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AN A POSTERIORI ERROR ESTIMATE FOR THE VARIABLE-DEGREE RAVIART-THOMAS METHOD

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ABSTRACT. We propose a new a posteriori error analysis of the variable-degree, hybridized version of the Raviart-Thomas method for second-order elliptic problems on conforming meshes made of simplexes. We establish both the reliability and efficiency of the estimator for the L_2-norm of the error of the flux. We also find the explicit dependence of the estimator on the order of the local spaces $k \geq 0$; the only constants that are not explicitly computed are those depending on the shape-regularity of the simplexes. In particular, the constant of the local efficiency inequality is proven to behave like $(k+2)^{3/2}$. However, we present numerical experiments suggesting that such a constant is actually independent of k.

1. Introduction

In this paper, we establish the reliability and efficiency of a new a posteriori error estimator for the variable-degree Raviart-Thomas (RT) method applied to the following model problem:

$$\boldsymbol{q} + \boldsymbol{\nabla}\, u = 0 \qquad \text{in } \Omega, \tag{1.1a}$$

$$\nabla\cdot \boldsymbol{q} = f \qquad \text{in } \Omega, \tag{1.1b}$$

$$u = 0 \qquad \text{on } \Gamma, \tag{1.1c}$$

where $\Omega \in R^d$ $(d = 2, 3)$ is a polyhedral domain with Lipschitz boundary Γ, and f. We restrict ourselves to exploring the use of a new discontinuous postprocessing to define the a posteriori error estimator.

Let us compare our result with the relevant papers in the available literature [1–3, 5, 9, 16, 17, 20]. In 1996, Alonso [3] considered the RT and Brezzi-Douglas-Marini (BDM) mixed methods in two space dimensions and obtained an a posteriori error estimate for the L^2-error of the flux,

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)},$$

for which he proved its reliability and efficiency by using a Helmholtz decomposition. Independently and almost at the same time, Braess and Verfürth [5] considered the RT method in two space dimensions and obtained two a posteriori error estimates. One for what they called the *natural norm*, namely,

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{H(div,\Omega)} + \|u - u_h\|_{L^2(\Omega)}.$$

By using a *saturation* assumption, they proved the reliability and efficiency of the estimator. However, the effectivity index, that is, the ratio of the estimator to the actual error, for the natural norm was shown to be suboptimal, as it was of the

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order of h^{-1}. In 1997, Carstensen [9] showed how to obtain efficient and reliable a posteriori error estimates for the natural norm by bypassing the use of the saturation assumption. His results applied to the RT and BDM methods.

The second a posteriori error estimates obtained by Braess and Verfürth [5] are for the mesh-dependent norm

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,h} + |u - u_h|_{1,h},$$

where the first norm is, roughly speaking, a weighted L^2-norm, and the second is what has become the classical H^1-seminorm for discontinuous functions. The authors proved the reliability and efficiency of the estimator; the effectivity index they obtained was bounded independently of the meshsize h. Unfortunately, since in the case of very smooth solutions, $|u - u_h|_{1,h}$ converges with an order *less* than $\|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,h}$, the estimate was not suitable to be used to control the error in the flux. To overcome this problem, in 2006, Lovadina and Stenberg [17] considered the RT (and BDM) method and constructed an error estimator for the error

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,h} + |u - u_h^*|_{1,h},$$

where u_h^* is a suitably defined postprocessing. The efficiency followed very easily, but the reliability was more involved. It was obtained with a saturation assumption for two and three space dimensions, and by means of a Hemlholtz decomposition for two space dimensions.

In 2008, Larson and Målqvist [16] switched back to working with the error

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)},$$

and proved the reliability of their estimator without using the above-mentioned saturation assumption or the Helmholtz decomposition. Their result holds for methods including the RT and BDM methods in two and three space dimensions. Their result is closely related to that of Lovadina and Stenberg [17], but applies to *any* postprocessing u_h^*. However, no efficiency was obtained due to the fact that their error was different from that used by Lovadina and Stenberg [17].

In 2007, Ainsworth [1] considered the lowest order RT method and obtained a reliable and efficient estimator for the error

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)}.$$

Just as for the results of Lovadina and Stenberg [17], and Larson and Målqvist [16], his estimate is expressed in terms of a locally postprocessed approximation u_h^* which in his case continuous and piecewise quadratic. Remarkably enough, all the constants in his estimator are explicitly computed. Moreover, his result applies to the case in which $\boldsymbol{q} = -A\nabla u$. In the same year, Vohralik [20] obtained an a posteriori error estimate for convection-diffusion-reaction equations for schemes using the RT method to discretize the second-order term. In the purely diffusive case the estimator was shown to be reliable and efficient, and, moreover, asymptotically exact as the meshsize goes to zero. He also used a discontinuous local postprocessing but, in contrast, it is not fully quadratic on each element. Finally, in 2012, Ainsworth and Ma [2] extended the estimate obtained by Ainsworth [1] to the variable-degree BDM method on conforming meshes.

In this paper, we consider the variable-degree RT method on conforming meshes and propose a *new* a posteriori error estimate for the error

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)}.$$

We build upon the results obtained by Ainsworth [1] and keep his orthogonal decomposition of the error in $\boldsymbol{q}$, [1, Theorem 1], his estimation of the data error, [1, Theorem 2], and his characterization of the remaining error, [1, Lemma 1]. However, we do not take our postprocessing u_h^* to be a continuous function obtained by a suitable modification of the solution of a local Neumann problem; see [1, 2]. Instead, we take it to be a *discontinuous* function. In order to focus on this issue, we have considered homogeneous boundary conditions and are taking the matrix A in the equation $\boldsymbol{q} + A\,\boldsymbol{\nabla}\, u = 0$ to be the identity.

The postprocessing we take is defined essentially in the same manner as the one used by Lovadina and Stenberg [17], that is, by using a simple projection. There the only difference between our postprocessing and that used by Lovadina and Stenberg [17] is that, when the local RT space is of order k, our postprocessed solution is a polynomial of degree $k+2$, not a polynomial of degree $k+1$. Our main contribution is to show that, thanks to this slight difference, all the terms of our a posteriori error estimator can be controlled by the $L^2(\Omega)$-error in the flux only.

Moreover, the new definition of the postprocessing facilitates the analysis on the effect of polynomial degree on the effectivity index. Thus, we prove that the error is bounded by the error estimator up to a constant depending only on the shape-regularity of the elements. We also show that the estimator (associated to the face e) is bounded locally by the error up to a constant of order $(k(e)+1)^{3/2}$ where $k(e)$ denotes the polynomial degree on the face e; in 2011, a similar result was obtained by Zhu *et al.* [22] for the interior penalty method. However, our numerical experiments for the uniform-order case suggest that the effectivity index is actually independent of the polynomial degree.

The paper is organized as follows. In Section 2, we introduce the hybridized RT methods, the a posteriori error estimator and the main result. In Section 3, we carry out the analysis of its reliability and efficiency. In Section 4, we present the numerical experiments.

2. Main Result

2.1. **Notation.** Let $\mathcal{T}_h = \{K\}$ be a conforming triangulation of the domain Ω made of simplexes K. Let h_K denote the diameter of K and h_e the diameter of the face e of an element. We associate this triangulation to the set of interior faces $\mathcal{E}_h^i$ and the set of boundary faces $\mathcal{E}_h^\partial$. We say that $e \in \mathcal{E}_h^i$ if there are two elements K^+ and K^- in $\mathcal{T}_h$ such that $e = \partial K^+ \cap \partial K^-$, and we say that $e \in \mathcal{E}_h^\partial$ if there is an element K in $\mathcal{T}_h$ such that $e = \partial K \cap \Gamma$. We set $\mathcal{E}_h := \mathcal{E}^i \cup \mathcal{E}^\partial$. Finally, for any face e in $\mathcal{E}_h$, we set

$$\mathcal{K}(e) := \{K \in \mathcal{T}_h,\, m_{d-1}(e \cap \partial K) > 0\}.$$

We use the conventional notation $[\![\boldsymbol{v}]\!]$ to denote the jump of vector-valued function $\boldsymbol{v}$ across the face, and $[\![\varphi]\!]$ to denote the jump of scalar-valued function φ, that is,

$$[\![\boldsymbol{v}]\!] = \begin{cases} \boldsymbol{v}^- \cdot \boldsymbol{n}^- + \boldsymbol{v}^+ \cdot \boldsymbol{n}^+, & e \in \mathcal{E}_h^i, \\ \boldsymbol{v} \cdot \boldsymbol{n}, & e \in \mathcal{E}_h^\partial, \end{cases}$$

$$[\![\varphi]\!] = \begin{cases} \varphi^- \boldsymbol{n}^- + \varphi^+ \boldsymbol{n}^+, & e \in \mathcal{E}_h^i, \\ \varphi \boldsymbol{n}, & e \in \mathcal{E}_h^\partial. \end{cases}$$

We also use the standard notation $(\cdot\,,\,\cdot)_D$, $\langle\cdot\,,\cdot\rangle_\Gamma$ to denote the L_2 inner product on the elements D and faces Γ, respectively, that is,

$$\begin{aligned}
(\boldsymbol{\sigma}, \boldsymbol{v})_D &:= \sum_{K\in D}\int_K \boldsymbol{\sigma}(x)\cdot\boldsymbol{v}(x)\,dx,\\
(\zeta,\omega)_D &:= \sum_{K\in D}\int_K \zeta(x)w(x)dx,\\
\langle\zeta,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma &:= \sum_{e\in\Gamma}\int_e \zeta(x)\boldsymbol{v}(x)\cdot\boldsymbol{n}dx,
\end{aligned}$$

The outward unit normal vector to ∂K is denoted by $\boldsymbol{n}$.

Finally, the $L^2(D)$-norm of a function ζ will be denoted by $\|\zeta\|_D$.

2.2. **Assumption on the meshes.** Here, we state our assumption on the meshes.

A. *The meshes $\mathcal{T}_h$ are shape-regular, that is, there is a constant $\sigma > 0$, such that*

$$h_K/\rho_K \le \sigma, \tag{2.1}$$

for any element $K \in \mathcal{T}_h$, where ρ_K denotes the diameter of the largest ball inside K.

Note that this assumption implies that $h_e \le h_K \le \sigma h_e$, for any face e of the element $K \in \mathcal{T}_h$.

2.3. **The hybridized Raviart-Thomas method.** The hybridized version of the RT method is a finite element method which seeks an approximation to the exact solution $(\boldsymbol{q}|_\Omega, u|_\Omega, u|_{\partial\mathcal{E}_h})$, $(\boldsymbol{q}_h, u_h, \lambda_h)$, in the set $\boldsymbol{V}_h\times W_h\times \mathsf{M}_h$ where

$$\boldsymbol{V}_h := \{\boldsymbol{v}\in L^2(\Omega) \quad : \boldsymbol{v}|_k \in \boldsymbol{V}(K) \quad \forall K\in\Omega_h\}, \tag{2.2a}$$

$$W_h := \{\omega\in L^2(\Omega) \quad : \omega|_k \in W(K) \quad \forall K\in\Omega_h\}, \tag{2.2b}$$

$$\mathsf{M}_h := \{m\in L^2(\mathcal{E}_h) \quad : m|_e \in \mathsf{M}(e) \quad \forall e\in\mathcal{E}_h\}. \tag{2.2c}$$

The approximate solution $(\boldsymbol{q}_h, u_h)$ is determined on each element $K\in\mathcal{T}_h$ in terms of λ_h by the so-called local solver

$$(\boldsymbol{q}_h,\boldsymbol{v})_K - (u_h, \nabla\cdot\boldsymbol{v})_K = -\langle\lambda_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K}, \tag{2.3a}$$

$$(\nabla\cdot\boldsymbol{q}_h, w)_K = (f,w)_K, \tag{2.3b}$$

for all $(\boldsymbol{v}, w)\in\boldsymbol{V}(K)\times W(K)$. The numerical trace λ_h is then determined by the transmission condition

$$\langle\boldsymbol{q}_h\cdot\boldsymbol{n},\mu\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0 \qquad \forall\,\mu\in\mathsf{M}_h, \tag{2.4a}$$

and the Dirichlet boundary condition

$$\langle\lambda_h,\mu\rangle_\Gamma = 0 \qquad \forall\,\mu\in\mathsf{M}_h. \tag{2.4b}$$

To complete the definition of the method, we need to specify the local spaces $\boldsymbol{V}(K)$, $W(K)$ and $\mathsf{M}(e)$. We take $\boldsymbol{V}(K)\times W(K)$ as the Raviart-Thomas space of degree $k(K)$:

$$\boldsymbol{V}(K) := \mathcal{P}_{k(K)}(K)^d + \boldsymbol{x}\,\mathcal{P}_{k(K)}(K), \quad W(K) := \mathcal{P}_{k(K)}(K), \quad k(K)\ge 0,$$

where $\mathcal{P}_{k(K)}(K)$ denotes the space of the polynomial with degree less than or equal to $k(K)$ on the element K and $\mathcal{P}_{k(K)}(K)^d$ denotes the set of vector functions whose components are in $\mathcal{P}_{k(K)}(K)$. We define the space of approximate traces as:

$$\mathsf{M}(e) := \mathcal{P}_{k(e)}(e), \quad \text{where} \quad k(e) := \max\{k(K^+), k(K^-)\}.$$

The variable-degree RT-H method just described is uniquely solvable [10].

2.4. **Local postprocessing.** It is well known [4,6,7,13,18,19] that the accuracy of the approximate scalar variable of mixed methods can be improved by an element-by-element computation of a new approximation. Here, we use a new variation of the postprocessing [18,19] used by Lovadina and Stenberg [17].

We take the *new* postprocessing u_h^* in the space

$$W_h^* := \{w \in L^2(K), w|_K \in \mathcal{P}_{k(K)+2}(K) \quad \forall K \in \mathcal{T}_h\}, \tag{2.5}$$

and define it as follows. On the element K the function u_h^* is the element of $\mathcal{P}_{k(K)+2}(K)$ satisfying the following equations:

$$(u_h^* - u_h, 1)_K = 0, \tag{2.6a}$$

$$(\nabla u_h^*, \nabla w)_K = -(\boldsymbol{q}_h, \nabla w)_K \quad \text{for all } w \in \mathcal{P}_{k(K)+2}(K). \tag{2.6b}$$

It is easy to see that u_h^* is well defined.

2.5. **The a posteriori error estimate.** Our main result provides upper and local lower bounds of the error $\| \boldsymbol{q} - \boldsymbol{q}_h \|_\Omega$ in terms of the data oscillation

$$osc_h(f, \mathcal{T}_h) := \Big(\sum_{K\in\mathcal{T}_h} h_K^2 \| f - \mathsf{P}_W f \|_K^2\Big)^{1/2}, \tag{2.7}$$

and the error estimators

$$\eta_{i,h}^2 := \sum_{e\in\mathcal{E}_h} \eta_{i,h}^2(e) \quad \text{for } i = 1, 2, \tag{2.8a}$$

where

$$\eta_{1,h}^2(e) := \sum_{K\in\mathcal{K}(e)} \| \boldsymbol{q}_h + \nabla u_h^* \|_K^2, \tag{2.8b}$$

$$\eta_{2,h}^2(e) := (k(e)+2)^2 h_e^{-1} \| [\![u_h^*]\!] \|_e^2. \tag{2.8c}$$

We have the following result for establishing the reliability and efficiency of the error estimator in terms of the $L^2(\Omega)$ error of the flux only.

Theorem 2.1 (Reliability and local efficiency of the error estimator)**.** *Suppose that the mesh shape-regularity assumption* **A** *is satisfied. Then there are three constants C_1, C_2 and C_3 depending only on the shape-regularity constant σ such that*

$$\| \boldsymbol{q} - \boldsymbol{q}_h \|_\Omega^2 \le \frac{1}{\pi^2} osc_h^2(f, \mathcal{T}_h) + \frac{1}{d+1} \eta_{1,h}^2 + C_1 \eta_{2,h}^2.$$

Moreover, for each face $e \in \mathcal{E}_h$,

$$C_2 \eta_{1,h}^2(e) \le (k(e)+2) \sum_{K\in\mathcal{K}(e)} \| \boldsymbol{q} - \boldsymbol{q}_h \|_K^2,$$

$$C_3 \eta_{2,h}^2(e) \le (k(e)+2)^3 \sum_{K\in\mathcal{K}(e)} \| \boldsymbol{q} - \boldsymbol{q}_h \|_K^2.$$

Let us briefly discuss the case in which f lies in W_h, that is, in the case in which $osc_h(f, \mathcal{T}_h) = 0$, and when the local polynomial degree is the same in all elements, that is, when $k(K) = k(e) = k$. We see that the ratio $\eta_{1,h}^2 / \| \boldsymbol{q} - \boldsymbol{q}_h \|_\Omega^2$ is bounded by $(k+2)/C_2$, and that the ratio $\eta_{2,h}^2 / \| \boldsymbol{q} - \boldsymbol{q}_h \|_\Omega^2$ is bounded by $(d+1)\,(k+2)^3/C_3$. An upper bound of order $(k+2)^3$ was also found for the interior penalty method in [22]. However, our numerical experiments suggest that the ratios are in fact fairly insensitive to the change in the polynomial degree.

3. Proof

In this section, we provide a proof of the efficiency and reliability of the estimator given in Theorem 2.1.

3.1. **Proof of the reliability.** We begin with the following result.

Lemma 3.1 ([1]). *We have that*

$$\| \boldsymbol{q} - \boldsymbol{q}_h \|_\Omega^2 = \| \boldsymbol{q} - \boldsymbol{q}_{P_W f} \|_\Omega^2 + \| \boldsymbol{q}_{P_W f} - \boldsymbol{q}_h \|_\Omega^2,$$

where $(\boldsymbol{q}_{P_W f}, u_{P_W f})$ *is the solution of the model problem* (1.1) *with* f *replaced by its* L^2*-projection into the space* W_h*. Moreover,*

$$\| \boldsymbol{q} - \boldsymbol{q}_{P_W f} \|_\Omega^2 \leq \frac{1}{\pi^2}\, osc_h^2(f, \mathcal{T}_h),$$
$$\| \boldsymbol{q}_{P_W f} - \boldsymbol{q}_h \|_\Omega^2 \leq \min_{\nu \in H_0^1(\Omega)} \|\nabla \nu + \boldsymbol{q}_h\|_\Omega^2.$$

Although this result was proven for the two-dimensional case in [1], its proof for the three-dimensional case is similar.

Next, we get an estimate of $\min_{\nu \in H_0^1(\Omega)} \|\nabla \nu + \boldsymbol{q}_h\|_\Omega^2$. In what follows, we use the notation $\|\boldsymbol{\zeta}\|_{\mathcal{T}_h}^2 := \sum_{K \in \mathcal{T}_h} \|\boldsymbol{\zeta}\|_K^2$ and $(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2)_K$.

Lemma 3.2. *We have that*

$$\min_{\nu \in H_0^1(\Omega)} \|\nabla \nu + \boldsymbol{q}_h\|_\Omega^2 \leq \min_{\tilde{u}_h^* \in W_h^* \cap H_0^1(\Omega)} \|\nabla(\tilde{u}_h^* - u_h^*)\|_{\mathcal{T}_h}^2 + \|\nabla u_h^* + \boldsymbol{q}_h\|_{\mathcal{T}_h}^2.$$

Proof. Since

$$\min_{\nu \in H_0^1(\Omega)} \|\nabla \nu + \boldsymbol{q}_h\|_\Omega^2 \leq \min_{\tilde{u}_h^* \in W_h^* \cap H_0^1(\Omega)} \|\nabla \tilde{u}_h^* + \boldsymbol{q}_h\|_{\mathcal{T}_h}^2,$$

the result follows from the fact that

$$\|\nabla \tilde{u}_h^* + \boldsymbol{q}_h\|_\Omega^2 = \|\nabla(\tilde{u}_h^* - u_h^*)\|_{\mathcal{T}_h}^2 + \|\nabla u_h^* + \boldsymbol{q}_h\|_{\mathcal{T}_h}^2,$$

since $(\nabla(\tilde{u}_h^* - u_h^*), \nabla u_h^* + \boldsymbol{q}_h)_{\mathcal{T}_h} = 0$ by definition of u_h^*, (2.6b). This completes the proof. $\square$

Finally, we obtain an estimate of $\min_{\tilde{u}_h^* \in W_h^* \cap H_0^1(\Omega)} \|\nabla(\tilde{u}_h^* - u_h^*)\|_\Omega^2$ by using the following result. It was proved in [14,15,22] for nonconforming meshes with uniform polynomial degree. The extension to the variable-degree case we present here is straightforward.

Lemma 3.3. *For any* $w_h \in W_h^*$*, there exists a function* $\tilde{w}_h \in W_h^* \cap H_0^1(\Omega)$ *such that*

$$\| \boldsymbol{\nabla}(w_h - \tilde{w}_h) \|_{\mathcal{T}_h}^2 \leq C_1 \sum_{e \in \mathcal{E}_h} (k(e) + 2)^2\, h_e^{-1} \| [\![w_h]\!] \|_e^2,$$

where the constant C_1 *only depends on the shape regularity constant* σ*.*

We are now ready to prove the reliability of the error estimator. We have, by Lemma 3.1,

$$\begin{aligned}\|\,\boldsymbol{q}-\boldsymbol{q}_h\,\|_\Omega^2 &\le \frac{1}{\pi^2}\,osc_h^2(f,\mathcal{T}_h)+\min_{\nu\in H_0^1(\Omega)}\|\nabla\nu+\boldsymbol{q}_h\|_\Omega^2\\ &\le \frac{1}{\pi^2}\,osc_h^2(f,\mathcal{T}_h)+\min_{\tilde u_h^*\in W_h^*\cap H_0^1(\Omega)}\|\nabla(\tilde u_h^*-u_h^*)\|_{\mathcal{T}_h}^2+\|\nabla u_h^*+\boldsymbol{q}_h\|_{\mathcal{T}_h}^2\end{aligned}$$

by Lemma 3.2. Moreover, by Lemma 3.3,

$$\begin{aligned}\|\,\boldsymbol{q}-\boldsymbol{q}_h\,\|_\Omega^2 &\le \frac{1}{\pi^2}\,osc_h^2(f,\mathcal{T}_h)+C_1\sum_{e\in\mathcal{E}_h}(k(e)+2)^2\,h_e^{-1}\,\|\,[\![u_h^*]\!]\,\|_e^2+\|\nabla u_h^*+\boldsymbol{q}_h\|_{\mathcal{T}_h}^2\\ &\le \frac{1}{\pi^2}\,osc_h^2(f,\mathcal{T}_h)+C_1\,\eta_{2,h}^2+\frac{1}{d+1}\sum_{K\in\mathcal{K}(e)}\|\nabla u_h^*+\boldsymbol{q}_h\|_K^2,\end{aligned}$$

and the result follows. This completes the proof of the reliability of the estimator.

3.2. **Proof of the local efficiency.** To complete the proof of Theorem 2.1, it only remains to prove the following local lower bound for the error $\|\,\boldsymbol{q}-\boldsymbol{q}_h\,\|_{\mathcal{K}(e)}$. We proceed in three steps.

Step 1. First, we establish a relation between the residual in the element and the residual across the faces.

Lemma 3.4. *For any face $e\in\mathcal{E}_h$, we have that*

$$h_e^{-1/2}\|\,\mathsf{P}_{\mathcal{M}_0}\,[\![u_h^*]\!]\,\|_e\le\frac{\sigma}{\sqrt{2d}}\sum_{K\in\mathcal{K}(e)}\|\,\boldsymbol{q}_h+\boldsymbol{\nabla}\,u_h^*\,\|_K.$$

Proof. From the equation defining the hybridized mixed method (2.3) and those defining the postprocessing (2.6), we have that, for any simplex $K\in\mathcal{T}_h$,

$$(\boldsymbol{q}_h,\boldsymbol{v})_K-(u_h^*,\nabla\cdot\boldsymbol{v})_K=-\langle\lambda_h,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K},$$

for all functions $\boldsymbol{v}$ in the lowest order Raviart-Thomas space, $\mathrm{RT}_0(K)$. Integrating by parts, we get

$$(\boldsymbol{q}_h+\boldsymbol{\nabla}\,u_h^*,\boldsymbol{v})_K=-\langle\lambda_h-u_h^*,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K}.$$

Now take $\boldsymbol{v}$ in $\boldsymbol{H}(\mathrm{div},\mathcal{K}(e))$. Since λ_h is single valued on the faces, summing over the elements $K\in\mathcal{K}(e)$ yields:

$$\sum_{K\in\mathcal{K}(e)}(\boldsymbol{q}_h+\boldsymbol{\nabla}\,u_h^*,\boldsymbol{v})_K=-\sum_{K\in\mathcal{K}(e)}\sum_{F\in\partial K\setminus e}\langle\lambda_h-u_h^*,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_F+\langle\,[\![u_h^*]\!],\boldsymbol{v}\rangle_e.$$

Taking $\boldsymbol{v}\in\mathrm{RT}_0(K)$, such that for each $K\in\mathcal{K}(e)$,

$$\begin{aligned}&\int_e\boldsymbol{v}\cdot\boldsymbol{n}=\int_e\mathsf{P}_{\mathcal{M}_0}\,[\![u_h^*]\!] &&\text{for the face } e,\\ &\int_F\boldsymbol{v}\cdot\boldsymbol{n}=0 &&\text{for all } F\in\partial K\setminus e,\end{aligned}$$

we obtain

$$
\begin{aligned}
\| \mathsf{P}_{\mathcal{M}_0} [\![u_h^*]\!] \|_e^2 &= \sum_{K \in \mathcal{K}(e)} (\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*, \boldsymbol{v})_K \\
&\le \sum_{K \in \mathcal{K}(e)} \|\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*\|_K \, \|\boldsymbol{v}\|_K \\
&\le \frac{\sigma}{\sqrt{2d}} \sum_{K \in \mathcal{K}(e)} \|\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*\|_K \, h_e^{1/2} \, \|\boldsymbol{v}\|_e \qquad \text{by Lemma A.1,} \\
&= \frac{\sigma}{\sqrt{2d}} \sum_{K \in \mathcal{K}(e)} \|\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*\|_K \, h_e^{1/2} \, \| \mathsf{P}_{\mathcal{M}_0} [\![u_h^*]\!] \|_e
\end{aligned}
$$

by definition of the function $\boldsymbol{v}$. This completes the proof. □

Step 2. Next, we obtain the following estimate.

Lemma 3.5. *For each face $e \in \mathcal{E}_h$, we have*

$$
h_e^{-1} \| (\mathsf{Id} - \mathsf{P}_{\mathcal{M}_0}) [\![u_h^*]\!] \|_e^2 \le 2 C_{tr} \sum_{K \in \mathcal{K}(e)} \| \boldsymbol{\nabla} (u - u_h^*) \|_K^2,
$$

where C_{tr} is a constant depending only on the shape regularity constant σ.

Proof. Adding and subtracting u, we have

$$
h_e^{-1} \| (\mathsf{Id} - \mathsf{P}_{\mathcal{M}_0}) [\![u_h^*]\!] \|_e^2 \le 2 h_e^{-1} \sum_{K \in \mathcal{K}(e)} \| (\mathsf{Id} - \mathsf{P}_{\mathcal{M}_0}) (u - u_h^*|_K) \|_e^2,
$$

where $u_h^*|_K$ denotes the trace on e from the interior of the element K. To prove the lemma, we only need to show that, for any $K \in \mathcal{K}(e)$,

$$
h_e^{-1} \| (\mathsf{Id} - \mathsf{P}_{\mathcal{M}_0}) (u - u_h^*|_K) \|_e^2 \le C_{tr} \| \boldsymbol{\nabla} (u - u_h^*) \|_K^2.
$$

To do so, we note that

$$
\| (\mathsf{Id} - \mathsf{P}_{\mathcal{M}_0}) (u - u_h^*|_K) \|_e^2 \le \| (\mathsf{Id} - \mathsf{P}_{W_0}) (u - u_h^*|_K) \|_e^2,
$$

where P_{W_0} denotes the L_2-orthogonal projection into the space of piecewise constant functions on each element, we have

$$
\begin{aligned}
h_e^{-1} \| (\mathsf{Id} - \mathsf{P}_{\mathcal{M}_0}) (u - u_h^*|_K) \|_e^2 &\le h_e^{-1} \| (\mathsf{Id} - \mathsf{P}_{W_0}) (u - u_h^*|_K) \|_e^2 \\
&\le C_{tr} \| \boldsymbol{\nabla} (u - u_h^*) \|_K^2
\end{aligned}
$$

by the trace theorem and Poincaré's inequality. This completes the proof. □

Step 3. In view of Lemma 3.4 and Lemma 3.5, to establish a lower bound for the estimator, we only need to show that $\| \boldsymbol{q}_h + \boldsymbol{\nabla} u_h^* \|_K^2$ is bounded by $\| \boldsymbol{q} - \boldsymbol{q}_h \|_K^2$. To do so, we are going to use the following orthogonal decomposition.

Lemma 3.6 ([11])**.** *Let T be a simplex. Then any function μ in the space*

$$\mathcal{P}_{k+2}(\partial T) := \{\mu \in L^2(\partial T) : \ \mu|_e \in \mathcal{P}_{k+2}(e) \text{ for each } e \in \partial T\},$$

can be written in a unique manner as

$$\mu = w + \boldsymbol{v} \cdot \boldsymbol{n},$$

for some $w \in \mathcal{P}_{k+2}(T)^\perp$ and $\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k+2}(T)^\perp$, where

$$\mathcal{P}_{k+2}(T)^\perp := \{w \in \mathcal{P}_{k+2}(T),\ (w, \tilde{w})_T = 0 \quad \text{for all } \tilde{w} \in \mathcal{P}_{k+1}(T)\},$$
$$\boldsymbol{\mathcal{P}}_{k+2}(T)^\perp := \{\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k+2}(T),\ (\boldsymbol{v}, \tilde{\boldsymbol{v}})_T = 0 \quad \text{for all } \tilde{\boldsymbol{v}} \in \boldsymbol{\mathcal{P}}_{k+1}(T)\}.$$

Moreover, $\langle w, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial T} = 0$.

We are now ready to prove the following result.

Lemma 3.7. *For each element $K \in \mathcal{T}_h$, we have that*

$$\| \boldsymbol{q}_h + \boldsymbol{\nabla} u_h^* \|_K^2 \leq C_4 (1 + D_k) \| \boldsymbol{q} - \boldsymbol{q}_h \|_K^2,$$

where $C_4 := 2d\,(1 + \sqrt{d})^2\, \sigma^2$ and $D_k := \frac{d+1}{d+4}\,(k+2)$.

Proof. Since $(\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*) \cdot \boldsymbol{n} \in \mathcal{P}_{k+2}(\partial T)$, by Lemma 3.6, we can write

$$(\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*) \cdot \boldsymbol{n} = \delta_w + \boldsymbol{\delta_v} \cdot \boldsymbol{n},$$

where $\delta_w \in \mathcal{P}_{k+2}^\perp(K)$ and $\boldsymbol{\delta_v} \in \boldsymbol{\mathcal{P}}_{k+2}^\perp(K)$.

Let us show that $\delta_w = 0$. We have

$$\begin{aligned}
\langle \delta_w, \delta_w \rangle_{\partial K} &= \langle \delta_w + \boldsymbol{\delta_v} \cdot \boldsymbol{n}, \delta_w \rangle_{\partial K} && \text{by Lemma 3.6,} \\
&= \langle (\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*) \cdot \boldsymbol{n}, \delta_w \rangle_{\partial K} \\
&= \big(\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*, \boldsymbol{\nabla}\, \delta_w\big) + \big(\nabla \cdot (\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*), \delta_w\big) \\
&= \big(\nabla \cdot (\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*), \delta_w\big) && \text{by definition of } u_h^*, \text{ (2.6b),} \\
&= 0,
\end{aligned}$$

since $\nabla \cdot (\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*) \in \mathcal{P}_k(K)$, and since δ_w is $L_2(\Omega)$-orthogonal to polynomials of degree $k+1$. This implies that $\delta_w = 0$.

So, if we set $\mathbf{v} := \boldsymbol{q}_h + \boldsymbol{\nabla} u_h^* - \boldsymbol{\delta_v}$, we immediately have that $\mathbf{v} \cdot \boldsymbol{n} = 0$ on ∂K. Let us now show that we also have $\nabla \cdot \mathbf{v} = 0$.

Indeed, since $\mathbf{v} \cdot \boldsymbol{n}$ vanishes on ∂K, we have, by integration by parts, that

$$\begin{aligned}
(\nabla \cdot \mathbf{v}, w) &= -\,(\mathbf{v}, \boldsymbol{\nabla}\, w) \\
&= -\,(\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*, \boldsymbol{\nabla}\, w) + (\boldsymbol{\delta_v}, \boldsymbol{\nabla}\, w) && \text{by definition of } \mathbf{v}, \\
&= (\boldsymbol{\delta_v}, \boldsymbol{\nabla}\, w) && \text{by definition of } u_h^*, \text{ (2.6b),} \\
&= 0,
\end{aligned}$$

provided $w \in \mathcal{P}_{k+1}(K)$, since $\boldsymbol{\delta_v}$ is $\mathbf{L}^2(\Omega)$-orthogonal to vector-valued polynomials of degree $k+1$. This implies that $\nabla \cdot \mathbf{v} = 0$.

Since $\mathbf{v}$ is a divergence-free function with zero normal traces, we have that

$$(\mathbf{v}, \boldsymbol{\nabla}\, \phi) = 0$$

for all $\phi \in H^1(K)$. Thus, we have

$$\begin{aligned}
\| \boldsymbol{q}_h + \boldsymbol{\nabla} u_h^* \|_K^2 &= (\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*, \mathbf{v}) + (\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*, \boldsymbol{\delta_v}) \\
&= (\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*, \mathbf{v}) \\
&= (\boldsymbol{q}_h + \boldsymbol{\nabla} u, \mathbf{v}) \\
&\le \|\boldsymbol{q}_h + \boldsymbol{\nabla} u\|_K \, \|\mathbf{v}\|_K,
\end{aligned}$$

and the result follows if we prove that

$$\| \mathbf{v} \|_K \le \sqrt{C_4(1+D_k)} \| \boldsymbol{q}_h + \boldsymbol{\nabla} u_h^* \|_K.$$

But since $\mathbf{v} \in \boldsymbol{\mathcal{P}}_{k+2}(K)$ and $\mathbf{v} \cdot \boldsymbol{n} = 0$ on ∂K, by Corollary A.1, we have

$$\begin{aligned}
\| \mathbf{v} \|_K &\le \sqrt{C_4(1+D_k)} \| \mathsf{P}_{k+1} \mathbf{v} \|_K \\
&= \sqrt{C_4(1+D_k)} \| \mathsf{P}_{k+1} (\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^* - \boldsymbol{\delta_v}) \|_K \\
&= \sqrt{C_4(1+D_k)} \| \boldsymbol{q}_h + \boldsymbol{\nabla} u_h^* \|_K.
\end{aligned}$$

This completes the proof. ☐

Step 4. We are now ready to prove the local efficiency estimate. We have that

$$\begin{aligned}
\eta_{1,h}^2(e) &= \sum_{K \in \mathcal{K}(e)} \|\boldsymbol{q}_h + \nabla u_h^*\|_K^2 \\
&\le C_4 \sum_{K \in \mathcal{K}(e)} (1 + D_{k(K)}) \, \|\boldsymbol{q}_h + \nabla u\|_K^2 \\
&\le C_4 \, (1 + D_{k(e)}) \sum_{K \in \mathcal{K}(e)} \|\boldsymbol{q}_h + \nabla u\|_K^2 \\
&\le \frac{(k(e)+2)}{C_2} \sum_{K \in \mathcal{K}(e)} \|\boldsymbol{q}_h + \nabla u\|_K^2,
\end{aligned}$$

by Lemma 3.7 and since $k(e) := \max\{k(K) : \ K \in \mathcal{K}(e)\}$.

We also have that

$$\begin{aligned}
\eta_{2,h}^2(e) &= (k(e)+2)^2 \, h_e^{-1} \, \| \, [\![u_h^*]\!] \, \|_e^2 \\
&= (k(e)+2)^2 \, h_e^{-1} \left(\|\mathsf{P}_{\mathcal{M}_0} \, [\![u_h^*]\!]\|_e^2 + \|(\mathsf{Id} - \mathsf{P}_{\mathcal{M}_0}) \, [\![u_h^*]\!]\|_e^2 \right) \\
&\le (k(e)+2)^2 \left(\frac{\sigma^2}{d} + 2C_{tr} \right) \sum_{K \in \mathcal{K}(e)} \| \, \boldsymbol{\nabla}(u - u_h^*) \, \|_K^2,
\end{aligned}$$

by Lemmas 3.4 and 3.5. Finally, by Lemma 3.7, we get

$$\begin{aligned}
\eta_{2,h}^2(e) &\le (k(e)+2)^2 \left(\frac{\sigma^2}{d} + 2C_{tr} \right) C_4 \, (1 + D_{k(e)}) \sum_{K \in \mathcal{K}(e)} \|\boldsymbol{q} - \boldsymbol{q}_h\|_K^2 \\
&\le \frac{(k(e)+2)^3}{C_3} \sum_{K \in \mathcal{K}(e)} \|\boldsymbol{q} - \boldsymbol{q}_h\|_K^2.
\end{aligned}$$

This completes the proof of the local efficiency estimate.

4. Numerical Results

In this section, we present numerical experiments devised to verify the reliability and efficiency properties of the a posteriori estimate predicted by our theoretical result, Theorem 2.1, in the two-dimensional case. We restrict ourselves to exploring how the constants relating the estimators with the actual error behave in terms of the meshsize and the polynomial degree.

4.1. **Preliminaries.** We do this by using a test problem with a smooth solution and one with a solution presenting a singularity. The exact solutions are:

$$u = \sin(\pi x)\sin(\pi y), \qquad \Omega := (0,1)^2, \tag{4.1}$$

$$u = r^{2/3}\sin(2\theta/3), \qquad \Omega := (-1,1)^2 \setminus [0,1]\times[-1,0]. \tag{4.2}$$

In both test problems, we set the Dirichlet boundary data g equal to the exact solution u. Note that for the first problem we have

$$f = \pi^2 \sin(\pi x)\sin(\pi y), \quad g = 0.$$

For the second, we have $f = 0$ but that g is not identically zero on the boundary $\partial\Omega$. This implies that the a posteriori error estimate we propose does not apply to this problem. However, since most of the error will be concentrated around the re-entrant corner $(0,0)$ and $g = 0$ on the edges of the boundary $\partial\Omega$ touching it, the oscillation term associated to the nonpolynomial nature of g elsewhere should not play a significant role, as we actually see.

We test the behavior of the a posteriori estimate with uniform mesh refinement as well as with an adaptive mesh refinement.

The *computational order of convergence* of each of these quantities is calculated as follows. Suppose that $e(N)$ and $e(\tilde{N})$ is any of the above quantities for two consecutive triangulations with N and $\tilde{N}$ number of triangles, respectively. Then, the computational rate of convergence is given by

$$-2\frac{\log(e(N)/e(\tilde{N}))}{\log(N/\tilde{N})}.$$

We also define the effective index as the ratio of the estimator over the error.

4.2. **Uniform refinement.** In Table 1, we present the numerical results for the smooth solution. We observe that the first effectivity index, $\eta_1/\|\boldsymbol{q}-\boldsymbol{q}_h\|_\Omega$, remains fairly constant as the meshsize decreases and that it grows slightly as k increases. We also see that the second effectivity, $\eta_2/\|\boldsymbol{q}-\boldsymbol{q}_h\|_\Omega$, index slightly increases with respect to mesh size and stays between 1.12 and 1.95 in the asymptotic regime. As for the effect of the polynomial degree k, we see that the index is slightly decreasing with respect to k. Thus, we can conclude that both estimators are robust with respect to the meshsize and the polynomial degree.

In Table 2, we display the behavior of the a posteriori error estimators for the nonsmooth solution, (4.2). We observe that, for all meshes and $k \geq 1$, the first effectivity index stays between 0.43 and 0.5 while the second effectivity index stays between 1.12 and 1.18. The estimators perform even better for the nonsmooth solution than for the smooth solution.

4.3. **Adaptive refinement.** Next, we present numerical results obtained for the following adaptive algorithm:

(1) Start with an initial mesh $\mathcal{T}_h$.
(2) Solve the discrete problem (2.3), (2.4a) and (2.4b).
(3) Compute the estimators.
(4) Decide to stop or go to the next step.
(5) Mark the elements to be refined.
(6) Modify the mesh with the so-called *red-green-blue* procedure.
(7) Denote the new mesh by $\mathcal{T}_h$ and return to step 2.

The marking strategy [12] consists in selecting a subset M of Ω_h such that

$$\alpha^2 \sum_{K \in \mathcal{T}_h} \eta_K^2 \leq \sum_{K \in M} \eta_K^2,$$

where the parameter α is taken in $(0, 1)$. Here, we take $\alpha = 1/2$.

We present the numerical results for the smooth solution in Table 3. We observe a behavior similar to the one seen for the uniform meshes, that is, the effectivity indices of both estimators are very stable with respect to changes in the meshes and in the polynomial degree. The first one grows slightly with respect to k while the second one decreases slightly with respect to k. Thus, we conclude that the robustness of the estimators holds also for the adaptive algorithm.

Finally, we display the results for the nonsmooth solution in Table 4. We see once more that both effectivity indices are extremely stable under changes of meshes and polynomial degrees. Note that the variation of the second effectivity index is practically negligible for $k \geq 1$. Moreover, compared with the smooth solution example, both estimators show even better performance in the sense that the corresponding effective indices become fairly constant. Thus, these results further confirm the conclusions of the previous examples, namely, that the estimators are robust with respect to the meshes and the polynomial degree.

TABLE 1. History of convergence of the error $e := \| \boldsymbol{q} - \boldsymbol{q}_h \|_\Omega$, the error estimators $\eta_1 := \| \boldsymbol{q}_h + \boldsymbol{\nabla} u_h^* \|_{\mathcal{T}_h}$ and $\eta_2 := \sqrt{\sum \frac{(k+2)^2}{h_e} \| [\![u_h^*]\!] \|_e^2}$, the oscillation $osc := \sqrt{\sum h_K^2 \| f - \mathsf{P}_W f \|_K^2}$ and the effectivity index $eff := \sqrt{osc^2/\pi^2 + \eta_1^2 + \eta_2^2}/e$. Uniform mesh refinement, smooth solution.

$k = 0$

N	e	order	η_1	η_1/e	η_2	η_2/e	osc	order	osc/e	eff
16.	0.99E+00	-	0.53E-15	0.00	0.10E+01	1.04	0.89E+00	-	0.90	1.08
64.	0.50E+00	0.98	0.55E-15	0.00	0.79E+00	1.57	0.23E+00	1.97	0.45	1.58
256.	0.25E+00	1.00	0.54E-15	0.00	0.45E+00	1.80	0.57E-01	1.99	0.23	1.80
1024.	0.13E+00	1.00	0.54E-15	0.00	0.24E+00	1.90	0.14E-01	2.00	0.11	1.90
4096.	0.63E-01	1.00	0.55E-15	0.00	0.12E+00	1.95	0.36E-02	2.00	0.06	1.95

$k = 1$

N	e	order	η_1	η_1/e	η_2	η_2/e	osc	order	osc/e	eff
16.	0.15E+00	-	0.80E-01	0.55	0.16E+00	1.07	0.17E+00	-	1.17	1.82
64.	0.37E-01	1.97	0.21E-01	0.57	0.56E-01	1.50	0.22E-01	2.97	0.59	1.61
256.	0.94E-02	1.99	0.53E-02	0.57	0.16E-01	1.67	0.27E-02	2.99	0.29	1.77
1024.	0.23E-02	2.00	0.13E-02	0.57	0.41E-02	1.74	0.34E-03	3.00	0.15	1.83
4096.	0.59E-03	2.00	0.33E-03	0.57	0.10E-02	1.77	0.43E-04	3.00	0.07	1.84

$k = 2$

N	e	order	η_1	η_1/e	η_2	η_2/e	osc	order	osc/e	eff
16.	0.17E-01	-	0.12E-01	0.68	0.14E-01	0.81	0.22E-01	-	1.29	1.22
64.	0.22E-02	2.97	0.15E-02	0.68	0.25E-02	1.13	0.14E-02	3.97	0.64	1.32
256.	0.28E-03	2.99	0.19E-03	0.68	0.35E-03	1.28	0.89E-04	3.99	0.32	1.45
1024.	0.34E-04	3.00	0.23E-04	0.68	0.47E-04	1.35	0.56E-05	4.00	0.16	1.52
4096.	0.43E-05	3.00	0.29E-05	0.68	0.60E-05	1.38	0.35E-06	4.00	0.08	1.54

$k = 3$

N	e	order	η_1	η_1/e	η_2	η_2/e	osc	order	osc/e	eff
16.	0.16E-02	-	0.12E-02	0.77	0.12E-02	0.77	0.22E-02	-	1.37	1.17
64.	0.10E-03	3.98	0.78E-04	0.78	0.11E-03	1.11	0.69E-04	4.98	0.69	1.37
256.	0.63E-05	4.00	0.49E-05	0.78	0.78E-05	1.24	0.22E-05	4.99	0.34	1.47
1024.	0.39E-06	4.00	0.31E-06	0.78	0.51E-06	1.30	0.68E-07	5.00	0.17	1.52
4096.	0.36E-07	3.46	0.27E-07	0.76	0.57E-07	1.59	0.21E-08	5.00	0.06	1.76

$k = 4$

N	e	order	η_1	η_1/e	η_2	η_2/e	osc	order	osc/e	eff
16.	0.12E-03	-	0.98E-04	0.82	0.78E-04	0.65	0.17E-03	-	1.42	1.14
64.	0.38E-05	4.98	0.31E-05	0.82	0.35E-05	0.92	0.27E-05	5.98	0.71	1.25
256.	0.12E-06	4.99	0.97E-07	0.82	0.12E-06	1.04	0.42E-07	5.99	0.36	1.33
1024.	0.37E-08	5.00	0.30E-08	0.82	0.41E-08	1.10	0.66E-09	6.00	0.18	1.37
4096.	0.12E-09	5.00	0.95E-10	0.82	0.13E-09	1.12	0.10E-10	6.00	0.09	1.39

TABLE 2. History of convergence of the error $e := \| \boldsymbol{q} - \boldsymbol{q}_h \|_\Omega$, the error estimators $\eta_1 := \| \boldsymbol{q}_h + \boldsymbol{\nabla} u_h^* \|_{\mathcal{T}_h}$ and $\eta_2 := \sqrt{\sum \frac{(k+2)^2}{h_e} \| [\![u_h^*]\!] \|_e^2}$, and the effectivity index $eff := \sqrt{osc^2/\pi^2 + \eta_1^2 + \eta_2^2}/e$. Uniform mesh refinement, nonsmooth solution. Note that in this case, the oscillation osc is zero.

$k = 0$

N	e	order	η_1	η_1/e	η_2	η_2/e	eff
12.	0.30E+00	-	0.30E-15	0.00	0.66E+00	2.23	2.23
48.	0.21E+00	0.53	0.34E-15	0.00	0.48E+00	2.31	2.31
192.	0.14E+00	0.59	0.31E-15	0.00	0.33E+00	2.40	2.40
768.	0.89E-01	0.63	0.32E-15	0.00	0.22E+00	2.52	2.52
3072.	0.57E-01	0.64	0.33E-15	0.00	0.15E+00	2.66	2.66

$k = 1$

N	e	order	η_1	η_1/e	η_2	η_2/e	eff
12.	0.14E+00	-	0.61E-01	0.44	0.16E+00	1.17	1.25
48.	0.89E-01	0.65	0.38E-01	0.43	0.10E+00	1.15	1.23
192.	0.56E-01	0.66	0.24E-01	0.43	0.64E-01	1.14	1.22
768.	0.35E-01	0.67	0.15E-01	0.43	0.40E-01	1.14	1.22
3072.	0.22E-01	0.67	0.96E-02	0.43	0.25E-01	1.14	1.22

$k = 2$

N	e	order	η_1	η_1/e	η_2	η_2/e	eff
12.	0.86E-01	-	0.41E-01	0.47	0.97E-01	1.13	1.22
48.	0.55E-01	0.66	0.26E-01	0.47	0.61E-01	1.12	1.21
192.	0.34E-01	0.67	0.16E-01	0.47	0.39E-01	1.12	1.21
768.	0.22E-01	0.67	0.10E-01	0.47	0.24E-01	1.12	1.21
3072.	0.14E-01	0.67	0.65E-02	0.47	0.15E-01	1.12	1.21

$k = 3$

N	e	order	η_1	η_1/e	η_2	η_2/e	eff
12.	0.60E-01	-	0.30E-01	0.50	0.70E-01	1.18	1.28
48.	0.38E-01	0.66	0.19E-01	0.50	0.44E-01	1.18	1.28
192.	0.24E-01	0.67	0.12E-01	0.50	0.28E-01	1.18	1.28
768.	0.15E-01	0.67	0.75E-02	0.50	0.18E-01	1.18	1.28
3072.	0.94E-02	0.67	0.47E-02	0.50	0.11E-01	1.18	1.28

$k = 4$

N	e	order	η_1	η_1/e	η_2	η_2/e	eff
12.	0.50E-01	-	0.23E-01	0.47	0.56E-01	1.11	1.21
48.	0.31E-01	0.67	0.15E-01	0.47	0.35E-01	1.11	1.21
192.	0.20E-01	0.67	0.93E-02	0.47	0.22E-01	1.11	1.21
768.	0.12E-01	0.67	0.59E-02	0.47	0.14E-01	1.12	1.21
3072.	0.79E-02	0.67	0.37E-02	0.47	0.88E-02	1.12	1.21

TABLE 3. History of convergence of the error $e := \|\boldsymbol{q} - \boldsymbol{q}_h\|_\Omega$, the error estimators $\eta_1 := \|\boldsymbol{q}_h + \nabla u_h^*\|_{\mathcal{T}_h}$ and $\eta_2 := \sqrt{\sum \frac{(k+2)^2}{h_e} \| [\![u_h^*]\!] \|_e^2}$, the oscillation $osc := \sqrt{\sum h_K^2 \| f - \mathsf{P}_W f\|_K^2}$ and the effectivity index $eff := \sqrt{osc^2/\pi^2 + \eta_1^2 + \eta_2^2}/e$. Adaptive mesh refinement, smooth solution.

$k = 0$

N	e	order	η_1	η_1/e	η_2	η_2/e	osc	order	osc/e	eff
16.	0.99E+00	-	0.53E-15	0.00	0.10E+01	1.04	0.89E+00	-	0.90	1.09
56.	0.50E+00	1.10	0.59E-15	0.00	0.93E+00	1.86	0.28E+00	1.87	0.55	1.87
120.	0.29E+00	1.47	0.55E-15	0.00	0.51E+00	1.80	0.25E+00	0.22	0.89	1.82
184.	0.22E+00	1.14	0.60E-15	0.00	0.43E+00	1.93	0.15E+00	2.32	0.69	1.94
352.	0.16E+00	1.05	0.58E-15	0.00	0.29E+00	1.80	0.15E+00	0.14	0.93	1.82

$k = 1$

N	e	order	η_1	η_1/e	η_2	η_2/e	osc	order	osc/e	eff
16.	0.15E+00	-	0.80E-01	0.55	0.16E+00	1.07	0.17E+00	-	1.17	1.26
56.	0.38E-01	2.14	0.27E-01	0.72	0.42E-01	1.11	0.39E-01	2.36	1.02	1.36
88.	0.29E-01	1.20	0.19E-01	0.66	0.35E-01	1.20	0.19E-01	3.27	0.64	1.38
136.	0.18E-01	2.11	0.14E-01	0.76	0.24E-01	1.32	0.18E-01	0.24	0.96	1.55
176.	0.13E-01	2.73	0.94E-02	0.73	0.18E-01	1.36	0.13E-01	2.39	1.00	1.58

$k = 2$

N	e	order	η_1	η_1/e	η_2	η_2/e	osc	order	osc/e	eff
16.	0.17E-01	-	0.12E-01	0.68	0.14E-01	0.81	0.22E-01	-	1.29	1.13
56.	0.28E-02	2.89	0.23E-02	0.81	0.25E-02	0.88	0.40E-02	2.75	1.41	1.28
88.	0.14E-02	3.05	0.13E-02	0.89	0.11E-02	0.75	0.13E-02	5.05	0.90	1.20
176.	0.62E-03	2.40	0.43E-03	0.70	0.56E-03	0.91	0.66E-03	1.90	1.06	1.20
216.	0.41E-03	4.07	0.36E-03	0.87	0.33E-03	0.81	0.26E-03	9.13	0.63	1.21

$k = 3$

N	e	order	η_1	η_1/e	η_2	η_2/e	osc	order	osc/e	eff
16.	0.16E-02	-	0.12E-02	0.77	0.12E-02	0.77	0.22E-02	-	1.37	1.17
56.	0.18E-03	3.45	0.17E-03	0.92	0.12E-03	0.65	0.34E-03	2.98	1.84	1.27
88.	0.82E-04	3.55	0.69E-04	0.85	0.66E-04	0.81	0.56E-04	7.96	0.68	1.19
168.	0.23E-04	4.00	0.21E-04	0.95	0.12E-04	0.53	0.38E-04	1.14	1.71	1.22
200.	0.14E-04	5.56	0.13E-04	0.94	0.93E-05	0.67	0.15E-04	11.09	1.05	1.20

$k = 4$

N	e	order	η_1	η_1/e	η_2	η_2/e	osc	order	osc/e	eff
16.	0.12E-03	-	0.98E-04	0.82	0.78E-04	0.65	0.17E-03	-	1.42	1.14
56.	0.10E-04	3.96	0.94E-05	0.94	0.47E-05	0.46	0.24E-04	3.14	2.37	1.29
88.	0.26E-05	5.91	0.25E-05	0.94	0.16E-05	0.60	0.24E-05	10.19	0.90	1.15
160.	0.85E-06	3.79	0.65E-06	0.76	0.67E-06	0.78	0.13E-05	2.08	1.50	1.19
200.	0.43E-06	6.18	0.41E-06	0.97	0.22E-06	0.51	0.47E-06	8.94	1.10	1.15

Table 4. History of convergence of the error $e := \|\boldsymbol{q} - \boldsymbol{q}_h\|_\Omega$, the error estimators $\eta_1 := \|\boldsymbol{q}_h + \boldsymbol{\nabla} u_h^*\|_{\mathcal{T}_h}$ and $\eta_2 := \sqrt{\sum \frac{(k+2)^2}{h_e} \| [\![u_h^*]\!] \|_e^2}$, and the effectivity index $eff := \sqrt{osc^2/\pi^2 + \eta_1^2 + \eta_2^2}/e$. Adaptive mesh refinement, nonsmooth solution. Note that in this case, the oscillation osc is zero.

$k = 0$

N	e	order	η_1	η_1/e	η_2	η_2/e	eff
12.	0.30E+00	-	0.30E-15	0.00	0.66E+00	2.23	2.23
26.	0.26E+00	0.35	0.29E-15	0.00	0.58E+00	2.24	2.24
46.	0.20E+00	0.86	0.36E-15	0.00	0.47E+00	2.28	2.28
74.	0.20E+00	0.07	0.30E-15	0.00	0.41E+00	2.05	2.05
114.	0.15E+00	1.41	0.32E-15	0.00	0.36E+00	2.44	2.44
164.	0.12E+00	1.16	0.31E-15	0.00	0.32E+00	2.65	2.65

$k = 1$

N	e	order	η_1	η_1/e	η_2	η_2/e	eff
12.	0.14E+00	-	0.61E-01	0.44	0.16E+00	1.17	1.25
28.	0.88E-01	1.11	0.44E-01	0.50	0.10E+00	1.19	1.29
48.	0.58E-01	1.55	0.31E-01	0.53	0.73E-01	1.26	1.37
68.	0.40E-01	2.12	0.24E-01	0.59	0.55E-01	1.37	1.49
98.	0.28E-01	1.89	0.19E-01	0.66	0.36E-01	1.29	1.45
150.	0.21E-01	1.34	0.16E-01	0.75	0.28E-01	1.30	1.50
178.	0.15E-01	3.99	0.11E-01	0.75	0.21E-01	1.40	1.59

$k = 2$

N	e	order	η_1	η_1/e	η_2	η_2/e	eff
12.	0.86E-01	-	0.41E-01	0.47	0.97E-01	1.13	1.22
28.	0.52E-01	1.18	0.27E-01	0.52	0.62E-01	1.19	1.30
48.	0.33E-01	1.70	0.17E-01	0.53	0.39E-01	1.19	1.30
68.	0.21E-01	2.60	0.11E-01	0.53	0.25E-01	1.20	1.31
88.	0.14E-01	3.42	0.74E-02	0.55	0.17E-01	1.22	1.34
108.	0.90E-02	4.03	0.52E-02	0.58	0.11E-01	1.26	1.39
128.	0.63E-02	4.19	0.40E-02	0.64	0.84E-02	1.33	1.48

$k = 3$

N	e	order	η_1	η_1/e	η_2	η_2/e	eff
12.	0.60E-01	-	0.30E-01	0.50	0.70E-01	1.18	1.28
28.	0.36E-01	1.22	0.20E-01	0.56	0.46E-01	1.28	1.40
48.	0.22E-01	1.71	0.13E-01	0.56	0.29E-01	1.28	1.40
68.	0.14E-01	2.65	0.79E-02	0.56	0.18E-01	1.28	1.40
88.	0.89E-02	3.57	0.50E-02	0.56	0.11E-01	1.28	1.40
108.	0.57E-02	4.47	0.32E-02	0.57	0.72E-02	1.28	1.40
128.	0.36E-02	5.31	0.21E-02	0.58	0.46E-02	1.27	1.40

$k = 4$

N	e	order	η_1	η_1/e	η_2	η_2/e	eff
12.	0.50E-01	-	0.23E-01	0.47	0.56E-01	1.11	1.21
28.	0.30E-01	1.16	0.16E-01	0.51	0.36E-01	1.18	1.29
48.	0.19E-01	1.71	0.98E-02	0.51	0.23E-01	1.18	1.29
68.	0.12E-01	2.65	0.62E-02	0.51	0.14E-01	1.18	1.29
88.	0.76E-02	3.58	0.39E-02	0.51	0.90E-02	1.18	1.29
108.	0.48E-02	4.51	0.24E-02	0.51	0.57E-02	1.18	1.29
128.	0.30E-02	5.43	0.15E-02	0.51	0.36E-02	1.18	1.29

Appendix A. Two auxiliary inequalities

In this appendix, we prove two inequalities used in the proof of the local efficiency of the estimator. The first inequality was used in the proof of Lemma 3.4.

Lemma A.1. *Let $\boldsymbol{v} \in \mathrm{RT}_0(K)$ with $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on all the faces of the simplex K except on the face e. Then*

$$\Theta := \frac{h_e^{-1/2} \| \boldsymbol{v} \|_K}{\| \boldsymbol{v} \cdot \boldsymbol{n}_e \|_e} \leq \frac{\sigma}{\sqrt{2d}}.$$

Proof. Let $\boldsymbol{x}_e$ be the node off the face e. Since $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on all the remaining faces, the function $\boldsymbol{v}$ must be of the form $\boldsymbol{v} = \alpha(\boldsymbol{x} - \boldsymbol{x}_e)$. Then

$$\| \boldsymbol{v} \cdot \boldsymbol{n}_e \|_e^2 = \int_e \alpha^2 |(\boldsymbol{x} - \boldsymbol{x}_e) \cdot \boldsymbol{n}_e|^2 d\boldsymbol{x} = \alpha^2 (h_e^{\perp})^2 |e|,$$

where $h_e^{\perp} := (\boldsymbol{x} - \boldsymbol{x}_e) \cdot \boldsymbol{n}_e$ for any $\boldsymbol{x} \in e$. We also have

$$\| \boldsymbol{v} \|_K^2 = \int_K \alpha^2 |\boldsymbol{x} - \boldsymbol{x}_e|^2 d\boldsymbol{x} \leq \alpha^2 h_K^2 |K|.$$

Hence

$$\Theta \leq \sqrt{\frac{h_e^{-1} \alpha^2 h_K^2 |K|}{\alpha^2 (h_e^{\perp})^2 |e|}} = \sqrt{\frac{h_K^2}{d\, h_e\, h_e^{\perp}}} \leq \sqrt{\frac{\sigma\, h_K}{d\, h_e^{\perp}}}.$$

The lemma follows immediately from the fact that $h_K \leq \sigma \rho_K \leq \frac{1}{2} \sigma h_e^{\perp}$. This completes the proof. ∎

The second inequality, was used in the proof of Lemma 3.7. To obtain it, we begin by proving the following result.

Lemma A.2. *Let K be a d-dimensional simplex. Then, for all $w \in \mathcal{P}_{k+2}(K)$ such that $w = 0$ on a face e of K, we have*

$$\| w \|_K^2 \leq (1 + D_k) \| \mathsf{P}_k w \|_K^2,$$

where D_k is defined in Lemma 3.7.

Proof. We begin by considering the standard reference simplex K, and showing that

$$\| (\mathsf{Id} - \mathsf{P}_k) w \|_K^2 \leq D_k \| \mathsf{P}_{k+1} w \|_K^2,$$

provided that $w \in \mathcal{P}_{k+2}(K)$ with $w = 0$ on one face e where $e \in \partial K$.

We proceed as in [21] to get that

$$\begin{aligned} \| (\mathsf{Id} - \mathsf{P}_{k+1}) w \|_K^2 &= \frac{1}{k + 2 + d/2} \| (\mathsf{Id} - \mathsf{P}_{k+1}) w \|_e^2 \\ &= \frac{1}{k + 2 + d/2} \| \mathsf{P}_{k+1} w \|_e^2 \qquad \text{since } w|_e = 0, \\ &= \frac{(k+2)(k+1+d)}{2k+4+d} \| \mathsf{P}_{k+1} w \|^2, \end{aligned}$$

by the inequality following equation (2) in [21]. Since

$$\frac{(k+2)(k+1+d)}{2k+4+d} \leq \frac{d+1}{d+4} (k+2) = D_k,$$

this completes the proof for a reference element. Using a standard scaling argument, we immediately obtain the result for arbitrary simplexes. ∎

Corollary A.1. *For all* $\mathbf{v} \in \mathcal{P}_{\mathbf{k+2}}(\mathbf{K})$ *with* $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ *on all the faces of* K *with the exception of one, we have*

$$\| \mathbf{v} \|_K^2 \leq C_4 \, (1 + D_k) \| \, \mathsf{P}_{k+1} \mathbf{v} \, \|_K^2.$$

Proof. If $\widehat{K}$ is the reference simplex, the inequality

$$\| \mathbf{v} \|_{\widehat{K}}^2 \leq (1 + D_k) \| \, \mathsf{P}_{k+1} \mathbf{v} \, \|_{\widehat{K}}^2$$

follows from the previous lemma.

For an arbitrary simplex K, we consider the Piola transform

$$\mathbf{v}(\mathbf{x}) = \frac{1}{det(B)} B \widehat{\mathbf{v}}(\widehat{\mathbf{x}}),$$

where $\mathbf{x} = B\widehat{\mathbf{x}} + \mathbf{b}$. Here B is the linear mapping from the reference simplex $\widehat{K}$ to the physical element K and $\mathbf{b}$ is the vertex of K mapped to the origin. Since the Piola transformation preserves the normal trace (see [8]), we immediately have that

$$\| \, \widehat{\mathbf{v}} \, \|_K^2 \leq (1 + D_k) \| \, \mathsf{P}_{k+1} \widehat{\mathbf{v}} \, \|_K^2.$$

Hence, we obtain

$$\begin{aligned} \| \, \mathbf{v}(\mathbf{x}) \, \|_K^2 &\leq \frac{\|B\|^2}{det(B)^2} \| \, \widehat{\mathbf{v}} \, \|_{\widehat{K}}^2 \\ &\leq (1 + D_k) \frac{\|B\|^2}{det(B)^2} \| \, \mathsf{P}_{k+1} \widehat{\mathbf{v}} \, \|_{\widehat{K}}^2 \\ &\leq \big(\|B\| \, \|B^{-1}\| \big)^2 (1 + D_k) \| \, \mathsf{P}_{k+1} \mathbf{v} \, \|_K^2 \\ &\leq \big(\frac{h_K}{\rho_{\widehat{K}}} \frac{h_{\widehat{K}}}{\rho_K} \big)^2 (1 + D_k) \| \, \mathsf{P}_{k+1} \mathbf{v} \, \|_K^2 \\ &\leq \big(\frac{h_{\widehat{K}}}{\rho_{\widehat{K}}} \big)^2 \sigma^2 \, (1 + D_k) \| \, \mathsf{P}_{k+1} \mathbf{v} \, \|_K^2. \end{aligned}$$

The result follows from the fact that, $\frac{h_{\widehat{K}}}{\rho_{\widehat{K}}} = \sqrt{2d}\,(1 + \sqrt{d})$, for $d = 2, 3$ and from the definition of C_4. This completes the proof. $\square$

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