of the measures $\mu_{1/4}$ and $\mu_{1/8}$, see Figures 2 and 3 below.

Details on this material can be found in [7], [8], and [17].

Definition 3.1. Suppose μ is a probability measure defined on the Borel sigma-algebra $\mathcal{B}$ of $\mathbb{R}$. Assume further that

$$f(x+1) = f(x) \text{ holds } \mu\text{-a.e. on } \mathbb{R} \text{ for all } f \in L^2(\mu). \tag{3.6}$$

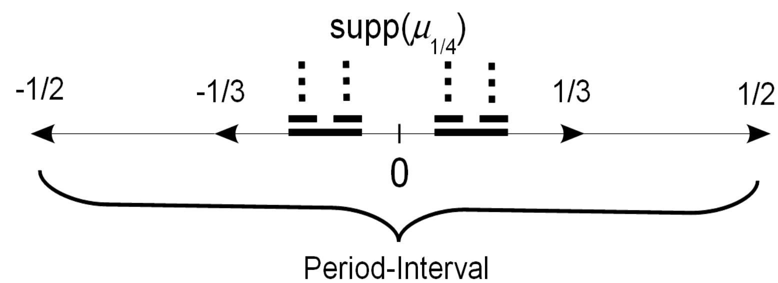

FIGURE 2. The support of $\mu_{1/4}$ is a fractal contained in $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

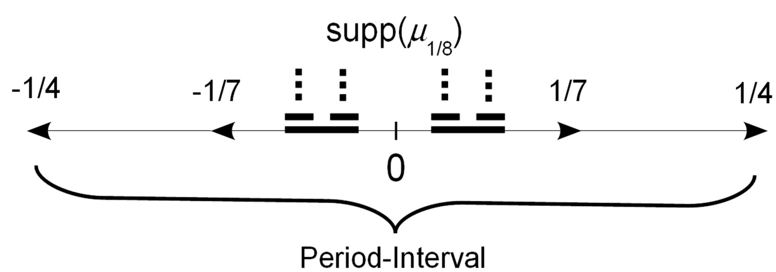

FIGURE 3. The support of $\mu_{1/8}$ is a fractal contained in $\left[-\frac{1}{7}, \frac{1}{7}\right]$.

Then for every $p \in \mathbb{Z}_+$, the transformation $\sigma_p(x) := px \bmod \mathbb{Z}$ induces an endomorphism of $X(\mu) := \operatorname{supp}(\mu) =$ the support of the measure μ, and $X(\mu)$ naturally embeds in $\mathbb{R}/\mathbb{Z} \simeq$ the circle group $\simeq$ the unit-interval. See Figures 2 and 3.

The resulting measure space will be denoted $(X(\mu), \mathcal{B}, \mu)$, and we will say that σ_p is an *endomorphism* of $(X(\mu), \mathcal{B}, \mu)$, or of $(X(\mu), \mu)$ for short.

We will say that σ_p *preserves* μ if

$$\mu \circ \sigma_p^{-1} = \mu \tag{3.7}$$

holds on $\mathcal{B}$.

If σ_p: $X(\mu) \to X(\mu)$ has a measurable inverse ρ_p: $X(\mu) \to X(\mu)$ we then say that σ_p is a measure preserving *automorphism* of the measure space $(X(\mu), \mu)$. The geometry of these mappings is sketched in Figures 2 and 3.

Corollary 3.2. *The mapping*

$$\sigma_3(x) := 3x \bmod \mathbb{Z} \tag{3.8}$$

is a measure-preserving endomorphism *of* $\left(X(\mu_{1/4}), \mu_{1/4}\right)$, *but it is* not *an automorphism.*

If $m > 2$, *then* σ_3 *is a measure-preserving* automorphism *of* $(X(\mu_{1/2m}), \mu_{1/2m})$.

Proof. The ideas in the proof draw on the theorem in Section 2 above, on the paper [17], a recent preprint of [16], and the following *a priori* estimate. □

Consider $\mu := \mu_{1/2m}$, i.e., scale $s = 1/2m,\ m \in \mathbb{Z}_+,\ p \in \mathbb{Z}_+$ fixed and odd, and

$$\Lambda_m := \left\{ \sum_{k=0}^{\text{finite}} b_k \left(2m\right)^k \mid b_k \in \left\{0, \frac{m}{2}\right\} \right\}.$$

Then the set $\Gamma := p\Lambda_m$ satisfies $\Gamma = 2m\Gamma \cup (\frac{pm}{2} + 2m\Gamma)$. We now compute

$$\begin{aligned} \sigma_\Gamma\left(t\right) &= \sum_{\gamma\in\Gamma} \left|\hat{\mu}\left(t-\gamma\right)\right|^2 \\ &= \sum_{\lambda\in\Gamma} \left|\hat{\mu}\left(t-2m\lambda\right)\right|^2 + \sum_{\lambda\in\Gamma} \left|\hat{\mu}\left(t-\frac{pm}{2}-2m\lambda\right)\right|^2 \\ &= \cos^2\left(\frac{\pi t}{m}\right) \sum_{\lambda\in\Gamma} \left|\hat{\mu}\left(\frac{t}{2m}-\lambda\right)\right|^2 \\ &\quad + \sin^2\left(\frac{\pi t}{m}\right) \sum_{\lambda\in\Gamma} \left|\hat{\mu}\left(\frac{t}{2m}-\frac{p}{4}-\lambda\right)\right|^2 \\ &= \cos^2\left(\frac{\pi t}{m}\right) \sigma_\Gamma\left(\frac{t}{2m}\right) + \sin^2\left(\frac{\pi t}{m}\right) \sigma_\Gamma\left(\frac{t}{2m}-\frac{p}{4}\right); \end{aligned}$$

$$\text{and contractivity constant } c\left(m,p\right) = \frac{1+p\frac{\pi}{2}}{2m}.$$

The following results were proved in [16] and [17].

Lemma 3.3.

(a) *The set* Λ_2 *in* (3.3) *is an ONB in* $L^2\left(\mu_{1/4}\right)$, *while* $3\Lambda_2$ *is orthogonal in* $L^2\left(\mu_{1/4}\right)$ *but not total; in fact,*

$$\left\langle e_{-1}, e_{3\lambda}\right\rangle_{\mu_{1/4}} = 0 \text{ for all } \lambda \in \Lambda_2. \tag{3.9}$$

(b) *For* $m \geq 4$, *the set* $3\Lambda_m$ *is an ONB in* $L^2\left(\mu_{1/2m}\right)$.

Proof of Corollary 3.2 *concluded.* We see that the isometry V_3 is a unitary operator in $L^2\left(\mu_{1/2m}\right)$ if and only if σ_3 is a measure-preserving automorphism in the measure space $\left(X\left(\mu_{1/2m}\right), \mu_{1/2m}\right)$. From (3.9) in Lemma 3.3, we see that σ_3 is not an automorphism in $\left(X\left(\mu_{1/4}\right), \mu_{1/4}\right)$, but it is an automorphism in $\left(X\left(\mu_{1/2m}\right), \mu_{1/2m}\right)$ when $m \geq 4$. □

4. The Gelfand Space

Fix an affine selfsimilar measure with compact support in $\mathbb{R}^d$. Assume that it has a spectrum, i.e., that there is an ONB of complex exponentials in $L^2(\mu)$ defined from a discrete subset Λ in $\mathbb{R}^d$. The pair (μ, Λ) is called a spectral pair. Acting on the Hilbert space $L^2(\mu)$, there is then an abelian algebra of multiplication operators. From the spectral assumption placed on μ we may pick the multiplication operators to be continuous and having a period lattice. With this restriction we compute the corresponding Gelfand space, and show that it coincides with the (compact) support of μ.

Working with $\Lambda(1/4) = \{\sum_0^{\text{finite}} a_i 4^i |\ a_i \in \{0,1\}\}$, and $\Lambda(1/8)$ we note that the functions $f \in L^2(\mu_{1/4})$ satisfy $f(x+1) = f(x)$, $\mu_{1/4}$-a.e., and the functions $f \in L^2(\mu_{1/8})$ satisfy $f(x+1/2) = f(x)$, $\mu_{1/8}$-a.e. on $\mathbb{R}$.

Indeed [11], there is a general periodicity result applying to all *affine spectral pairs* (μ, Λ). If $\operatorname{supp}(\mu) \subseteq \mathbb{R}^d$, and $\Lambda \subset \mathbb{R}^d$ (discrete) are such that $E(\Lambda)$ is an ONB in $L^2(\mu)$, then we say that (μ, Λ) is a *spectral pair*. We restrict attention here to the following additional constraints:

(i) $\operatorname{supp}(\mu)$ is compact; and
(ii) the mappings which define μ in the sense of [14] are affine.

Theorem 4.1. *Let (μ, Λ) be a spectral pair, and assume in addition that (i)–(ii) hold. Set*

$$\mathfrak{A} := \text{ operator-norm closure of } \{M(\varphi)|\ \varphi \text{ is continuous on } \mathbb{R}^d, \text{ and}$$

$$\varphi(x+\xi) = \varphi(x), \ \forall x \in \mathbb{R}^d, \ \forall \xi \in \Lambda^\circ\} \tag{4.1}$$

where

$$\Lambda^\circ = \left\{\xi \in \mathbb{R}^d |\ \xi \cdot \lambda \in \mathbb{Z}, \ \forall \lambda \in \Lambda\right\} \tag{4.2}$$

and

$$M(\varphi) f = \varphi f, \ \varphi \text{ as in } (1), \text{ and } f \in L^2(\mu). \tag{4.3}$$

Then the Gelfand space of the abelian C^*-algebra $\mathfrak{A}$ is $\operatorname{supp}(\mu)$ = the support of the measure μ.

Proof. In essence, the idea is that the quotient of $\mathbb{R}^d$ by the lattice Λ° is a compact metric space, hence the algebra of continuous functions with supremum norm and lattice invariance in the sense of (4.1) has Gelfand space precisely equal to this metric space. If instead we take the norm to be the operator norm as a multiplier on $L^2(\mu)$, then we get a larger algebra, so we get a smaller Gelfand space. The problem is then to identify a subset of our metric space which corresponds uniquely to this smaller Gelfand space. We stress the fact that distinct points of the support of μ give distinct point evaluation functionals because Λ° is discrete. We now turn to the details one by one.

First note that the C^*-algebra $\mathfrak{A}$ in (1) depends on both items (μ, Λ) in the given spectral pair since $M(\varphi)$ is a multiplication operator in the Hilbert space $L^2(\mu)$. We may assume WLG that $0 \in \Lambda$ so that the constant function $e_0 \in L^2(\mu)$ is one of the basis functions in $E(\Lambda)$. Since μ is a Borel measure by [14], we conclude that the subspace $\{M(\varphi) e_0\} \subset L^2(\mu)$ is dense in $L^2(\mu)$ with respect to the $L^2(\mu)$-norm.

The assertion of the theorem amounts to the fact that every multiplicative functional on $\mathfrak{A}$ has the form

$$m_x : M(\varphi) \longmapsto \varphi(x) \tag{4.4}$$

for a unique point $x \in \operatorname{supp}(\mu)$; i.e., the multiplicative functionals of $\mathfrak{A}$ are point-evaluation by points in the compact space $X := X(\mu) = \operatorname{supp}(\mu) \subset \mathbb{R}^d$. Equivalently, the Gelfand transform $\widehat{\ }$ is $\widehat{M(\varphi)}(x) = m_x(M(\varphi)) = \varphi(x), \ x \in X(\mu)$.

Note that every functional in (4.4) corresponding to points in X is multiplicative, i.e., satisfies

$$m(M(\varphi_1) M(\varphi_2)) = m(M(\varphi_1)) m(M(\varphi_2)) \tag{4.5}$$

and

$$m\left(M\left(\mathbb{1}\right)\right)=1 \tag{4.6}$$

with $\mathbb{1}$ denoting the constant function "one" on $\mathbb{R}^d$.

To prove the converse, we may use the specific representation (4.1)–(4.3) of the operators in $\mathfrak{A}$.

Specifically, we must prove that $z \in \mathbb{C} \setminus X$ and if $x \longmapsto \varphi\left(z\right) - \varphi\left(x\right)$ is non-zero on X, then the difference-operator

$$\varphi\left(z\right) I_{L^2(\mu)} - M\left(\varphi\right) \tag{4.7}$$

has a bounded inverse $L^2\left(\mu\right) \to L^2\left(\mu\right)$. The latter fact follows from two observations:

(a) Multiplication by $x \to \left(\varphi\left(z\right) - \varphi\left(x\right)\right)^{-1}$ serves as an inverse to the operator in (4.7); and

(b) this operator is bounded in $L^2\left(\mu\right)$ since the function in (a) is bounded on X. □

Corollary 4.2. *Let (μ, Λ) be as in the theorem, and let P be a $d \times d$ matrix over $\mathbb{Z}$ such that $\det P \neq 0$, and $E\left(P\Lambda\right)$ is an ONB in $L^2\left(\mu\right)$.*

Then there are:

(a) *a unitary operator $U = U_P : L^2\left(\mu\right) \to L^2\left(\mu\right)$ determined by*

$$U e_\lambda = e_{P\lambda},\ \forall \lambda \in \Lambda \tag{4.8}$$

and

(b) *a measurable transformation $\tau : X \to X$ (where $X := \operatorname{supp}\left(\mu\right)$) satisfying*

$$Uf = f \circ \tau\ ,\ \forall f \in L^2\left(\mu\right). \tag{4.9}$$

Proof. We saw in the proof of the theorem that $L^2\left(\mu\right)$ is the Hilbert space which arises from an application of the Gelfand-Naimark-Segal (GNS) theorem, applied to the state m_μ defined by

$$m_\mu\left(M\left(\varphi\right)\right) := \int \varphi\left(x\right)\ d\mu\left(x\right) \tag{4.10}$$

(with $M\left(\varphi\right)$ referring to the C^*-algebra $\mathfrak{A}$ in (4.1)). Since

$$e_{P\lambda}\left(x\right) = e_\lambda\left(P^{tr} x\right),\ \forall \lambda \in \Lambda, \tag{4.11}$$

we may consider the action

$$\varphi \longmapsto \varphi\left(P^{tr} x\right) \tag{4.12}$$

on $\mathfrak{A}$. □

Lemma 4.3. *We have*

$$m_\mu\left(M\left(\varphi \circ P^{tr}\right)\right) = m_\mu\left(M\left(\varphi\right)\right). \tag{4.13}$$

Proof. We claim that (4.12) leaves invariant m_μ in (4.10). To see this, expand

$$\varphi = \sum_{\lambda \in \Lambda} c\left(\lambda\right) e_\lambda \tag{4.14}$$

according to the ONB $E\left(\Lambda\right)$, so

$$c\left(\lambda\right) = \int \overline{e_\lambda} \varphi\ d\mu,\ \forall \lambda \in \Lambda. \tag{4.15}$$

Since $0 \in \Lambda$, in particular,

$$c(0) = \int \varphi \, d\mu = m_\mu(\varphi) . \tag{4.16}$$

Substitution of (4.12) into (4.14) yields

$$\varphi\left(P^{tr}x\right) = \sum_{\lambda\in\Lambda} c(\lambda)\, e_{P\lambda}(x) \tag{4.17}$$

and hence

$$\begin{aligned} m_\mu\left(\varphi\left(P^{tr}\cdot\right)\right) &= \sum_{\lambda\in\Lambda} c(\lambda) \left\langle e_0, e_{P\lambda}\right\rangle_{L^2(\mu)} \\ &= c(0) \\ &=_{(\text{by } (4.16))} m_\mu(\varphi) . \end{aligned} \tag{4.18}$$

Note we used that $E(P\Lambda)$ is an ONB and that $0 \in \Lambda$ by assumption.

Using (4.18), we see that (4.12) transforms the Gelfand space X of $\mathfrak{A}$ into itself. Hence the unitary operator $U = U_P$ in (4.8) must be induced by a transformation $\tau : X \to X$, but by the theorem $X = \operatorname{supp}(\mu)$, and the desired conclusion holds; see (4.9). □

Definition 4.4. A measure-preserving automorphism σ in a measure space $(X, \mathcal{B}, \mu)$ is said to be *ergodic* if the following implication holds:

$$E \in \mathcal{B} \text{ and } \sigma E = E \Rightarrow \mu(E) \in \{0, 1\} . \tag{4.19}$$

In specific cases we can sometimes say more is possible in the setting of Corollary 4.2. The following corollary gives an example in which the transformation τ can be seen to be ergodic.

Corollary 4.5. *Let (μ, Λ) be the spectral pair in $\mathbb{R}$ given by*

$$\hat{\mu}(t) = \prod_{k=1}^{\infty} \cos\left(\frac{2\pi t}{8^k}\right) \tag{4.20}$$

and

$$\Lambda = \left\{ \sum_0^{\text{finite}} b_i 8^i \middle|\ b_i \in \{0, 2\} \right\} .$$

Let $U : L^2(\mu) \to L^2(\mu)$ be determined by

$$U e_\lambda = e_{3\lambda}. \tag{4.21}$$

Let

$$X = \left\{ \sum_{k=1}^{\infty} \frac{(\pm 1)}{8^k} \right\} \tag{4.22}$$

be the Cantor fractal $\subset \left[-\frac{1}{7}, \frac{1}{7}\right]$ with Hausdorff dimension $1/3$. Then there is a measurable transformation $\tau : X \to X$ such that

$$U f = f \circ \tau,\ \forall f \in L^2(\mu) ;$$

i.e., a measurable bijection.

Proof. We now turn to the details. A helpful tool is a bijective correspondence between operations in a symbol space Ω on the one side and iteration of the $\tau_\pm$ mappings on the other:

$$\begin{aligned}
&\tau_\pm(x) = \frac{1}{8}(x \pm 1), \text{ and} \\
&\tau_w(x) = (\tau_{w_1} \circ \tau_{w_2} \circ \cdots \circ \tau_{w_k})(x) \text{ where } w = (w_1 w_2 \cdots w_k) \text{ and} \\
&\quad w_i \in \{\pm\}, 1 \leq i \leq k
\end{aligned}$$

The result follows from an application of Corollary 4.2 above to the results from the previous section. ☐

Let (μ, Λ) be a spectral pair specified as in Corollary 4.2. Assume that $\operatorname{supp}(\mu) =: X(\mu)$ satisfies

$$\tau_j(X(\mu)) \cap \tau_k(X(\mu)) = \varnothing \quad \forall j \neq k \tag{4.23}$$

for an affine iterated function system (τ_j) in $\mathbb{R}^d$ given by a $d \times d$ matrix R over $\mathbb{Z}$ with $|\lambda| > 1$ for all $\lambda \in \operatorname{spec}(R)$. Then there is a finite set A and a homeomorphism

$$\Omega := \prod_1^\infty A \xrightarrow{\pi} X(\mu) \tag{4.24}$$

given by

$$\pi(\underbrace{j_1\ j_2\ j_3\ \cdots}_{\in\Omega}) = \bigcap_{n=1}^\infty \tau_{j_1} \circ \cdots \circ \tau_{j_n}(X(\mu)) \tag{4.25}$$

where the intersection in (4.25) is a singleton. If $\omega = (j_1\ j_2\ j_3\ \cdots)$, then

$$s(j_1\ j_2\ \cdots) := (j_2\ j_3\ j_4\ \cdots)$$

defines the shift operation $s = s_\Omega$ in (4.24), and R denotes matrix-multiplication in $\mathbb{R}^d$ modulo $\mathbb{Z}^d$, and then

$$R \circ \tau = \tau \circ s \text{ holds on } \Omega. \tag{4.26}$$

Since π in (4.24) is a homeomorphism, the mapping τ from (4.9) in Corollary 4.2(b) corresponds to a shift-invariant transformation in Ω, i.e.,

$$\tau \circ s = s \circ \tau \text{ on } \Omega. \tag{4.27}$$

The next corollary summarizes the dynamical systems-properties of the particular class of affine iteration fractals considered in detail above.

Corollary 4.6. *Let $\mu, \Lambda, X(\mu), s, \tau$ be as above and as introduced in Corollary 4.2. Then there is a function $t : \Omega \to A$ such that*

$$\tau(\omega) = \left(t(\omega),\ t(s\omega),\ t\left(s^2\omega\right), \cdots\right),\ \forall\omega \in \Omega \tag{4.28}$$

or, equivalently,

$$\tau(\omega) = (\tau_j(\omega))_{j=1}^\infty$$

where

$$\tau_j := \tau \circ s^{j-1},\ j \in \mathbb{Z}_+. \tag{4.29}$$

When A and $\{\tau_a\}_{a\in A}$ are fixed, then the measure ρ_Ω on Ω which corresponds to μ on $X(\mu)$ in (4.24) is an infinite-product measure, i.e., we have

$$\mu(E) = \rho_\Omega\left(\pi^{-1}(E)\right) \text{ for all Borel sets } E \subset X(\mu).$$

For additional details regarding infinite product measures and their connection to Bernoulli convolutions, see e.g., [5], [20], and the references cited there.

5. Ergodicity

In case some spectral pair (μ, Λ) has a second dilated set also serving as spectrum. We have then shown that there is an associated unitary operator U in $L^2(\mu)$ implementing the dilation. We further show that U is induced by a measure preserving transformation τ in $\mathrm{supp}(\mu)$ and, in fact, we show that τ is ergodic. Now we will show that, in fact, σ_3 is also ergodic.

The result in this section applies to the measure-preserving automorphisms σ_p from section 3 above, i.e.,

$$\sigma_p(x) = px \bmod \mathbb{Z}, \tag{5.1}$$

viewed as an automorphism in the measure space $\left(X\left(\mu_{1/2m}\right), \mu_{1/2m}\right)$. However, to simplify the arguments we will restrict attention to the case $p = 3$ and $m = 4$, so σ_3 is not an automorphism in this space. This is to recall the distinction between $p = 3$ and $\Lambda(1/4)$ and $\Lambda(1/8)$. Nonetheless, both $X\left(\mu_{1/4}\right)$ and $X\left(\mu_{1/8}\right)$ are Cantor sets, the first with Hausdorff dimension $1/2$ and the second $1/3$.

Theorem 5.1. *The automorphism σ_3 in the measure space $(X(\mu_{1/8}), \mu_{1/8})$ is ergodic.*

Proof. In view of Definition 4.4 and Corollary 3.2 the conclusion follows if we check that the unitary operator

$$V_3 f := f \circ \sigma_3 \tag{5.2}$$

in $L^2\left(\mu_{1/8}\right)$ has one-dimensional eigenspace

$$\left\{f \in L^2\left(\mu_{1/8}\right) \mid V_3 f = f\right\} = \mathbb{C}e_0 \tag{5.3}$$

where e_0 is the constant function 1 in $L^2\left(\mu_{1/8}\right)$.

As a result we must check that if $f \in L^2\left(\mu_{1/8}\right) \ominus (e_0)$ and $V_3 f = f$, then $f = 0$ in $L^2\left(\mu_{1/8}\right)$.

Let

$$\Lambda := \Lambda_8 = \Lambda\left(\mu_{1/8}\right) = \left\{\sum_{k=0}^{\text{finite}} a_k 8^k \mid a_k \in \{0, 2\}\right\}, \ \mu := \mu_{1/8} \tag{5.4}$$

and set

$$\Lambda^* := \Lambda \diagdown \{0\}, \tag{5.5}$$

and

$$L^2_*(\mu) := L^2(\mu) \ominus (e_0). \tag{5.6}$$

Then Λ^* is an ONB in $L^2_*(\mu)$ by Corollary 3.2 and [17]. □

We will need the following.

Lemma 5.2. *We have*

$$\hat{\mu}\,(2\lambda) \neq 1 \text{ for all } \lambda \in \Lambda^*. \tag{5.7}$$

Proof. By the product formula (3.2) for $\hat{\mu} = \widehat{\mu_{1/8}}$ we have

$$\hat{\mu}\,(t) = \prod_{n=1}^{\infty} \cos\left(\frac{2\pi t}{8^n}\right). \tag{5.8}$$

By (5.4), for $\lambda \in \Lambda^*$, we have the representation

$$\lambda = 2 \cdot \left(8^k + b_1 \cdot 8^{k+1} + \cdots + b_m \cdot 8^{k+m}\right) \tag{5.9}$$

where $k \in \{0, 1, 2, \cdots\}$ and $b_i \in \{0, 1\}$ are fixed.

Using (5.7) for $t = 2\lambda$, we get

$$\hat{\mu}\,(2\lambda) = \cos\left(\frac{2\pi \cdot 2\lambda}{8}\right) \prod_{n=2}^{\infty} \cos\left(\frac{4\pi\lambda}{8^n}\right).$$

Substituting (5.9), we then get for the first factor $\cos(\frac{4\pi\lambda}{8}) = \cos(\pi 8^k(1 + b_1 8 + \cdots + b_m 8^m)) = 1$, while $\cos(\frac{4\pi\lambda}{8^{k+1}}) = -1$.

For the remaining factors $\Pi_{n=k+2}^{\infty} \cos(\frac{4\pi\lambda}{8^n})$, we use that $\cos\,(x) > 1 - \frac{x^2}{2}$.

$$\prod_{k+2}^{\infty} (\cdots) > 1 - \frac{\pi}{8^{k+1}}, \tag{5.10}$$

and as a result

$$\hat{\mu}\,(2\lambda) > 1 - \frac{\pi}{8^{k+1}}, \tag{5.11}$$

which is stronger than the desired conclusion (5.7) of the lemma. □

Proof of Theorem 5.1 *continued.* We show that

$$\left\{f \in L^2_*\,(\mu) \mid V_3 f = f\right\} = \{0\}\,. \tag{5.12}$$

Since Λ^* is an ONB in $L^2\,(\mu)$ a function f from (5.12) must have the representation

$$f = \sum_{\lambda \in \Lambda^*} c\,(\lambda)\, e_\lambda. \tag{5.13}$$

Fix some $\lambda \in \Lambda^*$, then

$$\langle e_\lambda, V_3 e_\lambda\rangle_\mu = \langle e_\lambda, e_{3\lambda}\rangle_\mu = \hat{\mu}\,(2\lambda) < 1,$$

by the lemma.

Since $V_3 e_0 = V_3^* e_0 = e_0$, we conclude that

$$V_3 e_\lambda = e_{3\lambda} \in L^2_*\,(\mu)\,. \tag{5.14}$$

As a result there is some $g \in$ closed span$\{e_\gamma \mid \gamma \in \Lambda \diagdown \{0, \lambda\}\}$ and

$$V_3 e_\lambda = \hat{\mu}\,(2\lambda)\, e_\lambda + g. \tag{5.15}$$

We get $0 = \langle e_\lambda, f - V_3 f\rangle_\mu = c\,(\lambda) \cdot (1 - \hat{\mu}(2\lambda))$. Now invoke Lemma 5.2 and conclude that $c(\lambda) = 0$. Using (5.13), we get the desired conclusion (5.12).

Now assume $E \in \mathcal{B}$ satisfies $\sigma_3 E = E$. Then $\chi_E \in L^2\,(\mu)$ satisfies $V_3 \chi_E = \chi_E$, and we conclude that $\chi_E \in \mathbb{C} e_0$, i.e., is represented by the constant function in $L^2\,(\mu)$. Hence $\mu\,(E) = \|\chi_E\|_\mu^2$ can attain only the two values 0 or 1, proving that σ_3 is ergodic in $(X(\mu), \mu)$ as claimed. □

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Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242-1419
E-mail address: `jorgen@math.uiowa.edu`