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A FINITE ELEMENT METHOD FOR REISSNER-MINDLIN PLATES

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ABSTRACT. A finite element method is proposed and analyzed for the Reissner-Mindlin plate problem subject to various boundary conditions. Rotation and transverse displacement variables are approximated by continuous linear elements (enriched with local bubbles) and an L^2 projector is applied to the shear energy term onto the Raviart-Thomas $H(\mathrm{div};\Omega)$ finite element. Stability and optimal error bounds hold uniformly in the plate thickness.

1. INTRODUCTION

The Reissner-Mindlin plate model is widely employed for both thick and thin plates, due to incorporating independent assumptions for the transverse displacement and normal rotation. This allows the use of simple continuous elements in the displacement-based approach. However, such an approach usually suffers from the so-called shear locking phenomenon, as the plate thickness becomes numerically small (cf. [9], [13]). Another source of trouble is boundary layer effects (cf. [6], [5]). Therefore, how to avoid these problems has been and still remains an active research subject in seeking the finite element solution of the Reissner-Mindlin plate problem; see [14], [4], [16], [39], [25], [42], [29], [24], [38], [44], [7], [31], [32], [35], [12], [40], [26], [17], [11], [23], [22], [21], [20], [2], [3], [36], [37].

In this paper, we discretize the Reissner-Mindlin plate problem by a new finite element method, using linear C^0 elements (enriched with local element bubble functions) for rotation and transverse displacement, with an L^2 projector applied to the shear energy term onto the lowest-order Raviart-Thomas $H(\mathrm{div};\Omega)$ element as depicted in [13]. Clamped, free and soft simply-supported boundary conditions are considered. Note that the boundary layer effects are the strongest in the last two boundary conditions. By a link-interpolation trick, we show that this new method is uniformly convergent with respect to the plate thickness: $\mathcal{O}(h)$ and $\mathcal{O}(h^{\frac{1}{2}})$ are obtained for clamped and free, soft simply-supported plates, respectively.

The well-known MITC (mixed interpolation of tensorial components) method [11], [14], [38] uses $H(\mathrm{curl};\Omega)$ finite elements to approximate the shear strain for clamped plate and admits a uniform convergence with respect to the plate thickness. Nevertheless, when applying to both soft simply-supported and free plates, the error bound in the natural energy norm of transverse displacement and rotation is only

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$\mathcal{O}(t^{-\frac{1}{2}}\, h)$, to our knowledge (see [40]), where t stands for the plate thickness. Similar error bounds are given therein for the Arnold-Brezzi element [2] and the Arnold-Falk element [4]. We would like also to mention the recent papers [33] and [34] on the current topic, where in [33] a refined error analysis is presented for the clamped plate in terms of the load regularity with interior and boundary error contributions and in [34] superconvergence and postprocessing approaches are investigated.

Another disadvantage is that no theoretical results for MITC method hold for the Inf-Sup condition (see [31], [30]), which is stated in the $H^{-1}(\mathrm{div};\Omega)$ norm for clamped plate. Note that, as pointed out in [13], [19], the space $H^{-1}(\mathrm{div};\Omega)$ is the only correct one for the shear strain when passing the Reissner-Mindlin plate to the Kirchhoff-Love thin plate as the plate thickness t tends to zero. From the classical theory in [13] for the saddle-point problem and the general comments in [30] for the Kirchhoff-Love thin plate, such Inf-Sup conditions play a critical role in obtaining uniform stability and uniform optimal error estimates of the underlying finite element method. On the one hand, the theory in [20] indicates that to obtain the Inf-Sup condition the normal component of the shear strain should be controlled from the transverse displacement. However, since the MITC method uses the $H_0(\mathrm{curl};\Omega)$ conforming element as the approximating space for the shear strain (considering the clamped plate, for example), it is not possible to control the normal component of the shear strain in only the $H_0(\mathrm{curl};\Omega)$-conforming element. Meanwhile, for the limit plate (i.e., the Kirchhoff-Love thin plate, the limit of the Reissner-Mindlin plate with the plate thickness $t = 0$), the shear strain no longer belongs to $H_0(\mathrm{curl};\Omega)$. As commented in [13, page 303], this is the reason why the boundary layer effect occurs. For this reason, there has been lacking a mathematical justification on how to pass the plate thickness to zero directly in the finite element formulation, stability, and error estimates of the MITC method in order to solve the limit plate. In other words, the MITC method performs well for the plate thickness $t > 0$, but a direct application of the MITC method including the related theory cannot cover the limit plate. As a matter of fact, when applying MITC elements, the theory does not exist on whether or not the Inf-Sup condition in the $H^{-1}(\mathrm{div};\Omega)$ space holds, although some mesh-dependent Inf-Sup condition indeed holds [30].[1] In addition, although the MITC method may be recasted into a saddle-point form (but the reduction operator cannot be removed because it is not an L^2 projection), it is not easy to design an iterative algorithm with shear strain $H(\mathrm{curl};\Omega)$ conforming, and even the well-known simplest Uzawa algorithm cannot be directly employed, since there does not hold the mentioned Inf-Sup condition (which should be fulfilled uniformly with respect to the plate thickness t and the mesh size to ensure a convergence rate uniform in t and h for the Uzawa algorithm). Also, as pointed out in [8], the positive semi-definiteness of the upper left matrix in the saddle-point form of the MITC method would introduce additional complications in implementation.

On the other hand, our method satisfies the Inf-Sup condition uniformly in the plate thickness t and in the mesh size in the $H^{-1}(\mathrm{div};\Omega)$ space (considering clamped plate, for example). Note that we use the $H(\mathrm{div};\Omega)$ element for the shear strain, and the jump of the normal component of the $H(\mathrm{div};\Omega)$ element across inter-element

[1]See "*So far no analytical results exist regarding the norm equivalence (36) and regarding the inf-sup condition (15). Such results would be very hard, if not impossible, to obtain for general distorted meshes, and we therefore resort to a numerical treatment*" [30, p. 3647].

boundaries is trivially zero. We also point out that since the shear strain is in $H_0(\mathrm{div};\Omega)$ for the free plate, it is most natural to employ the $H(\mathrm{div};\Omega)$ element for approximating the shear strain. Moreover, the upper left matrix in the saddle-point form of our method is positive definite. Note that the L^2 projection does not exist in the equivalent saddle-point problem. The Inf-Sup condition greatly facilitates the use of the well-known (inexact/preconditioned) Uzawa iterative method and the multigrid method (see [38], [28], [43] and the references therein) for our method in the saddle-point form. In this paper, we describe the iterated penalty method in [43] for our method, and following the general theory therein we can obtain a geometrical convergence which indicates that our finite element method can be efficiently solved in its equivalent saddle-point form by the proposed iterated penalty method behaving like a direct method. So, although in the displacement-based form the L^2 projection in our method is a little bit more complex than the local finite element interpolation in the MITC method, our method can be more readily realized than the latter in the saddle-point form. Note that the displacement-based form of the finite element method of the Reissner-Mindlin plate is usually solved in the saddle-point form, since the factor t^{-2} in the former is transformed into the factor t^2 in the latter so that a preconditioned iterative approach may be more easily designed. More importantly, due to the mentioned Inf-Sup condition, our method is applicable to both the Reissner-Mindlin plate and the Kirchhoff-Love limit plate, and the standard mathematical theory for saddle-point problems can be directly applied to obtain optimal error bounds $\mathcal{O}(h)$ for the limit plate. This means that our method will not "lock" under any circumstances. We emphasize that, at least in theory, the MITC method cannot be directly applied to the limit plate itself, since the mentioned Inf-Sup condition could not hold.

We also mention the recently developed discontinuous Galerkin method in [3]. Note that the earlier method of discontinuous Galerkin type for the Reissner-Mindlin plate should be due to our work in [20]. In [3], it is not surprising that the reduction operator is not needed because it is a saddle-point method. The discontinuous Galerkin mixed method in [3] seems to be not favorable for implementation issues because of the number of jumps across inter-element boundaries. In addition, in [3], the simplest element for transverse displacement is quadratic. Similar to the MITC element, the method in [3] employs the $H(\mathrm{curl};\Omega)$ element for the shear strain. As we remarked earlier, the Inf-Sup condition in the space $H^{-1}(\mathrm{div};\Omega)$ (for clamped plate, for example) could not be analytically established. In the case where the discontinuous linear element is used for the shear strain, from the theory in [20] we infer that the transverse displacement quadratic element in [3] is not "big" enough to control the normal component of the shear strain, and as a consequence, the Inf-Sup condition in the $H^{-1}(\mathrm{div};\Omega)$ space may not be established, either. Therefore, likewise to the MITC method, no theory could exist to support the direct application of the discontinuous Galerkin mixed method in [3] to the limit plate and it is left as an unknown whether there are immediately available highly efficient iterative solvers for the resultant saddle-point system since the Inf-Sup condition in the $H^{-1}(\mathrm{div};\Omega)$ space could not hold.

Essentially, our method is the linear element method, except for the element bubbles, which may be statically eliminated at element levels, and for a few edge bubbles along the domain boundary for dealing with the curved boundary. Those

edge bubbles disappear when the domain boundary is piecewise straight-sided. Note that the MITC method adopts at least the quadratic element for rotation.

The outline of this paper is as follows: in Section 2 the Reissner-Mindlin plate model, including regularity result, is recalled. In Section 3, the finite element method is defined. In Section 4, error bounds are derived. In Section 5, the Inf-Sup condition and the error bound are established for the shear strain. In the last section, an implementation issue is addressed, including a concluding remark.

2. The Reissner-Mindlin plate model

Let $\Omega \subset \mathbb{R}^2$ be the midplane occupied by the plate, with boundary $\partial\,\Omega$. Denote by w and $\boldsymbol{\phi} = (\phi_1, \phi_2)$ the transverse displacement of Ω, and the rotation of the normals to Ω, respectively. As described in [13], the Reissner-Mindlin plate model is to find w and $\boldsymbol{\phi}$ such that

$$-\mathbf{div}\,\mathsf{C}\,\boldsymbol{\mathcal{E}}(\boldsymbol{\phi}) - G\,t^{-2}\,(\nabla\,w - \boldsymbol{\phi}) = 0, \tag{2.1}$$

$$-G\,t^{-2}\,\mathrm{div}\,(\nabla\,w - \boldsymbol{\phi}) = f, \tag{2.2}$$

where f is the load, t is the plate thickness, $G = \dfrac{E\kappa}{2(1+\nu)}$ is the shear modulus with E the Young modulus, ν the Poisson ratio, and κ the shear correction factor, $\boldsymbol{\mathcal{E}}(\boldsymbol{\phi}) = (\nabla\,\boldsymbol{\phi} + (\nabla\,\boldsymbol{\phi})^T)/2$, and the fourth-order tensor C is defined by $\mathsf{C}\boldsymbol{\mathcal{T}} = D\,[(1-\nu)\,\boldsymbol{\mathcal{T}} + \nu\,\mathrm{tr}\,(\boldsymbol{\mathcal{T}})\,\boldsymbol{\mathcal{I}}]$, $\quad D = \dfrac{E}{12\,(1-\nu^2)}$, for any 2×2 matrix $\boldsymbol{\mathcal{T}}$ ($\boldsymbol{\mathcal{I}}$ denotes the 2×2 identity matrix). On $\partial\,\Omega$, we consider homogeneous clamped, soft simply supported and free boundary conditions [6]:

$$\boldsymbol{\phi}\cdot\boldsymbol{n} = \boldsymbol{\phi}\cdot\boldsymbol{s} = w = 0 \quad \text{(hard) clamped}, \tag{2.3}$$

$$M_{\boldsymbol{n}}(\boldsymbol{\phi}) = M_{\boldsymbol{s}}(\boldsymbol{\phi}) = w = 0 \quad \text{soft simply supported}, \tag{2.4}$$

$$M_{\boldsymbol{n}}(\boldsymbol{\phi}) = M_{\boldsymbol{s}}(\boldsymbol{\phi}) = \boldsymbol{\gamma}\cdot\boldsymbol{n} = 0 \quad \text{free}, \tag{2.5}$$

where $\boldsymbol{n}$ and $\boldsymbol{s}$ denote the unit normal and counterclockwise tangent vectors, respectively, and $M_{\boldsymbol{n}}(\boldsymbol{\phi}) = \boldsymbol{n}\cdot\mathsf{C}\,\boldsymbol{\mathcal{E}}(\boldsymbol{\phi})\boldsymbol{n}$, $M_{\boldsymbol{s}}(\boldsymbol{\phi}) = \boldsymbol{s}\cdot\mathsf{C}\,\boldsymbol{\mathcal{E}}(\boldsymbol{\phi})\boldsymbol{n}$, and $\boldsymbol{\gamma} := Gt^{-2}\,(\nabla\,w - \boldsymbol{\phi})$ is the shear strain. Note that for free plates f is assumed to be orthogonal to $\mathbb{L}$ in $L^2(\Omega)$, with $\mathbb{L}$ the three-dimensional space of linear polynomial functions on Ω; see [6].

For the analysis, we introduce some Hilbert spaces [1], [27]. Hilbert spaces $H^s(\Omega)$ for the real number $s \in \mathbb{R}$, and $H^0(\Omega) = L^2(\Omega)$, are equipped with norm $||\cdot||_s$, and Hilbert space $H^1(\Omega)/\mathbb{R} = \{v \in H^1(\Omega) : \int_\Omega v = 0\}$, $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\,\Omega} = 0\}$, $H(\mathrm{div};\Omega) = \{\boldsymbol{q} \in (L^2(\Omega))^2 : \mathrm{div}\,\boldsymbol{q} \in L^2(\Omega)\}$ and $H_0(\mathrm{div};\Omega) = \{\boldsymbol{q} \in H(\mathrm{div};\Omega) : \boldsymbol{q}\cdot\boldsymbol{n}|_{\partial\,\Omega} = 0\}$. Let $(\cdot,\cdot)$ denote the L^2 inner product. Here, we recall Green's formulae of integrating by parts:

$$(\mathrm{curl}\,\boldsymbol{\psi}, q) = (\boldsymbol{\psi}, \mathbf{curl}\,q) - \int_{\partial\,\Omega} \boldsymbol{\psi}\cdot\boldsymbol{s}\,q, \quad (\mathrm{div}\,\boldsymbol{\psi}, q) = -(\boldsymbol{\psi}, \nabla\,q) + \int_{\partial\,\Omega} \boldsymbol{\psi}\cdot\boldsymbol{n}\,q,$$

where

$$\mathrm{curl}\,\boldsymbol{\psi} = \frac{\partial\psi_1}{\partial y} - \frac{\partial\psi_2}{\partial x},\ \mathrm{div}\,\boldsymbol{\psi} = \frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y},\ \mathbf{curl}\,q = \Big(-\frac{\partial q}{\partial y}, \frac{\partial q}{\partial x}\Big)^T.$$

Assume that the plate thickness t satisfies $0 < G^{-1/2}t < 1$. Putting $\alpha = (1 - G^{-1}t^2)^{-1}$ and $g = \alpha f$, we may write a variational problem of the Reissner-Mindlin plate model: Find $(\boldsymbol{\phi}, w) \in \Theta \times W$ such that

$$A((\boldsymbol{\phi}, w); (\boldsymbol{\psi}, v)) + Gt^{-2}(\nabla w - \boldsymbol{\phi}, \nabla v - \boldsymbol{\psi}) = (g, v) \quad \forall (\boldsymbol{\psi}, v) \in \Theta \times W, \tag{2.6}$$

where $\Theta \times W = (H_0^1(\Omega))^2 \times H_0^1(\Omega)$, $(H^1(\Omega))^2 \times H_0^1(\Omega)$ and $((H^1(\Omega))^2 \times H^1(\Omega))/(\mathbb{R}^2 \times \mathbb{L})$, correspond to clamped, soft simply-supported and free boundary conditions, respectively, and the bilinear form A is defined by

$$A((\boldsymbol{\phi}, w); (\boldsymbol{\psi}, v)) = \alpha \{a(\boldsymbol{\phi}, \boldsymbol{\psi}) + (\nabla w - \boldsymbol{\phi}, \nabla v - \boldsymbol{\psi})\}, \tag{2.7}$$

with $a(\boldsymbol{\phi}, \boldsymbol{\psi}) = \dfrac{E}{12(1-\nu^2)} \int_\Omega [(1-\nu)\boldsymbol{\mathcal{E}}(\boldsymbol{\phi}) : \boldsymbol{\mathcal{E}}(\boldsymbol{\psi}) + \nu \mathrm{div}\boldsymbol{\phi}\, \mathrm{div}\boldsymbol{\psi}]$. Problem (2.7) admits a unique solution $(\boldsymbol{\phi}, w) \in \Theta \times W$ for any $g \in L^2(\Omega)$, since

$$A((\boldsymbol{\psi}, v); (\boldsymbol{\psi}, v)) \geq C\{||\boldsymbol{\psi}||_1^2 + ||v||_1^2\} \quad \forall (\boldsymbol{\psi}, v) \in \Theta \times W. \tag{2.8}$$

We remark that such coercivity (2.8) uniform in t is preserved for conforming elements so that a uniform stability for the transverse displacement can hold. This is why we split the shear energy term into two parts, having one part included into the bending energy term as in (2.7). Now, the saddle-point form of (2.6) is a typical form in the framework of saddle-point problem, in the sense that the upper left operator is coercive over the whole space $\Theta \times W$. Namely, by the introduction of the shear strain

$$\boldsymbol{\gamma} = Gt^{-2}(\nabla w - \boldsymbol{\phi}), \tag{2.9}$$

we have the following saddle-point form of (2.6): to find $(\boldsymbol{\phi}, w, \boldsymbol{\gamma})$ such that

$$\begin{cases} A((\boldsymbol{\phi}, w); (\boldsymbol{\psi}, v)) + (\boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}) = (g, v) \quad \forall (\boldsymbol{\psi}, v), \\ (\nabla w - \boldsymbol{\phi}, \boldsymbol{\chi}) - G^{-1}t^2(\boldsymbol{\gamma}, \boldsymbol{\chi}) = 0 \quad \forall \boldsymbol{\chi}. \end{cases}$$

The corresponding block form is

$$\begin{pmatrix} \mathtt{A} & \mathtt{B}^T \\ \mathtt{B} & \mathtt{C} \end{pmatrix},$$

where $\mathtt{A}$ is $\Theta \times W$ elliptic due to (2.8) and $\mathtt{B}$ satisfies the Inf-Sup condition in the space $H^{-1}(\mathrm{div}; \Omega)$ for clamped plate; for example, see [13]. Both ellipticity and Inf-Sup are required to hold in the underlying mixed finite element method in order to have uniform stability and uniform optimal convergence. In the spirit of the stabilization method in [13], here the split of the shear energy term introduces a *stabilization* which acts on the transverse displacement so that A is coercive over the whole space $\Theta \times W$. This remark is also applied to the finite element method that we will develop.

Proposition 2.1 ([13]). *Any $\boldsymbol{q} \in (L^2(\Omega))^2$ can be written as the Helmholtz decomposition*

$$\boldsymbol{q} = \nabla u + \mathbf{curl}\, p, \tag{2.10}$$

with $u \in H_0^1(\Omega)$ and $p \in H^1(\Omega)/\mathbb{R}$, or $u \in H^1(\Omega)/\mathbb{R}$ and $p \in H_0^1(\Omega)$.

Proposition 2.2 ([6], [5]). *Let $(\boldsymbol{\phi}, w)$ be the exact solution and the shear strain $\boldsymbol{\gamma}$ be defined by (2.9) with its Helmholtz decomposition $\boldsymbol{\gamma} = \nabla u + \mathbf{curl}\, p$ as in (2.10).*

Assuming that Ω and g are smooth enough. For clamped, soft simply supported and free plates we have

$$||w||_s \le C, \quad s \in \mathbb{R}. \tag{2.11}$$

For clamped plate we have

$$||\boldsymbol{\phi}||_s \le C\, t^{\min(0,5/2-s)}, \quad ||\boldsymbol{\gamma}||_s \le C\, t^{\min(0,1/2-s)}, \quad s \in \mathbb{R}, \tag{2.12}$$

$$||u||_s \le C, \quad ||p||_s \le C\, t^{\min(0,3/2-s)}, \quad s \in \mathbb{R}. \tag{2.13}$$

For soft simply supported and free plates we have

$$||\boldsymbol{\phi}||_s \le C\, t^{\min(0,3/2-s)}, \quad ||\boldsymbol{\gamma}||_s \le C\, t^{\min(0,-1/2-s)}, \quad s \in \mathbb{R}, \tag{2.14}$$

$$||u||_s \le C, \quad ||p||_s \le C\, t^{\min(0,1/2-s)}, \quad s \in \mathbb{R}. \tag{2.15}$$

Above and below the letter C (with or without subscripts) denotes generic positive constants, taking different values at different occurrences, which is independent of t and the mesh parameter h but may depend on f.

3. The finite element method

In this section we define the finite element method for different boundary conditions as listed in (2.3)–(2.5), by using different finite element spaces.

Let $\mathcal{C}_h$ be a family of regular partitions of Ω [15], [41] into triangles, where some triangles $K \in \mathcal{C}_h$ could have one curved side coinciding with part of $\partial\,\Omega$. Let $h = \max_{K\in\mathcal{C}_h} h_K$, where h_K is the diameter of K.

Suppose that $(\Theta_h, W_h) \subset (H^1(\Omega))^2 \times H^1(\Omega)$ and $X_h \subset H(\mathrm{div};\Omega)$, which are to be determined later, are spaces of piecewise polynomials on $\mathcal{C}_h$. Let $R_h : (L^2(\Omega))^2 \to X_h$ be an L^2 projector. Then the finite element approximation to (2.6) is to find $(\boldsymbol{\phi}_h, w_h) \in \Theta_h \times W_h$ such that

$$A((\boldsymbol{\phi}_h, w_h); (\boldsymbol{\psi}, v)) + Gt^{-2}\,(R_h\,(\nabla\, w_h - \boldsymbol{\phi}_h), R_h(\nabla\, v - \boldsymbol{\psi})) = (g, v) \tag{3.1}$$

for all $(\boldsymbol{\psi}, v) \in \Theta_h \times W_h$. Thanks to (2.8), problem (3.1) would admit a unique solution $(\boldsymbol{\phi}_h, w_h)$, while the shear strain is defined as follows:

$$\boldsymbol{\gamma}_h = Gt^{-2}\, R_h(\nabla w_h - \boldsymbol{\phi}_h) \in X_h. \tag{3.2}$$

Note that the role of the L^2 projection R_h which is only applied to the shear energy term factored with t^{-2} is referred to as "reduction". The reduction operator has been extensively employed in elasticity and plate problems to circumvent locking phenomenon. The essence is to eliminate the factor t^{-1} that would appear in the error estimate in energy norm between the exact solution and the finite element solution, so that the "locking" phenomenon would not take place. Moreover, with the reduction operator, error estimates uniform in t may be obtained, so that the boundary layer effects are removed. We do not apply the L^2 projection R_h to the shear energy part of the bilinear form A in (2.7), besides the reason given after equation (2.8), since no t^{-1} appears therein.

Let K_s denote the straight-sided triangle whose vertices coincide with those of K, and Ω_s the union of such straight-sided triangles. Throughout this paper we always assume that the gap between $\partial\, K_s$ and $\partial\, K$ is of order $\mathcal{O}(h_K^2)$, so that for every curved edge $e \in \partial\,\Omega$, with its corresponding straight-side e_s, we have $\boldsymbol{n} = \boldsymbol{n}_0 + \boldsymbol{n}_1$ on e, where $\boldsymbol{n}_0$ is the constant outward unit normal vector to the straight side e_s and $\boldsymbol{n}_1 = \mathcal{O}(h_K)$. Similarly, the tangential vector $\boldsymbol{s}$ along e satisfies

a similar decomposition. We also assume that h is small enough so that every curved side $e \in \partial\Omega$ can be represented as a graph, so that we can use the finite element interpolation results and related results in [41].

On $K_s \subset \Omega_s$, let $S(K_s) = \text{span}\{(1,0)^T, (0,1)^T, (x,y)^T\}$ be the lowest-order Raviart-Thomas triangular element and $\mathcal{P}_1(K_s)$ the space of linear polynomials. On $K \in \mathcal{C}_h$ we then obtain $\mathcal{P}_1(K)$ and $S(K)$ as the extension/restriction of $\mathcal{P}_1(K_s)$ and $S(K_s)$ to K, respectively. We also obtain $\mathcal{P}_1(e)$ as the restriction of $\mathcal{P}_1(K)$ to e where e belongs to the boundary ∂K of K.

Denote by $\mathcal{C}_{\partial\Omega}$ the set of all the triangles sharing one edge with $\partial\Omega$, and by $E(\mathcal{C}_{\partial\Omega})$ the set of all edges of $\mathcal{C}_{\partial\Omega}$. For any $e \in E(\mathcal{C}_{\partial\Omega})$ we introduce an edge bubble in the following way. Let $e = K_1 \cap K_2$, where $K_1, K_2 \in \mathcal{C}_{\partial\Omega}$ and e is a straight-sided edge, and let ς_e^K denote the local basis associated with $e \in \partial K$ of $\mathcal{P}_2(K)$, where $\mathcal{P}_2(K)$ denotes the extension/restriction to K of the space of quadratic polynomials $\mathcal{P}_2(K_s)$ on K_s. Then, we define the edge bubble $b_e \in H^1(K_1 \cup K_2)$ by setting $b_e(x,y) = \varsigma_e^{K_1}$ if $(x,y) \in K_1$ and $b_e(x,y) = \varsigma_e^{K_2}$ if $(x,y) \in K_2$ and $b_e(x,y) = 0$ elsewhere. We can easily see that $b_e|_e \in H_0^1(e)$ where $b_e|_e$ stands for the restriction to e of b_e. Let e be the edge of K on $\partial\Omega$, we similarly define an edge bubble $b_e \in H^1(K)$ satisfying $b_e|_e \in H_0^1(e)$, which also satisfies $b_e|_{e'} = 0$ for $e' \neq e \in \partial K$. In addition, we denote by $b_K \in H_0^1(K)$ the element-bubble function on K, which is zero elsewhere and satisfies $\int_K b_K \approx C\, h_K^2$.

Denote by $E^\partial(\mathcal{C}_{\partial\Omega}) \subset E(\mathcal{C}_{\partial\Omega})$ the set of all edges on $\partial\Omega$. We define

$$M_h^0 = \text{span}\{\mathcal{P}_0(e)\, b_e; e \in E^\partial(\mathcal{C}_{\partial\Omega})\}, \tag{3.3}$$

$$M_h^1 = \text{span}\{\mathcal{P}_1(e)\, b_e; e \in E^\partial(\mathcal{C}_{\partial\Omega})\}, \tag{3.4}$$

$$B_h^0 = \text{span}\{\mathcal{P}_0(K)\, b_K; K \in \mathcal{C}_h\}, \tag{3.5}$$

$$\boldsymbol{B}_h^{RT} = \text{span}\{S(K)\, b_K; K \in \mathcal{C}_h\}, \tag{3.6}$$

$$\boldsymbol{B}_h^1 = \text{span}\{(\mathcal{P}_1(K))^2\, b_K; K \in \mathcal{C}_h\}. \tag{3.7}$$

We define

$$V_h = \{v \in H^1(\Omega); v|_K \in \mathcal{P}_1(K), \forall K \in \mathcal{C}_h\}, \tag{3.8}$$

$$S_h^{RT} = \{\boldsymbol{q} \in H(\text{div};\Omega); \boldsymbol{q}|_K \in S(K), \forall K \in \mathcal{C}_h\}, \tag{3.9}$$

$$S_h^1 = \{\boldsymbol{q} \in H(\text{div};\Omega); \boldsymbol{q}|_K \in (\mathcal{P}_1(K))^2, \forall K \in \mathcal{C}_h\}, \tag{3.10}$$

$$V_{0,h} = \{v \in V_h; v = 0 \quad \text{at vertices on } \partial\Omega\}, \tag{3.11}$$

$$S_{0,h}^{RT} = \left\{\boldsymbol{q} \in S_h^{RT}; \int_e \boldsymbol{q}\cdot\boldsymbol{n} = 0, \forall e \in E^\partial(\mathcal{C}_{\partial\Omega})\right\}. \tag{3.12}$$

Thus, for various boundary conditions, we define finite element spaces in the following:

Clamped plate (Θ_h, W_h, X_h):

$$\Theta_h = (V_{0,h})^2 \oplus \boldsymbol{B}_h^{RT}, \tag{3.13}$$

$$W_h = \left\{ w \in V_h \oplus B_h^0 \oplus M_h^0;\ \begin{array}{l} w = 0 \quad \text{at all vertices on } \partial\Omega, \\ \int_e w = 0 \quad \forall e \in E^\partial(\mathcal{C}_{\partial\Omega}) \end{array} \right\}, \tag{3.14}$$

$$X_h = S_h^{RT}. \tag{3.15}$$

Soft simply-supported plate (Θ_h, W_h, X_h):

$$\Theta_h = (V_h)^2 \oplus \boldsymbol{B}_h^1, \tag{3.16}$$

$$W_h = \left\{ w \in V_h \oplus B_h^0 \oplus M_h^1;\ \begin{array}{l} w = 0 \quad \text{at all vertices on } \partial\,\Omega, \\ \int_e w\, v = 0 \quad \forall v \in \mathcal{P}_1(e), \forall e \in E^{\partial}(\mathcal{C}_{\partial\,\Omega}) \end{array} \right\}, \tag{3.17}$$

$$X_h = S_h^1. \tag{3.18}$$

Free plate (Θ_h, W_h, X_h):

$$\Theta_h = ((V_h)^2 \oplus \boldsymbol{B}_h^{RT})/\mathbb{R}^2, \tag{3.19}$$

$$W_h = (V_h \oplus B_h^0 \oplus M_h^0)/\mathbb{L}, \tag{3.20}$$

$$X_h = S_{0,h}^{RT}. \tag{3.21}$$

Remark 3.1. Note that the role of edge bubble is to deal with the curved boundary, and for piecewise straight-sided boundary, the edge bubbles obviously disappear for clamped and soft simply-supported plates while the edge bubbles can be dropped as well for free plates (see Remark 4.1 in the next section). The finite element spaces for clamped plate and free plate are the same, up to boundary conditions. So we can use the same finite element space to treat the boundary condition mixed with clamped and free conditions. The finite element space for soft simply-supported plate is a little bit bigger, with some higher-order edge bubbles along the domain boundary. Obviously, for straight-sided boundaries, the finite element space for three various plates is the same. On the other hand, whatever straight-sided or curved boundary, we can always use the finite element space of soft simply-supported plate for the other two clamped and free plates, including the plate with the boundary condition mixed with clamped, soft simply-supported and free conditions; see Remark 4.2 in the next section. □

Remark 3.2. The role of the element bubble is for the link interpolation that is established in the next section. As usual, one may statically eliminate element bubbles at element levels. With Remark 3.1, our method is essentially a linear element method. □

The L^2 projection R_h in (3.1) is defined for any $\boldsymbol{\psi} \in (L^2(\Omega))^2$ by: $R_h(\boldsymbol{\psi}) \in X_h$ satisfies

$$(R_h(\boldsymbol{\psi}), \boldsymbol{\chi}) = (\boldsymbol{\psi}, \boldsymbol{\chi}) \quad \forall \boldsymbol{\chi} \in X_h. \tag{3.22}$$

We can easily verify that this L^2 projection satisfies

$$||R_h(\boldsymbol{\psi}) - \boldsymbol{\psi}||_0 \le C\, h\, ||\boldsymbol{\psi}||_1, \quad \boldsymbol{\psi} \in (H^1(\Omega))^2. \tag{3.23}$$

4. Error estimates

In this section we analyze the errors between the exact solution of (2.6) and the finite element solution of (3.1). The error estimates consist of mainly the link interpolation of the transverse displacement and the rotation (see Lemma 4.1 below) and the consistency error estimates (see Lemma 4.3 below). These two very technical lemmas are elaborated in details for different boundary conditions.

Lemma 4.1. *Let* $(\boldsymbol{\phi}, w) \in ((H^{1+r}(\Omega))^2 \cap \Theta) \times (H^2(\Omega) \cap W)$, *where* $r = 1$ *for clamped plate and* $r = 1/2$ *for soft simply-supported and free plates. There exists* $(\tilde{\boldsymbol{\phi}}, \tilde{w}) \in \Theta_h \times W_h$ *such that*

$$R_h(\nabla\,(w - \tilde{w}) - (\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})) = \mathbf{0}, \tag{4.1}$$

$$||\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}||_1 \le C\,h^r\,(||\boldsymbol{\phi}||_{1+r} + ||w||_2), \quad ||w - \tilde{w}||_1 \le C\,h\,||w||_2. \tag{4.2}$$

Proof. We first consider clamped plate. Since $w \in H^2(\Omega) \cap H_0^1(\Omega)$, we have $w^* \in V_{0,h}$ such that ([41], [45], [10])

$$\left(\sum_{K \in \mathcal{C}_h} h_K^{-2}\,||w - w^*||_{0,K}^2\right)^{1/2} + \left(\sum_{e \in E_h} h_e^{-1}\,||w - w^*||_{0,e}^2\right)^{1/2} + ||w - w^*||_1 \le C\,h\,||w||_2, \tag{4.3}$$

where E_h denotes the set of all edges of $\mathcal{C}_h$. We then define $w^1 \in W_h$ by

$$w^1 = w^* + w^\partial, \quad w^\partial = \sum_{e \in E^\partial(\mathcal{C}_{\partial\,\Omega})} c_e\,b_e, \tag{4.4}$$

where we determine w^∂ by setting

$$\int_e w^1 = 0, \qquad \text{i.e.,} \quad c_e = \frac{-\int_e w^*}{\int_e b_e} \quad \forall e \in E^\partial(\mathcal{C}_{\partial\,\Omega}). \tag{4.5}$$

We see that $w^1 = w^* + w^\partial$ satisfies

$$w^1 = w^* = 0 \quad \text{at all vertices on } \partial\,\Omega, \tag{4.6}$$

$$\int_e w^1 = 0 \quad \forall e \in E^\partial(\mathcal{C}_{\partial\,\Omega}), \tag{4.7}$$

(4.8)

$$\left(\sum_{K \in \mathcal{C}_h} h_K^{-2}\,||w - w^1||_{0,K}^2\right)^{1/2} + ||w - w^1||_1 \le C\left(\sum_{K \in \mathcal{C}_h} h_K^{-2}\,||w - w^*||_{0,K}^2\right)^{1/2} + C\left(\sum_{e \in E^\partial(\mathcal{C}_{\partial\,\Omega})} h_e^{-1}\,||w - w^*||_{0,e}^2\right)^{1/2} + ||w - w^*||_1 \le C\,h\,||w||_2.$$

We finally define $\tilde{w} \in W_h$ by

$$\tilde{w} = w^1 + w^b, \quad w^b = \sum_{K \in \mathcal{C}_h} c_K\,b_K, \tag{4.9}$$

where we determine w^b by setting

$$\int_K (w - \tilde{w}) = 0, \qquad \text{i.e.,} \quad c_K = \frac{\int_K (w - w^1)}{\int_K b_K} \quad \forall K \in \mathcal{C}_h. \tag{4.10}$$

We see that

$$\tilde{w} = w^1 = 0 \quad \text{at all vertices on } \partial\,\Omega, \tag{4.11}$$

$$\int_e \tilde{w} = \int_e w^1 = 0 \quad \forall e \in E^{\partial}(\mathcal{C}_{\partial\,\Omega}), \tag{4.12}$$

$$\int_K (w - \tilde{w}) = 0 \quad \forall K \in \mathcal{C}_h, \tag{4.13}$$

$$\begin{aligned} ||w - \tilde{w}||_1 &\le ||w - w^1||_1 + C \left(\sum_{K \in \mathcal{C}_h} h_K^{-2} \, ||w - w^1||_{0,K}^2 \right)^{1/2} \\ &\le C\, h \, ||w||_2. \end{aligned} \tag{4.14}$$

In what follows we are to define $\tilde{\boldsymbol{\phi}}$. Letting $\boldsymbol{\phi}^* \in (V_{0,h})^2$ be the finite element interpolant to $\boldsymbol{\phi}$ such that

$$\left(\sum_{K \in \mathcal{C}_h} h_K^{-2} \, ||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_{0,K}^2 \right)^{1/2} + ||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_1 \le C\, h \, ||\boldsymbol{\phi}||_2, \tag{4.15}$$

we first define $\boldsymbol{\phi}^1 \in \Theta_h$ by

$$\boldsymbol{\phi}^1 = \boldsymbol{\phi}^* + \boldsymbol{\phi}^b, \quad \boldsymbol{\phi}^b = \sum_{K \in \mathcal{C}_h} \sum_{i=1}^{3} c_{Ki} \, \boldsymbol{\chi}_i \, b_K, \tag{4.16}$$

where $\boldsymbol{\chi}_i$, $1 \le i \le 3$, is the basis of $S(K)$, and $\boldsymbol{\phi}^b$ is determined by setting

$$\int_K (\boldsymbol{\phi} - \boldsymbol{\phi}^1) \, \boldsymbol{\chi} = 0 \quad \forall \boldsymbol{\chi} \in S(K), \forall K \in \mathcal{C}_h, \tag{4.17}$$

i.e.,

$$\sum_{i=1}^{3} c_{Ki} \int_K \boldsymbol{\chi}_i \, \boldsymbol{\chi}_j \, b_K = \int_K (\boldsymbol{\phi} - \boldsymbol{\phi}^*) \, \boldsymbol{\chi}_j, \quad 1 \le i \le 3, 1 \le j \le 3. \tag{4.18}$$

We see that

$$\boldsymbol{\phi}^1 = \mathbf{0} \quad \text{at all vertices on } \partial\,\Omega, \tag{4.19}$$

$$(\boldsymbol{\phi} - \boldsymbol{\phi}^1, \boldsymbol{\chi}) = 0 \quad \forall \boldsymbol{\chi} \in X_h = S_h^{RT}, \tag{4.20}$$

$$||\boldsymbol{\phi} - \boldsymbol{\phi}^1||_1 \le \left(\sum_{K \in \mathcal{C}_h} h_K^{-2} \, ||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_{0,K}^2 \right)^{1/2} + ||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_1 \le C\, h \, ||\boldsymbol{\phi}||_2. \tag{4.21}$$

We now define $\tilde{\boldsymbol{\phi}} \in \Theta_h$ by

$$\tilde{\boldsymbol{\phi}} = \boldsymbol{\phi}^1 + \boldsymbol{\phi}^2, \quad \boldsymbol{\phi}^2 = \sum_{K \in \mathcal{C}_{\partial\,\Omega}} \sum_{i=1}^{3} c_{Ki} \, \boldsymbol{\chi}_i \, b_K, \quad \boldsymbol{\phi}^2|_K = \mathbf{0} \quad \forall K \in \mathcal{C}_h \setminus \mathcal{C}_{\partial\,\Omega}, \tag{4.22}$$

where $\boldsymbol{\phi}^2$ is determined by setting

$$-\int_e \tilde{w} \, \boldsymbol{\chi} \cdot \boldsymbol{n} + (\boldsymbol{\phi}^2, \boldsymbol{\chi})_{0,K} = 0, \quad e \in \partial\, K \cap \partial\,\Omega, \forall \boldsymbol{\chi} \in S(K), \forall K \in \mathcal{C}_{\partial\,\Omega}, \tag{4.23}$$

i.e.,

$$\sum_{i=1}^{3} c_{Ki} \int_K \boldsymbol{\chi}_i \, \boldsymbol{\chi}_j \, b_K = \int_e \tilde{w} \, \boldsymbol{\chi}_j \cdot \boldsymbol{n} \quad 1 \le i \le 3, 1 \le j \le 3. \tag{4.24}$$

Noting that

$$\boldsymbol{n} = \boldsymbol{n}_0 + \boldsymbol{n}_1, \tag{4.25}$$

where $\boldsymbol{n}_0$ is the unit normal to the straight side e_s which has the same two vertices as the curved edge e, and $\boldsymbol{n}_1 = \mathcal{O}(h_K)$, and that

$$\boldsymbol{\chi} \cdot \boldsymbol{n}_0|_{e_s} = c_s \quad \text{constant}, \tag{4.26}$$

and that $\int_e \tilde{w} = 0$, we have

$$\int_e \tilde{w}\, \boldsymbol{\chi} \cdot \boldsymbol{n}_0 = \int_e \tilde{w}\, (\boldsymbol{\chi} \cdot \boldsymbol{n}_0 - c_s) \le ||\tilde{w}||_{0,e}\, ||\boldsymbol{\chi} \cdot \boldsymbol{n}_0 - c_s||_{0,e}, \tag{4.27}$$

where, since $\boldsymbol{\chi} \cdot \boldsymbol{n}_0 - c_s$ is zero at the two ends of e and $\boldsymbol{\chi} \cdot \boldsymbol{n}_0 - c_s$ is linear on K,

$$\begin{aligned} ||\boldsymbol{\chi} \cdot \boldsymbol{n}_0 - c_s||_{0,e} &\le C h_e^{-\frac{1}{2}} ||\boldsymbol{\chi} \cdot \boldsymbol{n}_0 - c_s||_{0,K} \\ &\le C h_e^{-\frac{1}{2}} h_K^2 ||\boldsymbol{\chi} \cdot \boldsymbol{n}_0 - c_s||_{2,K} \le C h_e^{-\frac{1}{2}} h_K^2 ||\boldsymbol{\chi}||_{1,K} \\ &\le C\, h_e^{-\frac{1}{2}}\, h_K ||\boldsymbol{\chi}||_{0,K} \le C\, h_e^{\frac{1}{2}}\, ||\boldsymbol{\chi}||_{0,K}, \end{aligned} \tag{4.28}$$

and similarly, $\tilde{w}$ is zero at the two ends of e, we have

$$||\tilde{w}||_{0,e} \le C\, h_e^{\frac{3}{2}}\, ||\tilde{w}||_{2,K}. \tag{4.29}$$

On the other hand,

$$\int_e \tilde{w}\, \boldsymbol{\chi} \cdot \boldsymbol{n}_1 \le C\, h_e\, ||\tilde{w}||_{0,e}\, ||\boldsymbol{\chi}||_{0,e} \le C\, h_e^{\frac{1}{2}}\, ||\tilde{w}||_{0,e}\, ||\boldsymbol{\chi}||_{0,K}. \tag{4.30}$$

From (4.23), (4.25), (4.27)–(4.30) we then obtain

$$||\boldsymbol{\phi}^2||_{0,K} \le C\, h_e^{\frac{1}{2}}\, ||\tilde{w}||_{0,e} \le C\, h_e^2\, ||\tilde{w}||_{2,K} \le C\, h_K^2\, ||\tilde{w}||_{2,K}, \tag{4.31}$$

from which, by a local inverse estimate, we have

$$||\boldsymbol{\phi}^2||_{1,K} \le C h_K^{-1} ||\boldsymbol{\phi}^2||_{0,K} \le C\, h_K\, ||\tilde{w}||_{2,K} \le C\, h_K\, ||w||_{2,K}. \tag{4.32}$$

We therefore see that

$$||\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}||_1 \le ||\boldsymbol{\phi} - \boldsymbol{\phi}^1||_1 + ||\boldsymbol{\phi}^2||_1 \le C\, h\, (||\boldsymbol{\phi}||_2 + ||w||_2). \tag{4.33}$$

We further see that

$$R_h(\nabla\, (w - \tilde{w}) - (\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})) = \mathbf{0}, \tag{4.34}$$

since from (4.13), (4.17) and (4.23) we have

$$\begin{aligned} (R_h(\nabla\, (w - \tilde{w}) - (\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})), \boldsymbol{\chi}) &= (\nabla\, (w - \tilde{w}) - (\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}), \boldsymbol{\chi}) \\ &= (w - \tilde{w}, \operatorname{div} \boldsymbol{\chi}) - (\boldsymbol{\phi} - \boldsymbol{\phi}^1, \boldsymbol{\chi}) \\ &+ \int_{\partial\,\Omega} (w - \tilde{w})\, \boldsymbol{\chi} \cdot \boldsymbol{n} + (\boldsymbol{\phi}^2, \boldsymbol{\chi}) \\ &= \sum_{e \in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} \int_e (w - \tilde{w})\, \boldsymbol{\chi} \cdot \boldsymbol{n} + \sum_{K \in \mathcal{C}_{\partial\,\Omega}} (\boldsymbol{\phi}^2, \boldsymbol{\chi})_{0,K} \\ &= 0, \quad \forall \boldsymbol{\chi} \in X_h, \end{aligned} \tag{4.35}$$

where we have used the fact that $\operatorname{div} \boldsymbol{\chi}|_K \in \mathcal{P}_0(K)$ for all $\boldsymbol{\chi} \in X_h$ and for all $K \in \mathcal{C}_h$.

Next we consider soft simply-supported plate. The argument is similar. In fact, let $w^* \in V_{0,h}$ be the finite element interpolant to $w \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\,||w-w^*||_{0,K}^2\right)^{1/2} + \left(\sum_{e\in E_h} h_e^{-1}\,||w-w^*||_{0,e}^2\right)^{1/2} + ||w-w^*||_1 \le C\,h\,||w||_2. \tag{4.36}$$

We then define $w^1 \in W_h$ by

$$w^1 = w^* + w^\partial, \quad w^\partial = \sum_{e\in E^\partial(\mathcal{C}_{\partial\,\Omega})} \sum_{i=1}^{2} c_{ei}\,v_i\,b_e, \tag{4.37}$$

where v_i, $1 \le i \le 2$, is the basis of $\mathcal{P}_1(e)$, and we determine w^∂ by setting

$$\int_e w^1\,v = 0 \quad \forall v \in \mathcal{P}_1(e), \forall e \in E^\partial(\mathcal{C}_{\partial\,\Omega}), \tag{4.38}$$

with

$$\sum_{i=1}^{2} c_{ei} \int_e v_i\,v_j\,b_e = -\int_e w^*\,v_j, \quad 1 \le i \le 2, 1 \le j \le 2. \tag{4.39}$$

We see that $w^1 = w^* + w^\partial$ satisfies

$$w^1 = w^* = 0 \quad \text{at all verices on } \partial\,\Omega, \tag{4.40}$$

$$\int_e w^1\,v = 0 \quad \forall v \in \mathcal{P}_1(e), \forall e \in E^\partial(\mathcal{C}_{\partial\,\Omega}), \tag{4.41}$$

(4.42)

$$\begin{aligned}\left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\,||w-w^1||_{0,K}^2\right)^{1/2} &+ ||w-w^1||_1 \\ &\le \left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\,||w-w^*||_{0,K}^2\right)^{1/2} + C\left(\sum_{e\in E^\partial(\mathcal{C}_{\partial\,\Omega})} h_e^{-1}\,||w-w^*||_{0,e}^2\right)^{1/2} \\ &+ ||w-w^*||_1 \le C\,h\,||w||_2.\end{aligned}$$

We finally define $\tilde{w} \in W_h$ by

$$\tilde{w} = w^1 + w^b, \quad w^b = \sum_{K\in\mathcal{C}_h} c_K\,b_K, \tag{4.43}$$

where we determine w^b by setting

$$\int_K (w-\tilde{w}) = 0, \qquad \text{i.e.,} \quad c_K = \frac{\int_K (w-w^1)}{\int_K b_K} \quad \forall K \in \mathcal{C}_h. \tag{4.44}$$

We see that

$$\tilde{w} = w^1 = 0 \quad \text{at all verices on } \partial\,\Omega, \tag{4.45}$$

$$\int_e \tilde{w}\,v = \int_e w^1\,v = 0 \quad \forall v \in \mathcal{P}_1(e), \quad \forall e \in E^\partial(\mathcal{C}_{\partial\,\Omega}), \tag{4.46}$$

$$\int_K (w - \tilde{w}) = 0 \quad \forall K \in \mathcal{C}_h, \tag{4.47}$$

$$||w - \tilde{w}||_1 \leq C \left(\sum_{K \in \mathcal{C}_h} h_K^{-2} \, ||w - w^1||_{0,K}^2 \right)^{1/2} + ||w - w^1||_1 \leq C \, h \, ||w||_2. \tag{4.48}$$

To define a $\tilde{\boldsymbol{\phi}}$, letting $\boldsymbol{\phi}^* \in (V_h)^2$ be the finite element interpolant to $\boldsymbol{\phi} \in (H^{\frac{3}{2}}(\Omega))^2$ such that

$$\left(\sum_{K \in \mathcal{C}_h} h_K^{-2} \, ||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_{0,K}^2 \right)^{1/2} + ||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_1 \leq C \, h^{1/2} \, ||\boldsymbol{\phi}||_{3/2}, \tag{4.49}$$

we first define $\boldsymbol{\phi}^1 \in \Theta_h$ by

$$\boldsymbol{\phi}^1 = \boldsymbol{\phi}^* + \boldsymbol{\phi}^b, \quad \boldsymbol{\phi}^b = \sum_{K \in \mathcal{C}_h} \sum_{i=1}^{6} c_{Ki} \, \boldsymbol{\chi}_i \, b_K, \tag{4.50}$$

where $\boldsymbol{\chi}_i$, $1 \leq i \leq 6$, is the basis of $(\mathcal{P}_1(K))^2$, and $\boldsymbol{\phi}^b$ is determined by

$$\int_K (\boldsymbol{\phi} - \boldsymbol{\phi}^1) \, \boldsymbol{\chi} = 0 \quad \forall \boldsymbol{\chi} \in (\mathcal{P}_1(K))^2, \forall K \in \mathcal{C}_h, \tag{4.51}$$

with

$$\sum_{i=1}^{6} c_{Ki} \int_K \boldsymbol{\chi}_i \, \boldsymbol{\chi}_j \, b_K = \int_K (\boldsymbol{\phi} - \boldsymbol{\phi}^*) \, \boldsymbol{\chi}_j, \quad 1 \leq i \leq 6, 1 \leq j \leq 6. \tag{4.52}$$

We see that

$$(\boldsymbol{\phi} - \boldsymbol{\phi}^1, \boldsymbol{\chi}) = 0 \quad \forall \boldsymbol{\chi} \in X_h, \tag{4.53}$$

$$||\boldsymbol{\phi} - \boldsymbol{\phi}^1||_1 \leq C \left(\sum_{K \in \mathcal{C}_h} h_K^{-2} \, ||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_{0,K}^2 \right)^{1/2} + ||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_1 \leq C \, h^{1/2} \, ||\boldsymbol{\phi}||_{3/2}. \tag{4.54}$$

We now define $\tilde{\boldsymbol{\phi}} \in \Theta_h$ by

$$\tilde{\boldsymbol{\phi}} = \boldsymbol{\phi}^1 + \boldsymbol{\phi}^2, \quad \boldsymbol{\phi}^2 = \sum_{K \in \mathcal{C}_{\partial\,\Omega}} \sum_{i=1}^{6} c_{Ki} \, \boldsymbol{\chi}_i \, b_K, \quad \boldsymbol{\phi}^2|_K = \mathbf{0} \quad \forall K \in \mathcal{C}_h / \mathcal{C}_{\partial\,\Omega}, \tag{4.55}$$

where $\boldsymbol{\phi}^2$ is determined by setting

$$-\int_e \tilde{w} \, \boldsymbol{\chi} \cdot \boldsymbol{n} + (\boldsymbol{\phi}^2, \boldsymbol{\chi})_{0,K} = 0, \; e \in \partial\, K \cap \partial\, \Omega, \; \forall \boldsymbol{\chi} \in (\mathcal{P}_1(K))^2, \forall K \in \mathcal{C}_{\partial\,\Omega}, \tag{4.56}$$

with

$$\sum_{i=1}^{6} c_{Ki} \int_K \boldsymbol{\chi}_i \, \boldsymbol{\chi}_j \, b_K = \int_e \tilde{w} \, \boldsymbol{\chi}_j \cdot \boldsymbol{n} \quad 1 \leq i \leq 6, 1 \leq j \leq 6. \tag{4.57}$$

Noting that $\boldsymbol{n} = \boldsymbol{n}_0 + \boldsymbol{n}_1$ and that

$$\boldsymbol{\chi} \cdot \boldsymbol{n}_0|_e \in \mathcal{P}_1(e), \tag{4.58}$$

$$\int_e \tilde{w} \, \boldsymbol{\chi} \cdot \boldsymbol{n}_0 = 0, \tag{4.59}$$

similar to (4.31), we can obtain

$$||\boldsymbol{\phi}^2||_{0,K} \leq C\, h_e^{\frac{1}{2}}\, ||\tilde{w}||_{0,e} \leq C\, h_K^2\, ||\tilde{w}||_{2,K}, \tag{4.60}$$

from which, by a local inverse estimate,

$$||\boldsymbol{\phi}^2||_{1,K} \leq C\, h_K\, ||\tilde{w}||_{2,K} \leq C h_K ||w||_{2,K}. \tag{4.61}$$

We therefore see that

$$||\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}||_1 \leq ||\boldsymbol{\phi} - \boldsymbol{\phi}^1||_1 + ||\boldsymbol{\phi}^2||_1 \leq C\, h^{1/2}\, (||\boldsymbol{\phi}||_{3/2} + ||w||_2). \tag{4.62}$$

We further see that

$$R_h(\nabla\,(w - \tilde{w}) - (\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})) = \mathbf{0}, \tag{4.63}$$

since

$$\begin{aligned} (R_h(\nabla\,(w - \tilde{w}) - (\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})), \boldsymbol{\chi}) &= (\nabla\,(w - \tilde{w}) - (\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}), \boldsymbol{\chi}) \\ &= (w - \tilde{w}, \operatorname{div} \boldsymbol{\chi}) - (\boldsymbol{\phi} - \boldsymbol{\phi}^1, \boldsymbol{\chi}) \\ &+ \int_{\partial\,\Omega} (w - \tilde{w})\, \boldsymbol{\chi} \cdot \boldsymbol{n} + (\boldsymbol{\phi}^2, \boldsymbol{\chi}) \\ &= \sum_{e \in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} \int_e (w - \tilde{w})\, \boldsymbol{\chi} \cdot \boldsymbol{n} + \sum_{K \in \mathcal{C}_{\partial\,\Omega}} (\boldsymbol{\phi}^2, \boldsymbol{\chi})_{0,K} \\ &= 0, \quad \forall \boldsymbol{\chi} \in X_h, \end{aligned} \tag{4.64}$$

where we have used the fact that $\operatorname{div} \boldsymbol{\chi}|_K \in \mathcal{P}_0(K)$ for all $K \in \mathcal{C}_h$ and for all $\boldsymbol{\chi} \in X_h$.

Then we consider free plate. The argument is still similar. In fact, let $w^* \in V_h/\mathbb{L}$ be the finite element interpolant to $w \in H^2(\Omega) \cap W$ such that

$$\begin{aligned} \left(\sum_{K \in \mathcal{C}_h} h_K^{-2}\, ||w - w^*||_{0,K}^2 \right)^{1/2} &+ \left(\sum_{e \in E_h} h_e^{-1}\, ||w - w^*||_{0,e}^2 \right)^{1/2} \\ &+ ||w - w^*||_1 \leq C\, h\, ||w||_2. \end{aligned} \tag{4.65}$$

We then define $w^1 \in W_h$ by

$$w^1 = w^* + w^{\partial}, \quad w^{\partial} = \sum_{e \in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} c_e\, b_e, \tag{4.66}$$

where we determine w^{∂} by setting

$$\int_e (w - w^1) = 0, \qquad \text{i.e.,} \quad c_e = \frac{\int_e (w - w^*)}{\int_e b_e} \quad \forall e \in E^{\partial}(\mathcal{C}_{\partial\,\Omega}). \tag{4.67}$$

We see that $w^1 = w^* + w^{\partial}$ satisfies

$$\int_e (w - w^1) = 0 \quad \forall e \in E^{\partial}(\mathcal{C}_{\partial\,\Omega}), \tag{4.68}$$

$$\left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\,||w-w^1||_{0,K}^2\right)^{1/2} + \left(\sum_{e\in E^\partial(\mathcal{C}_{\partial\,\Omega})} h_e^{-1}\,||w-w^1||_{0,e}^2\right)^{1/2} + ||w-w^1||_1 \le \left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\,||w-w^*||_{0,K}^2\right)^{1/2} + C\left(\sum_{e\in E^\partial(\mathcal{C}_{\partial\,\Omega})} h_e^{-1}\,||w-w^*||_{0,e}^2\right)^{1/2} + ||w-w^*||_1 \le C\,h\,||w||_2. \tag{4.69}$$

We finally define $\tilde{w}\in W_h$ by

$$\tilde{w} = w^1 + w^b, \quad w^b = \sum_{K\in\mathcal{C}_h} c_K\,b_K, \tag{4.70}$$

where we determine w^b by setting

$$\int_K (w-\tilde{w}) = 0, \qquad \text{i.e.,} \quad c_K = \frac{\int_K (w-w^1)}{\int_K b_K} \quad \forall K\in\mathcal{C}_h. \tag{4.71}$$

We see that

$$\int_e (w-\tilde{w}) = \int_e (w-w^1) = 0 \quad \forall e\in E^\partial(\mathcal{C}_{\partial\,\Omega}), \tag{4.72}$$

$$\int_K (w-\tilde{w}) = 0 \quad \forall K\in\mathcal{C}_h, \tag{4.73}$$

$$\left(\sum_{e\in E^\partial(\mathcal{C}_{\partial\,\Omega})} h_e^{-1}\,||w-\tilde{w}||_{0,e}^2\right)^{1/2} + ||w-\tilde{w}||_1 \le C\left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\,||w-w^1||_{0,K}^2\right)^{1/2} + C\left(\sum_{e\in E^\partial(\mathcal{C}_{\partial\,\Omega})} h_e^{-1}\,||w-w^1||_{0,e}^2\right)^{1/2} + ||w-w^1||_1 \le C\,h\,||w||_2. \tag{4.74}$$

Now we define $\tilde{\boldsymbol{\phi}}$. Let $\boldsymbol{\phi}^*\in(V_h)^2/\mathbb{R}^2$ be the finite element interpolant to $\boldsymbol{\phi}\in(H^{\frac{3}{2}}(\Omega))^2\cap\Theta$ such that

$$\left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\,||\boldsymbol{\phi}-\boldsymbol{\phi}^*||_{0,K}^2\right)^{1/2} + ||\boldsymbol{\phi}-\boldsymbol{\phi}^*||_1 \le C\,h^{1/2}\,||\boldsymbol{\phi}||_{3/2}. \tag{4.75}$$

We then define $\boldsymbol{\phi}^1\in\Theta_h$ by

$$\boldsymbol{\phi}^1 = \boldsymbol{\phi}^* + \boldsymbol{\phi}^b, \quad \boldsymbol{\phi}^b = \sum_{K\in\mathcal{C}_h}\sum_{i=1}^{3} c_{Ki}\,\boldsymbol{\chi}_i\,b_K, \tag{4.76}$$

where $\boldsymbol{\chi}_i$, $1\le i\le 3$, is the basis of $S(K)$, and $\boldsymbol{\phi}^b$ is determined by

$$\int_K (\boldsymbol{\phi}-\boldsymbol{\phi}^1)\,\boldsymbol{\chi} = 0 \quad \forall\boldsymbol{\chi}\in S(K),\ \forall K\in\mathcal{C}_h, \tag{4.77}$$

i.e.,

$$\sum_{i=1}^{3} c_{Ki} \int_K \boldsymbol{\chi}_i \, \boldsymbol{\chi}_j \, b_K = \int_K (\boldsymbol{\phi} - \boldsymbol{\phi}^*) \, \boldsymbol{\chi}_j, \quad 1 \le i \le 3, 1 \le j \le 3. \tag{4.78}$$

We see that

$$(\boldsymbol{\phi} - \boldsymbol{\phi}^1, \boldsymbol{\chi}) = 0 \quad \forall \boldsymbol{\chi} \in S_h, \tag{4.79}$$

$$||\boldsymbol{\phi} - \boldsymbol{\phi}^1||_1 \le C \left(\sum_{K \in \mathcal{C}_h} h_K^{-2} \, ||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_{0,K}^2 \right)^{1/2} + ||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_1 \le C \, h^{1/2} \, ||\boldsymbol{\phi}||_{3/2}. \tag{4.80}$$

We now define $\tilde{\boldsymbol{\phi}} \in \Theta_h$ by

$$\tilde{\boldsymbol{\phi}} = \boldsymbol{\phi}^1 + \boldsymbol{\phi}^2, \quad \boldsymbol{\phi}^2 = \sum_{K \in \mathcal{C}_{\partial\,\Omega}} \sum_{i=1}^{3} c_{Ki} \, \boldsymbol{\chi}_i \, b_K, \quad \boldsymbol{\phi}^2|_K = \mathbf{0} \quad \forall K \in \mathcal{C}_h / \mathcal{C}_{\partial\,\Omega}, \tag{4.81}$$

where $\boldsymbol{\phi}^2$ is determined by setting

$$\int_e (w - \tilde{w}) \, \boldsymbol{\chi} \cdot \boldsymbol{n} + (\boldsymbol{\phi}^2, \boldsymbol{\chi})_{0,K} = 0, \quad e \in \partial\, K \cap \partial\, \Omega, \forall \boldsymbol{\chi} \in S(K), \forall K \in \mathcal{C}_{\partial\,\Omega}, \tag{4.82}$$

i.e.,

$$\sum_{i=1}^{3} c_{Ki} \int_K \boldsymbol{\chi}_i \, \boldsymbol{\chi}_j \, b_K = - \int_e (w - \tilde{w}) \, \boldsymbol{\chi}_j \cdot \boldsymbol{n} \quad 1 \le i \le 3, 1 \le j \le 3. \tag{4.83}$$

Noting that $\boldsymbol{n} = \boldsymbol{n}_0 + \boldsymbol{n}_1$, and that

$$\boldsymbol{\chi} \cdot \boldsymbol{n}_0|_{e_s} = c_s \quad \text{constant}, \tag{4.84}$$

and that

$$\int_e (w - \tilde{w}) \, \boldsymbol{\chi} \cdot \boldsymbol{n}_0 = \int_e (w - \tilde{w}) \, (\boldsymbol{\chi} \cdot \boldsymbol{n}_0 - c_s) \le C \, h_e^{\frac{1}{2}} \, ||w - \tilde{w}||_{0,e} \, ||\boldsymbol{\chi}||_{0,K}, \tag{4.85}$$

$$\int_e (w - \tilde{w}) \, \boldsymbol{\chi} \cdot \boldsymbol{n}_1 \le C \, h_e^{\frac{1}{2}} \, ||w - \tilde{w}||_{0,e} \, ||\boldsymbol{\chi}||_{0,K},$$

from (4.82) and the above two estimates we can obtain

$$||\boldsymbol{\phi}^2||_{0,K} \le C \, h_K^{\frac{1}{2}} \, ||\tilde{w} - w||_{0,e}, \tag{4.86}$$

from which, by a local inverse estimate,

$$||\boldsymbol{\phi}^2||_{1,K} \le C \, h_K^{-\frac{1}{2}} \, ||\tilde{w} - w||_{0,e}. \tag{4.87}$$

We therefore see that

$$\begin{aligned} ||\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}||_1 &\le ||\boldsymbol{\phi} - \boldsymbol{\phi}^1||_1 + ||\boldsymbol{\phi}^2||_1 \\ &\le C \, h^{\frac{1}{2}} \, ||\boldsymbol{\phi}||_{3/2} + C \left(\sum_{e \in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} h_e^{-1} \, ||w - \tilde{w}||_{0,e}^2 \right)^{1/2} \\ &\le C \, h^{1/2} \, (||\boldsymbol{\phi}||_{3/2} + ||w||_2). \end{aligned} \tag{4.88}$$

We further see that

$$R_h(\nabla \, (w - \tilde{w}) - (\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})) = \mathbf{0}, \tag{4.89}$$

since

$$
\begin{aligned}
(4.90)\qquad (R_h(\nabla(w-\tilde{w})-(\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}})),\boldsymbol{\chi}) &= (\nabla(w-\tilde{w})-(\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}),\boldsymbol{\chi})\\
&= (w-\tilde{w},\operatorname{div}\boldsymbol{\chi})-(\boldsymbol{\phi}-\boldsymbol{\phi}^1,\boldsymbol{\chi})\\
&+\int_{\partial\Omega}(w-\tilde{w})\,\boldsymbol{\chi}\cdot\boldsymbol{n}+(\boldsymbol{\phi}^2,\boldsymbol{\chi})\\
&=\sum_{e\in E^{\partial}(\mathcal{C}_{\partial\Omega})}\int_e(w-\tilde{w})\,\boldsymbol{\chi}\cdot\boldsymbol{n}+\sum_{K\in\mathcal{C}_{\partial\Omega}}(\boldsymbol{\phi}^2,\boldsymbol{\chi})_{0,K}\\
&=0,\quad\forall\boldsymbol{\chi}\in X_h,
\end{aligned}
$$

where we have used the fact that $\operatorname{div}\boldsymbol{\chi}|_K\in\mathcal{P}_0(K)$ for all $K\in\mathcal{C}_h$ and for all $\boldsymbol{\chi}\in X_h$. □

Remark 4.1. For free plate we can alternatively define $X_{0,h}$ as follows:

$$
X_{0,h}=\{\boldsymbol{\chi}\in S_h^{RT};\boldsymbol{\chi}\cdot\boldsymbol{n}_0(a_1)=\boldsymbol{\chi}\cdot\boldsymbol{n}_0(a_2)=0\quad\text{with } a_1 \text{ and } a_2 \text{ two ends of } e_s,\ \forall e\in E^{\partial}(\mathcal{C}_{\partial\Omega})\},
$$

where e_s is the straight-side having the same two vertices of e and $\boldsymbol{n}_0$ is the unit normal to e_s. Then we can drop the edge bubbles from W_h along $\partial\Omega$. In fact, in that case, since w^{∂} in (4.66) is dropped, only the way in which we estimate the left-most side of (4.85) is different. Noting that we have $\tilde{w}=w^1+w^b=w^*+w^b$, and

$$
\int_e(w-\tilde{w})\,\boldsymbol{\chi}\cdot\boldsymbol{n}=\int_e(w-\tilde{w})\,\boldsymbol{\chi}\cdot\boldsymbol{n}_0+\int_e(w-\tilde{w})\,\boldsymbol{\chi}\cdot\boldsymbol{n}_1,
$$

where we estimate as follows:

$$
\begin{aligned}
\int_e(w-\tilde{w})\,\boldsymbol{\chi}\cdot\boldsymbol{n}_0 &\le C\,||\boldsymbol{\chi}\cdot\boldsymbol{n}_0||_{0,e}\,||w-\tilde{w}||_{0,e}\le C\,h_K^{\frac{3}{2}}\,||\boldsymbol{\chi}||_{1,K}\,||w-\tilde{w}||_{0,e}\\
&\le C\,h_K^{\frac{1}{2}}\,||\boldsymbol{\chi}||_{0,K}\,||w-\tilde{w}||_{0,e},
\end{aligned}
$$

where $\boldsymbol{\chi}\cdot\boldsymbol{n}_0$ is zero at two ends of e (noting that e and e_s have the same ends) and $\boldsymbol{\chi}\cdot\boldsymbol{n}_0|_K\in\mathcal{P}_1(K)$ is a linear polynomial. □

Remark 4.2. We have seen that the procedures for the construction of the pair $(\tilde{w},\tilde{\boldsymbol{\phi}})$ for clamped, soft simply-supported and free plates are essentially the same, up to minor adaptations corresponding to different boundary conditions, although the finite element spaces are different. In addition, we have used the different finite element spaces for different boundary conditions. However, this may not be necessary. As a matter of fact, we may use the same finite element spaces, only up to boundary conditions, for clamped, soft simply-supported and free plates. Concretely, we introduce the following three finite element spaces

$$
\begin{aligned}
\Theta_h^{\sharp}&=(V_h)^2\oplus\boldsymbol{B}_h^1,\\
W_h^{\sharp}&=V_h\oplus B_h^0\oplus M_h^1,\\
X_h^{\sharp}&=S_h^1.
\end{aligned}
$$

With the same finite element spaces as above, for clamped, soft simply-supported and free plates, we need only to impose the corresponding boundary conditions to them. Specifically, for clamped plate we have

$$
\Theta_h=\{\boldsymbol{\psi}\in\Theta_h^{\sharp}:\boldsymbol{\psi}=0\quad\text{at all vertices on }\partial\Omega\},
$$

$$W_h = \left\{ w \in W_h^\sharp;\ \begin{array}{l} w = 0 \quad \text{at all vertices on } \partial\,\Omega, \\ \int_e w\,v = 0 \quad \forall v \in \mathcal{P}_1(e), \forall e \in E^\partial(\mathcal{C}_{\partial\,\Omega}) \end{array} \right\},$$

$$X_h = X_h^\sharp,$$

while for soft simply-supported plate, we have (3.16), (3.17) and (3.18) unchanged, and regarding free plate we have

$$\Theta_h = \Theta_h^\sharp / \mathbb{R}^2,$$

$$W_h = W_h^\sharp / \mathbb{L},$$

$$X_h = \left\{ \boldsymbol{q} \in X_h^\sharp : \int_e \boldsymbol{q} \cdot \boldsymbol{n} v = 0 \quad \forall v \in \mathcal{P}_1(e), \forall e \in E^\partial(\partial\Omega) \right\}.$$

Note that the finite element space for soft simply-supported plate is not changed, and thus so are the same argument and the same conclusion in Lemma 4.1. Just following the same argument as proving Lemma 4.1, with some suitable modifications corresponding to the above new choice of finite element spaces, we still have the same conclusions for clamped and soft simply-supported plates. We also remark that if we are to deal with the boundary condition mixed with clamped, soft simply-supported and free boundary conditions, we need only to impose the essential part of the mixed boundary condition in the above triplet $(\Theta_h^\sharp, W_h^\sharp, X_h^\sharp)$. □

Remark 4.3. For inhomogeneous boundary conditions, i.e., $\boldsymbol{\phi}|_{\partial\Omega} = \boldsymbol{\phi}_0$, $w|_{\partial\Omega} = w_0$, for smooth boundary data $\boldsymbol{\phi}_0$ and w_0 we may follow the classical"interpolated" method in [41] to define the finite element solution $\boldsymbol{\phi}_h$ and w_h on $\partial\Omega$, i.e., $\boldsymbol{\phi}_h = \boldsymbol{\phi}_0$ at vertices on $\partial\Omega$, $w_h = w_0$ at vertices on $\partial\Omega$ and $\int_e w_h v = 0$ for all $v \in \mathcal{P}_r(e)$, for all $e \in E^\partial(\partial\Omega)$, where $r = 0$ for clamped plate and $r = 1$ for soft simply-supported plate, respectively. □

In what follows, we will turn to the error estimates between the finite element solution and the exact solution. We will first formulate the inconsistency errors in Lemma 4.2 and Lemma 4.3 and then give the main theorem on the error estimates in Theorem 4.1. Let

$$\mathcal{L}((\boldsymbol{\phi}, w); (\boldsymbol{\psi}, v)) = A((\boldsymbol{\phi}, w); (\boldsymbol{\psi}, v)) + Gt^{-2}\left(R_h\left(\nabla w - \boldsymbol{\phi}\right), R_h\left(\nabla v - \boldsymbol{\psi}\right)\right), \tag{4.91}$$

$$|||(\boldsymbol{\psi}, v)|||^2 := \mathcal{L}((\boldsymbol{\psi}, v); (\boldsymbol{\psi}, v)). \tag{4.92}$$

For free plate, since Θ_h and W_h are conforming, from (2.8) we have the following coercivity:

$$|||(\boldsymbol{\psi}, v)||| \geq C(||v||_1 + ||\boldsymbol{\psi}||_1).$$

For soft simply-supported plate, Θ_h is conforming and W_h has a little bit of nonconformity along $\partial\Omega$, while for clamped plate, both Θ_h and W_h have a little bit of nonconformity along $\partial\Omega$, from [41] we can show that there holds for a sufficiently small h on $\Theta_h \times W_h$,

$$A((\boldsymbol{\psi}, v); (\boldsymbol{\psi}, v)) \geq C(||v||_1^2 + ||\boldsymbol{\psi}||_1^2);$$

thus, we still have the above coercivity property with $|||\cdot|||$. Moreover, we have the generalized Cauchy-Schwarz inequality

$$|\mathcal{L}((\boldsymbol{\phi}, w); (\boldsymbol{\psi}, v))| \leq |||(\boldsymbol{\phi}, w)|||\,|||(\boldsymbol{\psi}, v)|||.$$

Lemma 4.2. *Let $(\boldsymbol{\phi}, w)$ and $(\boldsymbol{\phi}_h, w_h)$ be the exact and finite element solutions, with the shear strain $\boldsymbol{\gamma}$ being given by* (2.9). *For clamped plate, we have, for all $(\boldsymbol{\psi}, v) \in \Theta_h \times W_h$,*

$$\mathcal{L}((\boldsymbol{\phi}-\boldsymbol{\phi}_h, w-w_h); (\boldsymbol{\psi}, v)) = (R_h(\boldsymbol{\gamma}) - \boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}) + \int_{\partial\Omega} \boldsymbol{\gamma}\cdot\boldsymbol{n}\, v + \frac{1}{1-G^{-1}t^2}\int_{\partial\Omega} \boldsymbol{\psi}\cdot \mathsf{C}\boldsymbol{\mathcal{E}}(\boldsymbol{\phi})\,\boldsymbol{n} + \frac{G^{-1}t^2}{1-G^{-1}t^2}\int_{\partial\Omega} \boldsymbol{\gamma}\cdot\boldsymbol{n}\, v. \tag{4.93}$$

For soft simply-supported plate, we have, for all $(\boldsymbol{\psi}, v) \in \Theta_h \times W_h$,

$$\mathcal{L}((\boldsymbol{\phi}-\boldsymbol{\phi}_h, w-w_h); (\boldsymbol{\psi}, v)) = (R_h(\boldsymbol{\gamma}) - \boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}) + \int_{\partial\Omega} \boldsymbol{\gamma}\cdot\boldsymbol{n}\, v + \frac{G^{-1}t^2}{1-G^{-1}t^2}\int_{\partial\Omega} \boldsymbol{\gamma}\cdot\boldsymbol{n}\, v. \tag{4.94}$$

For free plate, we have, for all $(\boldsymbol{\psi}, v) \in \Theta_h \times W_h$,

$$\mathcal{L}((\boldsymbol{\phi}-\boldsymbol{\phi}_h, w-w_h); (\boldsymbol{\psi}, v)) = (R_h(\boldsymbol{\gamma}) - \boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}). \tag{4.95}$$

Proof. Let $(\boldsymbol{\phi}, w)$ be the exact solution which solves (2.1) and (2.2), together with either of the boundary conditions in (2.3)–(2.5). Taking $(\boldsymbol{\psi}, v) \in \Theta_h \times W_h$, noting $\alpha = \dfrac{1}{1-G^{-1}t^2}$ and $g = \alpha f$, from (3.1), (3.22), (2.7), (2.9), and (2.1), (2.2) we have

$$\begin{aligned}
\mathcal{L}((\boldsymbol{\phi}, w); (\boldsymbol{\psi}, v)) &= \frac{1}{1-G^{-1}t^2}\{a(\boldsymbol{\phi}, \boldsymbol{\psi}) + (G^{-1}t^2\boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi})\} + (R_h(\boldsymbol{\gamma}), \nabla v - \boldsymbol{\psi}) \\
&= \frac{1}{1-G^{-1}t^2}\{(-\mathbf{div}\,\mathsf{C}\boldsymbol{\mathcal{E}}(\boldsymbol{\phi}), \boldsymbol{\psi}) + \int_{\partial\Omega} \boldsymbol{\psi}\cdot\mathsf{C}\boldsymbol{\mathcal{E}}(\boldsymbol{\phi})\,\boldsymbol{n} + (G^{-1}t^2\boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi})\} \\
&\quad + (R_h(\boldsymbol{\gamma}) - \boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}) + (\boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}) \\
&= \frac{1}{1-G^{-1}t^2}\{(\boldsymbol{\gamma}, \boldsymbol{\psi}) + \int_{\partial\Omega} \boldsymbol{\psi}\cdot\mathsf{C}\boldsymbol{\mathcal{E}}(\boldsymbol{\phi})\,\boldsymbol{n} + (G^{-1}t^2\boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi})\} \\
&\quad + (R_h(\boldsymbol{\gamma}) - \boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}) + (f, v) + \int_{\partial\Omega} \boldsymbol{\gamma}\cdot\boldsymbol{n}\, v - (\boldsymbol{\gamma}, \boldsymbol{\psi}) \\
&= (g, v) + (R_h(\boldsymbol{\gamma}) - \boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}) + \frac{1}{1-G^{-1}t^2}\int_{\partial\Omega} \boldsymbol{\psi}\cdot\mathsf{C}\boldsymbol{\mathcal{E}}(\boldsymbol{\phi})\,\boldsymbol{n} \\
&\quad + \frac{G^{-1}t^2}{1-G^{-1}t^2}\int_{\partial\Omega} \boldsymbol{\gamma}\cdot\boldsymbol{n}\, v + \int_{\partial\Omega} \boldsymbol{\gamma}\cdot\boldsymbol{n}\, v,
\end{aligned}$$

which, together with (3.1), leads to (4.93)–(4.95) corresponding to boundary conditions (2.3)–(2.5). □

Lemma 4.3. *We have for all $(\boldsymbol{\psi}, v) \in \Theta_h \times W_h$,*

$$|\mathcal{L}((\boldsymbol{\phi}-\boldsymbol{\phi}_h, w-w_h); (\boldsymbol{\psi}, v))| \le C\, h^r\, |||(\boldsymbol{\psi}, v)|||, \tag{4.96}$$

where $r = 1$ for clamped plate and $r = 1/2$ for soft simply-supported and free plates.

Proof. We first consider clamped plate. From (4.93) we respectively estimate the following three terms:

$$\frac{1}{1-G^{-1}t^2}\int_{\partial\Omega} \boldsymbol{\psi}\cdot\mathsf{C}\boldsymbol{\mathcal{E}}(\boldsymbol{\phi})\,\boldsymbol{n}, \qquad (R_h(\boldsymbol{\gamma}) - \boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}) + \int_{\partial\Omega} \boldsymbol{\gamma}\cdot\boldsymbol{n}\, v,$$
$$\frac{G^{-1}t^2}{1-G^{-1}t^2}\int_{\partial\Omega} \boldsymbol{\gamma}\cdot\boldsymbol{n}\, v.$$

Taking $(\boldsymbol{\psi}, v) \in \Theta_h \times W_h$, where Θ_h and W_h are given by (3.13) and (3.14), writing $\boldsymbol{\psi} = \boldsymbol{\psi}^1 + \boldsymbol{\psi}^b$, with $\boldsymbol{\psi}^1 \in (V_{0,h})^2$ and $\boldsymbol{\psi}^b \in \boldsymbol{B}_h^{RT}$ and noting that $\boldsymbol{\psi}^1$ is zero at the two ends of $e \in E^{\partial}(\mathcal{C}_{\partial\,\Omega})$ and is linear on K and $\boldsymbol{\psi}^b|_e = 0$, we have $|\boldsymbol{\psi}|_{1,K}^2 = |\boldsymbol{\psi}^1|_{1,K}^2 + 2(\nabla\,\boldsymbol{\psi}^1, \nabla\,\boldsymbol{\psi}^b)_{0,K} + |\boldsymbol{\psi}^b|_{1,K}^2 = |\boldsymbol{\psi}^1|_{1,K}^2 + |\boldsymbol{\psi}^b|_{1,K}^2$ and $||\boldsymbol{\psi}^b||_{0,K} \le Ch_K|\boldsymbol{\psi}^b|_{1,K} \le Ch_K|\boldsymbol{\psi}|_{1,K}$. By noting that $||\boldsymbol{\psi}^1||_{0,K} = ||\boldsymbol{\psi} - \boldsymbol{\psi}^b||_{0,K} \le ||\boldsymbol{\psi}||_{0,K} + ||\boldsymbol{\psi}^b||_{0,K}$, we then have

$$\begin{aligned} ||\boldsymbol{\psi}^1||_{1,K} &\le C\,(||\boldsymbol{\psi}^1||_{0,K} + |\boldsymbol{\psi}^1|_{1,K}) \\ &\le C\,(||\boldsymbol{\psi}||_{0,K} + ||\boldsymbol{\psi}^b||_{0,K} + |\boldsymbol{\psi}|_{1,K}) \le C\,||\boldsymbol{\psi}||_{1,K}, \end{aligned}$$

$$\begin{aligned} ||\boldsymbol{\psi}||_{0,e} = ||\boldsymbol{\psi}^1||_{0,e} &\le C\,h_e^{\frac{3}{2}}\,||\boldsymbol{\psi}^1||_{2,K} = C\,h_e^{\frac{3}{2}}\,||\boldsymbol{\psi}^1||_{1,K} \\ &\le C\,h_e^{\frac{3}{2}}\,||\boldsymbol{\psi}||_{1,K} \quad \text{with } e \in \partial\,K \cap \partial\,\Omega, \end{aligned} \tag{4.97}$$

and we have

$$\frac{1}{1 - G^{-1}t^2} \int_{\partial\,\Omega} \boldsymbol{\psi} \cdot \mathsf{C}\,\boldsymbol{\mathcal{E}}(\boldsymbol{\phi})\,\boldsymbol{n} \le C\,h\,||\boldsymbol{\psi}||_1\,||\boldsymbol{\phi}||_2. \tag{4.98}$$

Let $\boldsymbol{\gamma} = \nabla\,u + \mathbf{curl}\,p$, with $u \in H_0^1(\Omega)$ and $p \in H^1(\Omega)/\mathbb{R}$, denote the Helmholtz decomposition of the shear strain $\boldsymbol{\gamma}$; see Proposition 2.1. We have $R_h(\boldsymbol{\gamma}) = R_h(\nabla\,u) + R_h(\mathbf{curl}\,p)$. Let $\tilde{p} \in V_h$ be the finite element interpolant of p, satisfying ([18], [10], [15])

$$||p - \tilde{p}||_0 + h\,||p - \tilde{p}||_1 \le C\,h^{1+r}\,||p||_{1+r}, \tag{4.99}$$

$$||\tilde{p} - p||_0 \le C\,h^r\,||p||_r, \quad ||\tilde{p}||_r \le C\,||p||_r, \tag{4.100}$$

where we take $r = 1$ for clamped plate and $r = 1/2$ for both soft simply-supported plate and free plate,

$$R_h(\mathbf{curl}\,p) = \varepsilon_h + \mathbf{curl}\,\tilde{p}, \tag{4.101}$$

with $\varepsilon_h := R_h(\mathbf{curl}\,p) - \mathbf{curl}\,\tilde{p}$. Thus, by the definition (3.22) of the L^2 projection R_h,

$$||\varepsilon_h||_0^2 = (R_h(\mathbf{curl}\,p) - \mathbf{curl}\,\tilde{p}, \varepsilon_h) = (\mathbf{curl}\,(p - \tilde{p}), \varepsilon_h) \le C\,h^r\,||p||_{1+r}\,||\varepsilon_h||_0, \tag{4.102}$$

that is,

$$||\varepsilon_h||_0 \le C\,h^r\,||p||_{1+r}. \tag{4.103}$$

Hence,

$$\begin{aligned} &(R_h(\boldsymbol{\gamma}) - \boldsymbol{\gamma}, \nabla\,v - \boldsymbol{\psi}) + \int_{\partial\,\Omega} \boldsymbol{\gamma} \cdot \boldsymbol{n}\,v \\ &\quad = (R_h(\nabla\,u) - \nabla\,u, \nabla\,v - \boldsymbol{\psi}) + \int_{\partial\,\Omega} \frac{\partial u}{\partial n}\,v \\ &\quad + (\mathbf{curl}\,(\tilde{p} - p), \nabla\,v - \boldsymbol{\psi}) + \int_{\partial\,\Omega} \mathbf{curl}\,p \cdot \boldsymbol{n}\,v \\ &\quad + (\varepsilon_h, R_h(\nabla\,v - \boldsymbol{\psi})), \end{aligned} \tag{4.104}$$

where from (4.103) we have

$$\begin{aligned} (\varepsilon_h, R_h(\nabla\,v - \boldsymbol{\psi})) &\le ||\varepsilon_h||_0\,||R_h(\nabla\,v - \boldsymbol{\psi})||_0 \\ &\le C\,h^r\,t\,||p||_{1+r}\,t^{-1}\,||R_h(\nabla\,v - \boldsymbol{\psi})||_0, \end{aligned} \tag{4.105}$$

while from (3.23) we have

$$(R_h(\nabla u) - \nabla u, \nabla v - \boldsymbol{\psi}) \le ||R_h(\nabla u) - \nabla u||_0 \, ||\nabla v - \boldsymbol{\psi}||_0 \le C\, h\, ||u||_2 \, \{||\boldsymbol{\psi}||_0 + |v|_1\}. \tag{4.106}$$

We are left to estimate the other two terms in the right-hand side of (4.104): $\int_{\partial\Omega} \dfrac{\partial u}{\partial n} v$ and $(\mathbf{curl}\,(\tilde{p} - p), \nabla v - \boldsymbol{\psi}) + \int_{\partial\Omega} \mathbf{curl}\, p \cdot \boldsymbol{n}\, v$. First, we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v = \sum_{e \in E^{\partial}(\mathcal{C}_{\partial\Omega})} \int_e \nabla u \cdot \boldsymbol{n}_0 \, v + \sum_{e \in E^{\partial}(\mathcal{C}_{\partial\Omega})} \int_e \nabla u \cdot \boldsymbol{n}_1 \, v, \tag{4.107}$$

where $\boldsymbol{n} = \boldsymbol{n}_0 + \boldsymbol{n}_1$, $\boldsymbol{n}_0$ is the unit normal to the straight-side e_s corresponding to the curved edge $e \in \partial\Omega$ and $\boldsymbol{n}_1 = \mathcal{O}(h_K)$. Take

$$\widetilde{\nabla u}|_K = \frac{1}{|K|} \int_K \nabla u \quad \forall K \in \mathcal{C}_{\partial\Omega},$$

which is the L^2 projection of ∇u on K, satisfying

$$||\widetilde{\nabla u} - \nabla u||_{0,K} + h_K |\widetilde{\nabla u} - \nabla u|_{1,K} \le C h_K ||u||_{2,K}.$$

From the definition (3.14) of W_h we have

$$\begin{aligned} \sum_{e \in E^{\partial}(\mathcal{C}_{\partial\Omega})} \int_e \nabla u \cdot \boldsymbol{n}_0 \, v &= \sum_{e \in E^{\partial}(\mathcal{C}_{\partial\Omega})} \int_e (\nabla u - \widetilde{\nabla u}) \cdot \boldsymbol{n}_0 \, v \\ &\le C \sum_{e \in E^{\partial}(\mathcal{C}_{\partial\Omega})} h_e^{\frac{3}{2}} \, ||v||_{2,K} \, h_e^{-\frac{1}{2}} \, (||\nabla u - \widetilde{\nabla u}||_{0,K} + h_K \, |\nabla u - \widetilde{\nabla u}|_{1,K}) \\ &\le C\, h\, ||u||_2 \, ||v||_1, \end{aligned} \tag{4.108}$$

where we used $||v||_{0,e} \le C\, h_K^{\frac{3}{2}} ||v||_{2,K}$ and the inverse estimate $h_K ||v||_{2,K} \le C ||v||_{1,K}$ for all $v \in W_h$, while

$$\sum_{e \in E^{\partial}(\mathcal{C}_{\partial\Omega})} \int_e \nabla u \cdot \boldsymbol{n}_1 \, v \le C \sum_{e \in E^{\partial}} h_K \, h_e^{-\frac{1}{2}} ||u||_{2,K} \, h_e^{\frac{3}{2}} \, ||v||_{2,K} \le C\, h\, ||u||_2 \, ||v||_1. \tag{4.109}$$

That is to say, we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v \le C h ||u||_2 ||v||_1. \tag{4.110}$$

For the other term in the right-hand side of (4.104), we have

$$\begin{aligned} &(\mathbf{curl}\,(\tilde{p} - p), \nabla v - \boldsymbol{\psi}) + \textstyle\int_{\partial\Omega} \mathbf{curl}\, p \cdot \boldsymbol{n}\, v \\ &= (\mathbf{curl}\,(\tilde{p} - p), \nabla v) - (\mathbf{curl}\,(\tilde{p} - p), \boldsymbol{\psi}) + \textstyle\int_{\partial\Omega} \mathbf{curl}\, p \cdot \boldsymbol{n}\, v \\ &= -(\tilde{p} - p, \operatorname{curl} \boldsymbol{\psi}) - \textstyle\int_{\partial\Omega} (\tilde{p} - p)\, \boldsymbol{\psi} \cdot \boldsymbol{s} + \int_{\partial\Omega} \mathbf{curl}\, \tilde{p} \cdot \boldsymbol{n}\, v, \end{aligned} \tag{4.111}$$

where

$$(\tilde{p} - p, \operatorname{curl} \boldsymbol{\psi}) \le C\, h^r \, ||p||_r \, ||\boldsymbol{\psi}||_1, \tag{4.112}$$

$$\begin{aligned}(4.113)\qquad &\int_{\partial\,\Omega}(\tilde{p}-p)\,\boldsymbol{\psi}\cdot\boldsymbol{s}\\ &\le \sum_{e\in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} h_e^{-1/2}\,(||\tilde{p}-p||_{0,K}+h_K^r\,\|\tilde{p}-p\|_{r,K})\,h_e^{\frac{3}{2}}\,||\boldsymbol{\psi}||_{2,K}\\ &\le C\,h^r\,||p||_r\,||\boldsymbol{\psi}||_1,\end{aligned}$$

$$\int_{\partial\,\Omega}\mathbf{curl}\,\tilde{p}\cdot\boldsymbol{n}\,v=\int_{\partial\,\Omega}\mathbf{curl}\,\tilde{p}\cdot\boldsymbol{n}_0\,v+\int_{\partial\,\Omega}\mathbf{curl}\,\tilde{p}\cdot\boldsymbol{n}_1\,v=\int_{\partial\,\Omega}\mathbf{curl}\,\tilde{p}\cdot\boldsymbol{n}_1\,v,$$

where we have used the fact that $\mathbf{curl}\,\tilde{p}\cdot\boldsymbol{n}_0$ is constant and $\int_e v=0$, while

$$\begin{aligned}\textstyle\int_{\partial\,\Omega}\mathbf{curl}\,\tilde{p}\cdot\boldsymbol{n}_1\,v\quad &\le C\sum_{e\in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} h_K\,h_e^{-\frac{1}{2}}\,||\tilde{p}||_{1,K}\,h_e^{\frac{3}{2}}\,||v||_{2,K}\\ &\le C\sum_{e\in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} h_K^r\,||\tilde{p}||_{r,K}\,||v||_{1,K}\\ &\le C\,h^r\,||\tilde{p}||_r\,||v||_1\le C\,h^r\,||p||_r\,||v||_1.\end{aligned}\tag{4.114}$$

Thus, we have

$$(\mathbf{curl}\,(\tilde{p}-p),\nabla\,v-\boldsymbol{\psi})+\int_{\partial\,\Omega}\mathbf{curl}\,p\cdot\boldsymbol{n}\,v\le Ch^r||p||_r(||v||_1+||\boldsymbol{\psi}||_1).\tag{4.115}$$

Combining (4.104), (4.105), (4.106), (4.110) and (4.115) to have

(4.116)

$$\begin{aligned}&(R_h(\boldsymbol{\gamma})-\boldsymbol{\gamma},\nabla\,v-\boldsymbol{\psi})+\int_{\partial\,\Omega}\boldsymbol{\gamma}\cdot\boldsymbol{n}\,v\\ &\quad\le Ch^r(||u||_2+t||p||_{1+r}+||p||_r)(||v||_1+||\boldsymbol{\psi}||_1+t^{-1}||R_h(\nabla\,v-\boldsymbol{\psi})||_0)\\ &\quad\le Ch^r(||u||_2+t||p||_{1+r}+||p||_r)|||(\boldsymbol{\psi},v)|||.\end{aligned}$$

We now estimate the last term $\dfrac{G^{-1}t^2}{1-G^{-1}t^2}\int_{\partial\,\Omega}\boldsymbol{\gamma}\cdot\boldsymbol{n}\,v$. With $\boldsymbol{\gamma}=\nabla\,u+\mathbf{curl}\,p$, we estimate it following the same argument as for (4.110) and we obtain

$$\frac{G^{-1}t^2}{1-G^{-1}t^2}\int_{\partial\,\Omega}\boldsymbol{\gamma}\cdot\boldsymbol{n}\,v\le Ct^2(h||u||_2+h^r||p||_{1+r})||v||_1.\tag{4.117}$$

Therefore, summarizing (4.93) and (4.98), (4.116), (4.117) we have

(4.118)

$$|\mathcal{L}((\boldsymbol{\phi}-\boldsymbol{\phi}_h,w-w_h);(\boldsymbol{\psi},v))|\le Ch^r(||\boldsymbol{\phi}||_2+||u||_2+||p||_r+t||p||_{1+r})|||(\boldsymbol{\psi},v)|||.$$

From Proposition 2.2 we know that

$$||\boldsymbol{\phi}||_2\le C,\quad ||p||_1\le C,\quad t||p||_2\le Ct^{1/2},\quad ||u||_2\le C,$$

therefore we obtain (4.96), with the choice $r=1$, for clamped plate.

We next consider soft simply-supported plate with (4.94). We estimate the following two terms:

$$(R_h\,(\boldsymbol{\gamma})-\boldsymbol{\gamma},\nabla\,v-\boldsymbol{\psi})+\int_{\partial\,\Omega}\boldsymbol{\gamma}\cdot\boldsymbol{n}\,v,\qquad \frac{G^{-1}t^2}{1-G^{-1}t^2}\int_{\partial\,\Omega}\boldsymbol{\gamma}\cdot\boldsymbol{n}\,v.$$

The argument for estimating them is similar to the one in the case of clamped plate.

In fact, all the estimates are the same as those for clamped plate, except that the term $\int_{\partial\,\Omega}(\tilde{p}-p)\,\boldsymbol{\psi}\cdot\boldsymbol{s}$ appearing in (4.111) needs a different treatment, because

$\boldsymbol{\psi}$ is no longer zero at the two ends of edges (see the definition (3.16) of Θ_h). For that goal we introduce a finite dimensional space which is defined by

$$V_h^2 = \{v \in H^1(\Omega); v|_K \in \mathcal{P}_2(K), \forall K \in \mathcal{C}_h\}. \tag{4.119}$$

Clearly, $\mathbf{curl}\, V_h^2 \subset X_h$, note that X_h, which is given by (3.18), is an $H(\mathrm{div};\Omega)$ conforming linear element. We take $\tilde{p} \in V_h^2$ as the interpolant of p, satisfying the same interpolation property as in (4.99) and (4.100) and the following additional property:

$$\int_e \tilde{p} = \int_e p, \quad \text{for all edges.} \tag{4.120}$$

We then estimate as follows:

$$\int_{\partial\,\Omega} (\tilde{p} - p)\, \boldsymbol{\psi} \cdot \boldsymbol{s} = \int_{\partial\,\Omega} (\tilde{p} - p)\, \boldsymbol{\psi} \cdot \boldsymbol{s}_0 + \int_{\partial\,\Omega} (\tilde{p} - p)\, \boldsymbol{\psi} \cdot \boldsymbol{s}_1, \tag{4.121}$$

where $\boldsymbol{s} = \boldsymbol{s}_0 + \boldsymbol{s}_1$, $\boldsymbol{s}_0$ is a constant tangent vector to the straight-side e_s corresponding to the curved edge $e \subset \partial\,\Omega$ and $\boldsymbol{s}_1 = \mathcal{O}(h_K)$. Take $\tilde{\boldsymbol{\psi}} = \dfrac{1}{|K|} \int_K \boldsymbol{\psi}$, which satisfies

$$||\tilde{\boldsymbol{\psi}} - \boldsymbol{\psi}||_{0,K} + h_K |\tilde{\boldsymbol{\psi}} - \boldsymbol{\psi}|_{1,K} \le C\, h_K |\boldsymbol{\psi}|_{1,K}.$$

From (4.120) and (4.100) we then have

$$\int_{\partial\,\Omega} (\tilde{p} - p)\, \boldsymbol{\psi} \cdot \boldsymbol{s}_0 = \sum_{e \in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} \int_e (\tilde{p} - p)\, (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}) \cdot \boldsymbol{s}_0 \le C\, h^r\, ||p||_r\, ||\boldsymbol{\psi}||_1, \tag{4.122}$$

while

$$\int_{\partial\,\Omega} (\tilde{p} - p)\, \boldsymbol{\psi} \cdot \boldsymbol{s}_1 \le C\, h^r\, ||p||_r\, ||\boldsymbol{\psi}||_0. \tag{4.123}$$

We therefore have an estimation for (4.94) as follows:

$$|\mathcal{L}((\boldsymbol{\phi} - \boldsymbol{\phi}_h, w - w_h); (\boldsymbol{\psi}, v))| \le C h^r (||u||_2 + ||p||_r + t||p||_{1+r}) |||(\boldsymbol{\psi}, v)|||. \tag{4.124}$$

From Proposition 2.2 we know that

$$||u||_2 \le C, \quad ||p||_{1/2} \le C, \quad t||p||_{3/2} \le C,$$

and we obtain (4.96) with $r = 1/2$ for soft simply connected plate.

We finally consider free plate with (4.95). The argument is similar. To estimate the term

$$(R_h(\boldsymbol{\gamma}) - \boldsymbol{\gamma}, \nabla\, v - \boldsymbol{\psi}),$$

we take the Helmholtz decomposition for $\boldsymbol{\gamma}$ by $\boldsymbol{\gamma} = \nabla\, u + \mathbf{curl}\, p$ with $u \in H^1(\Omega)/\mathbb{R}$ and $p \in H_0^1(\Omega)$, and we take $\tilde{p} \in V_{0,h}$ as the interpolant of p, satisfying the same interpolation property as in (4.99) and (4.100). Just by following the argument in proving clamped plate (cf. (4.104)) we have

$$\begin{aligned} (R_h(\boldsymbol{\gamma}) - \boldsymbol{\gamma}, \nabla\, v - \boldsymbol{\psi}) &= (R_h(\nabla\, u) - \nabla\, u, \nabla\, v - \boldsymbol{\psi}) \\ &\quad + (\mathbf{curl}\,(\tilde{p} - p), \nabla\, v - \boldsymbol{\psi}) \\ &\quad + (\varepsilon_h, R_h(\nabla\, v - \boldsymbol{\psi})), \end{aligned} \tag{4.125}$$

where the first term and the last term in the right-hand side of (4.125) are estimated in the same way as (4.106) and (4.105), while the second term in the right-hand side of (4.125) is

$$\begin{aligned}(\mathbf{curl}\,(\tilde{p}-p), \nabla\, v - \boldsymbol{\psi}) &= (\tilde{p}-p, \operatorname{curl}\boldsymbol{\psi}) + (\mathbf{curl}\,\tilde{p}, \nabla\, v - \boldsymbol{\psi}) \\ &= (\tilde{p}-p, \operatorname{curl}\boldsymbol{\psi}) + \int_{\partial\,\Omega} \tilde{p}\,(\nabla\, v - \boldsymbol{\psi})\cdot \boldsymbol{s},\end{aligned}$$

since $p \in H_0^1(\Omega)$. The term $(\tilde{p}-p, \operatorname{curl}\boldsymbol{\psi})$ can be estimated in the same way as (4.112). We estimate the term $\int_{\partial\,\Omega} \tilde{p}\,(\nabla\, v - \boldsymbol{\psi})\cdot \boldsymbol{s}$ in the following:

$$\begin{aligned}\int_{\partial\,\Omega} \tilde{p}\,(\nabla\, v - \boldsymbol{\psi})\cdot \boldsymbol{s} &= \sum_{e\in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} \int_e \tilde{p}\,(\nabla\, v - \boldsymbol{\psi})\cdot \boldsymbol{s} \\ &\le \sum_{e\in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} ||\tilde{p}||_{0,e}\, h_e^{-\frac{1}{2}}\,(||\nabla\, v - \boldsymbol{\psi}||_{0,K} + h_K\,|\nabla\, v - \boldsymbol{\psi}|_{1,K}) \\ &\le \sum_{e\in E^{\partial}} h_e^{\frac{3}{2}}\,||\tilde{p}||_{1,K}\, h_e^{-\frac{1}{2}}\,(||v||_{1,K} + ||\boldsymbol{\psi}||_{0,K}) \\ &\le C\,h\,||\tilde{p}||_1\,(||v||_1 + ||\boldsymbol{\psi}||_0) \\ &\le C\,h^r\,||\tilde{p}||_r\,(||v||_1 + ||\boldsymbol{\psi}||_0) \\ &\le C\,h^r\,||p||_r\,(||v||_1 + ||\boldsymbol{\psi}||_0).\end{aligned} \tag{4.126}$$

Therefore, we have

$$\begin{aligned}|\mathcal{L}((\boldsymbol{\phi}-\boldsymbol{\phi}_h, w-w_h); (\boldsymbol{\psi}, v))| &= |(R_h(\boldsymbol{\gamma}) - \boldsymbol{\gamma}, \nabla\, v - \boldsymbol{\psi})| \\ &\le Ch^r(||u||_2 + ||p||_r + t||p||_{1+r})|||(\boldsymbol{\psi}, v)|||,\end{aligned} \tag{4.127}$$

from which we obtain (4.96) for free plate, since $||p||_r + t||p||_{1+r}$ is bounded uniformly in t for $r = 1/2$. □

Theorem 4.1. *Let $(\boldsymbol{\phi}, w)$ and $(\boldsymbol{\phi}_h, w_h)$ be the exact and finite element solutions. We have*

$$||\boldsymbol{\phi}-\boldsymbol{\phi}_h||_1 + ||w-w_h||_1 \le C\,h^r, \tag{4.128}$$

with $r = 1$ for clamped plate and $r = 1/2$ for free and soft simply-supported plates.

Proof. Let $(\tilde{\boldsymbol{\phi}}, \tilde{w}) \in \Theta_h \times W_h$ be constructed in Lemma 4.2. We have

$$\begin{aligned}|||(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}_h, \tilde{w}-w_h)|||^2 &= \mathcal{L}((\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}_h, \tilde{w}-w_h); (\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}_h, \tilde{w}-w_h)) \\ &= \mathcal{L}((\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}, \tilde{w}-w); (\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}_h, \tilde{w}-w_h)) \\ &\quad + \mathcal{L}((\boldsymbol{\phi}-\boldsymbol{\phi}_h, w-w_h); (\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}_h, \tilde{w}-w_h))\end{aligned} \tag{4.129}$$

where

$$\mathcal{L}((\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}, \tilde{w}-w); (\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}_h, \tilde{w}-w_h)) \le |||(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}, \tilde{w}-w)|||\,|||(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}_h, \tilde{w}-w_h)|||, \tag{4.130}$$

$$\begin{aligned}|||(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}, \tilde{w}-w)|||^2 &= \mathcal{L}((\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}, \tilde{w}-w); (\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}, \tilde{w}-w)) \\ &= A((\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}, \tilde{w}-w); (\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}, \tilde{w}-w)) \\ &\le C\,h^{2\,r}(||\boldsymbol{\phi}||_{1+r} + ||w||_2)^2,\end{aligned} \tag{4.131}$$

where $r = 1$ for clamped plate and $r = 1/2$ for soft simply-supported and free plates. Combining the above and Lemma 4.3 and Proposition 2.2 we have

$$|||(\tilde{\boldsymbol{\phi}} - \boldsymbol{\phi}_h, \tilde{w} - w_h)||| \leq Ch^r. \tag{4.132}$$

By the coercivity of $|||\cdot|||$ we have

$$||\tilde{\boldsymbol{\phi}} - \boldsymbol{\phi}_h||_1 + ||\tilde{w} - w_h||_1 \leq C|||(\tilde{\boldsymbol{\phi}} - \boldsymbol{\phi}_h, \tilde{w} - w_h)||| \leq Ch^r. \tag{4.133}$$

Finally, by the triangle inequality, from Lemma 4.1 and Proposition 2.2 we obtain

$$||\boldsymbol{\phi} - \boldsymbol{\phi}_h||_1 + ||w - w_h||_1 \leq ||\tilde{\boldsymbol{\phi}} - \boldsymbol{\phi}_h||_1 + ||\tilde{w} - w_h||_1 + ||\tilde{\boldsymbol{\phi}} - \boldsymbol{\phi}||_1 + ||\tilde{w} - w||_1 \leq Ch^r, \tag{4.134}$$

where $r = 1$ for clamped plate and $r = 1/2$ for soft simply-supported and free plates. □

5. Error estimates for shear strain

In this section the error estimates for the shear strain is given by the establishment of an inf-sup inequality that holds uniformly in the plate thickness. The inf-sup inequality also ensures the convergence of the iterated penalty method defined in next section.

Let

$$||(\boldsymbol{\psi}, v)||_1^2 := A((\boldsymbol{\psi}, v); (\boldsymbol{\psi}, v)),$$

which is equivalent to $||v||_1^2 + ||\boldsymbol{\psi}||_1^2$ from (2.8). On X_h we introduce

$$\|\boldsymbol{q}\|_h^2 = \sum_{K \in \mathcal{C}_h} h_K^2 \{\|\boldsymbol{q}\|_{0,K}^2 + \|\mathrm{div}\, \boldsymbol{q}\|_{0,K}^2\}, \tag{5.1}$$

$$|||\boldsymbol{q}|||_{*,h}^2 = \|\boldsymbol{q}\|_h^2 + \|\boldsymbol{q}\|_*^2, \tag{5.2}$$

where

$$||\boldsymbol{q}||_* = \sup_{(\boldsymbol{\psi}, v) \in \Theta \times W} \frac{(\boldsymbol{q}, \nabla v - \boldsymbol{\psi})}{||(\boldsymbol{\psi}, v)||_1} \tag{5.3}$$

is defined for all $\boldsymbol{q} \in (L^2(\Omega))^2$. Note that for clamped plate the norm $||\cdot||_*$ is none other than $||\cdot||_{H^{-1}(\mathrm{div};\Omega)}$, with $H^{-1}(\mathrm{div};\Omega) := \{\boldsymbol{q} \in (H^{-1}(\Omega))^2; \mathrm{div}\, \boldsymbol{q} \in H^{-1}(\Omega)\}$, the dual of $H_0(\mathrm{curl};\Omega) = \{\boldsymbol{q} \in (L^2(\Omega))^2; \mathrm{curl}\, \boldsymbol{q} \in L^2(\Omega), \boldsymbol{q} \cdot \boldsymbol{s}|_{\partial\,\Omega} = 0\}$, and $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$; see [1], [13]

Lemma 5.1. *We have*

$$\sup_{(\boldsymbol{\psi}, v) \in \Theta_h \times W_h} \frac{(\boldsymbol{q}, \nabla v - \boldsymbol{\psi})}{||\boldsymbol{\psi}||_1 + ||v||_1} \geq C\, ||\boldsymbol{q}||_h \quad \forall \boldsymbol{q} \in X_h. \tag{5.4}$$

Proof. The proof is divided into two steps.

Step 1. Choosing $v = -\sum_{K \in \mathcal{T}_h} |K|\, \mathrm{div}\, \boldsymbol{q}\, b_K$, we have

$$(\nabla v, \boldsymbol{q}) = -(v, \mathrm{div}\, \boldsymbol{q}) = \sum_{K \in \mathcal{C}_h} ||\mathrm{div}\, \boldsymbol{q}||_{0,K}^2 \int_K b_K \;\geq\; C \sum_{K \in \mathcal{T}_h} h_K^2\, ||\mathrm{div}\, \boldsymbol{q}||_{0,K}^2, \tag{5.5}$$

$$||v||_1^2 \leq\; C \sum_{K \in \mathcal{C}_h} h_K^2\, ||\mathrm{div}\, \boldsymbol{q}||_{0,K}^2. \tag{5.6}$$

Step 2. Choosing $\boldsymbol{\psi} = -\sum_{K\in\mathcal{T}_h} |K|\, \boldsymbol{q}\, b_K$, we have

$$(-\boldsymbol{\psi}, \boldsymbol{q}) = \sum_{K\in\mathcal{C}_h} |K|\, (\boldsymbol{q} b_K, \boldsymbol{q})_{0,K} \geq C \sum_{K\in\mathcal{T}_h} h_K^2\, ||\boldsymbol{q}||_{0,K}^2, \tag{5.7}$$

$$||\boldsymbol{\psi}||_1^2 \leq C \sum_{K\in\mathcal{C}_h} h_K^2\, ||\boldsymbol{q}||_{0,K}^2. \tag{5.8}$$

Combining the two steps, we conclude that (5.4) holds. □

Lemma 5.2. *We have*

$$\sup_{(\boldsymbol{\psi},v)\in\Theta_h\times W_h} \frac{(\boldsymbol{q}, \nabla v - \boldsymbol{\psi})}{||\boldsymbol{\psi}||_1 + ||v||_1} \geq C\, |||\boldsymbol{q}|||_{*,h} \quad \forall \boldsymbol{q} \in X_h. \tag{5.9}$$

Proof. Owing to (5.3), for any given $\boldsymbol{q} \in X_h$, we can find a $(\boldsymbol{\psi}^*, v^*) \in \Theta \times W$ such that

$$(\boldsymbol{q}, \nabla v^* - \boldsymbol{\psi}^*) = ||\boldsymbol{q}||_*^2, \quad ||\boldsymbol{\psi}^*||_1 + ||v^*||_1 \leq C\, ||\boldsymbol{q}||_*. \tag{5.10}$$

In fact, letting $(\boldsymbol{\psi}^*, v^*) \in \Theta \times W$ solve

$$A((\boldsymbol{\psi}^*, v^*); (\boldsymbol{\psi}, v)) = (\boldsymbol{q}, \nabla v - \boldsymbol{\psi}) \quad \forall (\boldsymbol{\psi}, v) \in \Theta \times W,$$

by the definition (5.3) we have

$$||(\boldsymbol{\psi}^*, v^*)||_1 = ||\boldsymbol{q}||_*;$$

on the other hand, we have

$$||(\boldsymbol{\psi}^*, v^*)||_1^2 = A((\boldsymbol{\psi}^*, v^*); (\boldsymbol{\psi}^*, v^*)) = (\boldsymbol{q}, \nabla v^* - \boldsymbol{\psi}^*).$$

Hence (5.10) holds.

Let $(\boldsymbol{\psi}_h^*, v_h^*) \in \Theta_h \times W_h$ be the interpolant to $(\boldsymbol{\psi}^*, v^*)$, satisfying

$$\begin{aligned} &\left(\sum_{K\in\mathcal{T}_h} h_K^{-2} \{ ||\boldsymbol{\psi}^* - \boldsymbol{\psi}_h^*||_{0,K}^2 + ||v^* - v_h^*||_{0,K}^2 \} \right)^{1/2} \\ &\qquad + ||\boldsymbol{\psi}_h^*||_1 + ||v_h^*||_1 \leq C \{ ||\boldsymbol{\psi}^*||_1 + ||v^*||_1 \}. \end{aligned} \tag{5.11}$$

We can obtain

$$(\boldsymbol{q}, \nabla (v^* - v_h^*) - (\boldsymbol{\psi}^* - \boldsymbol{\psi}_h^*)) \leq C\, ||\boldsymbol{q}||_*\, ||\boldsymbol{q}||_h. \tag{5.12}$$

In fact, we need only consider

$$(\boldsymbol{q}, \nabla (v^* - v_h^*)) = -(\operatorname{div} \boldsymbol{q}, v^* - v_h^*) + \int_{\partial\Omega} \boldsymbol{q}\cdot\boldsymbol{n}\, (v^* - v_h^*). \tag{5.13}$$

From (5.11) we first easily see that

$$-(\operatorname{div} \boldsymbol{q}, v^* - v_h^*) \leq C \left(\sum_{K\in\mathcal{C}_h} h_K^2\, ||\operatorname{div} \boldsymbol{q}||_{0,K}^2 \right)^{\frac{1}{2}} ||v_h^*||_1. \tag{5.14}$$

For clamped and soft simply-supported plates, since $v^* \in H_0^1(\Omega)$, following the same argument as in the previous section, we have

$$\int_{\partial\Omega} \boldsymbol{q}\cdot\boldsymbol{n}\, (v^* - v_h^*) = -\sum_{e\in E^\partial(\mathcal{C}_{\partial\Omega})} \int_e \boldsymbol{q}\cdot\boldsymbol{n}\, v_h^* \leq C \left(\sum_{K\in\mathcal{C}_h} h_K^2\, ||\boldsymbol{q}||_{0,K}^2 \right)^{\frac{1}{2}} ||v_h^*||_1. \tag{5.15}$$

For free plate, still following the same argument, i.e., (4.85), in the previous section, we get

$$\int_{\partial\,\Omega} \boldsymbol{q}\cdot\boldsymbol{n}\,(v^*-v_h^*) = \sum_{e\in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} \int_e \boldsymbol{q}\cdot\boldsymbol{n}\,(v^*-v_h^*) \le C\left(\sum_{K\in\mathcal{C}_h} h_K^2\,||\boldsymbol{q}||_{0,K}^2\right)^{\frac{1}{2}} ||v^*||_1. \tag{5.16}$$

If we consider the definition in Remark 4.1, then following the argument in Remark 4.1 we have

$$\int_{\partial\,\Omega} \boldsymbol{q}\cdot\boldsymbol{n}\,(v^*-v_h^*) = \int_{\partial\,\Omega} \boldsymbol{q}\cdot\boldsymbol{n}_0\,(v^*-v_h^*) + \int_{\partial\,\Omega} \boldsymbol{q}\cdot\boldsymbol{n}_1\,(v^*-v_h^*),$$

$$\begin{aligned}\sum_{e\in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} \int_e \boldsymbol{q}\cdot\boldsymbol{n}_0\,(v^*-v_h^*) &\le \sum_{e\in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} C\,h_e^{\frac{3}{2}}\,||\nabla\,\boldsymbol{q}\cdot\boldsymbol{n}_0||_{0,K}\,||v^*-v_h^*||_{0,e}\\ &\le C\left(\sum_{K\in\mathcal{C}_h} h_K^2\,||\boldsymbol{q}||_{0,K}^2\right)^{\frac{1}{2}} ||v^*||_1,\end{aligned} \tag{5.17}$$

$$\begin{aligned}\sum_{e\in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} \int_e \boldsymbol{q}\cdot\boldsymbol{n}_1\,(v^*-v_h^*) &\le \sum_{e\in E^{\partial}(\mathcal{C}_{\partial\,\Omega})} C\,h_K\,||\boldsymbol{q}||_{0,e}\,||v^*-v_h^*||_{0,e}\\ &\le C\left(\sum_{K\in\mathcal{C}_h} h_K^2\,||\boldsymbol{q}||_{0,K}^2\right)^{\frac{1}{2}} ||v^*||_1.\end{aligned} \tag{5.18}$$

Therefore, (5.12) holds, using

$$||\boldsymbol{\psi}_h^*||_1 + ||v_h^*||_1 + ||\boldsymbol{\psi}^*||_1 + ||v^*||_1 \le C\,||\boldsymbol{q}||_*. \tag{5.19}$$

We then have

$$\begin{aligned}\sup_{(\boldsymbol{\psi},v)\in\Theta_h\times W_h} \frac{(\boldsymbol{q},\nabla\,v-\boldsymbol{\psi})}{||\boldsymbol{\psi}||_1+||v||_1} &\ge \frac{(\boldsymbol{q},\nabla\,v_h^*-\boldsymbol{\psi}_h^*)}{||\boldsymbol{\psi}_h^*||_1+||v_h^*||_1}\\ &= \frac{(\boldsymbol{q},\nabla\,v^*-\boldsymbol{\psi}^*)}{||\boldsymbol{\psi}_h^*||_1+||v_h^*||_1} + \frac{(\boldsymbol{q},\nabla\,(v_h^*-v^*)-(\boldsymbol{\psi}_h^*-\boldsymbol{\psi}^*))}{||\boldsymbol{\psi}_h^*||_1+||v_h^*||_1}\\ &\ge C_1\,||\boldsymbol{q}||_* - C_2||\boldsymbol{q}||_h.\end{aligned} \tag{5.20}$$

Combining (5.20) and (5.4), we obtain (5.9). □

Remark 5.1. Theorem 5.1 indicates that in our method, there holds a uniform inf-sup condition in the norm $||\cdot||_*$ for the shear strain. If we are to consider the limit plate, with clamped boundary condition, for example, to find $(\boldsymbol{\phi}^0, w^0, \boldsymbol{\gamma}^0) \in \Theta\times W\times H^{-1}(\mathrm{div};\Omega)$ such that

$$\begin{cases} A((\boldsymbol{\phi}^0,w^0);(\boldsymbol{\psi},v)) + (\boldsymbol{\gamma}^0,\nabla\,v-\boldsymbol{\psi}) = (g,v) \quad \forall(\boldsymbol{\psi},v)\in\Theta\times W,\\ (\nabla\,w^0-\boldsymbol{\phi}^0,\boldsymbol{\chi}) = 0 \quad \forall\boldsymbol{\chi}\in H^{-1}(\mathrm{div};\Omega),\end{cases}$$

the direct application of our finite element method reads: to find $(\boldsymbol{\phi}_h^0, w_h^0, \boldsymbol{\gamma}_h^0) \in \Theta_h\times W_h\times X_h$ such that

$$\begin{cases} A((\boldsymbol{\phi}_h^0,w_h^0);(\boldsymbol{\psi},v)) + (\boldsymbol{\gamma}_h^0,\nabla\,v-\boldsymbol{\psi}) = (g,v) \quad \forall(\boldsymbol{\psi},v)\in\Theta_h\times W_h,\\ (\nabla\,w_h^0-\boldsymbol{\phi}_h^0,\boldsymbol{\chi}) = 0 \quad \forall\boldsymbol{\chi}\in X_h.\end{cases}$$

Then, thanks to the established Inf-Sup condition over $H^{-1}(\mathrm{div};\Omega)$ and the ellipticity of A over the whole $\Theta_h \times W_h$, a direct application of the standard classical theory for the saddle-point problem gives the following:

$$||\boldsymbol{\phi}^0 - \boldsymbol{\phi}_h^0||_1 + ||w^0 - w_h^0||_1 + ||\boldsymbol{\gamma}^0 - \boldsymbol{\gamma}_h^0||_{H^{-1}(\mathrm{div};\Omega)} \leq C\, h.$$

For simply-supported and free limit plates, in the above we should instead use the norm $||\cdot||_*$ defined by (5.3) for $\boldsymbol{\gamma}^0$, and we can have the optimal error bound $\mathcal{O}(h)$. □

Theorem 5.1. *Let $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_h$ be the exact and finite element solutions, respectively given by* (2.9) *and* (3.2)*. We have the uniform error estimate*

$$||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_* \leq C\, h^r, \tag{5.21}$$

where $r = 1$ for clamped plate and $r = 1/2$ for both soft simply-supported plate and free plate.

Proof. Corresponding to Helmholtz decomposition of $\boldsymbol{\gamma}$, we have

$$\boldsymbol{\gamma} = \nabla\, u + \mathbf{curl}\, p, \tag{5.22}$$

we take

$$\tilde{\gamma} = \widetilde{\nabla\, u} + \mathbf{curl}\, \tilde{p} \in X_h, \tag{5.23}$$

where $\widetilde{\nabla\, u} \in X_h$ is the standard finite element interpolant to $\nabla\, u$, e.g., we may choose $\widetilde{\nabla\, u} := R_h(\nabla\, u)$, satisfying (3.23), and $\tilde{p} \in V_h$ is the finite element interpolant to p, satisfying (4.99) and (4.100). We have

$$\begin{aligned} C\,||\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_h||_* &\leq \sup_{(\boldsymbol{\psi},v)\in\Theta_h\times W_h} \frac{(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_h, \nabla\, v - \boldsymbol{\psi})}{||\boldsymbol{\psi}||_1 + ||v||_1} \\ &\leq \sup_{(\boldsymbol{\psi},v)\in\Theta_h\times W_h} \frac{(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}, \nabla\, v - \boldsymbol{\psi}) + (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \nabla\, v - \boldsymbol{\psi})}{||\boldsymbol{\psi}||_1 + ||v||_1}, \end{aligned} \tag{5.24}$$

where by following the same argument as in the proof of Lemma 4.2 we have

$$\begin{aligned} (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \nabla\, v - \boldsymbol{\psi}) &= \textstyle\int_{\partial\,\Omega} \boldsymbol{\gamma}\cdot\boldsymbol{n}\, v + \dfrac{G^{-1}t^2}{1 - G^{-1}t^2} \int_{\partial\,\Omega} \boldsymbol{\gamma}\cdot\boldsymbol{n}\, v \\ &\quad + \textstyle\int_{\partial\,\Omega} \boldsymbol{\psi}\cdot\boldsymbol{\mathcal{C}}\,\boldsymbol{\mathcal{E}}(\boldsymbol{\phi})\,\boldsymbol{n} + \dfrac{1}{1 - G^{-1}t^2} \int_{\partial\,\Omega} \boldsymbol{\psi}\cdot\boldsymbol{\mathcal{C}}\,\boldsymbol{\mathcal{E}}(\boldsymbol{\phi})\,\boldsymbol{n} \\ &\quad + A((\boldsymbol{\phi}_h - \boldsymbol{\phi}, w_h - w); (\boldsymbol{\psi}, v)), \end{aligned} \tag{5.25}$$

$$\begin{aligned} (\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}, \nabla\, v - \boldsymbol{\psi}) &= (\widetilde{\nabla\, u} - \nabla\, u, \nabla\, v - \boldsymbol{\psi}) + \textstyle\int_{\partial\,\Omega} \mathbf{curl}\,(\tilde{p} - p)\cdot\boldsymbol{n}\, v \\ &\quad - (\tilde{p} - p, \mathrm{curl}\,\boldsymbol{\psi}) - \textstyle\int_{\partial\,\Omega} (\tilde{p} - p)\,\boldsymbol{\psi}\cdot\boldsymbol{s}. \end{aligned} \tag{5.26}$$

We thus have

$$\begin{aligned} (\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_h, \nabla\, v - \boldsymbol{\psi}) &= (\widetilde{\nabla\, u} - \nabla\, u, \nabla\, v - \boldsymbol{\psi}) + \textstyle\int_{\partial\,\Omega} \mathbf{curl}\,\tilde{p}\cdot\boldsymbol{n}\, v + \int_{\partial\,\Omega} \dfrac{\partial u}{\partial n}\, v \\ &\quad - (\tilde{p} - p, \mathrm{curl}\,\boldsymbol{\psi}) - \textstyle\int_{\partial\,\Omega} (\tilde{p} - p)\,\boldsymbol{\psi}\cdot\boldsymbol{s} \\ &\quad + \dfrac{G^{-1}t^2}{1 - G^{-1}t^2} \textstyle\int_{\partial\,\Omega} \boldsymbol{q}\cdot\boldsymbol{n}\, v + \dfrac{1}{1 - G^{-1}t^2} \int_{\partial\,\Omega} \boldsymbol{\psi}\cdot\boldsymbol{\mathcal{C}}\,\boldsymbol{\mathcal{E}}(\boldsymbol{\phi})\,\boldsymbol{n} \\ &\quad + A((\boldsymbol{\phi}_h - \boldsymbol{\phi}, w_h - w); (\boldsymbol{\psi}, v)). \end{aligned} \tag{5.27}$$

Following the same argument as in the proof of Lemma 4.3, from Theorem 4.1 we get

$$(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_h, \nabla v - \boldsymbol{\psi}) \leq C\, h^r\, (||\boldsymbol{\psi}||_1 + ||v||_1), \tag{5.28}$$

and we obtain

$$||\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_h||_* \leq C\, h^r. \tag{5.29}$$

On the other hand,

$$C\, ||\boldsymbol{\gamma} - \tilde{\boldsymbol{\gamma}}||_* \leq \sup_{(\boldsymbol{\psi}, v) \in \Theta \times W} \frac{(\boldsymbol{\gamma} - \tilde{\boldsymbol{\gamma}}, \nabla v - \boldsymbol{\psi})}{||\boldsymbol{\psi}||_1 + ||v||_1}, \tag{5.30}$$

$$\begin{aligned}(\boldsymbol{\gamma} - \tilde{\boldsymbol{\gamma}}, \nabla v - \boldsymbol{\psi}) &= (\nabla u - \widetilde{\nabla u}, \nabla v - \boldsymbol{\psi}) \\ &\quad + (\tilde{p} - p, \operatorname{curl} \boldsymbol{\psi}) + \textstyle\int_{\partial \Omega} (\tilde{p} - p)\, (\nabla v - \boldsymbol{\psi}) \cdot \boldsymbol{s},\end{aligned} \tag{5.31}$$

noting that, for clamped and soft simply-supported plates, $v \in H_0^1(\Omega)$; for free plate, $p \in H_0^1(\Omega)$, following the same argument as in the previous section, we have

$$(\boldsymbol{\gamma} - \tilde{\boldsymbol{\gamma}}, \nabla v - \boldsymbol{\psi}) \leq C\, h^r\, (||v||_1 + ||\boldsymbol{\psi}||_1), \tag{5.32}$$

and we hence obtain

$$||\boldsymbol{\gamma} - \tilde{\boldsymbol{\gamma}}||_* \leq C\, h^r. \tag{5.33}$$

Finally, (5.21) is concluded by the triangle inequality from (5.33) and (5.29). □

Remark 5.2. Following the same argument as in Theorem 5.1, we can further obtain

$$||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_h + t||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_0 \leq C\, h^r,$$

where $r = 1$ for clamped plate and $r = 1/2$ for both soft simply-supported plate and free plate. □

6. Implementation by an iterated penalty method

To solve (3.1) iteratively, we may use (preconditioned) iterative methods (multigrid/multilevel methods) for saddle-point problems. As an example, we describe an iterated penalty method which behaves like a direct method.

We first formulate (3.1) in the form of the saddle-point problem: Find $(\boldsymbol{\phi}_h, w_h, \boldsymbol{\gamma}_h) \in \Theta_h \times W_h \times X_h$ such that

$$\left\{ \begin{array}{rcl} A((\boldsymbol{\phi}_h, w_h); (\boldsymbol{\psi}, v)) + B((\boldsymbol{\psi}, v); \boldsymbol{\gamma}_h) & = & (g, v) \quad \forall (\boldsymbol{\psi}, v) \in \Theta_h \times W_h, \\ B((\boldsymbol{\phi}_h, w_h); \boldsymbol{\chi}) - t^2\, D(\boldsymbol{\gamma}_h, \boldsymbol{\chi}) & = & 0 \quad \forall \boldsymbol{\chi} \in X_h, \end{array} \right. \tag{6.1}$$

where

$$B((\boldsymbol{\psi}, v); \boldsymbol{\chi}) = (\nabla v - \boldsymbol{\psi}, \boldsymbol{\chi}), \quad D(\boldsymbol{\gamma}, \boldsymbol{\chi}) = G^{-1}(\boldsymbol{\gamma}, \boldsymbol{\chi}). \tag{6.2}$$

We next introduce a mesh-dependent bilinear form as follows:

$$E(\boldsymbol{\gamma}, \boldsymbol{\chi}) = \sum_{K \in \mathcal{T}_h} h_K^2\, (\boldsymbol{\gamma}, \boldsymbol{\chi})_{0,K}. \tag{6.3}$$

Next, let ϵ be a small positive number and $\boldsymbol{\gamma}_0 \in X_h$ an initial guess, find a sequence $\{\boldsymbol{\phi}_m, w_m, \boldsymbol{\gamma}_m\} \in \Theta_h \times W_h \times X_h$ such that, for $m = 1, 2, \ldots,$

(6.4)
$$\begin{cases} A((\boldsymbol{\phi}_m, w_m); (\boldsymbol{\psi}, v)) + B((\boldsymbol{\psi}, v); \boldsymbol{\gamma}_m) = (g, v), \quad \forall (\boldsymbol{\psi}, v) \in \Theta_h \times W_h, \\ B((\boldsymbol{\phi}_m, w_m); \boldsymbol{\chi}) - t^2\, D(\boldsymbol{\gamma}_m, \boldsymbol{\chi}) - \epsilon E(\boldsymbol{\gamma}_m, \boldsymbol{\chi}) = -\epsilon E(\boldsymbol{\gamma}_{m-1}, \boldsymbol{\chi}), \quad \forall \boldsymbol{\chi} \in X_h. \end{cases}$$

Note that B satisfies the Inf-Sup condition in the norm $||\cdot||_*$ (uniform in t and h, cf. Remark 5.1) and A, D are symmetric, positive definite (uniform in t and h), from a general argument in [43] one can obtain

$$\begin{aligned} ||\boldsymbol{\phi}_h - \boldsymbol{\phi}_m||_1 + ||w_h - w_m||_1 + ||\boldsymbol{\gamma}_h - \boldsymbol{\gamma}_m||_* + t||\boldsymbol{\gamma}_h - \boldsymbol{\gamma}_m||_0 \\ + ||\boldsymbol{\gamma}_h - \boldsymbol{\gamma}_m||_h \le C\,\epsilon^m\, ||\boldsymbol{\gamma}_h - \boldsymbol{\gamma}_0||_h \end{aligned} \tag{6.5}$$

for $m = 1, 2, \ldots,$ where $\epsilon > 0$ is a small number, independently of h and t. Just taking $t = 0$, we obtain the iterated penalty method for solving the finite element method for the limit plate.

Remark 6.1. Due to a geometrical convergence $C\epsilon^m$ in (6.5), the iterative method (6.4) behaves like a direct method. Thus, problem (3.1) can be efficiently solved in its saddle-point form in which no L^2 projection exists. In addition, a preconditioned version of (6.4) can be developed; e.g., see [8], [28]. Neither of these iterative methods will be discussed further here. □

In conclusion, in this paper we provide a new lower-order finite element method for the Reissner-Mindlin plate problem with various types of boundary conditions: clamped, free and soft simply-supported. Optimal error estimates are obtained, consistent with the uniform regularity of the exact solution. An efficient iterative method is also presented so that the finite element method can be solved the same as for a direct method. The method yields stability and error bounds, both of which are uniform in the plate thickness, and thus the method is locking-free and does not suffer from boundary-layer effects. The method and the theory cover the limit plate as well, since the Inf-Sup condition is satisfied in the space $H^{-1}(\mathrm{div}; \Omega)$ (e.g., for clamped plate).

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