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THE SMOOTHING EFFECT OF INTEGRATION IN $\mathbb{R}^d$ AND THE ANOVA DECOMPOSITION

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ABSTRACT. This paper studies the ANOVA decomposition of a d-variate function f defined on the whole of $\mathbb{R}^d$, where f is the maximum of a smooth function and zero (or f could be the absolute value of a smooth function). Our study is motivated by option pricing problems. We show that under suitable conditions all terms of the ANOVA decomposition, except the one of highest order, can have unlimited smoothness. In particular, this is the case for arithmetic Asian options with both the standard and Brownian bridge constructions of the Brownian motion.

1. INTRODUCTION

In this paper we study the ANOVA decomposition of d-variate real-valued functions f defined on the whole of $\mathbb{R}^d$, where f fails to be smooth because it is the maximum of a smooth function and zero. That is, we consider

$$f(\boldsymbol{x}) = \phi(\boldsymbol{x})_+ := \max(\phi(\boldsymbol{x}), 0), \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{1.1}$$

with ϕ a smooth function on $\mathbb{R}^d$. The conclusions will apply equally to the absolute value of ϕ, since

$$|\phi(\boldsymbol{x})| = \phi(\boldsymbol{x})_+ + (-\phi(\boldsymbol{x}))_+.$$

Our study is motivated by option pricing problems, which take the form of (1.1) because a financial option is considered to be worthless once its value drops below a specified 'strike price'.

In a previous paper [8] we considered the smoothness of the terms of the ANOVA decomposition when a d-variate function such as (1.1) is mapped to the unit cube in a suitable way. There we found, under suitable conditions, that the low-order terms of the ANOVA decomposition can be reasonably smooth, even though f itself has a 'kink' arising from the max function in (1.1). Essentially, this occurs because the process of integrating out the 'other' variables has a smoothing effect. The smoothness matters if quasi-Monte Carlo [13, 14] or sparse grid [4] methods are used to estimate the expected values of financial options expressed as high dimensional integrals, because the convergence theory for both of these methods assumes that the integrands have (at least) square integrable mixed first derivatives [7, 10], a property that is manifestly not true for the 'kink' function. But a rigorous error analysis becomes thinkable if, on the one hand, the higher order ANOVA terms are small (as is often speculated to be the case—this is the notion of 'low superposition dimension' introduced by [6]), and on the other hand, if the low-order ANOVA terms all have the required smoothness property.

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In the present paper we avoid the mapping to the unit cube, and instead treat the problem as one posed on the whole of $\mathbb{R}^d$. In this case the results turn out to be more surprising, in that the effect of integrating out a single variable can be unlimited smoothness with respect to the other variables, in contrast to an increase in smoothness of just one degree in the case of the unit cube. These results are expected to lay the foundation for a future rigorous error analysis of direct numerical methods for option pricing integrals over $\mathbb{R}^d$, methods that do not involve mapping $\mathbb{R}^d$ to the unit cube.

The structure of the paper is as follows. In Section 2 we establish the mathematical background, including the definition of the ANOVA decomposition, and define the notation. In Section 3 we demonstrate the smoothing effect produced by integrating out a single variable. In Section 4 we apply the results to the problem of pricing Asian options, with the striking result that, in the case of both the standard and Brownian bridge constructions, *every term of the ANOVA decomposition except for the very highest one has unlimited smoothness.* Numerical examples in Section 5 complete the paper.

2. Background

Let ρ be a continuous and strictly positive univariate probability density function, i.e., $\rho(t) > 0$ for all $t \in \mathbb{R}$ and $\int_{-\infty}^{\infty} \rho(t)\,\mathrm{d}t = 1$. From this we construct a d-variate probability density

$$\rho_d(\boldsymbol{x}) := \prod_{j=1}^{d} \rho(x_j) \qquad \text{for} \quad \boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

For $p \in [1, \infty]$, we consider the *weighted* $\mathcal{L}_p$ space defined over $\mathbb{R}^d$, denoted by $\mathcal{L}_{p,\rho_d}(\mathbb{R}^d)$, with the weighted norm

$$\|f\|_{\mathcal{L}_{p,\rho_d}} = \begin{cases} \left(\int_{\mathbb{R}^d} |f(\boldsymbol{x})|^p \, \rho_d(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{1/p} & \text{if } p \in [1, \infty), \\ \operatorname{ess\,sup}_{\boldsymbol{x} \in \mathbb{R}^d} |f(\boldsymbol{x})| & \text{if } p = \infty. \end{cases} \tag{2.1}$$

It can be verified using Hölder's inequality that $\|f\|_{\mathcal{L}_{p,\rho_d}} \le \|f\|_{\mathcal{L}_{p',\rho_d}}$ for $p \le p'$, and hence

$$\mathcal{L}_{p',\rho_d}(\mathbb{R}^d) \subseteq \mathcal{L}_{p,\rho_d}(\mathbb{R}^d) \subseteq \mathcal{L}_{1,\rho_d}(\mathbb{R}^d) \qquad \text{for} \quad 1 \le p \le p' \le \infty. \tag{2.2}$$

If a function f defined on $\mathbb{R}^d$ is integrable with respect to ρ_d, i.e., if $f \in \mathcal{L}_{1,\rho_d}$, we write

$$I_d f := \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \rho_d(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Then

$$|I_d f| \le \int_{\mathbb{R}^d} |f(\boldsymbol{x})|\, \rho_d(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \|f\|_{\mathcal{L}_{1,\rho_d}}.$$

Throughout this paper we assume that the dimension d is fixed, and we write

$$\mathfrak{D} := \{1, 2, \ldots, d\}.$$

2.1. **Univariate integration and the ANOVA decomposition.** For $j \in \mathfrak{D}$ and $f \in \mathcal{L}_{1,\rho_d}(\mathbb{R}^d)$, let P_j be the projection defined by

$$(P_j f)(\boldsymbol{x}) \;=\; \int_{-\infty}^{\infty} f(x_1, \ldots, x_{j-1}, t_j, x_{j+1}, \ldots, x_d)\, \rho(t_j)\, \mathrm{d}t_j$$

for $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Thus $P_j f$ is the function obtained by integrating out the jth component of $\boldsymbol{x}$ with respect to the weight function ρ, and so is a function that is constant with respect to x_j. For convenience we often say that $P_j f$ does not depend on this component x_j, and we write interchangeably

$$(P_j f)(\boldsymbol{x}) \;=\; (P_j f)(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}),$$

where $\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}$ denotes the $d-1$ components of $\boldsymbol{x}$ apart from x_j, and we express the corresponding $(d-1)$-dimensional Euclidean space by $\mathbb{R}^{\mathfrak{D}\setminus\{j\}}$. By Fubini's theorem [5, Section 5.4], $P_j f$ exists for *almost all* $\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}$ and belongs to $\mathcal{L}_{1,\rho_{\mathfrak{D}\setminus\{j\}}}(\mathbb{R}^{\mathfrak{D}\setminus\{j\}})$.

For $\mathbf{u} \subseteq \mathfrak{D}$ we write

$$P_{\mathbf{u}} \;=\; \prod_{j\in\mathbf{u}} P_j.$$

Here the ordering within the product is not important because, by Fubini's theorem, $P_j P_k = P_k P_j$ for all $j, k \in \mathfrak{D}$. Thus $P_{\mathbf{u}} f$ is the function obtained by integrating out all the components of $\boldsymbol{x}$ with indices in $\mathbf{u}$. Note that $P_{\mathbf{u}}^2 = P_{\mathbf{u}}$ and $P_{\mathfrak{D}} = I_d$.

The ANOVA decomposition of f (see, e.g., [6, 12]) is

$$f \;=\; \sum_{\mathbf{u}\subseteq\mathfrak{D}} f_{\mathbf{u}},$$

with $f_{\mathbf{u}}$ depending only on the variables x_j with indices $j \in \mathbf{u}$, and with $f_{\mathbf{u}}$ satisfying $P_j f_{\mathbf{u}} = 0$ for all $j \in \mathbf{u}$. The functions $f_{\mathbf{u}}$ satisfy the recurrence relation

$$f_{\emptyset} \;=\; I_d f \qquad \text{and} \qquad f_{\mathbf{u}} \;=\; P_{\mathfrak{D}\setminus\mathbf{u}} f - \sum_{\mathbf{v}\subsetneq\mathbf{u}} f_{\mathbf{v}}.$$

Often this recurrence relation is used as the defining property of the ANOVA terms $f_{\mathbf{u}}$. It is known, for example from the recent paper [11], that the ANOVA terms $f_{\mathbf{u}}$ are given explicitly by

$$f_{\mathbf{u}} \;=\; \sum_{\mathbf{v}\subseteq\mathbf{u}} (-1)^{|\mathbf{u}|-|\mathbf{v}|} P_{\mathfrak{D}\setminus\mathbf{v}} f \;=\; P_{\mathfrak{D}\setminus\mathbf{u}} f \;+\; \sum_{\mathbf{v}\subsetneq\mathbf{u}} (-1)^{|\mathbf{u}|-|\mathbf{v}|} P_{\mathbf{u}\setminus\mathbf{v}} (P_{\mathfrak{D}\setminus\mathbf{u}} f). \tag{2.3}$$

In the latter form it becomes plausible that *the smoothness of $f_{\mathbf{u}}$ is determined by $P_{\mathfrak{D}\setminus\mathbf{u}} f$*, since we do not expect the further integrations $P_{\mathbf{u}\setminus\mathbf{v}}$ in the terms of the second sum to reduce the smoothness of $P_{\mathfrak{D}\setminus\mathbf{u}} f$; this expectation is proved in Theorem 2.2 below.

2.2. **Sobolev spaces and weak derivatives.** For $j \in \mathfrak{D}$, let D_j denote the partial derivative operator

$$(D_j f)(\boldsymbol{x}) \;=\; \frac{\partial f}{\partial x_j}(\boldsymbol{x}).$$

Throughout this paper, the term *multi-index* refers to a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ whose components are nonnegative integers, and we use the notation $|\boldsymbol{\alpha}| = \alpha_1 +$

$\cdots + \alpha_d$ to denote the sum of its components. For any multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, we define

$$D^{\boldsymbol{\alpha}} = \prod_{j=1}^{d} D_j^{\alpha_j} = \prod_{j=1}^{d} \left(\frac{\partial}{\partial x_j}\right)^{\alpha_j} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\prod_{j=1}^{d} \partial x_j^{\alpha_j}}, \tag{2.4}$$

and we say that the derivative $D^{\boldsymbol{\alpha}} f$ is of order $|\boldsymbol{\alpha}|$.

Let $\mathcal{C}(\mathbb{R}^d) = \mathcal{C}^0(\mathbb{R}^d)$ denote the linear space of continuous functions defined on $\mathbb{R}^d$. For a nonnegative integer $r \geq 0$, we define $\mathcal{C}^r(\mathbb{R}^d)$ to be the space of functions whose classical derivatives of order $\leq r$ are all continuous at every point in $\mathbb{R}^d$, with no limitation on their behaviour at infinity. For example, the function $f(\boldsymbol{x}) = \exp(\sum_{j=1}^{d} x_j^2)$ belongs to $\mathcal{C}^r(\mathbb{R}^d)$ for all values of r. For convenience we write $\mathcal{C}^\infty(\mathbb{R}^d) = \bigcap_{r \geq 0} \mathcal{C}^r(\mathbb{R}^d)$.

In addition to classical derivatives, we shall consider also *weak* derivatives in this paper. By definition, the weak derivative $D^{\boldsymbol{\alpha}} f$ is a measurable function on $\mathbb{R}^d$ which satisfies

$$\int_{\mathbb{R}^d} (D^{\boldsymbol{\alpha}} f)(\boldsymbol{x})\, v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = (-1)^{|\boldsymbol{\alpha}|} \int_{\mathbb{R}^d} f(\boldsymbol{x})\, (D^{\boldsymbol{\alpha}} v)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \quad \text{for all } v \in \mathcal{C}_0^\infty(\mathbb{R}^d), \tag{2.5}$$

where $\mathcal{C}_0^\infty(\mathbb{R}^d)$ denotes the space of infinitely differentiable functions with compact support in $\mathbb{R}^d$, and where the derivatives on the right-hand side of (2.5) are classical partial derivatives. It follows that $D_j D_k = D_k D_j$ for all $j, k \in \mathfrak{D}$, that is, the ordering of the weak first derivatives that make up $D^{\boldsymbol{\alpha}}$ in (2.4) is irrelevant.

If f has classical continuous derivatives everywhere, then they satisfy (2.5), which in the classical sense is just the integration by parts formula. In principle all derivatives in this paper may be considered as weak derivatives, since classical derivatives are also weak derivatives.

For $p \in [1, \infty]$, we consider two kinds of Sobolev space: the *isotropic Sobolev space* with smoothness parameter $r \geq 0$, for r a nonnegative integer,

$$\mathcal{W}_{d,p,\rho_d}^r = \left\{ f \,:\, D^{\boldsymbol{\alpha}} f \in \mathcal{L}_{p,\rho_d}(\mathbb{R}^d) \quad \text{for all} \quad |\boldsymbol{\alpha}| \leq r \right\},$$

and the *mixed Sobolev space* with smoothness multi-index $\boldsymbol{r} = (r_1, \ldots, r_d)$,

$$\mathcal{W}_{d,p,\rho_d,\mathrm{mix}}^{\boldsymbol{r}} = \left\{ f \,:\, D^{\boldsymbol{\alpha}} f \in \mathcal{L}_{p,\rho_d}(\mathbb{R}^d) \quad \text{for all} \quad \boldsymbol{\alpha} \leq \boldsymbol{r} \right\},$$

where $\boldsymbol{\alpha} \leq \boldsymbol{r}$ is to be understood componentwise, and the derivatives are weak derivatives. See, e.g., [2, 3] for more details about Sobolev space weak derivatives. For convenience we also write $\mathcal{W}_{d,p,\rho_d}^0 = \mathcal{L}_{p,\rho_d}(\mathbb{R}^d)$ and $\mathcal{W}_{d,p,\rho_d}^\infty = \bigcap_{r \geq 0} \mathcal{W}_{d,p,\rho_d}^r$.

The norms corresponding to the two kinds of Sobolev space can be defined, for example, by

$$\|f\|_{\mathcal{W}_{d,p,\rho_d}^r} = \left(\sum_{|\boldsymbol{\alpha}| \leq r} \|D^{\boldsymbol{\alpha}} f\|_{\mathcal{L}_{p,\rho_d}}^2 \right)^{1/2} \quad \text{and} \quad \|f\|_{\mathcal{W}_{d,p,\rho_d,\mathrm{mix}}^{\boldsymbol{r}}} = \left(\sum_{\boldsymbol{\alpha} \leq \boldsymbol{r}} \|D^{\boldsymbol{\alpha}} f\|_{\mathcal{L}_{p,\rho_d}}^2 \right)^{1/2},$$

where $\|\cdot\|_{\mathcal{L}_{p,\rho_d}}$ denotes the weighted $\mathcal{L}_p$ norm (2.1). Due to (2.2), we have

$$\mathcal{W}_{d,p',\rho_d}^r \subseteq \mathcal{W}_{d,p,\rho_d}^r \quad \text{and} \quad \mathcal{W}_{d,p',\rho_d,\mathrm{mix}}^{\boldsymbol{r}} \subseteq \mathcal{W}_{d,p,\rho_d,\mathrm{mix}}^{\boldsymbol{r}} \quad \text{for} \quad 1 \leq p \leq p' \leq \infty.$$

Additionally, it is easily seen that the isotropic and the mixed Sobolev spaces are related by

$$\mathcal{W}_{d,p,\rho_d,\mathrm{mix}}^{\boldsymbol{r}} \subseteq \mathcal{W}_{d,p,\rho_d}^r \text{ iff } \min_{j \in \mathfrak{D}} r_j \geq r \quad \text{and} \quad \mathcal{W}_{d,p,\rho_d}^r \subseteq \mathcal{W}_{d,p,\rho_d,\mathrm{mix}}^{\boldsymbol{r}} \text{ iff } r \geq |\boldsymbol{r}|. \tag{2.6}$$

In particular, we have

$$\mathcal{W}^{(s,\ldots,s)}_{d,p,\rho_d,\mathrm{mix}} \subseteq \mathcal{W}^r_{d,p,\rho_d} \text{ iff } s \geq r \quad \text{and} \quad \mathcal{W}^r_{d,p,\rho_d} \subseteq \mathcal{W}^{(s,\ldots,s)}_{d,p,\rho_d,\mathrm{mix}} \text{ iff } r \geq s\,d.$$

We stress that there is no containment relation between $\mathcal{W}^r_{d,p,\rho_d}$ and $\mathcal{C}^r(\mathbb{R}^d)$. A smooth function from $\mathcal{C}^r(\mathbb{R}^d)$ may grow arbitrarily fast at infinity, and so may not have a finite weighted $\mathcal{L}_p$ norm with respect to the weight function ρ_d. On the other hand, for a function from $\mathcal{W}^r_{d,p,\rho_d}$, its derivatives of order even less than r might not be continuous. Some information on this question is given by the Sobolev embedding theorem: for a Sobolev space defined on a bounded open domain Ω (without a weight function ρ_d), the Sobolev embedding theorem tells us that $\mathcal{W}^r_{d,p}(\Omega) \subseteq \mathcal{C}(\overline{\Omega})$ if $r > d/p$, where $\overline{\Omega}$ denotes the closure of Ω. It follows that for a function $f \in \mathcal{W}^r_{d,p,\rho_d}$, its restriction to Ω belongs to $\mathcal{C}(\overline{\Omega})$ for $r > d/p$ (the continuous weight function ρ_d being irrelevant on a bounded domain Ω), and since this applies for arbitrary Ω, we see that $\mathcal{W}^r_{d,p,\rho_d} \subseteq \mathcal{C}(\mathbb{R}^d)$ if $r > d/p$. A similar argument applied to the derivatives of f yields

$$\mathcal{W}^r_{d,p,\rho_d} \subseteq \mathcal{C}^k(\mathbb{R}^d) \qquad \text{if} \qquad r > k + \frac{d}{p}. \tag{2.7}$$

2.3. **Notations.** We now introduce a number of notations used throughout the paper.

For a given index $j \in \mathfrak{D}$, we sometimes need to distinguish the jth component of a given vector $\boldsymbol{x} \in \mathbb{R}^d$. We achieve this by writing

$$\boldsymbol{x} = (x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}),$$

where as noted previously $\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}$ denotes the $d-1$ components of $\boldsymbol{x}$ apart from x_j. More generally, for a given set $\mathbf{u} \subseteq \mathfrak{D}$ we write

$$\boldsymbol{x}_{\mathbf{u}} = (x_j)_{j\in\mathbf{u}}$$

to denote the set of components x_j of $\boldsymbol{x}$ for which $j \in \mathbf{u}$. The cardinality of a set $\mathbf{u}$ is denoted by $|\mathbf{u}|$.

For $\mathbf{u} \subseteq \mathfrak{D}$, we write $\boldsymbol{r}_{\mathbf{u}} = (r_j)_{j\in\mathbf{u}}$ and $\rho_{\mathbf{u}}(\boldsymbol{x}_{\mathbf{u}}) = \prod_{j\in\mathbf{u}} \rho(x_j)$, and we define

$$\mathcal{W}^r_{\mathbf{u},p,\rho_{\mathbf{u}}} \qquad \text{and} \qquad \mathcal{W}^{\boldsymbol{r}_{\mathbf{u}}}_{\mathbf{u},p,\rho_{\mathbf{u}},\mathrm{mix}}$$

to be the *subspaces* of $\mathcal{W}^r_{d,p,\rho_d}$ and $\mathcal{W}^{\boldsymbol{r}}_{d,p,\rho_d,\mathrm{mix}}$, respectively, which contain those functions that are constant with respect to the components whose indices are outside of $\mathbf{u}$ (that is, functions that depend only on the variables $\boldsymbol{x}_{\mathbf{u}}$). To help to identify the relevant variables, we say that $\boldsymbol{x}_{\mathbf{u}}$ belongs to $\mathbb{R}^{\mathbf{u}}$. With this new notation, we have $\mathcal{W}^r_{\mathfrak{D},p,\rho_{\mathfrak{D}}} = \mathcal{W}^r_{d,p,\rho_d}$ and $\mathcal{W}^{\boldsymbol{r}}_{\mathfrak{D},p,\rho_{\mathfrak{D}},\mathrm{mix}} = \mathcal{W}^{\boldsymbol{r}}_{d,p,\rho_d,\mathrm{mix}}$.

2.4. **Useful theorems.** The classical Leibniz theorem allows us to swap the order of differentiation and integration. In this paper we need a more general form of the Leibniz theorem as given below.

Theorem 2.1 (The Leibniz Theorem)**.** *Let $p \in [1,\infty)$. For $f \in \mathcal{W}^1_{d,p,\rho_d}$ we have*

$$D_k P_j f = P_j D_k f \qquad \textit{for all} \quad j,k \in \mathfrak{D} \quad \textit{with} \quad j \neq k.$$

Proof. We wish to prove that $P_j D_k f$ is the weak derivative of $P_j f$ with respect to x_k. From the definition (2.5) it follows that we need to prove

$$-\int_{\mathbb{R}^d} (P_j f)(\boldsymbol{x})\,(D_k v)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} (P_j D_k f)(\boldsymbol{x})\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{for all } v \in \mathcal{C}_0^\infty(\mathbb{R}^d). \tag{2.8}$$

For arbitrary $v \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, we begin from the left-hand side of (2.8):

$$
\begin{aligned}
-\int_{\mathbb{R}^d} (P_j f)(\boldsymbol{x})(D_k v)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= -\int_{\mathbb{R}^d} \left(\int_{-\infty}^{\infty} f(t_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})\,\rho(t_j)\,\mathrm{d}t_j\right)(D_k v)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&= \int_{-\infty}^{\infty} \left(-\int_{\mathbb{R}^d} f(t_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})\,(D_k v)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)\rho(t_j)\,\mathrm{d}t_j, \qquad (2.9)
\end{aligned}
$$

where in the last step we used Fubini's theorem to interchange the order of integration. Fubini's theorem is applicable because the last integral is finite for $v \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, as follows from

$$
\begin{aligned}
&\left|\int_{-\infty}^{\infty}\int_{\mathbb{R}^d} f(t_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})\,(D_k v)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\rho(t_j)\,\mathrm{d}t_j\right| \\
&\le \int_{-\infty}^{\infty}\int_{\mathbb{R}^{\mathfrak{D}\setminus\{j\}}}\int_{-\infty}^{\infty} |f(t_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})|\,\rho(t_j)\,\rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \\
&\qquad\qquad \cdot \frac{|(D_k v)(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})|}{\rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})}\,\mathrm{d}x_j\,\mathrm{d}\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}\,\mathrm{d}t_j \\
&\le \int_{-\infty}^{\infty}\int_{\mathbb{R}^{\mathfrak{D}\setminus\{j\}}} |f(t_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})|\,\rho(t_j)\,\rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})\,\mathrm{d}\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}\,\mathrm{d}t_j \\
&\quad \cdot \left(\sup_{\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}\in V}\int_{-\infty}^{\infty} |(D_k v)(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})|\,\mathrm{d}x_j\right)\cdot \frac{1}{\inf_{\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}\in V}\rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})} \\
&< \infty,
\end{aligned}
$$

where $V = \{\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} \in \mathbb{R}^{\mathfrak{D}\setminus\{j\}} : (x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \in \operatorname{supp}(v) \text{ for some } x_j \in \mathbb{R}\}$ is a compact set because of the compactness of $\operatorname{supp}(v)$.

Now we use again the definition of weak derivative (2.5), this time in the inner integral of (2.9), followed again by Fubini's theorem, to obtain from (2.9),

$$
\begin{aligned}
-\int_{\mathbb{R}^d} (P_j f)(\boldsymbol{x})\,(D_k v)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^d} (D_k f)(t_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)\rho(t_j)\,\mathrm{d}t_j \\
&= \int_{\mathbb{R}^d}\left(\int_{-\infty}^{\infty} (D_k f)(t_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})\,\rho(t_j)\,\mathrm{d}t_j\right) v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}^d} (P_j D_k f)(\boldsymbol{x})\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},
\end{aligned}
$$

which is precisely the right-hand side of (2.8) as required. This proves for $j \neq k$ that $D_k P_j f$ exists, and is equal to $P_j D_k f$. □

The next theorem is an application of the Leibniz theorem; it establishes that $P_j f$ inherits the smoothness of f.

Theorem 2.2 (The Inheritance Theorem)**.** *Let $r \ge 0$ and $p \in [1, \infty)$. For $f \in \mathcal{W}^r_{d,p,\rho_d}$ we have*

$$
P_j f \in \mathcal{W}^r_{\mathfrak{D}\setminus\{j\},p,\rho_{\mathfrak{D}\setminus\{j\}}} \qquad \textit{for all} \quad j \in \mathfrak{D}.
$$

Proof. Consider first $r = 0$. For $f \in \mathcal{L}_{p,\rho_d}(\mathbb{R}^d)$ we want to show that $P_j f \in \mathcal{L}_{p,\rho_{\mathfrak{D}\setminus\{j\}}}(\mathbb{R}^{\mathfrak{D}\setminus\{j\}})$. We have

(2.10)

$$
\|P_j f\|_{\mathcal{L}_{p,\rho_{\mathfrak{D}\setminus\{j\}}}} = \left(\int_{\mathbb{R}^{\mathfrak{D}\setminus\{j\}}} \left|\int_{-\infty}^{\infty} f(\boldsymbol{x})\,\rho(x_j)\,\mathrm{d}x_j\right|^p \rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})\,\mathrm{d}\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}\right)^{1/p}.
$$

For q satisfying $1/p+1/q=1$, we estimate the inner integral in (2.10) using Hölder's inequality as follows:

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f(\boldsymbol{x})\,\rho(x_j)\,\mathrm{d}x_j\right|^p &\le \left(\int_{-\infty}^{\infty} |f(\boldsymbol{x})|\,(\rho(x_j))^{1/p}\,(\rho(x_j))^{1/q}\,\mathrm{d}x_j\right)^p \\ &\le \left(\int_{-\infty}^{\infty} |f(\boldsymbol{x})|^p\,\rho(x_j)\,\mathrm{d}x_j\right)\left(\int_{-\infty}^{\infty} \rho(x_j)\,\mathrm{d}x_j\right)^{p/q} \\ &= \int_{-\infty}^{\infty} |f(\boldsymbol{x})|^p\,\rho(x_j)\,\mathrm{d}x_j. \qquad (2.11)\end{aligned}$$

Substituting (2.11) into (2.10), we conclude that $\|P_j f\|_{\mathcal{L}_{p,\rho_{\mathfrak{D}\setminus\{j\}}}} \le \|f\|_{\mathcal{L}_{p,\rho_d}} < \infty$.

Consider now $r \ge 1$. Let $j \in \mathfrak{D}$ and let $\boldsymbol{\alpha}$ be any multi-index with $|\boldsymbol{\alpha}| \le r$ and $\alpha_j = 0$. Since $f \in \mathcal{W}^r_{d,p,\rho_d}$, we have $\|D^{\boldsymbol{\alpha}} f\|_{\mathcal{L}_{p,\rho_d}} < \infty$. To show that $P_j f \in \mathcal{W}^r_{\mathfrak{D}\setminus\{j\},p,\rho_{\mathfrak{D}\setminus\{j\}}}$ we need to show that $\|D^{\boldsymbol{\alpha}} P_j f\|_{\mathcal{L}_{p,\rho_{\mathfrak{D}\setminus\{j\}}}} < \infty$. We first observe that for any function g its weak derivative $D^{\boldsymbol{\alpha}} g$ can be written in the form

$$D^{\boldsymbol{\alpha}} g = \left(\prod_{i=1}^{|\boldsymbol{\alpha}|} D_{k_i}\right) g,$$

where $k_i \in \mathfrak{D}\setminus\{j\}$, and $k_1,\ldots,k_{|\boldsymbol{\alpha}|}$ need not be distinct. We then write successively

$$\begin{aligned}D^{\boldsymbol{\alpha}} P_j f &= \left(\prod_{i=1}^{|\boldsymbol{\alpha}|} D_{k_i}\right) P_j f = \left(\prod_{i=2}^{|\boldsymbol{\alpha}|} D_{k_i}\right) P_j D_{k_1} f \\ &= \cdots = D_{k_{|\boldsymbol{\alpha}|}} P_j \left(\prod_{i=1}^{|\boldsymbol{\alpha}|-1} D_{k_i}\right) f = P_j \left(\prod_{i=1}^{|\boldsymbol{\alpha}|} D_{k_i}\right) f = P_j D^{\boldsymbol{\alpha}} f,\end{aligned}$$

where each step involves a single differentiation under the integral sign, and is justified by the Leibniz theorem (Theorem 2.1) because $(\prod_{i=1}^{\ell} D_{k_i}) f \in \mathcal{W}^{r-\ell}_{d,p,\rho_d} \subseteq \mathcal{W}^1_{d,p,\rho_d}$ for all $\ell \le |\boldsymbol{\alpha}| - 1 \le r-1$. Therefore, we have

$$\begin{aligned}\|D^{\boldsymbol{\alpha}} P_j f\|_{\mathcal{L}_{p,\rho_{\mathfrak{D}\setminus\{j\}}}} &= \|P_j D^{\boldsymbol{\alpha}} f\|_{\mathcal{L}_{p,\rho_{\mathfrak{D}\setminus\{j\}}}} \\ &= \left(\int_{\mathbb{R}^{\mathfrak{D}\setminus\{j\}}} \left|\int_{-\infty}^{\infty} (D^{\boldsymbol{\alpha}} f)(\boldsymbol{x})\,\rho(x_j)\,\mathrm{d}x_j\right|^p \rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})\,\mathrm{d}\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}\right)^{1/p} \\ &\le \left(\int_{\mathbb{R}^d} |(D^{\boldsymbol{\alpha}} f)(\boldsymbol{x})|^p\,\rho_d(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{1/p} = \|D^{\boldsymbol{\alpha}} f\|_{\mathcal{L}_{p,\rho_d}} < \infty,\end{aligned}$$

where we applied Hölder's inequality as in (2.11). This completes the proof. □

The implicit function theorem stated below is crucial for the main results of this paper. In the following, $\overline{S}$ denotes the closure of the set S.

Theorem 2.3 (The Implicit Function Theorem)**.** *Let $j \in \mathfrak{D}$. Suppose $\phi \in \mathcal{C}^1(\mathbb{R}^d)$ satisfies*

$$(D_j\phi)(\boldsymbol{x}) \ne 0 \qquad \textit{for all} \quad \boldsymbol{x} \in \mathbb{R}^d. \qquad (2.12)$$

Let

$$U_j := \{\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} \in \mathbb{R}^{\mathfrak{D}\setminus\{j\}} : \phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) = 0 \textit{ for some (unique) } x_j \in \mathbb{R}\}. \qquad (2.13)$$

If U_j is not empty, then there exists a unique function $\psi_j \in \mathcal{C}^1(\overline{U_j})$ such that

$$\phi(\,\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}), \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \;=\; 0 \qquad \textit{for all} \quad \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} \in \overline{U_j},$$

and for all $k \neq j$ we have

$$(D_k\psi_j)(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \;=\; -\frac{(D_k\phi)(\boldsymbol{x})}{(D_j\phi)(\boldsymbol{x})}\,\bigg|_{x_j\,=\,\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})} \qquad \textit{for all} \quad \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} \in U_j. \tag{2.14}$$

If, in addition, $\phi \in \mathcal{C}^r(\mathbb{R}^d)$ for some $r \geq 2$, then $\psi_j \in \mathcal{C}^r(\overline{U_j})$.

Proof. If $\boldsymbol{x} = (x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \in \mathbb{R}^d$ satisfies $\phi(\boldsymbol{x}) = 0$ and $(D_j\phi)(\boldsymbol{x}) \neq 0$, then [9, Theorem 3.2.1] asserts the existence of an open set $A_{j,\boldsymbol{x}} \subseteq \mathbb{R}^{\mathfrak{D}\setminus\{j\}}$, depending on j and $\boldsymbol{x}$, such that $\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} \in A_{j,\boldsymbol{x}}$, and the existence of a unique continuously differentiable function $\psi_{j,\boldsymbol{x}} : A_{j,\boldsymbol{x}} \to \mathbb{R}$ such that $x_j = \psi_{j,\boldsymbol{x}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})$ and

$$\phi(\,\psi_{j,\boldsymbol{x}}(\boldsymbol{x}'_{\mathfrak{D}\setminus\{j\}}),\, \boldsymbol{x}'_{\mathfrak{D}\setminus\{j\}}\,) \;=\; 0 \qquad \text{for all} \quad \boldsymbol{x}'_{\mathfrak{D}\setminus\{j\}} \in A_{j,\boldsymbol{x}}.$$

If we take two points $\boldsymbol{x}$ and $\boldsymbol{y}$ such that the corresponding sets $A_{j,\boldsymbol{x}}$ and $A_{j,\boldsymbol{y}}$ overlap, then the functions $\psi_{j,\boldsymbol{x}}$ and $\psi_{j,\boldsymbol{y}}$ for each of these two domains must give the same values in the overlap region, by uniqueness. Given that (2.12) holds for all $\boldsymbol{x} \in \mathbb{R}^d$, we therefore have a globally defined, single unique continuously differentiable function $\psi_j : U_j \to \mathbb{R}$ such that for $\boldsymbol{x} \in \mathbb{R}^d$, we get

$$\phi(\boldsymbol{x}) \;=\; 0 \qquad \text{if and only if} \qquad x_j \;=\; \psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}).$$

Implicit differentiation of

$$\phi(\,\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}),\, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}\,) \;=\; 0, \qquad \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} \in U_j,$$

with respect to x_k then yields (2.14). For $\phi \in \mathcal{C}^r(\mathbb{R}^d)$, repeated differentiation then shows that $\psi_j \in \mathcal{C}^r(\overline{U_j})$. □

Note that the derivatives in the implicit function theorem are classical derivatives, and the condition (2.12) holds for *all* $\boldsymbol{x} \in \mathbb{R}^d$, as opposed to other results and definitions in this paper which hold for *almost all* $\boldsymbol{x} \in \mathbb{R}^d$. Note also that the condition $(D_j\phi)(\boldsymbol{x}) \neq 0$, when combined with the continuity of $D_j\phi$, means that $D_j\phi$ is either everywhere positive or everywhere negative. In this paper we will only use the implicit function theorem for functions ϕ for which $U_j = \mathbb{R}^{\mathfrak{D}\setminus\{j\}}$.

3. Smoothing for functions with kinks

In this section we consider a function of the form

$$f(\boldsymbol{x}) \;=\; \phi(\boldsymbol{x})_+, \qquad \boldsymbol{x} \in \mathbb{R}^d, \qquad \text{where} \quad \phi \in \mathcal{C}^\infty(\mathbb{R}^d). \tag{3.1}$$

We shall always assume that the equation $\phi(\boldsymbol{x}) = 0$ defines a smooth $(d-1)$-dimensional manifold. From the implicit function theorem (Theorem 2.3) this is the case if, for example, there exists at least one $j \in \mathfrak{D}$ such that $(D_j\phi)(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x} \in \mathbb{R}^d$. The function f is continuous but has a kink along the $(d-1)$-dimensional manifold $\phi(\boldsymbol{x}) = 0$. Clearly, f can be differentiated pointwise once with respect to any one of the d variables except on the manifold $\phi(\boldsymbol{x}) = 0$. Indeed, for $k \in \mathfrak{D}$ we have

$$(D_k f)(\boldsymbol{x}) \;=\; \begin{cases} (D_k\phi)(\boldsymbol{x}) & \text{if } \phi(\boldsymbol{x}) > 0, \\ 0 & \text{if } \phi(\boldsymbol{x}) < 0. \end{cases}$$

It can be easily verified that this is the weak derivative of f by checking the condition (2.5).

Following (3.1), we assume additionally that $\phi \in \mathcal{W}^r_{d,p,\rho_d}$ for some $r \geq 1$ and $p \in [1,\infty)$, that is, we assume that ϕ belongs to the intersection of $\mathcal{W}^r_{d,p,\rho_d}$ and $\mathcal{C}^\infty(\mathbb{R}^d)$. Then for any $k \in \mathfrak{D}$ we have

$$\begin{aligned}\|D_k f\|_{\mathcal{L}_{p,\rho_d}} &= \left(\int_{\mathbb{R}^d} |(D_k f)(\boldsymbol{x})|^p \, \rho_d(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}\right)^{1/p} \\ &= \left(\int_{\boldsymbol{x}\in\mathbb{R}^d \,:\, \phi(\boldsymbol{x})\geq 0} |(D_k \phi)(\boldsymbol{x})|^p \, \rho_d(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}\right)^{1/p} \leq \|D_k \phi\|_{\mathcal{L}_{p,\rho_d}} < \infty.\end{aligned}$$

Thus we conclude that

$$f \in \mathcal{W}^1_{d,p,\rho_d} \cap \mathcal{C}(\mathbb{R}^d). \tag{3.2}$$

It then follows from the inheritance theorem (Theorem 2.2) with $r = 1$ that $P_j f \in \mathcal{W}^1_{\mathfrak{D}\setminus\{j\},p,\rho_{\mathfrak{D}\setminus\{j\}}}$ for all $j \in \mathfrak{D}$.

In the following theorem we show that integration with respect to x_j can have a smoothing effect; we prove that $P_j f \in \mathcal{W}^r_{\mathfrak{D}\setminus\{j\},p,\rho_{\mathfrak{D}\setminus\{j\}}}$ provided that a number of conditions on ϕ are satisfied:

(i) $\phi \in \mathcal{W}^r_{d,p,\rho_d} \cap \mathcal{C}^\infty(\mathbb{R}^d)$.
(ii) $D_j\phi$ never changes sign; see (2.12).
(iii) For each $\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} \in \mathbb{R}^{\mathfrak{D}\setminus\{j\}}$ there exists $x_j \in \mathbb{R}$ such that $\phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) = 0$, that is, the set U_j defined by (2.13) is precisely $U_j = \mathbb{R}^{\mathfrak{D}\setminus\{j\}}$.
(iv) A special condition on the growth of the derivatives of ϕ holds for this value of r; see (3.4) below. (We will say more about (3.4) in the next section.)

Theorem 3.1. *Let $r \geq 1$, $p \in [1,\infty)$, and let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a strictly positive probability density function. Let*

$$f(\boldsymbol{x}) = \phi(\boldsymbol{x})_+, \qquad \text{where} \quad \phi \in \mathcal{W}^r_{d,p,\rho_d} \cap \mathcal{C}^\infty(\mathbb{R}^d).$$

Let $j \in \mathfrak{D}$ and suppose that

$$(D_j\phi)(\boldsymbol{x}) \neq 0 \qquad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^d,$$

and that

$$\text{for each } \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} \in \mathbb{R}^{\mathfrak{D}\setminus\{j\}} \text{ there exists } x_j \in \mathbb{R} \text{ such that } \phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) = 0. \tag{3.3}$$

Then there exists a unique function $\psi_j \in \mathcal{C}^\infty(\mathbb{R}^{\mathfrak{D}\setminus\{j\}})$ such that $\phi(\boldsymbol{x}) = 0$ if and only if $x_j = \psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})$ for all $\boldsymbol{x} \in \mathbb{R}^d$. Assume additionally that

$$\int_{\mathbb{R}^{\mathfrak{D}\setminus\{j\}}} \left| \frac{\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}}\phi)(\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}), \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})]}{[(D_j\phi)(\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}), \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})]^b} \, \rho^{(c)}(\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})) \right|^p \cdot \rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \, \mathrm{d}\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} < \infty \tag{3.4}$$

for all integers $2 \leq a \leq 2r-2$, $1 \leq b \leq 2r-3$, $0 \leq c \leq r-2$, $|\boldsymbol{\alpha}^{(i)}| \leq r-1$, where $\boldsymbol{\alpha}^{(i)}$ are multi-indices with d components. Then

$$P_j f \in \mathcal{W}^r_{\mathfrak{D}\setminus\{j\},p,\rho_{\mathfrak{D}\setminus\{j\}}}.$$

Proof. Given that $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$, $(D_j\phi)(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x} \in \mathbb{R}^d$, and (3.3) holds, it follows from the implicit function theorem (Theorem 2.3) with $U_j = \mathbb{R}^{\mathfrak{D}\setminus\{j\}}$ that there exists a unique function $\psi_j \in \mathcal{C}^\infty(\mathbb{R}^{\mathfrak{D}\setminus\{j\}})$ for which

$$\phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) = 0 \iff \psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) = x_j \quad \text{for all } (x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \in \mathbb{R}^d. \tag{3.5}$$

This justifies the existence of the function ψ_j as stated in the theorem.

For the function $f(\boldsymbol{x}) = \phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})_+$ we can write $P_j f$ as

$$(P_j f)(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) = \int_{x_j \in \mathbb{R}\,:\,\phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \geq 0} \phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})\, \rho(x_j)\, \mathrm{d}x_j. \tag{3.6}$$

Note that the condition $(D_j\phi)(\boldsymbol{x}) \neq 0$, when combined with the continuity of $D_j\phi$, means that $D_j\phi$ is either everywhere positive or everywhere negative. For definiteness we assume that

$$(D_j\phi)(\boldsymbol{x}) > 0 \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^d;$$

the other case is similar. It follows that, for fixed $\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}$, $\phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})$ is a strictly increasing function of x_j. Hence we can write (3.6) as

$$(P_j f)(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) = \int_{\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})}^{\infty} \phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})\, \rho(x_j)\, \mathrm{d}x_j.$$

To simplify our notation, for the remainder of this proof we write

$$\psi \equiv \psi_j \qquad \text{and} \qquad \boldsymbol{y} \equiv \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}.$$

Now we differentiate $P_j f$ with respect to x_k for any $k \neq j$, and we obtain from the chain rule that

$$(D_k P_j f)(\boldsymbol{y}) = \int_{\psi(\boldsymbol{y})}^{\infty} (D_k\phi)(x_j, \boldsymbol{y})\, \rho(x_j)\, \mathrm{d}x_j - \phi(\psi(\boldsymbol{y}), \boldsymbol{y}) \cdot \rho(\psi(\boldsymbol{y})) \cdot (D_k\psi)(\boldsymbol{y}). \tag{3.7}$$

(Note that all of the derivatives on the right-hand side of (3.7) are classical derivatives.) The second term on the right-hand side of (3.7) is zero, since it follows from (3.5) that $\phi(\psi(\boldsymbol{y}), \boldsymbol{y}) = 0$. Differentiating again with respect to x_ℓ for any $\ell \neq j$, allowing the possibility that $\ell = k$, we obtain

$$\begin{aligned} &(D_\ell D_k P_j f)(\boldsymbol{y}) \\ &= \int_{\psi(\boldsymbol{y})}^{\infty} (D_\ell D_k \phi)(x_j, \boldsymbol{y})\, \rho(x_j)\, \mathrm{d}x_j - (D_k\phi)(\psi(\boldsymbol{y}), \boldsymbol{y}) \cdot \rho(\psi(\boldsymbol{y})) \cdot (D_\ell\psi)(\boldsymbol{y}), \end{aligned} \tag{3.8}$$

and we see from (2.14) that $D_\ell\psi$ can be substituted by

$$(D_\ell\psi)(\boldsymbol{y}) = -\frac{(D_\ell\phi)(\psi(\boldsymbol{y}), \boldsymbol{y})}{(D_j\phi)(\psi(\boldsymbol{y}), \boldsymbol{y})}.$$

Note that, unlike the second term in (3.7), the second term in (3.8) does *not* vanish in general.

For every multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ with $|\boldsymbol{\alpha}| \leq r$ and $\alpha_j = 0$, we claim that

$$(D^{\boldsymbol{\alpha}} P_j f)(\boldsymbol{y}) = \int_{\psi(\boldsymbol{y})}^{\infty} (D^{\boldsymbol{\alpha}}\phi)(x_j, \boldsymbol{y})\, \rho(x_j)\, \mathrm{d}x_j + \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} g_{\boldsymbol{\alpha},m}(\boldsymbol{y}), \tag{3.9}$$

where $M_{|\boldsymbol{\alpha}|}$ is a nonnegative integer, and each function $g_{\boldsymbol{\alpha},m}$ is of the form

$$g_{\boldsymbol{\alpha}}(\boldsymbol{y}) := \beta\, \frac{\prod_{i=1}^{a} [(D^{\boldsymbol{\alpha}^{(i)}}\phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]}{[(D_j\phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]^b}\, \rho^{(c)}(\psi(\boldsymbol{y})), \tag{3.10}$$

with a, b, c being nonnegative integers, β being an integer, and each $\boldsymbol{\alpha}^{(i)}$ being a multi-index, satisfying

$$\begin{aligned}(3.11)\quad & 2 \le a \le 2|\boldsymbol{\alpha}| - 2, \quad 1 \le b \le 2|\boldsymbol{\alpha}| - 3, \quad 0 \le c \le |\boldsymbol{\alpha}| - 2, \quad |\boldsymbol{\alpha}^{(i)}| \le |\boldsymbol{\alpha}| - 1, \\ & \text{and} \quad |\beta| \le \text{the product of the first } (|\boldsymbol{\alpha}| - 2) \text{ odd numbers.}\end{aligned}$$

We will prove (3.9)–(3.11) by induction on $|\boldsymbol{\alpha}|$. The case $|\boldsymbol{\alpha}| = 1$ is shown in (3.7); there we have $M_1 = 0$. The case $|\boldsymbol{\alpha}| = 2$ is shown in (3.8); there we have $M_2 = 1$, and the function $g_{\boldsymbol{\alpha},1}$ is of the form (3.10), with $a = 2$, $b = 1$, $c = 0$, $\beta = 1$, $D^{\boldsymbol{\alpha}^{(1)}} = D_k$, $D^{\boldsymbol{\alpha}^{(2)}} = D_\ell$, and $|\boldsymbol{\alpha}^{(1)}| = |\boldsymbol{\alpha}^{(2)}| = 1$.

We now differentiate $D^{\boldsymbol{\alpha}} P_j f$ once more: for $\ell \ne j$ we have from (3.9)

$$\begin{aligned}(D_\ell D^{\boldsymbol{\alpha}} P_j f)(\boldsymbol{y}) \;=\; & \int_{\psi(\boldsymbol{y})}^{\infty} (D_\ell D^{\boldsymbol{\alpha}} \phi)(x_j, \boldsymbol{y})\, \rho(x_j)\, \mathrm{d}x_j \\ & - (D^{\boldsymbol{\alpha}} \phi)(\psi(\boldsymbol{y}), \boldsymbol{y}) \cdot \rho(\psi(\boldsymbol{y})) \cdot (D_\ell \psi)(\boldsymbol{y}) \;+\; \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} (D_\ell g_{\boldsymbol{\alpha},m})(\boldsymbol{y}).\end{aligned} \tag{3.12}$$

Clearly, the first term in (3.12) has the desired form when compared to the first term in (3.9). The second term in (3.12) is of the form (3.10), with $a = 2$, $b = 1$, $c = 0$, $\beta = 1$, $|\boldsymbol{\alpha}^{(1)}| = |\boldsymbol{\alpha}|$, and $|\boldsymbol{\alpha}^{(2)}| = 1$. For the remaining terms in (3.12), we have from (3.10)

$$\begin{aligned}(D_\ell g_{\boldsymbol{\alpha}})(\boldsymbol{y}) \;=\; & \beta\, \frac{D_\ell\big(\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}} \phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]\big)}{[(D_j \phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]^b}\, \rho^{(c)}(\psi(\boldsymbol{y})) \\ & + \beta\, \frac{\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}} \phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]}{[(D_j \phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]^b}\, \rho^{(c+1)}(\psi(\boldsymbol{y})) \cdot (D_\ell \psi)(\boldsymbol{y}) \\ & - \beta b\, \frac{\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}} \phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]}{[(D_j \phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]^{b+1}}\, \rho^{(c)}(\psi(\boldsymbol{y})) \\ & \qquad \cdot \Big[(D_\ell D_j \phi)(\psi(\boldsymbol{y}), \boldsymbol{y}) + (D_j D_j \phi)(\psi(\boldsymbol{y}), \boldsymbol{y}) \cdot (D_\ell \psi)(\boldsymbol{y})\Big],\end{aligned}$$

where

$$\begin{aligned}& D_\ell\Big(\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}} \phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]\Big) \\ & = \sum_{t=1}^a \Bigg(\Big[(D_\ell D^{\boldsymbol{\alpha}^{(t)}} \phi)(\psi(\boldsymbol{y}), \boldsymbol{y}) + (D_j D^{\boldsymbol{\alpha}^{(t)}} \phi)(\psi(\boldsymbol{y}), \boldsymbol{y}) \cdot (D_\ell \psi)(\boldsymbol{y})\Big] \\ & \qquad\qquad \cdot \prod_{\substack{i=1 \\ i \ne t}}^a (D^{\boldsymbol{\alpha}^{(i)}} \phi)(\psi(\boldsymbol{y}), \boldsymbol{y})\Bigg).\end{aligned}$$

Thus we conclude that $D_\ell g_{\boldsymbol{\alpha}}$ is a sum of functions of form (3.10), but with a increased by at most 2, b increased by at most 2, c increased by at most 1, $|\beta|$ multiplied by a factor of at most b, and with each $|\boldsymbol{\alpha}^{(i)}|$ increased by at most 1.

Hence, $D_\ell D^{\boldsymbol{\alpha}} P_j f$ consists of the first term in (3.12), plus a sum of functions of the form (3.10). This completes the induction proof for (3.9)–(3.11). In particular, the bounds in (3.11) can be deduced from the induction step.

We are now ready to consider

$$\|D^{\boldsymbol{\alpha}} P_j f\|_{\mathcal{L}_{p,\rho_{\mathfrak{D}\setminus\{j\}}}} = \left(\int_{\mathbb{R}^{\mathfrak{D}\setminus\{j\}}} |(D^{\boldsymbol{\alpha}} P_j f)(\boldsymbol{y})|^p \, \rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}\right)^{1/p}.$$

Using the special form of $D^{\boldsymbol{\alpha}} P_j f$ in (3.9), we have

$$\begin{aligned}
|(D^{\boldsymbol{\alpha}} P_j f)(\boldsymbol{y})|^p &= \left| \int_{\psi(\boldsymbol{y})}^{\infty} (D^{\boldsymbol{\alpha}}\phi)(x_j, \boldsymbol{y}) \, \rho(x_j) \, \mathrm{d}x_j + \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} g_{\boldsymbol{\alpha},m}(\boldsymbol{y}) \right|^p \\
&\le \left(\left| \int_{\psi(\boldsymbol{y})}^{\infty} (D^{\boldsymbol{\alpha}}\phi)(x_j, \boldsymbol{y}) \, \rho(x_j) \, \mathrm{d}x_j \right| + \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} |g_{\boldsymbol{\alpha},m}(\boldsymbol{y})| \right)^p \\
&\le (M_{|\boldsymbol{\alpha}|}+1)^{p-1} \left(\left| \int_{\psi(\boldsymbol{y})}^{\infty} (D^{\boldsymbol{\alpha}}\phi)(x_j, \boldsymbol{y}) \, \rho(x_j) \, \mathrm{d}x_j \right|^p + \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} |g_{\boldsymbol{\alpha},m}(\boldsymbol{y})|^p \right) \\
&\le (M_{|\boldsymbol{\alpha}|}+1)^{p-1} \left(\int_{\psi(\boldsymbol{y})}^{\infty} |(D^{\boldsymbol{\alpha}}\phi)(x_j, \boldsymbol{y})|^p \, \rho(x_j) \, \mathrm{d}x_j + \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} |g_{\boldsymbol{\alpha},m}(\boldsymbol{y})|^p \right),
\end{aligned}$$

where in the second to last step we used a generalized mean inequality (see [1, 3.2.4])

$$\frac{\sum_{i=1}^n a_i}{n} \le \left(\frac{\sum_{i=1}^n a_i^p}{n} \right)^{1/p}, \qquad a_i \ge 0, \quad p \in [1,\infty),$$

and in the last step we used Hölder's inequality as in (2.11). Thus

$$\begin{aligned}
\|D^{\boldsymbol{\alpha}} P_j f\|_{\mathcal{L}_{p,\rho_{\mathfrak{D}\setminus\{j\}}}} &\le (M_{|\boldsymbol{\alpha}|}+1)^{1-1/p} \left(\|D^{\boldsymbol{\alpha}}\phi\|^p_{\mathcal{L}_{p,\rho_d}} + \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} \|g_{\boldsymbol{\alpha},m}\|^p_{\mathcal{L}_{p,\rho_{\mathfrak{D}\setminus\{j\}}}} \right)^{1/p} \\
&< \infty,
\end{aligned}$$

where $\|D^{\boldsymbol{\alpha}}\phi\|_{\mathcal{L}_{p,\rho_d}} < \infty$ since $\phi \in \mathcal{W}^r_{d,p,\rho_d}$, and each $\|g_{\boldsymbol{\alpha},m}\|_{\mathcal{L}_{p,\rho_{\mathfrak{D}\setminus\{j\}}}} < \infty$ due to the assumption (3.4) (compare with (3.10) and (3.11)). This proves that $P_j f \in \mathcal{W}^r_{\mathfrak{D}\setminus\{j\},p,\rho_{\mathfrak{D}\setminus\{j\}}}$ as claimed. □

In the following theorem, the property $(D_j\phi)(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x} \in [0,1]^d$ and the conditions (3.3) and (3.4) are assumed to hold for all j in a subset $\mathbf{z} \subseteq \mathfrak{D}$.

Theorem 3.2. *Let $r \ge 1$, $p \in [1,\infty)$, and $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a strictly positive probability density function. Let $\mathbf{z}$ be a nonempty subset of $\mathfrak{D}$, and let*

$$(3.13) \quad f(\boldsymbol{x}) = \phi(\boldsymbol{x})_+, \quad \textit{with} \quad \begin{cases} \phi \in \mathcal{W}^r_{d,p,\rho_d} \cap \mathcal{C}^\infty(\mathbb{R}^d), \\ (D_j\phi)(\boldsymbol{x}) \neq 0 \quad \textit{for all } j \in \mathbf{z} \textit{ and all } \boldsymbol{x} \in \mathbb{R}^d, \\ (3.3) \textit{ holds for all } j \in \mathbf{z}, \\ (3.4) \textit{ holds for all } j \in \mathbf{z}. \end{cases}$$

Then $f \in \mathcal{W}^1_{d,p,\rho_d} \cap \mathcal{C}(\mathbb{R}^d)$, and

$$P_{\mathbf{u}} f \in \begin{cases} \mathcal{W}^1_{\mathfrak{D}\setminus\mathbf{u},p,\rho_{\mathfrak{D}\setminus\mathbf{u}}} & \textit{if} \quad \mathbf{z} \cap \mathbf{u} = \emptyset, \\ \mathcal{W}^r_{\mathfrak{D}\setminus\mathbf{u},p,\rho_{\mathfrak{D}\setminus\mathbf{u}}} & \textit{if} \quad \mathbf{z} \cap \mathbf{u} \neq \emptyset, \end{cases} \quad \textit{for all} \quad \mathbf{u} \subseteq \mathfrak{D}.$$

Moreover, the ANOVA terms of f satisfy

$$f_{\mathbf{u}} \in \begin{cases} \mathcal{W}^1_{\mathbf{u},p,\rho_{\mathbf{u}}} & \text{if} \quad \mathbf{z} \subseteq \mathbf{u}, \\ \mathcal{W}^r_{\mathbf{u},p,\rho_{\mathbf{u}}} & \text{if} \quad \mathbf{z} \not\subseteq \mathbf{u}, \end{cases} \qquad \text{for all} \quad \mathbf{u} \subseteq \mathfrak{D}.$$

Proof. The fact that $f \in \mathcal{W}^1_{d,p,\rho_d} \cap \mathcal{C}(\mathbb{R}^d)$ is already established in (3.2). For any $\mathbf{u} \subseteq \mathfrak{D}$, repeated application of the inheritance theorem (Theorem 2.2) with $r = 1$ yields $P_{\mathbf{u}} f \in \mathcal{W}^1_{\mathfrak{D}\setminus\mathbf{u},p,\rho_{\mathfrak{D}\setminus\mathbf{u}}}$. If for some $\mathbf{u} \subseteq \mathfrak{D}$ we have $\mathbf{z} \cap \mathbf{u} \neq \emptyset$, then there exists $j \in \mathbf{u}$ such that $D_j\phi$ never changes sign and (3.3) and (3.4) hold. In this case Theorem 3.1 applies and we have $P_j f \in \mathcal{W}^r_{\mathfrak{D}\setminus\{j\},p,\rho_{\mathfrak{D}\setminus\{j\}}}$. Repeated applications of the inheritance theorem then yield $P_{\mathbf{u}} f = P_{\mathbf{u}\setminus\{j\}}(P_j f) \in \mathcal{W}^r_{\mathfrak{D}\setminus\mathbf{u},p,\rho_{\mathfrak{D}\setminus\mathbf{u}}}$. The smoothness of the ANOVA terms then follows from the explicit formula (2.3). $\square$

To gain more insight into Theorem 3.2, suppose now that we have the best case $\mathbf{z} = \mathfrak{D}$. Then we see that smoothing occurs for all ANOVA terms except for the term with the highest order.

Corollary 3.3. *Suppose that Theorem* 3.2 *holds for* $\mathbf{z} = \mathfrak{D}$. *Then for* f *given by* (3.13) *we have*

$$f_{\mathfrak{D}} \in \mathcal{W}^1_{d,p,\rho_d} \qquad \text{and} \qquad f_{\mathbf{u}} \in \mathcal{W}^r_{\mathbf{u},p,\rho_{\mathbf{u}}} \quad \text{for all} \quad \mathbf{u} \subsetneq \mathfrak{D}.$$

As a simple illustration, consider now the function $\phi(\boldsymbol{x}) = x_1 + \cdots + x_d$ together with the standard Gaussian density $\rho(t) = e^{-t^2/2}/\sqrt{2\pi}$. Clearly, ϕ is smooth, $D_j\phi \equiv 1$ for all $j \in \mathfrak{D}$, and the condition (3.4) holds trivially. Moreover, the condition (3.3) holds for all $j \in \mathfrak{D}$ because we always have $\phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) = 0$ by taking $x_j = -\sum_{i\neq j} x_i$. Thus Theorem 3.2 holds for $\mathbf{z} = \mathfrak{D}$, and we conclude that all ANOVA terms of $f(\boldsymbol{x}) = \phi(\boldsymbol{x})_+$ are smooth except for the one with the highest order. This result can also be seen by evaluating $P_j f$ directly, to yield a function of the variables other than x_j that is manifestly smooth.

4. Option pricing problems

The assumption (3.4) in its current form is not easy to check due to the presence of the function ψ_j. However, a sufficient condition for (3.4) that is easier to check can be obtained if we know more about the weight function. In particular, when ρ is the Gaussian weight function we know that $\rho^{1/2}$ is integrable; in this case we can estimate the expression in (3.4) as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^{\mathfrak{D}\setminus\{j\}}} \left| \frac{\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}}\phi)(\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}), \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})]}{[(D_j\phi)(\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}), \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})]^b} \rho^{(c)}(\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})) \right|^p \\
&\qquad\qquad \cdot \rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})\, \mathrm{d}\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} \\
&\le \sup_{\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} \in \mathbb{R}^{\mathfrak{D}\setminus\{j\}}} \left(\left| \frac{\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}}\phi)(\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}), \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})]}{[(D_j\phi)(\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}), \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})]^b} \rho^{(c)}(\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})) \right|^p \right. \\
&\qquad\qquad \left. \cdot [\rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})]^{1/2} \right) \\
&\qquad \cdot \int_{\mathbb{R}^{\mathfrak{D}\setminus\{j\}}} [\rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})]^{1/2}\, \mathrm{d}\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}
\end{aligned}$$

$$(4.1) \quad \le \sup_{\boldsymbol{x}\in\mathbb{R}^d} \left(\left| \frac{\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}}\phi)(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})]}{[(D_j\phi)(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})]^b} \, \rho^{(c)}(x_j) \right|^p [\rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})]^{1/2} \right)$$
$$\cdot \int_{\mathbb{R}^{\mathfrak{D}\setminus\{j\}}} [\rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})]^{1/2} \, \mathrm{d}\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}.$$

Thus it suffices to check that the supremum in (4.1) is finite.

We now explain that these conditions hold for the *Asian option pricing problems*. We will not go into details about these problems here; see our previous paper [8, Section 5] for an elaborate discussion. (Note that in [8] we truncate the integrands near infinity and map the resulting functions into the unit cube; there is no need to carry out these steps here.) It suffices to say here that, after a change of variables by either (i) the *standard construction*, (ii) the *Brownian bridge construction*, or (iii) the *principal components construction*, we end up with the integral

$$\int_{\mathbb{R}^d} f(\boldsymbol{x}) \, \rho_d(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x},$$

with $\rho(t) = e^{-t^2/2}/\sqrt{2\pi}$ being the standard Gaussian density, $f(\boldsymbol{x}) = \phi(\boldsymbol{x})_+$, and

$$(4.2) \qquad \phi(\boldsymbol{x}) \;=\; \frac{S_0}{d} \sum_{\ell=1}^d \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) \ell\Delta t + \sigma \sum_{i=1}^d A_{\ell i} \, x_i \right) - K,$$

where S_0 is the initial stock price, K is the strike price, μ is the risk-free interest rate, σ is the volatility, d is the number of (equally-spaced) time steps, $\Delta t = T/d$ with T denoting the final time, and A is a $d \times d$ matrix which depends on the construction method (i)–(iii).

We see that ϕ is essentially a sum of exponential functions involving only linear combinations of $x_1, \ldots, x_d$ in the exponents. The derivatives of ϕ will contain at worst exponential functions of the same form, and the growth of these will be defeated by the Gaussian weight function (or even by the square root of the Gaussian weight function). More precisely, we have

$$\phi \in \mathcal{W}^r_{d,p,\rho_d} \cap \mathcal{C}^\infty(\mathbb{R}^d) \qquad \text{for all } r \ge 0 \text{ and all } p \in [1,\infty),$$

and it is straightforward to see that the supremum in (4.1) will be finite for all r and all finite p. Furthermore, we have

$$(D_j\phi)(\boldsymbol{x}) \;=\; \frac{\sigma\, S_0}{d} \sum_{\ell=1}^d \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) \ell\Delta t + \sigma \sum_{i=1}^d A_{\ell i} \, x_i \right) A_{\ell j}.$$

For the standard construction and the Brownian bridge construction, we already established in [8] that the elements $A_{\ell j}$ are always nonnegative, and thus $(D_j\phi)(\boldsymbol{x}) > 0$ for all $j \in \mathfrak{D}$ and all $\boldsymbol{x} \in \mathbb{R}^d$. Moreover, because the matrix elements are nonnegative, it is easy to see from (4.2) that for each j and each fixed $\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}$ we have

$$\phi(\boldsymbol{x}) \;=\; \phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \;\to\; \begin{cases} +\infty & \text{as } x_j \to +\infty, \\ -K & \text{as } x_j \to -\infty. \end{cases}$$

This ensures that ϕ changes sign and therefore that $\psi_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})$ exists for all $\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} \in \mathbb{R}^{\mathfrak{D}\setminus\{j\}}$. Hence Theorem 3.2 applies with $\boldsymbol{z} = \mathfrak{D}$, i.e., Corollary 3.3 holds, and we have

$$f_{\mathfrak{D}} \in \mathcal{W}^1_{d,p,\rho_d} \quad \text{and} \quad f_{\mathbf{u}} \in \mathcal{W}^r_{\mathbf{u},p,\rho_{\mathbf{u}}} \quad \text{for all} \quad \mathbf{u} \subsetneq \mathfrak{D}$$

for all $r \geq 0$ and all $p \in [1,\infty)$. Since the result holds for all values of r and all finite values of p, we conclude from (2.7) that

$$f_{\mathfrak{D}} \in \mathcal{W}^1_{d,p,\rho_d} \cap \mathcal{C}(\mathbb{R}^d) \quad \text{and} \quad f_{\mathbf{u}} \in \mathcal{W}^\infty_{\mathbf{u},p,\rho_{\mathbf{u}}} \cap \mathcal{C}^\infty(\mathbb{R}^{\mathbf{u}}) \quad \text{for all} \quad \mathbf{u} \subsetneq \mathfrak{D}. \tag{4.3}$$

Using (2.6), we conclude also that

$$f_{\mathbf{u}} \in \mathcal{W}^{\boldsymbol{r}_{\mathbf{u}}}_{\mathbf{u},p,\rho_{\mathbf{u}},\mathrm{mix}} \cap \mathcal{C}^\infty(\mathbb{R}^{\mathbf{u}}) \quad \text{for all} \quad \mathbf{u} \subsetneq \mathfrak{D}$$

for all $r_{\mathbf{u}} = (r_j)_{j\in\mathbf{u}}$ with nonnegative integers r_j.

For the principal components construction, we explained in [8] that although the elements $A_{\ell 1}$ are positive, the elements $A_{\ell j}$ for $j \geq 2$ can take both positive and negative values, and hence $(D_j\phi)(\boldsymbol{x})$ can change sign. We conclude in this case that

$$f_{\mathbf{u}} \in \begin{cases} \mathcal{W}^1_{\mathbf{u},p,\rho_{\mathbf{u}}} \cap \mathcal{C}(\mathbb{R}^{\mathbf{u}}) & \text{if} \quad \mathbf{z} \subseteq \mathbf{u}, \\ \mathcal{W}^\infty_{\mathbf{u},p,\rho_{\mathbf{u}}} \cap \mathcal{C}^\infty(\mathbb{R}^{\mathbf{u}}) & \text{if} \quad \mathbf{z} \not\subseteq \mathbf{u}, \end{cases} \qquad \text{for all} \quad \mathbf{u} \subseteq \mathfrak{D}, \tag{4.4}$$

where $\boldsymbol{z}$ denotes the set of indices j for which $D_j\phi$ never changes sign, and for which (3.3) holds. In particular, since $\mathbf{z}$ always contains the index 1, the ANOVA term $f_{\mathbf{u}}$ has maximum smoothness for all subsets $\mathbf{u}$ that do not involve the first coordinate. Again $\mathcal{W}^\infty_{\mathbf{u},p,\rho_{\mathbf{u}}}$ can be replaced by $\mathcal{W}^{\boldsymbol{r}_{\mathbf{u}}}_{\mathbf{u},p,\rho_{\mathbf{u}},\mathrm{mix}}$ for all $r_{\mathbf{u}} = (r_j)_{j\in\mathbf{u}}$ with nonnegative integers r_j.

5. Numerical results

The aim of our numerical experiment here is to illustrate the smoothing process in a low-dimensional example. We consider the arithmetic average Asian call option, combined with the standard construction, the Brownian bridge construction, and the principal components construction. In (4.2) we choose the parameters

$$d = 4, \quad S_0 = 100, \quad \sigma = 0.2, \quad \mu = 0.1, \quad T = 1, \quad \text{and} \quad K = 100.$$

See [8, Section 5.1] for the precise formulas for the elements of the matrix A under the three construction methods.

Before we present our numerical results, let us first summarize what our theory predicts.

- For the standard and Brownian bridge constructions, we know that all the ANOVA terms of the integrand $f(\boldsymbol{x}) = \phi(\boldsymbol{x})_+$ are smooth, with the exception of the final term $f_{\{1,2,3,4\}}$ which inherits the kink from f; see (4.3).
- For the principal components construction, we know that $D_1\phi$ never changes sign, and we can check numerically that $D_2\phi$, $D_3\phi$, and $D_4\phi$ all change sign somewhere in $\mathbb{R}^4$. This indicates that smoothing only occurs if integration with respect to x_1 takes place, see (4.4) with $\mathbf{z} = \{1\}$. Thus we expect that $f_{\{2\}}$, $f_{\{3\}}$, $f_{\{4\}}$, $f_{\{2,3\}}$, $f_{\{2,4\}}$, $f_{\{3,4\}}$, $f_{\{2,3,4\}}$ are all smooth, but $f_{\{1\}}$, $f_{\{1,2\}}$, $f_{\{1,3\}}$, $f_{\{1,4\}}$, $f_{\{1,2,3\}}$, $f_{\{1,2,4\}}$, $f_{\{1,3,4\}}$, $f_{\{1,2,3,4\}}$ all have kinks.

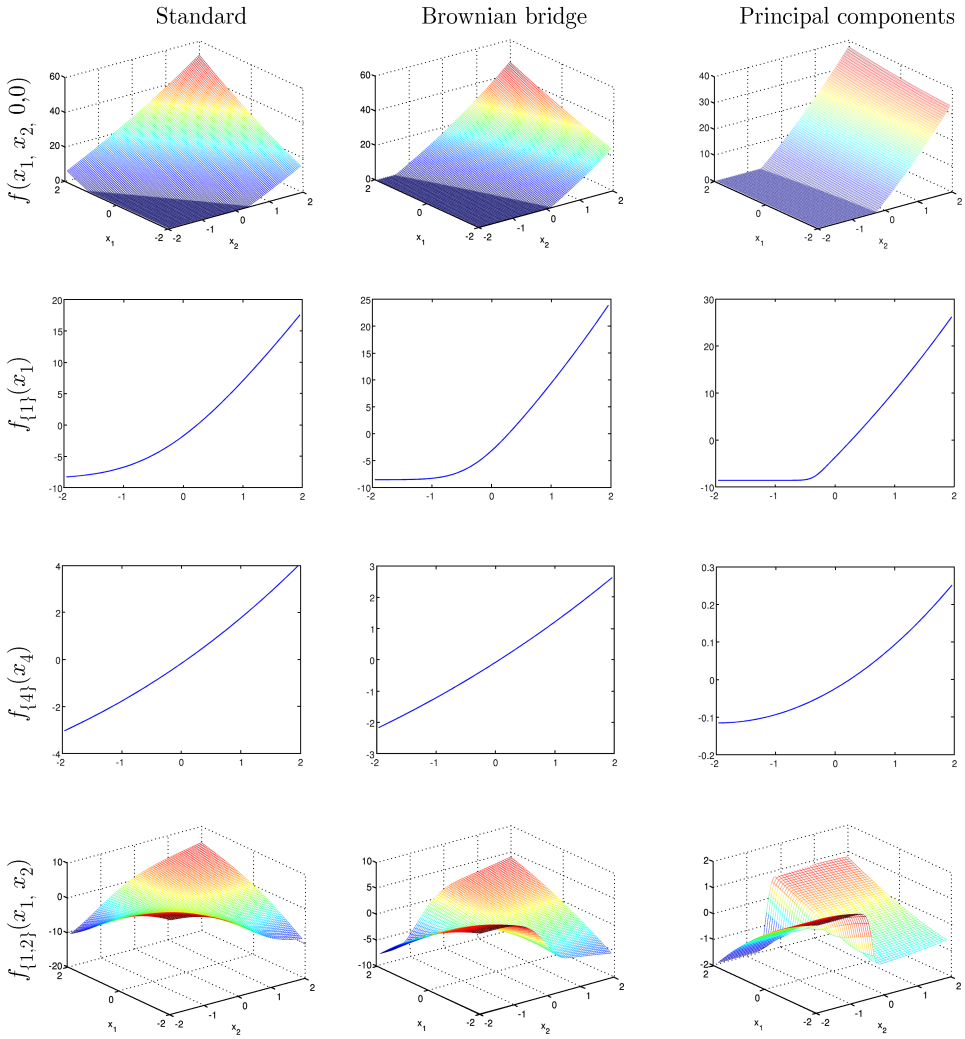

FIGURE 1. The integrand and its ANOVA terms for the arithmetic average Asian call option with $d = 4$ using different construction methods.

In Figures 1 and 2 we plot the integrand f and a selection of its ANOVA terms under the three construction methods. In those cases where the number of variables is greater than two, we plot just the two-dimensional projections of the first two variables and fix the remaining variables at the value 0. We restrict the view to $[-2, 2]^2$. It is remarkable that these figures completely agree with our theoretical predictions.

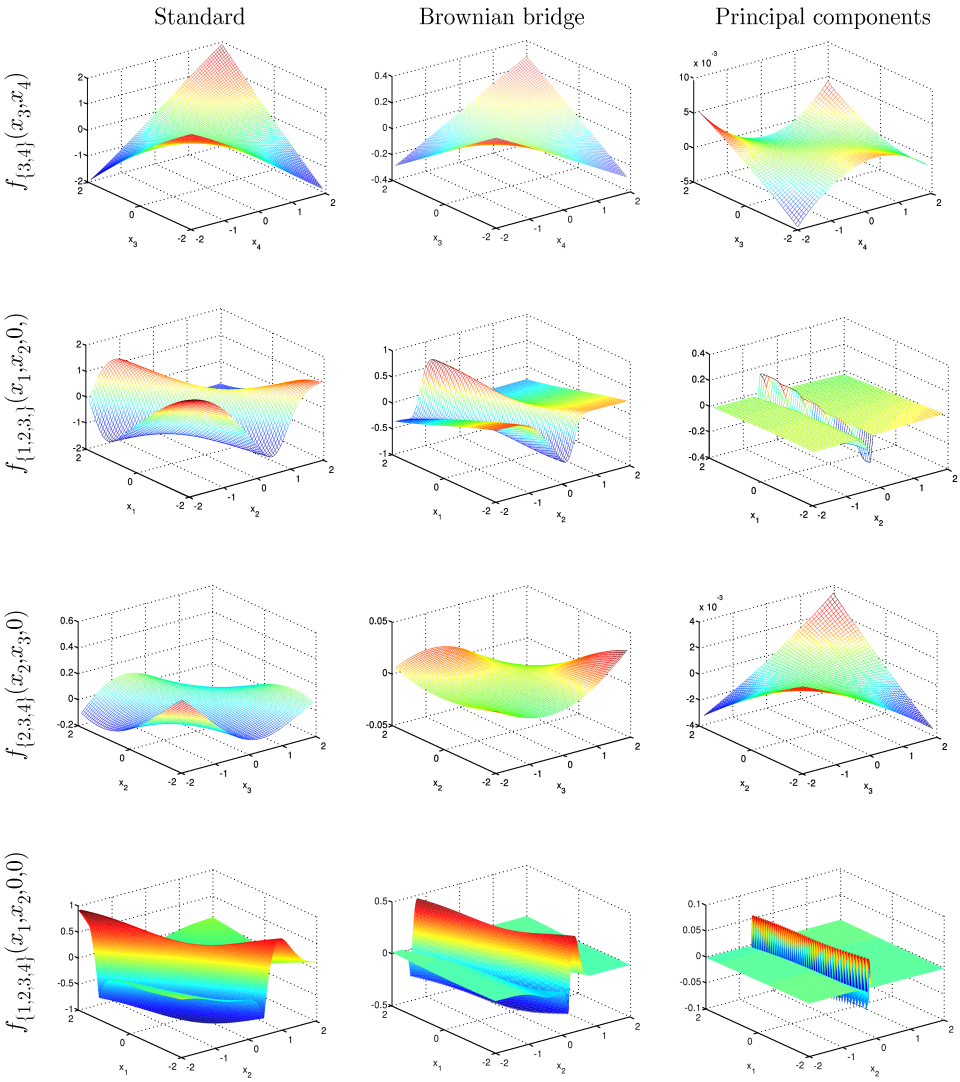

FIGURE 2. The integrand and its ANOVA terms for the arithmetic average Asian call option with $d = 4$ using different construction methods (continued).

Creating these figures turned out to be a difficult numerical task. The integrals in the ANOVA terms were approximated using $300,000$ Sobol$'$ points, and even then we observed substantial spikes with the order-4 terms $f_{\{1,2,3,4\}}$. A careful analysis showed that the spikes were not a problem of the numerics but of the graphics: the functions were evaluated pointwise on a grid of 100×100 (with the exception of the three graphs for $f_{\{1,2,3,4\}}$ which were evaluated on a grid of 500×500) and the graphs were plotted by Matlab via interpolation between the points; the kinks could not be represented correctly and this is the reason for the spiky behavior near the singularity lines. For successive finer grids the bumpiness does get better, but the principal problem remains. Another problem was the subtraction of terms

in the ANOVA decomposition. A naive implementation of the decomposition was numerically unstable and led to a serious roundoff error. It was necessary to sum up the positive and negative terms separately and then perform one single subtraction at the end.

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