

ON THE COMPUTATION OF LOCAL COMPONENTS OF A NEWFORM

DAVID LOEFFLER AND JARED WEINSTEIN

1. INTRODUCTION

1.1. The problem. Let f be a cuspidal newform for $\Gamma_1(N)$ with weight $k \geq 2$ and character ε . There are well-established methods for computing such forms using modular symbols; see [Ste07]. Let π_f be the corresponding automorphic representation of the adèle group $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$. (As there are numerous possible normalizations, we shall recall the construction in section 2.1 below.) This representation decomposes as a restricted tensor product over all places v of \mathbf{Q} :

$$\pi_f = \bigotimes_v \pi_{f,v},$$

where $\pi_{f,v}$ is an irreducible admissible representation of $\mathrm{GL}_2(\mathbf{Q}_v)$. The isomorphism class of the infinite component $\pi_{f,\infty}$ is determined by the weight k alone: it is the unique discrete series subrepresentation of the representation constructed via unitary induction from the character

$$\begin{pmatrix} t_1 & * \\ & t_2 \end{pmatrix} \mapsto \left| \frac{t_1}{t_2} \right|^{k/2} \mathrm{sgn}(t_1)^k$$

of the Borel subgroup of $\mathrm{GL}_2(\mathbf{R})$ [Gel75, prop. 5.21]. Similarly, the component $\pi_{f,p}$ for a prime $p \nmid N$ is the unramified principal series with Satake parameters equal to the roots of the polynomial $X^2 - a_p X + \varepsilon(p)p^{k-1}$. We are led naturally to the following problem:

Problem 1.1. Compute the local components $\pi_{f,p}$ for each prime $p \mid N$.

Problem 1.1 may be recast from an arithmetic perspective as a question about the local behavior of Galois representations. Suppose the Fourier coefficients of f lie in a number field E . Let

$$\rho_{f,\lambda}: \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow \mathrm{GL}_2(E_\lambda)$$

be the family of λ -adic representations attached to f by Deligne.

Problem 1.2. Compute the restriction of $\rho_{f,\lambda}$ to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ for every $p \mid N$ and every prime λ of E not dividing p .

Received by the editor August 16, 2010 and, in revised form, February 3, 2011.

2010 *Mathematics Subject Classification.* Primary 11F70; Secondary 11F11, 11Y99.

The first author's research is supported by EPSRC Postdoctoral Fellowship EP/F04304X/2.

The second author's research is supported by NSF Postdoctoral Fellowship DMS-0803089.

©2011 American Mathematical Society
Reverts to public domain 28 years from publication

Problems 1.1 and 1.2 are equivalent. This is a consequence of the theorem of Carayol [Car86], which we now summarize. For an irreducible admissible representation π_p of $\mathrm{GL}_2(\mathbf{Q}_p)$, with coefficients in an arbitrary algebraically closed field of characteristic zero, let $\sigma(\pi_p)$ be the corresponding two-dimensional Frobenius-semisimple Weil-Deligne representation of \mathbf{Q}_p under the Hecke correspondence. The Hecke correspondence is a modification of the local Langlands correspondence which has the convenient property of being invariant under automorphisms of the coefficient field; see [Car86], §0.5. If the coefficient field is the λ -adic field E_λ , where λ does not divide p , there is a further correspondence $\sigma(\pi_p) \mapsto \sigma^\lambda(\pi_p)$ between Frobenius-semisimple Weil-Deligne representations and continuous Frobenius-semisimple representations $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p) \rightarrow \mathrm{GL}_2(E_\lambda)$ due to Grothendieck; cf. [Tat79].

Let f be a cusp form as above. Carayol's theorem states that $\sigma^\lambda(\pi_{p,f})$ agrees with the restriction of $\rho_{f,\lambda}$ to $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$. Therefore if one has access to $\pi_{f,p}$ as in Problem 1.1 one could also compute $\rho_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)}$, at least if one has an explicit construction of the Hecke correspondence.

Our approach is as follows. When $\pi_{f,p}$ is a principal series or special representation of $G = \mathrm{GL}_2(\mathbf{Q}_p)$, the problem is not difficult: In these cases the isomorphism class of $\pi_{f,p}$ is determined by the p th Fourier coefficient of f and those of its twists by an explicit set of Dirichlet characters of p -power conductor. We are therefore reduced to the case of $\pi_{f,p}$ supercuspidal. Here $\pi_{f,p}$ is of the form $\mathrm{Ind}_K^G \tau$, for a finite-dimensional representation τ of a maximal compact-modulo-centre subgroup $K \subseteq G$. Using results of Casselman on the structure of irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$, we find a canonical model of τ inside $\pi_{f,p}$, which can be explicitly realized as a subspace of the cohomology of a modular curve. We then compute this space, or rather its dual, using the explicit presentation of the homology of modular curves provided by modular symbols.

An alternative approach is provided by the local converse theorem of [BH06, §27], which shows that an irreducible supercuspidal G -representation π is determined by the local epsilon factors of its twists by a certain set of characters. While this approach is of enormous theoretical importance, it is less convenient from a computational perspective, as the description of the representation π given by the epsilon factors of its twists is very inexplicit and hence difficult to relate to other more familiar parametrizations of the set of representations of G .

This article grew out of work done by the second author during his visit to the Magma Group in Sydney in 2009. A partial version of the algorithm described here, in which $\pi_{f,p}$ is computed up to a quadratic twist, is implemented in the computational algebra system Magma [BCP97]. The slightly more refined version presented here is the result of discussions between the authors at the workshop “Computer methods for L -functions and automorphic forms” hosted by the Centre de Recherches Mathématiques, Montréal, in March 2010, and an implementation by the first author will be included in a forthcoming version of the Sage system [Sag].

2. BASIC OBSERVATIONS

2.1. Cusp forms and automorphic representations. We begin by recalling how to construct automorphic representations from classical modular eigenforms, following [Bum97, §3.6]. Let $G_{\mathbf{Q}} = \mathrm{GL}_2(\mathbf{Q})$ and $G_{\mathbf{A}} = \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$, and let $Z_{\mathbf{Q}}$ and

$Z_{\mathbf{A}}$ be the centers of those groups. Let us fix a level N and a Dirichlet character ε modulo N . Let ω be the adelization of ε ; that is, ω is the unique character of $Z_{\mathbf{A}}/Z_{\mathbf{Q}}$ such that $\omega(\varpi_p) = \varepsilon(p)$ for primes $p \nmid N$, where $\varpi_p \in \mathbf{Q}_p^\times \subset Z_{\mathbf{A}}$ is a uniformizer at p . Let ω_p be the restriction of ω to \mathbf{Q}_p^\times .

Remark 2.1. Note that the restriction of ω_p to \mathbf{Z}_p^\times is the *inverse* of the character of \mathbf{Z}_p^\times inflated from the p -component of ε .

We write

$$K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) \mid c \equiv 0 \pmod{N} \right\},$$

so that $K_0(N)$ is an open compact subgroup of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}})$. By [Bum97, Theorem 3.3.1], we have $G_{\mathbf{A}} = G_{\mathbf{Q}} G_{\infty}^+ K_0(N)$, where

$$G_{\infty}^+ = \{g \in \mathrm{GL}_2(\mathbf{R}) \mid \det(g) > 0\}.$$

We define a character λ of $K_0(N)$ by

$$\lambda \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \prod_{p|N} \omega_p(d_p);$$

this clearly agrees with ω on $Z_{\mathbf{A}} \cap K_0(N)$.

Given an eigenform $f \in S_k(\Gamma_1(N), \varepsilon)$, we define a function ϕ_f on $G_{\mathbf{A}}$ by

$$\phi_f(\gamma g_{\infty} k_0) = f(g_{\infty}(i)) \cdot (ci + d)^{-k} \cdot (\det g_{\infty})^{k/2} \cdot \lambda(k_0),$$

for $\gamma \in G_{\mathbf{Q}}$, $g_{\infty} \in G_{\infty}^+$, and $k_0 \in K_0$. This is well defined, since $G_{\mathbf{Q}} \cap G_{\infty}^+ K_0 = \Gamma_0(N)$ and f satisfies $f(\frac{az+b}{cz+d}) = \varepsilon(d)(cz+d)^k f(z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$; it is clear that $\phi_f(gz) = \omega(z)\phi_f(g)$ for $z \in Z_{\mathbf{A}}$.

The function ϕ_f in fact lies in the space $\mathcal{A}_0(\omega)$ of automorphic forms on $G_{\mathbf{A}}$ of central character ω ; see [Gel75, §3]. We define π_f to be the linear span of the right translates of f under $G_{\mathbf{A}}$; then π_f is an admissible smooth representation of $G_{\mathbf{A}}$. If f is an eigenform, then it is well known that this representation is irreducible and decomposes as a restricted tensor product $\bigotimes'_v \pi_{f,v}$ of admissible irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_v)$.

2.2. New vectors. Let us fix a prime p , and let $G = \mathrm{GL}_2(\mathbf{Q}_p)$. We recall a theorem of Casselman on smooth representations of G :

Theorem 2.2 ([Cas73a]). *Let π be an irreducible smooth representation of G on a vector space V (over a field of characteristic 0), with central character ω . For large enough $r \geq 0$, there exists a nonzero vector $v \in V$ satisfying*

$$\pi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) v = \omega(d)v$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p)$ with $c \equiv 0 \pmod{p^r}$. If r is the minimal such exponent, then v is unique up to scalar.

Remark 2.3. Our conventions differ slightly from those of [Cas73a]. Casselman uses $\omega(a)$ rather than $\omega(d)$, but his theorem and Theorem 2.2 are clearly equivalent.

Definition 2.4. With the notation of Theorem 2.2, the *conductor* of π is $r = r(\pi)$, and a *new vector* for π is v .

One checks easily that the space of new vectors is preserved by the endomorphism

$$\mathbf{T}_p = \frac{1}{\sqrt{p}} \times \begin{cases} \sum_{i=0}^{p-1} \pi \left(\begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} \right) & (r \geq 1), \\ \sum_{i=0}^{p-1} \pi \left(\begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} \right) + \omega(p) \cdot \pi \left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right) & (r = 0). \end{cases}$$

Since the space of new vectors is one-dimensional, we must have $\mathbf{T}_p v = \lambda v$ for some scalar λ . We define $\lambda(\pi)$ to be this scalar.

Smooth irreducible representations of G (other than the trivial 1-dimensional representations factoring through the determinant) fall into three classes, which we summarize in Table 1 below (see for instance [BH06, §9.11])

TABLE 1. The three classes of irreducible infinite-dimensional smooth representations of $G = \mathrm{GL}_2(\mathbf{Q}_p)$

Class	Notation	Conductor	Central char.	$\lambda(\pi)$
Principal Series	$\pi(\chi_1, \chi_2)$	$f_1 + f_2$	$\chi_1 \chi_2$	$\chi_1^*(p) + \chi_2^*(p)$
Special	$\mathrm{St} \otimes \chi_1$	$\max(2f_1, 1)$	χ_1^2	$p^{-1/2} \chi_1^*(p)$
Supercuspidal	$\mathrm{Ind}_K^G \tau$	≥ 2	ω_τ	0

In the table, χ_1 and χ_2 are characters of \mathbf{Q}_p^\times of conductors p^{f_1} and p^{f_2} , and

$$\chi_i^*(p) = \begin{cases} \chi_i(p), & \chi_i \text{ is unramified,} \\ 0, & \chi_i \text{ is ramified.} \end{cases}$$

The representation denoted $\pi(\chi_1, \chi_2)$ is the principal series representation attached to the characters χ_1 and χ_2 , defined whenever $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$. The representation denoted St is the irreducible infinite-dimensional quotient of $\mathrm{Ind}_B^G 1$ (unnormalized induction), where $B \subset G$ is the subgroup of upper-triangular matrices. The supercuspidal representations of G are induced from certain finite-dimensional characters τ of compact-mod-center subgroups $K \subset G$; these will be reviewed in §3.1. We have written ω_τ for the central character of τ .

If f is a p -new eigenform, the automorphic form ϕ_f constructed above is a new vector for $\pi_{f,p}$ for every prime p . One checks that $\lambda(\pi_{f,p}) = a_p(f)/p^{(k-1)/2}$. In particular, if $p \nmid N$, then $\pi_{f,p} \cong \pi(\chi_1, \chi_2)$ with χ_1 and χ_2 the unramified characters whose values at p are $\{\alpha/p^{(k-1)/2}, \beta/p^{(k-1)/2}\}$, for α and β the roots of $X^2 - a_p X + \omega(p)p^{k-1}$.

2.3. Twisting. In this section we make an important reduction, which, in particular, will allow us to immediately dispose of the non-supercuspidal cases of Problem 1.1.

Definition 2.5. We say π is *primitive* if $r(\pi) \leq r(\pi \otimes (\psi \circ \det))$ for any character ψ of \mathbf{Q}_p^\times .

The principal series representation $\pi(\chi_1, \chi_2)$ is primitive if and only if (up to exchanging χ_1 and χ_2) the character χ_2 is unramified, in which case the conductor

of $\pi(\chi_1, \chi_2)$ equals the conductor of χ_1 . Similarly, $\text{St} \otimes \chi$ is primitive if and only if χ is unramified, in which case it has conductor 1 and unramified central character.

The notions of twisting by a character and of primitivity have the following global analogues. For $f \in S_k(\Gamma_1(N), \varepsilon)$ a new eigenform, and μ a Dirichlet character modulo M , the twist f_μ is the modular form (of level at most NM^2) whose Fourier coefficients are given by $a_n(f_\mu) = \mu(n)a_n(f)$. We write $f \otimes \mu$ for the unique *newform* whose p th Fourier coefficient is $\mu(p)a_p(f)$ for all but finitely many primes p ; this may have level smaller than NM^2 . Then $f \otimes \mu$ is the newform corresponding to the automorphic representation $\pi_f \otimes (\psi \circ \det)$, where ψ is the adelic character corresponding to μ as in §2.1.

Later we will need the following technical observation:

Lemma 2.6. *Let f be a newform of weight $k \geq 2$, and let p be prime. If there exists a nontrivial character χ of p -power conductor such that $f \otimes \chi = f$, then χ is quadratic with $\chi(-1) = -1$.*

Proof. If f is a newform of weight $k \geq 2$ and f does not have CM, then the set of primes ℓ such that $a_\ell(f) = 0$ is well known to have density zero. If $f \otimes \chi = f$, then we must have $a_\ell(f) = 0$ for all ℓ such that $\chi(\ell) \neq 1$; so if χ is nontrivial, f must be CM, and χ the unique quadratic character corresponding to the CM field of f . In particular, $\chi(-1) = -1$. \square

Definition 2.7 ([AL78, p. 236]). The newform f is *p -primitive* if the p -part of its level is minimal among all its twists.

Clearly, f is p -primitive if and only if $\pi_{f,p}$ is a primitive representation of G . If f is p -primitive, then its p th Fourier coefficient, its central character and its weight already determine $\pi_{f,p}$ completely, *unless* $\pi_{f,p}$ is supercuspidal:

Proposition 2.8. *Let $\pi = \pi_{f,p}$ be the local component at p of a p -primitive newform $f \in S_k(\Gamma_1(Np^r), \varepsilon)$, with $p \nmid N$ and $r \geq 1$. Let ω_p be the p -component of the character of $\mathbf{A}_{\mathbf{Q}}^\times/\mathbf{Q}^\times$ corresponding to ε .*

- (1) *If $r \geq 1$ and the conductor of ω_p is p^r , then $\pi \cong \pi(\chi_1, \chi_2)$, where χ_1 is unramified, $\chi_1(p) = a_p(f)/p^{(k-1)/2}$, and χ_2 is determined by $\chi_1\chi_2 = \omega_p$.*
- (2) *If $r = 1$ and ω_p is unramified, then $\pi_p \cong \text{St} \otimes \chi$, where χ is the unramified character with $\chi(p) = a_p(f)/p^{(k-2)/2}$.*
- (3) *If neither of the above conditions hold, then π is supercuspidal and the conductor of ω_p is at most $\lfloor r/2 \rfloor$.*

Proof. The only statement requiring proof is that a primitive supercuspidal representation of conductor r must have central character of conductor $\leq \lfloor r/2 \rfloor$; this is [AL78, Theorem 4.3']. The remaining statements are clear from Table 1 and the relation $\lambda(\pi_{f,p}) = a_p(f)/p^{(k-1)/2}$. \square

Corollary 2.9. *Suppose $\pi_{f,p}$ is not supercuspidal. Let χ be any character such that $f \otimes \chi$ is p -primitive. Then $\pi_{f,p}$ is determined up to isomorphism by ε , χ , k , and $a_p(f \otimes \chi)$.*

Hence it is the supercuspidal cases that are most interesting.

3. SUPERCUSPIDAL REPRESENTATIONS

3.1. Generalities on cuspidal types. Let π be a supercuspidal representation of $G = \mathrm{GL}_2(\mathbf{Q}_p)$. Then, by the induction theorem of [BH06, §15.5], π arises by induction from a finite-dimensional character of an (explicit) open subgroup $K \subset G$ which contains the center Z of G and for which K/Z is compact. The following definition is not standard but is well suited for our purposes:

Definition 3.1. A *cuspidal type* is a pair (K, τ) , where K is a maximal compact-mod-center subgroup of G and τ is a finite-dimensional character of K such that $\mathrm{Ind}_K^G \tau$ is an irreducible supercuspidal representation of G .

Up to conjugacy, there are two maximal compact-mod-center subgroups of G : one is $\mathbf{Q}_p^\times \mathrm{GL}_2(\mathbf{Z}_p)$, and the other is the normalizer in G of an Iwahori subgroup

$$I_0(p) = \begin{pmatrix} \mathbf{Z}_p^\times & \mathbf{Z}_p \\ p\mathbf{Z}_p & \mathbf{Z}_p^\times \end{pmatrix} \subset \mathrm{GL}_2(\mathbf{Z}_p).$$

We will refer to these as the *unramified* and *ramified* maximal compact-mod-center subgroups, respectively.

In either case, K comes along with a natural filtration by normal subgroups. Let $K_0 \subset K$ be the maximal compact subgroup, so that K_0 equals $\mathrm{GL}_2(\mathbf{Z}_p)$ or $I_0(p)$. Then K_0 is the group of units in a \mathbf{Z}_p -algebra $A \subset M_2(\mathbf{Z}_p)$ equal to $M_2(\mathbf{Z}_p)$ in the unramified case and $\begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p\mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix}$ in the ramified case. The Jacobson radical P of A is pA or $\begin{pmatrix} p\mathbf{Z}_p & \mathbf{Z}_p \\ p\mathbf{Z}_p & p\mathbf{Z}_p \end{pmatrix}$. Let $K_n = 1 + P^n$; then the K_n are normal in K .

Definition 3.2. Let (K, τ) be a cuspidal type. The *level* of (K, τ) is the least integer $n \geq 0$ for which τ factors through K/K_n . The type (K, τ) is *primitive* if $\pi = \mathrm{Ind}_K^G \tau$ is a primitive representation of G .

Theorem 3.3. Every irreducible supercuspidal representation of G arises from a cuspidal type (K, τ) . If K is ramified, then one may take the level of τ to be even.

Proof. For a proof that every irreducible supercuspidal representation arises from a cuspidal type, see [BH06], §15.8. There an exhaustive list of cuspidal types (K, τ) is given; these cuspidal types are classified according to which “simple strata” they contain. For the definition of simple stratum, see [BH06], §13.1 and §13.2. In the definition of ramified simple stratum (A, n, α) appearing in §13.1, the integer n is required to be odd; in our notation, the level of any τ containing such a stratum has level $n + 1$, which is therefore even. \square

We remark that in [BCDT01], a concise list of pairs (K, τ) is given for which $\mathrm{Ind}_K^G \tau$ exhausts the set of isomorphism classes of supercuspidal representations, at least in the case of p odd.

Proposition 3.4. Let (K, τ) be a primitive cuspidal type of level n . Let r be the conductor of $\mathrm{Ind}_K^G \tau$. Then $r = 2n$ if K is unramified and $r = n + 1$ if K is ramified.

Proof. The relationship between the level of (K, τ) and the conductor of $\mathrm{Ind}_K^G \tau$ is spelled out in Henniart’s appendix to [BM02], A3.2–A.3.8. \square

3.2. The restriction of a cuspidal representation to a compact-mod-center subgroup. Suppose f is a p -primitive cusp form for which one has determined (using Proposition 2.8, say) that the local component $\pi_{f,p}$ is supercuspidal. By Theorem 3.3, there exists a cuspidal type (K, τ) for which $\pi_{f,p} = \text{Ind}_K^G \tau$. The subgroup K is determined (up to conjugacy in $\text{GL}_2(\mathbf{Z}_p)$) by the parity of the exponent of p in the conductor of f . Therefore to complete the solution to Problem. 1.1 it suffices to compute a model for the K -module τ . We shall see how to compute τ in Theorem 4.6. In preparation for this we shall need to gather some purely local results on supercuspidal representations of $\text{GL}_2(\mathbf{Q}_p)$.

We will have use for an old result of Casselman; see Theorem 3 of [Cas73b] and its proof. For the moment, let $\Gamma(p^m) \subset \text{GL}_2(\mathbf{Z}_p)$ denote the group of matrices congruent to the identity modulo p^m (this agrees with K_m if K is unramified). For a ring R , let $N(R) \subset \text{GL}_2(R)$ be the unipotent subgroup $\begin{pmatrix} 1 & R \\ & 1 \end{pmatrix}$.

Theorem 3.5. *Let π be a primitive cuspidal representation of conductor p^r , and let $m = \lfloor (r+1)/2 \rfloor$. Then $\pi^{\Gamma(p^m)}$ is an irreducible representation of $\text{GL}_2(\mathbf{Z}/p^m\mathbf{Z})$. Furthermore, the restriction of $\pi^{\Gamma(p^m)}$ to $N(\mathbf{Z}/p^m\mathbf{Z}) \cong \mathbf{Z}/p^m\mathbf{Z}$ is determined as follows:*

- (1) *If r is even, then $\pi^{\Gamma(p^m)}|_{N(\mathbf{Z}/p^m\mathbf{Z})}$ is the direct sum of characters of conductor p^m . Each of these $p^{m-1}(p-1)$ characters appears with multiplicity one.*
- (2) *If r is odd, then $\pi^{\Gamma(p^m)}|_{N(\mathbf{Z}/p^m\mathbf{Z})}$ is the direct sum of characters of conductors p^{m-1} and p^m . Each of these $p^{m-2}(p^2-1)$ characters appears with multiplicity one.*

Let (K, τ) be a primitive cuspidal type, and let $\pi = \text{Ind}_K^G \tau$. We will use Casselman's theorem to construct a model of τ inside of π .

Theorem 3.6. *Let p^r be the conductor of π , let v be a new vector for π , and let w be the vector*

$$w = \pi \left(\begin{pmatrix} p^{\lfloor r/2 \rfloor} & 0 \\ 0 & 1 \end{pmatrix} \right) v.$$

Finally, let W be the span of the $N(\mathbf{Z}_p)$ -translates of w .

- (1) *We have $W = \pi^{K_n}$. In particular, since $K_n \subset K$ is normal, we have that W is stabilized by the action of K .*
- (2) *The action of K on W is irreducible.*
- (3) *There is an isomorphism of K -modules $W \approx \tau$.*

Proof. First observe that (1) and (2) implies (3). Indeed, since π is induced from τ , and τ is fixed by K_n , we see that π^{K_n} contains a K -module isomorphic to τ . But by (1) and (2), $\pi^{K_n} = W$ is already an irreducible K -module, which forces $W \approx \tau$. Therefore it suffices to prove (1) and (2).

First suppose $r = 2n$ is even, so that $K = \mathbf{Q}_p^\times \text{GL}_2(\mathbf{Z}_p)$ is unramified. Then $K_n = \Gamma(p^n)$. The vector w has the property that

$$\pi \left(\begin{pmatrix} a & p^n b \\ p^n c & d \end{pmatrix} \right) w = \omega(a)w,$$

where $a, d \in \mathbf{Z}_p^\times$, $b, c \in \mathbf{Z}_p$, and where ω is the central character of π . It immediately follows that $W \subset \pi^{K_n}$. Let $B(\mathbf{Z}_p) \subset \text{GL}_2(\mathbf{Z}_p)$ be the subgroup of upper-triangular

matrices. Since $B(\mathbf{Z}_p)$ preserves $\mathbf{C}w$ and normalizes $N(\mathbf{Z}_p)$ we see that $B(\mathbf{Z}_p)$ preserves W . Consider the decomposition of W into irreducible characters of $N(\mathbf{Z}_p)$; certainly each character appearing has conductor $\leq p^n$. On the other hand, by Theorem 3.5, part (1), $\pi^{K_n}|_{N(\mathbf{Z}_p)}$ contains exactly those characters of conductor p^n . Therefore W only contains characters of conductor p^n . But $B(\mathbf{Z}_p)$ preserves W : as a result, *every* character of conductor p^n occurs in W . In view of Theorem 3.5, we have shown that W is all of π^{K_n} . Furthermore, the action of $B(\mathbf{Z}_p)$ on W is irreducible (since it acts transitively on the set of characters of $N(\mathbf{Z}_p)$ inside W). Hence the action of K on W is also irreducible, as required.

Now suppose that K is ramified, so that r is odd: Write $r = 2m - 1$ and $n = 2m - 2$. We have that

$$\pi \left(\begin{pmatrix} a & p^{m-1}b \\ p^m c & d \end{pmatrix} \right) w = \omega(a)w,$$

for all $a, d \in \mathbf{Z}_p^\times$, $b, c \in \mathbf{Z}_p$. The central character ω has conductor $\leq p^{m-1}$. The group K_n is

$$K_n = \begin{pmatrix} 1 + p^{m-1}\mathbf{Z}_p & p^{m-1}\mathbf{Z}_p \\ p^m\mathbf{Z}_p & 1 + p^{m-1}\mathbf{Z}_p \end{pmatrix} \supset \Gamma(p^m),$$

from which we see that $W \subset \pi^{K_n} \subset \pi^{\Gamma(p^m)}$. Applying Theorem 3.5 shows that $\pi^{\Gamma(p^m)}$ is an irreducible representation of $\mathrm{GL}_2(\mathbf{Z}/p^m\mathbf{Z})$ containing exactly those characters of $N(\mathbf{Z}_p)$ of conductors p^{m-1} and p^m . Consider the restriction of W to $N(\mathbf{Z}_p)$; each character that appears must have conductor p^{m-1} . Since W is preserved by $B(\mathbf{Z}_p)$, we see that $W|_{N(\mathbf{Z}_p)}$ is the sum of exactly those characters of conductor p^{m-1} . Since no characters of conductor p^m could occur in $\pi^{K_n}|_{N(\mathbf{Z}_p)}$, we have that $W = \pi^{K_n}$. The action of K on W is again irreducible, for identical reasoning as in the previous paragraph. \square

Let $U_p \subset \mathrm{GL}_2(\mathbf{Z}_p)$ be the subgroup of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \equiv 0 \pmod{p^r}$ and $a, d \equiv 1 \pmod{p^{\lfloor r/2 \rfloor}}$. Observe that

$$(3.2.1) \quad U_p = \begin{pmatrix} p^{\lfloor r/2 \rfloor} & \\ & 1 \end{pmatrix}^{-1} K_n \begin{pmatrix} p^{\lfloor r/2 \rfloor} & \\ & 1 \end{pmatrix}.$$

Therefore if π is a primitive supercuspidal representation of conductor r , then π^{U_p} is stabilized by the group

$$(3.2.2) \quad K' = \begin{pmatrix} p^{\lfloor r/2 \rfloor} & \\ & 1 \end{pmatrix}^{-1} K \begin{pmatrix} p^{\lfloor r/2 \rfloor} & \\ & 1 \end{pmatrix}.$$

Note that K' contains the subgroup of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p)$ with $c \equiv 0 \pmod{p^r}$. We will need the following corollary to Theorem 3.6.

Corollary 3.7. *Let $\rho: K' \rightarrow \mathrm{Aut} V$ be a continuous linear representation of K' which is isomorphic to a direct sum of finitely many copies of π^{U_p} . Let $v \in V$ be a nonzero vector with the property that*

$$\rho \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) v = \omega(d)v$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p)$ with $c \equiv 0 \pmod{p^r}$. Let X be the span of the translates of v under $\begin{pmatrix} 1 & p^{-\lfloor r/2 \rfloor} \mathbf{Z}_p \\ & 1 \end{pmatrix}$. Then X is preserved by K' , and (K', X) is a cuspidal pair with $\mathrm{Ind}_{K'}^G X \cong \pi$.

Proof. Write $\rho = (\pi^{U_p})^{\oplus \mu}$ and suppose $v = (v_i)_{1 \leq i \leq \mu}$, where v_i is a vector in the space of π^{U_p} . Then by the hypothesis on v we see that v_i is a new vector for π . By uniqueness of the new vector up to scaling we must have $v = (a_i v_0)_{1 \leq i \leq \mu}$ for a particular new vector v_0 and a nonzero tuple $(a_i) \in \mathbf{C}^\mu$. Thus v is contained in $\{(a_i x) | x \in \pi^{U_p}\} \subset V$ which is itself a model for π^{U_p} . The result now follows from Theorem 3.6. \square

3.3. Admissible pairs. If p is odd, then there is an explicit parametrization of the primitive supercuspidal types.

Definition 3.8 ([BH06, §18]). An *admissible pair* is a pair (E, θ) , where E/\mathbf{Q}_p is a tamely ramified extension and θ is a character $E^\times \rightarrow \overline{\mathbf{Q}}^\times$. These must satisfy the conditions:

- (1) θ does not factor through the norm map $N_{E/\mathbf{Q}_p} : E^\times \rightarrow \mathbf{Q}_p^\times$.
- (2) If E/\mathbf{Q}_p is ramified, then the restriction $\theta|_{U_E^1}$ must not factor through N_{E/\mathbf{Q}_p} .

We say that (E, θ) and (E', θ') are equivalent if there is an isomorphism $\iota : E \rightarrow E'$ such that $\theta' \circ \iota = \theta$.

In [BH06, §20], a construction is given which associates to each admissible pair (E, θ) a cuspidal type (K, τ) , where K is ramified or unramified as E is. If χ is a character of \mathbf{Q}_p^\times , then $(E, \theta \times (\chi \circ N_{E/\mathbf{Q}_p}))$ corresponds to $(K, \tau \otimes (\chi \circ \det))$; hence (K, τ) is primitive if and only if for all characters χ of \mathbf{Q}_p^\times , we have $c(\theta \times \chi \circ N_{E/\mathbf{Q}_p}) \geq c(\theta)$.

For $p \neq 2$ this construction gives a bijection between admissible pairs (up to equivalence) and cuspidal types (up to isomorphism); if $p = 2$, then the admissible pairs correspond to a subset of the cuspidal types. We will need the following formula describing the trace of the representation (K, τ) :

Definition 3.9. An element $\alpha \in E^\times$ is minimal if:

- (1) E/\mathbf{Q}_p is unramified and $\alpha = p^k \alpha_0$ for a unit $\alpha_0 \in \mathcal{O}_E^\times$ whose image in $\mathcal{O}_E/\mathfrak{p}_E \cong \mathbf{F}_{p^2}$ does not lie in \mathbf{F}_p .
- (2) E/\mathbf{Q}_p is ramified and α has odd valuation.

Theorem 3.10. Let (K, τ) be a primitive cuspidal type of level n corresponding to an admissible pair (E, θ) , and let $\alpha \in E^\times$ be a minimal element. Let $s \in \mathrm{Gal}(E/\mathbf{Q}_p)$ be the nontrivial automorphism. Then

$$\mathrm{Tr} \tau(\alpha) = \iota(\theta(\alpha) + \theta(\alpha^s)),$$

where $\iota = (-1)^{(n-1)}$ if E/\mathbf{Q}_p is unramified and $\iota = 1$ otherwise.

Proof. See [Wei09], proof of Lemma 2.2. \square

Remark 3.11. Suppose (E, θ) is an admissible pair, and let $\pi_\theta = \mathrm{Ind}_K^G \tau$ be the corresponding supercuspidal representation of $\mathrm{GL}_2(\mathbf{Q}_p)$. By local class field theory one can view θ as a character of the Weil group W_E . Let $\sigma_\theta = \mathrm{Ind}_{E/\mathbf{Q}_p} \theta$; the

conditions on θ in Definition 3.8 ensure that σ is an irreducible representation of $W_{\mathbf{Q}_p}$.

It is *not* the case that π_θ and σ_θ correspond under the local Langlands correspondence. Rather, there is a character Δ_θ of E^\times of order dividing 4 for which $\pi_{\theta\Delta_\theta}$ and σ_θ correspond. If E/\mathbf{Q}_p is unramified, then Δ_θ is the unramified quadratic character. If E/\mathbf{Q}_p is ramified, then Δ_θ is a normalized Gauss sum; cf. [BH06], §34.4.

4. MODULAR SYMBOLS

4.1. Definitions. For $k \geq 2 \in \mathbf{Z}$ and Γ a finite-index subgroup of $\mathrm{SL}_2(\mathbf{Z})$, we let $\mathbf{S}_k(\Gamma, \mathbf{Q})$ denote the space of cuspidal modular symbols of level Γ over \mathbf{Q} . For the definition of this space, see [Ste07, §8.1]. For our purposes, it suffices to state that:

- (1) $\mathbf{S}_k(\Gamma, \mathbf{Q})$ is a finite-dimensional \mathbf{Q} -vector space, equipped with an action of the Hecke algebra $\mathcal{H} = \mathcal{H}(\mathrm{GL}_2^+(\mathbf{Q})/\Gamma)$.
- (2) Given an algorithm to determine whether or not a given $g \in \mathrm{SL}_2(\mathbf{Z})$ lies in Γ , there are algorithms that will calculate a basis for $\mathbf{S}_k(\Gamma, \mathbf{Q})$ and the matrix of the Hecke operator $[\Gamma x \Gamma]$ on $\mathbf{S}_k(\Gamma, \mathbf{Q})$ for any given $x \in \mathrm{GL}_2^+(\mathbf{Q})$.
- (3) There is a perfect pairing between $\mathbf{S}_k(\Gamma) = \mathbf{S}_k(\Gamma, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$ and $\mathbf{S}_k(\Gamma) \oplus \overline{\mathbf{S}_k(\Gamma)}$, with respect to which the action of \mathcal{H} is self-adjoint.
- (4) If a subgroup $\Gamma' \subset \mathrm{SL}_2(\mathbf{Q})$ normalizes Γ , there is an action

$$\Gamma' \rightarrow \mathrm{GL}(\mathbf{S}_k(\Gamma, \mathbf{Q}));$$

this action is denoted $(\gamma, \sigma) \mapsto \gamma \circ \sigma$.

The subgroups Γ we shall consider will all be of the form

$$\Gamma_H(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c = 0 \pmod{N}, d \in H \right\},$$

where H is some subgroup of $(\mathbf{Z}/N\mathbf{Z})^\times$. A highly optimized implementation¹ of modular symbol algorithms for these level groups is included in the software package [Sag].

If f is a new eigenform of level N , and the character ε of f is trivial on the subgroup H , then $f \in S_k(\Gamma_H(N))$, and hence there is a two-dimensional eigenspace $M_f \subset \mathbf{S}_k(\Gamma_H(N))$ with the same eigenvalues as f for the Hecke operators T_m . This space is preserved by the Hecke operator corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which has eigenvalues ± 1 on M_f . Thus there are corresponding basis vectors σ_f^+, σ_f^- of M_f , each unique up to scaling. We refer to these as the *plus eigensymbol* and *minus eigensymbol* for f .

4.2. Determination of the minimal twist. Let us now suppose f is a new eigenform of level Np^r , with $p \nmid N$, and character ε . Write $\varepsilon = \varepsilon_N \varepsilon_p$ as a product of characters of $(\mathbf{Z}/N\mathbf{Z})^\times$ and $(\mathbf{Z}/p^r\mathbf{Z})^\times$.

Proposition 4.1. *Let the conductor of ε_p be p^c , and let $u = \min(\lfloor r/2 \rfloor, r - c)$.*

- (1) *If χ is a Dirichlet character of conductor p^v with $v > u$, then f_χ is new of level $\mathrm{LCM}(Np^r, Np^{c+v}, Np^{2v}) > Np^r$.*

¹This implementation is the work of many authors, including William Stein, Jordi Quer, Craig Citro and the first author of this paper.

- (2) If χ is a character modulo p^u , then f_χ has level Np^r and is new at the primes $\ell \mid N$. Moreover, the space

$$X_f \subseteq S_k(\Gamma_1(Np^r))$$

spanned by the translates of f by $\begin{pmatrix} 1 & p^{-u}\mathbf{Z} \\ 0 & 1 \end{pmatrix}$ is equal to the span of the forms f_χ , for characters χ modulo p^u .

Proof. The first part follows from results in [AL78, §3].

For the second part, it is immediate that each twist f_χ is contained in X_f . For the reverse inclusion: If $u = 0$ there is nothing to prove; so we may assume that $r \geq 2$ and $c < r$. In this case, it is easy to see that U_p operator on $S_k(\Gamma_1(Np^r), \chi)$ factors through $S_k(\Gamma_1(Np^{r-1}), \chi)$, and hence is zero on $S_k(\Gamma_1(Np^r), \chi)_{\text{new}}$. So, since f is assumed to be a new eigenform, we must have $a_n(f) = 0$ whenever $p \mid n$.

Hence a basis for X_f is given by the series $\sum_{n=i \bmod p^u} a_n q^n$ where i runs through $(\mathbf{Z}/p^u\mathbf{Z})^\times$, and each such series is clearly a linear combination of twists of f by Dirichlet characters modulo p^u . This proves the reverse inclusion. \square

If g is a newform in $S_k(\Gamma_1(Np^s))$ for some $s \leq r$, we write $S_k(\Gamma_1(Np^r))[g]$ for the subspace of $S_k(\Gamma_1(Np^r))$ where the Hecke operators T_m for $(m, Np) = 1$ act as multiplication by $a_m(g)$. As is well known, a basis for $S_k(\Gamma_1(Np^r))[g]$ is given by the forms $g(p^j z)$ for $0 \leq j \leq r - s$, and we have a direct sum decomposition

$$S_k(\Gamma_1(Np^r))_{N\text{-new}} = \bigoplus_g S_k(\Gamma_1(Np^r))[g],$$

where the summands are eigenspaces for the prime-to- Np Hecke operators with distinct systems of eigenvalues.

Proposition 4.2. *Let $f \in S_k(\Gamma_1(Np^r))$ be a newform, and let g be a newform in $S_k(\Gamma_1(Np^s))$ for some $s \leq r$. The following are equivalent:*

- (1) *There exists a Dirichlet character χ of p -power conductor with $f = g \otimes \chi$,*
- (2) *$X_f \cap S_k(\Gamma_1(Np^r))[g] \neq 0$.*

In particular, f is p -primitive if and only if $X_f \subseteq S_k(\Gamma_1(Np^r))_{p\text{-new}}$.

Proof. In (1), if χ is a Dirichlet character whose conductor is a power of p larger than p^u , then $g \otimes \chi$ has level strictly bigger than Np^r , by part (1) of the previous proposition. So if such a χ exists, its conductor divides p^u .

Recall that X_f is spanned by the forms f_χ , for χ modulo p^u . Each of these is an eigenvector for the Hecke operators T_m for $(m, Np) = 1$, and hence lies in $S_k(\Gamma_1(Np^r))[g]$ for some newform g . Comparing eigenvalues for the T_m with $(m, Np) = 1$, we see that for each such m we must have $a_m(g) = a_m(f_\chi) = \chi(m)a_m(f)$ for each m with $(m, Np) = 1$.

Hence $X_f \cap S_k(\Gamma_1(Np^r))[g] \neq 0$ if and only if there is a Dirichlet character χ modulo p^u such that $a_m(g) = \chi(m)a_m(f)$ for m coprime to Np . Since f is new by assumption, this implies that $f = g \otimes \chi^{-1}$. \square

We now construct a space of modular symbols corresponding to X_f . Let H_N be any subgroup of $(\mathbf{Z}/N\mathbf{Z})^\times$ contained in the kernel of ε_N , and let H_p be the subgroup of $(\mathbf{Z}/p^r\mathbf{Z})^\times$ which is the kernel of reduction modulo $p^{\max(c, \lceil r/2 \rceil)}$. We let $H = H_N \times H_p \subseteq (\mathbf{Z}/Np^r\mathbf{Z})^\times$. Then $\begin{pmatrix} 1 & p^{-u} \\ 0 & 1 \end{pmatrix}$ normalizes $\Gamma = \Gamma_H(Np^r)$.

Definition 4.3. The *type space* X_f^+ of f is the subspace of $\mathbf{S}_k(\Gamma)$ spanned by the translates of the plus eigensymbol σ_f^+ under $\begin{pmatrix} 1 & p^{-u}\mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$.

For each newform g of level dividing Np^r , with character trivial on H , let us write $\mathbf{S}_k(\Gamma)[g]$ for the subspace of \mathbf{S}_k where the prime-to- Np Hecke operators T_m act as multiplication by $a_m(g)$. Just as with spaces of modular forms, we have a decomposition

$$\mathbf{S}_k(\Gamma) = \mathbf{S}_k(\Gamma)_{p\text{-new}} \oplus \mathbf{S}_k(\Gamma)_{p\text{-old}}$$

where the first summand is spanned by the spaces $\mathbf{S}_k(\Gamma)[g]$ with g having level exactly divisible by p^r , and the second by g of level divisible by some smaller power of p ; and these two spaces can be explicitly computed using degeneracy maps, as in [Ste07, §8.6]. Translating the previous proposition into the language of modular symbols, we obtain the following proposition, which allows us to algorithmically determine whether or not a given newform is p -primitive:

Proposition 4.4. *Let f be a newform of level Np^r , and g a newform whose level is Np^s for some $s \leq r$. Then the following are equivalent:*

- (1) *There exists a Dirichlet character χ of p -power conductor with $g = f \otimes \chi$.*
- (2) $X_f^+ \cap \mathbf{S}_k(\Gamma)[g] \neq 0$.

In particular, f is p -primitive if and only if $X_f^+ \subseteq \mathbf{S}_k(\Gamma)_{p\text{-new}}$.

Proof. Since each of the twists f_χ , for χ modulo p^u , has character trivial on H , the space X_f is contained in $S_k(\Gamma) \subseteq S_k(\Gamma_1(Np^r))$. Moreover, since the natural pairing between modular forms and modular symbols commutes with the action of the Hecke algebra on the two spaces, the same systems of eigenvalues for the prime-to- Np Hecke algebra appear in the spaces X_f and X_f^+ . Hence we have $X_f \cap S_k(\Gamma_1(Np^r))[g] \neq 0$ if and only if $X_f^+ \cap \mathbf{S}_k(\Gamma)[g] \neq 0$, and the result is now immediate from the previous proposition. \square

If f is in fact p -primitive, then either $u = 0$, or $u = \lfloor r/2 \rfloor$ and $\pi_{f,p}$ is supercuspidal. In the latter case, we can give the following more precise description of the space X_f^+ :

Lemma 4.5. *If f is p -primitive and supercuspidal at p , the space X_f^+ is the direct sum of the eigenspaces $\sigma_{f \otimes \chi}^{\chi(-1)}$ for characters χ modulo $p^{\lfloor r/2 \rfloor}$. In particular, it has dimension $\phi(p^{\lfloor r/2 \rfloor})$ and is stable under the operators T_m and $\langle d \rangle$ for all m and d , and under the Atkin-Lehner operator W_{p^r} .*

Proof. We define operators R_χ , Σ and $\langle m_p \rangle$ in $\text{End } \mathbf{S}_k(\Gamma_H(Np^r))$ as follows. For a Dirichlet character χ modulo $p^{\lfloor r/2 \rfloor}$, define

$$R_\chi = \sum_{u \in (\mathbf{Z}/p^{\lfloor r/2 \rfloor} \mathbf{Z})^\times} \chi(u)^{-1} \begin{pmatrix} 1 & p^{-\lfloor r/2 \rfloor} u \\ 0 & 1 \end{pmatrix}.$$

Let $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the star involution. Finally, for $m \in \mathbf{Z}$ relatively prime to Np , let $m_p \in (\mathbf{Z}/Np^{\lfloor r/2 \rfloor} \mathbf{Z})^\times$ denote the unique element congruent to 1 (mod N) and $m \pmod{p^{\lfloor r/2 \rfloor}}$.

The following commutation relations are elementary:

$$(4.2.1) \quad T_m \circ R_\chi = \chi(m) R_\chi \circ T_m,$$

$$(4.2.2) \quad \Sigma \circ R_\chi = \chi(-1) R_\chi \circ \Sigma,$$

$$(4.2.3) \quad T_m \circ W_{p^r} = W_{p^r} \circ T_m \circ \langle m_p \rangle^{-1},$$

$$(4.2.4) \quad \Sigma \circ W_{p^r} = \langle (-1)_p \rangle \circ W_{p^r} \circ \Sigma.$$

For any character χ modulo $p^{\lfloor r/2 \rfloor}$, we have $R_\chi(\sigma_f^+) \in X_f^+$. Hence X_f^+ contains the span of the $R_\chi(\sigma_f^+)$. In fact, X_f^+ equals the span of the $R_\chi(\sigma_f^+)$, by Fourier inversion. Relations (4.2.1) and (4.2.2) show that $R_\chi(\sigma_f^+)$ lies in the space $\mathbf{S}_k(\Gamma_H(Np^r))[f \otimes \chi]$ and is an eigenvector for Σ with eigenvalue $\chi(-1)$. This forces $R_\chi(\sigma_f^+)$ to be a scalar multiple of $\sigma_{f \otimes \chi}^{\chi(-1)}$, since the Σ -eigenspaces in $\mathbf{S}_k(\Gamma_H(Np^r))[f \otimes \chi]$ are 1-dimensional. Hence X_f^+ is stable under the operators T_m and $\langle d \rangle$ for all m and d . Note that the eigenspaces $\sigma_{f \otimes \chi}^{\chi(-1)}$ are all distinct, since by Lemma 2.6 we can only have $f \otimes \chi = f \otimes \chi'$ for distinct characters χ and χ' if $\chi(-1) = -\chi'(-1)$.

Moreover, by relation (4.2.3), for each χ , $W_{p^r}(\sigma_{f \otimes \chi}^{\chi(-1)})$ lies in the eigenspace $\mathbf{S}_k(\Gamma_H(Np^r))[f \otimes \varepsilon_p^{-1} \chi^{-1}]$ where ε_p is the p -part of the character of f , and (4.2.4) implies that it is a Σ -eigenvector with eigenvalue $\varepsilon_p(-1)\chi(-1)$. Thus X_f^+ is preserved by W_{p^r} . \square

4.3. The type space and cuspidal pairs. When f is p -primitive and $\pi_{f,p}$ is supercuspidal, then in fact X_f^+ tells us considerably more about $\pi_{f,p}$, as we shall show. As we have seen from Proposition 2.8, if $\pi_{f,p}$ is primitive and supercuspidal then we have $c \leq \lfloor r/2 \rfloor$, so H_p is the subgroup of elements of $\mathbf{Z}/p^r\mathbf{Z}$ which are 1 modulo $p^{\lfloor r/2 \rfloor}$.

As in §3.1, let K denote either the unramified or the ramified maximal compact-modulo-centre subgroup of $G = \mathrm{GL}_2(\mathbf{Q}_p)$, for r even or odd, respectively, and let $S(K)$ denote the intersection of K with $\mathrm{SL}_2(\mathbf{Q}_p)$. We observe that K is generated by $S(K)$ along with the following subgroups and elements:

- The torus $T = \begin{pmatrix} \hat{\mathbf{Z}}_p^\times & \\ & \hat{\mathbf{Z}}_p^\times \end{pmatrix}$.
- The matrix $pI_2 = \begin{pmatrix} p & \\ & p \end{pmatrix}$, if K is unramified.
- The matrix $\Pi = \begin{pmatrix} & 1 \\ -p & \end{pmatrix}$, if K is ramified.

Let $\tilde{S}(K)$ denote the subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ generated by $S(K)$ and the single matrix pI_2 or Π as K is unramified or ramified, respectively. Then $\tilde{S}(K)$ normalizes the subgroup $S(K_n) = K_n \cap \mathrm{SL}_2(\mathbf{Q}_p)$.

The closure of Γ in $\mathrm{SL}_2(\mathbf{Q}_p)$ is $\beta^{-1}S(K_n)\beta$, where $\beta = \begin{pmatrix} p^{\lfloor r/2 \rfloor} & \\ & 1 \end{pmatrix}$ is the matrix from Theorem 3.6 and n is the integer determined by Proposition 3.4. As $S(K_n)$ is normal in $\tilde{S}(K)$, this allows us to define an action $\rho: \tilde{S}(K) \rightarrow \mathrm{GL}(\mathbf{S}_k(\Gamma))$. This action may be described concretely as follows. For $\alpha \in \tilde{S}(K)$ and $\sigma \in \mathbf{S}_k(\Gamma)$, let $\tilde{\alpha} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q})$ be a matrix with integer entries satisfying:

- (1) The image of $\tilde{\alpha}$ in $\mathrm{GL}_2(\mathbf{Q}_p)$ lies in $\alpha S(K_n)$,
- (2) $c \equiv 0 \pmod{N}$, and $d \bmod N \in H$, and
- (3) $\det \tilde{\alpha}$ is a power of p .

Then the matrix $\beta^{-1}\tilde{\alpha}\beta$ normalizes Γ , and we may define

$$\rho(\alpha)\sigma = \beta^{-1}\tilde{\alpha}\beta \circ \sigma.$$

Note that in the case of K ramified, the operator $\rho(\Pi)$ is $\varepsilon_N(p^{\lfloor r/2 \rfloor}) \cdot W_{p^r}$, where W_{p^r} is the Atkin-Lehner operator; this follows from [AL78, Proposition 1.1].

The next theorem plays a crucial role in the determination of the local component of a newform when that component is supercuspidal.

Theorem 4.6. *Let f be a p -primitive newform of weight k , level Np^r and nebentypus ε . Assume that $\pi_{f,p}$ is a supercuspidal representation of $G_p = \mathrm{GL}_2(\mathbf{Q}_p)$. Let $X_f^+ \subset \mathbf{S}_k(\Gamma)$ be the subspace of modular symbols defined in Definition 4.3.*

- (1) *The space X_f^+ is preserved by the action ρ of $\tilde{S}(K)$ on $\mathbf{S}_k(\Gamma_H(Np^r))$.*
- (2) *In fact, there exists a unique action $\rho_f: K \rightarrow \mathrm{GL}(X_f^+)$ extending ρ for which*

$$\rho_f \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \sigma_f^+ = \omega_p(d)^{-1} \sigma_f^+, \quad a, d \in \mathbf{Z}_p^\times, \quad c \equiv 0 \pmod{p^r}$$

In particular, the central character of ρ_f is ω_p^{-1} .

- (3) *Let τ be the contragredient of ρ_f . The pair (K, τ) is a cuspidal type, and $\mathrm{Ind}_K^G \tau \cong \pi_{f,p}$.*

The proof of Theorem 4.6 will occupy the following three subsections.

4.4. Shimura curves. The type space X_f^+ admits an action of the group $\tilde{S}(K)$, which we intend to extend to an action of K , the maximal compact-mod-center subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$. For this we must perform a comparison between two sorts of objects. On the one hand, we have the space of modular symbols $\mathbf{S}_k(\Gamma_H(Np^r))$, which is amenable to computation and which admits an action by an “ SL_2 group”. On the other hand, there is the cohomology space of a (usually disconnected) Shimura curve, which has a direct interpretation in terms of automorphic representations and which admits an action by a “ GL_2 group”.

We recall the setup of Theorem 4.6. Let p be prime, $r \geq 2$ and integer, and $N \geq 1$ an integer prime to p . Let $H_N \subset (\mathbf{Z}/N\mathbf{Z})^\times$ be any subgroup, let $H_p \subset (\mathbf{Z}/p^r\mathbf{Z})^\times$ be the kernel of reduction modulo $p^{\lfloor r/2 \rfloor}$, and let $H = H_N \times H_p \subset (\mathbf{Z}/Np^r\mathbf{Z})^\times$. Finally let $\Gamma = \Gamma_H(Np^r)$.

We construct the Shimura curve analogue of the modular curve $Y(\Gamma)$. Let $U \subset \mathrm{GL}_2(\hat{\mathbf{Z}})$ be the open subgroup consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with

- (1) $c \equiv 0 \pmod{Np^r}$,
- (2) $d \bmod N \in H_N$,
- (3) $a, d \bmod p^r \in H_p$.

Thus if $U = \prod_\ell U_\ell$ is the obvious decomposition into open subgroups $U_\ell \subset \mathrm{GL}_2(\mathbf{Z}_\ell)$, then U_p is as in Eq. (3.2.1). That is, U_p is conjugate in $\mathrm{GL}_2(\mathbf{Q}_p)$ to K_n , where K is unramified (resp., ramified) and $r = 2n$ (resp., $n + 1$) as n is even (resp., odd).

We write \mathcal{H}^+ for the usual upper half-plane, and \mathcal{H}^\pm for $\mathbf{C} \setminus \mathbf{R}$. Let Sh_U be the Shimura curve of level U :

$$\mathrm{Sh}_U = \mathrm{GL}_2(\mathbf{Q}) \backslash (\mathrm{GL}_2(\mathbf{A}_f) \times \mathcal{H}^\pm) / \Gamma.$$

We have $U \cap \mathrm{GL}_2(\mathbf{Q}) = \Gamma$. Therefore the connected components of Sh_U are in bijection with $\hat{\mathbf{Z}}^\times / \det U = (\mathbf{Z}/p^{\lfloor n/2 \rfloor})^\times$, and each connected component is isomorphic to the modular curve $Y(\Gamma)$.

Fix a weight $k \geq 2$, and let $V = \mathrm{Sym}^{k-2} \mathbf{C}^2$ be the $(k-2)$ -fold symmetric power of the tautological representation of $\mathrm{GL}_2(\mathbf{Q})$. We write \mathcal{L}_k for the corresponding vector bundle on Sh_U . The following theorem relates the cohomology of \mathcal{L}_k to automorphic representations arising from newforms of level Γ :

Theorem 4.7. *The space $H_P^1(\mathrm{Sh}_U, \mathcal{L}_k)$ decomposes as a direct sum*

$$H_P^1(\mathrm{Sh}_U, \mathcal{L}_k) \cong \bigoplus_f H^1(\mathfrak{g}, K_\infty, \pi_{f,\infty} \otimes V) \otimes (\pi_{f,\mathrm{fin}})^U$$

over cuspidal newforms f of weight k . Here $\pi_f = \pi_{f,\mathrm{fin}} \otimes \pi_{f,\infty}$ is the automorphic representation corresponding to f , \mathfrak{g} is the Lie algebra of $\mathrm{GL}_2(\mathbf{R})$, and $K_\infty \subset \mathrm{GL}_2(\mathbf{R})$ is a maximal compact subgroup. For each f , the Lie algebra cohomology $H^1(\mathfrak{g}, K_\infty, \pi_{f,\infty} \otimes V)$ has dimension 2.

Proof. For the decomposition of $H_P^1(\mathrm{Sh}_U, \mathcal{L}_k)$ into subspaces indexed by automorphic representations, see [Car83], §2.2.4. For the statement about $H^1(\mathfrak{g}, K_\infty, \pi_{f,\infty} \otimes V)$, see [RT83], Proposition 1.5(b). \square

Recall the group K' of Eq. (3.2.2): K' contains U_p as a normal subgroup, so there is a natural action $K' \rightarrow \mathrm{Aut} \mathrm{Sh}_U$.

4.5. Modular symbols on Sh_U . We have a decomposition of Sh_U into connected components:

$$(4.5.1) \quad \mathrm{Sh}_U = \coprod_a Y(\Gamma)_a$$

where a runs over a set of representatives for $\hat{\mathbf{Z}}^\times / \det U$ and $Y(\Gamma)_a$ is the (isomorphic) image of $Y(\Gamma)$ under the map induced by $z \mapsto \left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}, z \right)$. The group K' acts on Sh_U , and the stabilizer of any component is

$$\tilde{S}(K)' = \left\{ g \in K' \mid \det g \in p^{\mathbf{Z}}(1 + p^{\lfloor r/2 \rfloor} \mathbf{Z}_p) \right\}.$$

Let $\mathbf{S}_k(U)$ be the space of cuspidal modular symbols of weight k on Sh_U . Thus $\mathbf{S}_k(U)$ is the direct sum of copies of $\mathbf{S}_k(\Gamma)$ indexed by a as in Eq. (4.5.1). The space $\mathbf{S}_k(U)$ admits an action $\rho: K' \rightarrow \mathrm{GL}(\mathbf{S}_k(U))$ as well as an action of the Hecke algebra.

Let $f \in \mathbf{S}_k(\Gamma)$ be a p -primitive newform of level Np^r , and let $\sigma_f^+ \in \mathbf{S}_k(\Gamma)$ be a plus eigensymbol for f . Let

$$\eta_f^+ = \sum_a \rho \left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} \right) \sigma_f^+ \in \mathbf{S}_k(U).$$

Then the Hecke operators act on η_f^+ with the same eigenvalues as they do on f (or on σ_f^+).

Lemma 4.8. *Let $W_f^+ \subset \mathbf{S}_k(U)$ be the span of the translates of η_f^+ under the $\begin{pmatrix} 1 & p^{-\lfloor r/2 \rfloor} \mathbf{Z}_p \\ & 1 \end{pmatrix}$. Then W_f^+ is preserved by K' . Let τ be the representation of K which is contragredient to the representation of K on W_f^+ . Then (K', τ) is a cuspidal pair and $\mathrm{Ind}_{K'}^G \tau \cong \pi_{f,p}$.*

Proof. The integration pairing

$$\mathbf{S}_k(U) \times H_P^1(\mathrm{Sh}_U, \mathcal{L}_k) \rightarrow \mathbf{C}$$

is perfect and equivariant for the actions of both the Hecke algebra and the group K' . By Proposition 4.7, η_f^+ lies in a K' -submodule of $\mathbf{S}_k(U)$ isomorphic to a direct sum of copies of $(\tilde{\pi}_{f,p})^{U_p}$. Furthermore, a calculation shows that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p)$ acts on η_f^+ as $\omega_p(d)^{-1}$ whenever $c \equiv 0 \pmod{p^r}$, where ω_p is the central character of $\pi_{f,p}$. The lemma now follows from Corollary 3.7. \square

4.6. Conclusion of the proof of Theorem 4.6. Consider the projection $\Phi: \mathbf{S}_k(U) \rightarrow \mathbf{S}_k(\Gamma)$ which restricts a modular symbol to the component $Y(\Gamma)_1 \subset \mathrm{Sh}_U$. Then Φ transforms the action of $\beta^{-1}\tilde{S}(K)\beta$ on $\mathbf{S}_k(U)$ into the action of $\tilde{S}(K)$ on $\mathbf{S}_k(U)$ described in §4.3. We have $\Phi(\eta_f^+) = \sigma_f^+$, and consequently $\Phi(W_f^+) = X_f^+$. On the other hand, $\dim W_f^+ = \dim X_f^+$ by Proposition 4.5. Therefore the restriction of Φ to W_f^+ furnishes an isomorphism $W_f^+ \rightarrow X_f^+$ compatible with the actions of $\tilde{S}(K)'$ and $\tilde{S}(K)$. This establishes part (1) of the theorem. The group K' preserves W_f^+ and therefore induces an action of K on X_f^+ . Part (2) follows from the corresponding fact concerning η_f^+ , and part (3) follows from Lemma 4.8.

5. THE ALGORITHM

We now use the above results to describe an explicit algorithm to determine $\pi_{f,p}$, given a new eigenform $f \in S_k(\Gamma_1(Np^r), \varepsilon)$ with $r \geq 1$ (specified by the sequence of Hecke eigenvalues $a_\ell(f)$ for $\ell \nmid N$, as elements of some number field).

Step 1: Check whether $a_p(f) = 0$. If $a_p(f) \neq 0$, then by Proposition 2.8, one of the following must hold:

- The p -part of the conductor of ε is p^r . In this case, $\pi_{f,p}$ is isomorphic to the principal series $\pi(\chi_1, \chi_2)$, where χ_1 is the unramified character with $\chi_1(p) = a_p(f)/p^{\frac{1-k}{2}}$, and χ_2 is the unique character such that $\chi_2|_{\mathbf{Z}_p} = \varepsilon_p^{-1}$ and $\chi_2(p) = \varepsilon_N(p)/\chi_1(p)$, where ε_p and ε_N are, respectively, the p -part and the prime-to- p -part of ε .
- $r = 1$ and the conductor of ε is prime to p . In this case, $a_p(f)^2 = p^{k-2}\varepsilon(p)$, and $\pi_{f,p}$ is isomorphic to the special representation $\mathrm{St} \otimes \chi$, where χ is the unramified character with $\chi(p) = a_p(f)/p^{\frac{k-2}{2}}$.

If $a_p(f) = 0$, we continue to step 2.

Step 2: Calculate a basis for the space of modular symbols $\mathbf{S}_k(\Gamma_H(Np^r))$ of §4.1. Calculate the matrices representing the action on this space of the star involution Σ and the Hecke operators T_ℓ for primes $\ell \neq p$ up to the Sturm bound. Use these to calculate the plus eigensymbol σ_f^+ attached to f , by intersecting the kernels of $\Sigma - 1$ and $T_\ell - a_\ell(f)$ until a 1-dimensional space is obtained.

- Step 3: Calculate the subspace $X_f^+ \subseteq \mathbf{S}_k(\Gamma_H(Np^r))$ spanned by the orbit of σ_f^+ under $\begin{pmatrix} 1 & p^{-u}\mathbf{Z} \\ & 1 \end{pmatrix}$.
- Step 4: Calculate the matrices of the two degeneracy maps from $\mathbf{S}_k(\Gamma_H(Np^r))$ to $\mathbf{S}_k(\Gamma_{H'}(Np^{r-1}))$, where H' is the image of H modulo Np^{r-1} . The sum of the images of X_f^+ is a Hecke-invariant subspace; if it is nonzero, then it contains an eigenform, which corresponds to a twist f' of f having level Np^{r-1} . In this case, go back to step 1 with f' in place of f . If this does not occur, we may continue to step 5 knowing that f is p -primitive.
- Step 5: Compute the matrix of $\rho(g)$ acting on X_f^+ for each g in a set of generators for the group $\tilde{S}(K)/S(K_n)$, where the groups $\tilde{S}(K)$ and $S(K_n)$ and the representation ρ are as defined in §4.3. One may take the set of generators to be

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \right\}$$

for K unramified, or

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \right\}$$

for K ramified (where a is any generator of $(\mathbf{Z}/p\mathbf{Z})^\times$); note that $\rho\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$ has been computed in the course of step 3.

- Step 6: For each a running through a set of generators of \mathbf{Z}_p^\times , determine the space of operators $\lambda_a \in \mathrm{GL}(X_f^+)$ satisfying $\lambda_a \rho(g) = \rho(\delta_a g \delta_a^{-1}) \lambda_a$, where g runs through a set of generators of $S(K)$ and where $\delta_a = \begin{pmatrix} a & \\ & 1 \end{pmatrix}$. By Theorem 4.6, there will be a unique system λ_a of such operators which each fix σ_f^+ . Let ρ_f be the unique extension of ρ to K which satisfies $\rho_f(\delta_a) = \lambda_a$ for each a .
- Step 7: Output the representation (K, ρ_f) . By Theorem 4.6 we have $\pi_{f,p} \cong \mathrm{Ind}_K^{\mathrm{GL}_2(\mathbf{Q}_p)} \check{\rho}_f$.

6. EXAMPLES

Example 1. Let f be the newform $q - q^2 + q^3 + q^4 + \dots$ of weight 2 and level 50, and take $p = 5$. Since $a_p(f) = 0$, we proceed to step 2. We find that the subgroup $\Gamma = \Gamma_H(50)$, where H consists of the elements of $(\mathbf{Z}/50\mathbf{Z})^\times$ that are 1 (mod 5), has index 360 in $\mathrm{SL}_2(\mathbf{Z})$, and the space $\mathbf{S}_2(\Gamma)$ has dimension 31. Continuing to step 3, we find that X_f^+ is 4-dimensional. At step 4, we find that X_f^+ is contained in the 18-dimensional subspace that is the intersection of the kernels of the two degeneracy maps to level 10. Thus f is indeed 5-primitive.

Since the exponent of 5 in the conductor of f is even, the group $K = \mathbf{Q}_p^\times \mathrm{GL}_2(Z_p)$ is unramified. In step 5, we compute the representation $\rho: S(K) \rightarrow \mathrm{GL}(X_f^+)$. The

matrices of $\rho\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$ and $\rho\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$ are, respectively,

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & -2 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -1 & 1 & -2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix}$$

with respect to a certain basis of X_f^+ whose first vector is σ_f^+ .

Since f has trivial character, $\rho\left(\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right)$ is the identity. In fact ρ is already irreducible as a representation of $S(K)$, as there are no nonscalar 4×4 matrices commuting with both A_1 and A_2 , so X_f^+ is irreducible as a representation of $\mathrm{SL}_2(\mathbf{Z}/5\mathbf{Z})$.

Continuing to step 6, we choose $a = 2$ as a generator of $(\mathbf{Z}/5\mathbf{Z})^\times$, and compute the action of the conjugated generators $\begin{pmatrix} 1 & a^{-1} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix}$. Either by a direct modular symbol computation, or (more efficiently) by expressing the conjugated generators as words in the original generators and using the matrices already computed, we find that these two elements act by

$$A'_1 = \begin{pmatrix} -2 & 1 & -1 & 1 \\ -2 & 2 & -2 & 1 \\ -1 & 2 & -1 & 0 \\ -2 & 2 & -1 & 0 \end{pmatrix} \quad \text{and} \quad A'_2 = \begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

A straightforward computation shows that the space of matrices such that $xA_i = A'_i x$ for $i = 1, 2$ is 1-dimensional, spanned by the matrix

$$B = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{pmatrix}.$$

As mentioned above, the first vector in our basis of X_f^+ is σ_f^+ . So B acts trivially on σ_f^+ , and the desired extension of ρ to a representation of $\mathrm{GL}_2(\mathbf{Z}/5\mathbf{Z})$ is given by $\rho\left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right) = B$. Since f has trivial character, ρ factors through $\mathrm{PGL}_2(\mathbf{Z}/5\mathbf{Z})$, and we see that the character of ρ is given by:

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$
4	0	0	-1	-2	1	1

In particular, using the trace relation of Theorem 3.10 we see that ρ corresponds to the admissible pair $(\mathbf{Q}_{25}, \theta)$, where \mathbf{Q}_{25} denotes the unramified quadratic extension of \mathbf{Q}_5 and θ is either of the two characters of \mathbf{Q}_{25}^\times trivial on \mathbf{Q}_5^\times and having order 3. (Since these two characters are interchanged by the nontrivial element of $\mathrm{Gal}(\mathbf{Q}_{25}/\mathbf{Q}_5)$, they correspond to equivalent admissible pairs.)

Example 2. Let λ be any root of $x^4 + 9 = 0$. Then $\lambda^2/3$ is a square root of -1 ; we denote this by i . Then there is an eigenform $f \in S_3(\Gamma_1(25))$ with q -expansion

$$f = q + \lambda q^2 + i\lambda q^3 - iq^4 - 3q^6 - 4\lambda q^7 + \dots$$

whose character is the unique character χ modulo 5 mapping 2 to i . Then, since χ generates the group of characters modulo 5, a basis for X_f^+ is given by the four eigensymbols $\sigma_f^+, \sigma_{f \otimes \chi}^-, \sigma_{f \otimes \chi^2}^+$ and $\sigma_{f \otimes \chi^3}^-$. All of the corresponding forms are in fact Galois-conjugate to f over \mathbf{Q} (they are inner twists of f).

One readily calculates that X_f^+ is irreducible as an $\mathrm{SL}_2(\mathbf{Z}/5\mathbf{Z})$ -representation, and the space of matrices intertwining X_f^+ with its conjugate by $\begin{pmatrix} 2 & \\ & 1 \end{pmatrix}$ is 1-dimensional; any such matrix has the four eigensymbols above as its eigenvectors, with eigenvalues $\pm t, \pm it$ for some scalar t . Thus, in particular, there is a unique extension of ρ to a representation of $\mathrm{GL}_2(\mathbf{Z}/5\mathbf{Z})$ with $\rho\left(\begin{pmatrix} 2 & \\ & 1 \end{pmatrix}\right) \sigma_f^+ = \sigma_f^+$. (Note that $g = f \otimes \chi^2$ has the same character as f , and moreover, X_f^+ and X_g^+ are the same space. The extensions to $\mathrm{GL}_2(\mathbf{F}_5)$ corresponding to f and g differ only in the choice of sign of $\rho\left(\begin{pmatrix} 2 & \\ & 1 \end{pmatrix}\right)$, and, in particular, they coincide as representations of the index 2 subgroup of $\mathrm{GL}_2(\mathbf{F}_5)$.)

One checks that \mathbf{F}_{25}^\times is generated by a root α of $x^2 - x + 2$, so the nonsplit torus in $\mathrm{GL}_2(\mathbf{Z}/5\mathbf{Z})$ is generated by $g = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$. If θ, θ^σ are the characters of \mathbf{Q}_{25}^\times corresponding to ρ , then we find that $\mathrm{Tr} \rho(g) = -\lambda$, so $\theta(\alpha) + \theta(\alpha^\sigma) = \lambda$. Since $\theta(\alpha)\theta(\alpha^\sigma) = \theta(\alpha\alpha^\sigma) = \theta(2) = i$, we see that $\theta(\alpha)$ and $\theta(\alpha^\sigma)$ are the roots of $x^2 - \lambda x + i$, which are roots of unity of order 24.

Remark 6.1. If θ is a character of \mathbf{Q}_{25}^\times of level 1 such that $(\mathbf{Q}_{25}, \theta)$ is an admissible pair, then θ is determined up to unramified twists by the image of a root α of $x^2 - x + 2$. This must be a root of unity whose order divides 24, and if $\theta \neq \theta^\sigma$, it cannot be a factor of 4. Hence it is one of $\{3, 6, 8, 12, 24\}$. Conversely, the order of $\theta(\alpha)$ determines θ uniquely up to Galois conjugacy and unramified twists. Table 2 gives examples of forms corresponding to each of these equivalence classes.

TABLE 2. Examples of forms realizing each equivalence class of supercuspidal representations of $\mathrm{GL}_2(\mathbf{Q}_5)$ of conductor 5^2 , modulo unramified twists and Galois conjugacy

Order of $\theta(\alpha)$	Weight	Level	q -expansion
3	2	50	$q - q^2 + q^3 + q^4 + \dots$
6	4	25	$q + q^2 + 7q^3 - 7q^4 + \dots$
8	3	50	$q + (i-1)q^2 + (3i+3)q^3 - 2iq^4 + \dots$
12	4	25	$q + iq^2 - 7iq^3 + 7q^4 + \dots$
24	3	25	$q + \lambda q^2 + i\lambda q^3 - iq^4 + \dots$ (where $\lambda^2 = 3i$)

Example 3. For the unique Galois orbit of newforms of weight 2, trivial character and level 81, of which a representative is $q + \sqrt{3}q^2 + q^4 - \sqrt{3}q^5 + \dots$, the space X_f^+ is a 6-dimensional representation of $\mathrm{PGL}_2(\mathbf{Z}/9\mathbf{Z})$, with $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ acting trivially.

One checks that the group

$$\frac{(\mathbf{Z}_9/9\mathbf{Z}_9)^\times}{(\mathbf{Z}_3/9\mathbf{Z}_3)^\times}$$

is cyclic of order 12 and is generated by a root α of $x^2 - x + 2$. This corresponds to the generator $g = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$ of the nonsplit torus in $\mathrm{PGL}_2(\mathbf{Z}/9\mathbf{Z})$. We find that $\rho(g)$ has trace $-\sqrt{3}$, so by Theorem 3.10, ρ corresponds to the pair of characters of \mathbf{Q}_9^\times mapping α to the roots of the polynomial $x^2 + \sqrt{3}x + 1$.

Example 4. Let f be the newform of weight 2, trivial character and level 27. Then X_f^+ is a 2-dimensional representation of the normalizer of the Iwahori subgroup of $\mathrm{GL}_2(\mathbf{Z}_3)$, with trivial central character. One checks that we have $\mathrm{Tr} \rho(g) = 0$ whenever $\det g$ does not lie in the index 2 subgroup of $\mathbf{Q}_3^\times / \mathbf{Q}_3^{\times 2}$ generated by 3. Hence $\rho \cong \rho \otimes \chi$, where χ is the character of \mathbf{Q}_3^\times corresponding via class field theory to the extension $E = \mathbf{Q}_3(\sqrt{-3})$, and we deduce that ρ corresponds to the admissible pair (E, θ) for some character θ of E^\times of level 2, trivial on \mathbf{Q}_3^\times . Using Theorem 3.10 one checks that $\theta(\sqrt{-3}) = -1$ and $\theta(1 + \sqrt{-3})$ is a root of unity of order 3. (In fact, f has CM by the field $\mathbf{Q}(\sqrt{-3})$, so one can obtain the same result by global methods.)

Similarly, one checks that both of the forms of weight 2, trivial character and level 54 correspond to characters of $\mathbf{Q}_3(\sqrt{3})$.

Example 5. In our final example, we show how our algorithm can give a complete description of the 2-adic Weil-Deligne representation $\sigma_{f,2} = \sigma(\pi_{f,2}): W_{\mathbf{Q}_2} \rightarrow \mathrm{GL}_2(\mathbf{C})$ attached to a cusp form f when the local component $\pi_{f,2}$ is supercuspidal but does not arise from an admissible pair. This means exactly that the image of $\sigma_{f,2}(W_{\mathbf{Q}_2})$ in $\mathrm{PGL}_2(\mathbf{C})$ is not a dihedral group: it must instead be isomorphic to A_4 or S_4 . Weil showed in [Wei74] that the A_4 (tetrahedral) case cannot occur, and that the S_4 (octahedral) cases all arise from the 3-torsion in a handful of elliptic curves defined over \mathbf{Q}_2 .

Our goal here is to calculate $\sigma_{f,2}$ for those representations of octahedral type. Of course, $\sigma_{f,2}$ is determined by $\pi_{f,2}$ in light of Carayol's theorem. In practice, however, it seems very difficult to pass from $\pi_{f,2}$ to $\sigma_{f,2}$ in an explicit manner. It is much easier to find a form g arising from one of Weil's elliptic curves for which $\pi_{f,2}$ and $\pi_{g,2}$ differ by an (explicit) character χ of \mathbf{Q}_2^\times . Since g arises from an elliptic curve, $\sigma_{g,2}$ may be computed directly. Then $\sigma_{f,2}$ can be calculated because it differs from $\sigma_{g,2}$ by the same character χ .

For example, let $f = q - 4q^3 - 2q^5 + 24q^7 + \dots$ be the unique newform of level 8, weight 4 and trivial character. Our algorithm shows that the type space X_f^+ is the 1-dimensional space spanned by σ_f^+ , and this admits an action of the ramified maximal compact-mod-center subgroup $K \subset \mathrm{GL}_2(\mathbf{Q}_p)$ through a character ρ_f of K . The restriction of ρ_f to the Iwahori subgroup is given by $\rho_f \left(\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \right) = (-1)^{b+c}$. As

for the uniformizer $\Pi = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$, we have $\rho_f(\Pi) = 1$. Since ρ_f is 1-dimensional, there are no nontrivial characters χ such that $\rho \cong \rho \otimes \chi$; thus $\pi_{f,2}$ is an “exceptional” representation of $\mathrm{GL}_2(\mathbf{Q}_2)$ in the sense of [BH06, §44.1]. This means that the irreducible Weil-Deligne representation $\sigma_{f,2}: W_{\mathbf{Q}_2} \rightarrow \mathrm{GL}_2(\mathbf{C})$ has projective image S_4 .

Now let $g = q - q^3 - 2q^5 + \dots$ be the newform corresponding to the unique isogeny class of elliptic curves over \mathbf{Q} of conductor 24. One such elliptic curve has Weierstrass equation $y^2 = x^3 - x^2 - 4x + 4$. We find that ρ_g is also one-dimensional;

the character ρ_g differs from ρ_f only in that $\rho_g(\Pi) = -1$. Thus $\pi_{f,2} = \pi_{g,2} \otimes \chi$, where χ is the unramified quadratic character of \mathbf{Q}_2^\times . This reduces the calculation of $\rho_{f,2}$ to that of $\rho_{g,2}$. Let $K = \mathbf{Q}_2(E[3])$. Then $\text{Gal}(K/\mathbf{Q}_2) \cong \text{GL}_2(\mathbf{F}_3) \cong \tilde{S}_4$ is the nonsplit double cover of S_4 . The \tilde{S}_4 -extensions of \mathbf{Q}_2 are classified in [Wei74], §36. The exponent of 2 in the conductor of E is 3, and this is already enough to determine K completely: By the calculations appearing in §6 of [Rio06], K/\mathbf{Q}_2 is the unique \tilde{S}_4 -extension whose discriminant has 2-adic valuation 64.

Since $\rho_{g,2}$ is an irreducible representation of the Weil-Deligne group, we may write $\sigma_{g,2} = \sigma \otimes \psi$, where ψ is an unramified character and σ is of “Galois type” (meaning it has finite image, or equivalently that it extends to a continuous representation of $\text{Gal}(\overline{\mathbf{Q}_2}/\mathbf{Q}_2)$); see [Tat79], (2.2.1). Here we may take σ to be a representation of $W_{\mathbf{Q}_2}$ which factors through a faithful representation $\text{Gal}(K/\mathbf{Q}_2) \cong \text{GL}_2(\mathbf{F}_3) \hookrightarrow \text{GL}_2(\mathbf{C})$. Then σ is unique up to isomorphism. The character ψ sends a Frobenius element Fr to $\sqrt{-2}$: this is forced by the relation $\det \sigma_{g,2}(\text{Fr}) = 2$. These observations determine $\sigma_{g,2}$ completely; hence they determine $\sigma_{f,2}$ completely as well.

REFERENCES

- [AL78] A. O. L. Atkin and Wen Ch'ing Winnie Li, *Twists of newforms and pseudo-eigenvalues of W -operators*, Invent. Math. **48** (1978), no. 3, 221–243. MR508986 (80a:10040)
- [BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor, *On the modularity of elliptic curves over \mathbf{Q} : Wild 3-adic exercises*, Journal of the Amer. Math. Soc. **14** (2001), 843–939. MR1839918 (2002d:11058)
- [BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24**(3-4) (1997), 235–265. MR1484478
- [BH06] Colin J. Bushnell and Guy Henniart, *The local Langlands conjecture for $\text{GL}(2)$* , Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006. MR2234120 (2007m:22013)
- [BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de $\text{GL}_2(\mathbf{Z}_p)$ et de $\text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ en $\ell = p$* , Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart. MR1944572 (2004i:11052)
- [Bum97] Daniel Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997. MR1431508 (97k:11080)
- [Car83] Henri Carayol, *Sur les représentations ℓ -adiques attachées aux formes modulaires de Hilbert*, C. R. Acad. Sci. Paris. **296** (1983), no. 15, 629–632. MR705677 (85e:11039)
- [Car86] ———, *Sur les représentations ℓ -adiques associées aux formes modulaires de Hilbert*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 3, 409–468. MR870690 (89c:11083)
- [Cas73a] William Casselman, *On some results of Atkin and Lehner*, Math. Ann. **201** (1973), 301–314. MR0337789 (49:2558)
- [Cas73b] ———, *The restriction of a representation of $\text{GL}_2(k)$ to $\text{GL}_2(\mathfrak{o})$* , Math. Ann. **206** (1973), 311–318. MR0338274 (49:3040)
- [Gel75] Stephen S. Gelbart, *Automorphic forms on adèle groups*, Princeton University Press, Princeton, N.J., 1975, Annals of Mathematics Studies, No. 83. MR0379375 (52:280)
- [Rio06] Anna Rio, *Dyadic exercises for octahedral extensions. II*, J. Number Theory **118** (2006), no. 2, 172–188. MR2223979 (2007c:11132)
- [RT83] J. D. Rogawski and J. B. Tunnell, *On Artin L -functions associated to Hilbert modular forms of weight one*, Invent. Math. **74** (1983), no. 1, 1–42. MR722724 (85i:11044)
- [Sag] *Sage mathematics software, version 4.4.2*, <http://www.sagemath.org/>.
- [Ste07] William Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, vol. 79, American Mathematical Society, Providence, RI, 2007, With an appendix by Paul E. Gunnells. MR2289048 (2008d:11037)

- [Tat79] John Tate, *Number theoretic background*, Automorphic forms, representations and L -functions (Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 3–26. MR546607 (80m:12009)
- [Wei74] André Weil, *Exercices dyadiques*, Invent. Math. **27** (1974), 1–22. MR0379445 (52:350)
- [Wei09] Jared Weinstein, *Hilbert modular forms with prescribed ramification*, Int. Math. Res. Not. (2009), no. 8, 1388–1420. MR2496768 (2010f:11070)

WARWICK MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL, UNITED KINGDOM

E-mail address: `d.a.loeffler@warwick.ac.uk`

DEPARTMENT OF MATHEMATICS, UCLA, LOS ANGELES, CALIFORNIA 90095-1555

E-mail address: `jared@math.ucla.edu`