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AN INFINITE FAMILY OF PERFECT PARALLELEPIPEDS

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ABSTRACT. A perfect parallelepiped has edges, face diagonals, and body diagonals all of integer length. We prove the existence of an infinite family of dissimilar perfect parallelepipeds with two nonparallel rectangular faces. We also show that we can obtain perfect parallelepipeds of this form with the angle of the nonrectangular face arbitrarily close to 90°. Finally, we discuss the implications that this family has on the famous open problem concerning the existence of a perfect cuboid. This leads to two conjectures that would imply no perfect cuboid exists.

1. INTRODUCTION

The question of the existence of a 3-dimensional rectangular box with integer length sides, face diagonals, and body diagonals is an open question referred to as the perfect cuboid problem. Richard Guy asked a more general question [1]: "Is there a parallelepiped with all edges, face diagonals, and body diagonals rational?" Multiplying the lengths of a rational parallelepiped by the LCM of their denominators gives a parallelepiped with integer length edges, face diagonals, and body diagonals, which we call a perfect parallelepiped. The first example of a perfect parallelepiped was discovered in 2009 by Sawyer and Reiter [2]. Their initial discovery produced a limited list of specific examples of perfect parallelepipeds which had no known structure and no insight as to whether or not infinitely many dissimilar examples exist. Here we are able to prove the existence of infinitely many perfect parallelepipeds by presenting parameterizations for an infinite family of dissimilar perfect parallelepipeds with two nonparallel rectangular faces; that is, at least four of the six parallelepiped faces are rectangles. We can obtain perfect parallelepipeds such that the angle of the nonrectangular face is arbitrarily close to 90°. We offer two conjectures that together would imply that no perfect cuboid exists.

2. AN INFINITE FAMILY

2.1. **Preliminaries.** A rational parallelepiped is determined by three edge vectors, $\vec{u}$, $\vec{v}$, and $\vec{w}$. We let the lengths $\|\vec{u}\| = |e_1|$, $\|\vec{v}\| = |e_2|$, $\|\vec{w}\| = |e_3|$, $\|\vec{u}-\vec{v}\| = |d_{12}|$, $\|\vec{u}-\vec{w}\| = |d_{13}|$, and $\|\vec{v}-\vec{w}\| = |d_{23}|$. We restrict ourselves to cases where $\vec{u}$ is perpendicular to both $\vec{v}$ and $\vec{w}$. To do so, we parameterize e_1, e_2, and e_3 in the form of two Pythagorean triples with e_1 shared. In particular, let

$$e_1 = 2p_1p_2,\ e_2 = p_2^2 - p_1^2, e_3 = s((\frac{p_1p_2}{s})^2 - 1), \tag{2.1}$$

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where s is some rational scaling factor which allows for the Pythagorean triple containing e_1 and e_3 to be nonprimitive. This means we allow it to have some common factor among its terms. Direct computation [6] verifies $\sqrt{e_1^2+e_2^2} = p_1^2+p_2^2$ and $\sqrt{e_1^2+e_3^2} = |s|\left(1+(\frac{p_1p_2}{s})^2\right)$. In this paper, *direct computation* refers to a computation made in a *Mathematica*® notebook found in [6]. Thus, we let

$$d_{12} = p_1^2+p_2^2 \text{ and } d_{13} = s\left(1+(\tfrac{p_1p_2}{s})^2\right),$$

and we obtain two edge-matched rectangles with rational diagonals. We allow for rational values for p_1, p_2, s, and m with the understanding that the resulting lengths can be multiplied by the LCM of their denominators to obtain integer values. Also, we are not concerned with the sign of the e_i since the edges are length $|e_i|$. The only remaining freedom is the choice of the angle between $\vec{v}$ and $\vec{w}$. We are able construct the diagonals of a rational parallelogram given e_2 and e_3 as its edge lengths by using parameterizations defined by Wyss [4]. Specifically, Wyss defines such diagonals to be

$$x_1 = (e_2+e_3)\frac{2m}{1+m^2} + (e_2-e_3)\frac{1-m^2}{1+m^2},$$

$$x_2 = (e_2+e_3)\frac{1-m^2}{1+m^2} - (e_2-e_3)\frac{2m}{1+m^2},$$

where m is rational. There are additional constraints on the variables to ensure Wyss's formulas in conjunction with our formulas for e_2 and e_3 will produce realizable parallelograms. We will discuss these constraints later in this section when we explore the embedding of the parallelepipeds in $\mathbb{R}^3$. By direct computation [6], we verify that these formulas satisfy the parallelogram law:

$$2e_2^2+2e_3^2 = x_1^2+x_2^2.$$

For our initial parameterization we assume $x_1 = d_{23}$ and obtain:

$$\begin{aligned} e_1 &= 2p_1p_2, \quad e_2 = p_2^2-p_1^2, \quad e_3 = s((\frac{p_1p_2}{s})^2-1), \\ d_{12} &= p_1^2+p_2^2, \quad d_{13} = s\left(1+(\frac{p_1p_2}{s})^2\right), \\ d_{23} &= (e_2+e_3)\,\frac{2m}{1+m^2}+(e_2-e_3)\,\frac{1-m^2}{1+m^2}. \end{aligned} \tag{2.2}$$

We now want to identify conditions so that the body diagonals are rational.

We know from [2] that the body diagonals have the form

$$\begin{aligned} D_1^2 &= -e_1^2+e_2^2+e_3^2+d_{12}^2+d_{13}^2-d_{23}^2, D_2^2 = e_1^2-e_2^2+e_3^2+d_{12}^2-d_{13}^2+d_{23}^2, \\ D_3^2 &= e_1^2+e_2^2-e_3^2-d_{12}^2+d_{13}^2+d_{23}^2, D_4^2 = 3e_1^2+3e_2^2+3e_3^2-d_{12}^2-d_{13}^2-d_{23}^2. \end{aligned}$$

In the case where the parallelepiped has two rectangular faces, i.e., $d_{12}^2 = e_1^2+e_2^2$ and $d_{13}^2 = e_1^2+e_2^3$, direct computation [6] verifies that these quantities reduce to

$$\begin{aligned} D_1^2 = D_4^2 &= -d_{23}^2+e_1^2+2\left(e_2^2+e_3^2\right), \\ D_2^2 = D_3^2 &= d_{23}^2+e_1^2. \end{aligned} \tag{2.3}$$

Figure 1 shows a realizable perfect parallelepiped with the edges shown in heavy black, the face diagonals with dotting, and the body diagonals in light black.

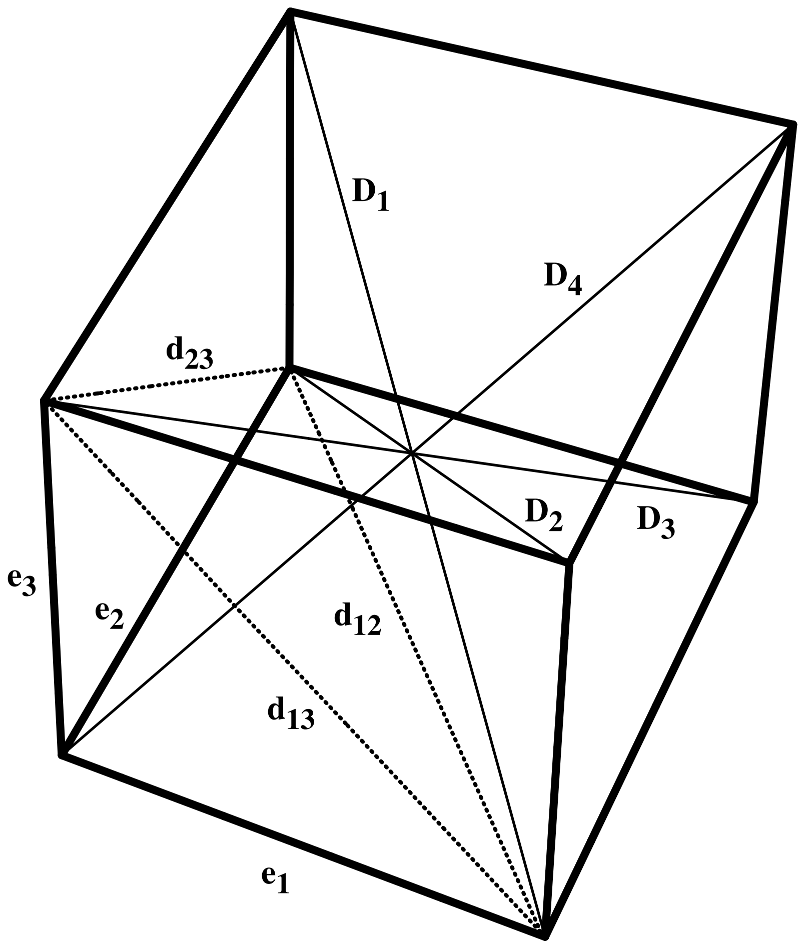

FIGURE 1. The edges, face diagonals, and body diagonals for a realizable perfect parallelepiped

2.2. **The infinite parameterized family.**

Theorem 2.1. *In the parameterization described in equation* (2.2), *if we let* $p_1 = p - 1$, $p_2 = p + 1$, *and* $s = \frac{((1-m^2)^2-4m^2)p}{m(1-m^2)}$ *for rational* p *and* m *and avoid zero denominators, then* e_1, e_2, e_3, d_{12}, d_{13}, d_{23}, D_1, D_2, D_3, *and* D_4 *will all be rational.*

Proof. Direct computation [6] verifies

$$D_1 = \sqrt{\frac{\left(p^2-4mp^2-4m^3p^2+4m^5p^2+4m^7p^2+m^8p^2+m^2(1+p^2)^2+m^6(1+p^2)^2-2m^4(1-5p^2+p^4)\right)^2}{m^2(-1+2m+m^2+m^4-2m^5-m^6)^2p^2}}$$

$$D_2 = \sqrt{\frac{\left(p^2+4mp^2+4m^3p^2-4m^5p^2-4m^7p^2+m^8p^2+m^2(1+p^2)^2+m^6(1+p^2)^2-2m^4(1-5p^2+p^4)\right)^2}{m^2(-1-2m+m^2+m^4+2m^5-m^6)^2p^2}}$$

$D_3 = D_2$
$D_4 = D_1$. □

Similarly, it can be shown that if we let $x_2 = d_{23}$, we will still obtain rational body diagonals for rational p and m (avoiding zero in the denominators). In fact, direct computation [6] verifies that the body diagonals obtained with this change will be the same as those obtained when $x_1 = d_{23}$. Thus, we have the following parameterization of edges and diagonals:

(2.4)

$$\begin{aligned}
e_1 &= 2(p^2-1),\\
e_2 &= 4p,\\
e_3 &= \frac{p^2+m^8p^2-m^2\left(1+10p^2+p^4\right)-m^6\left(1+10p^2+p^4\right)+2m^4\left(1+17p^2+p^4\right)}{m\left(-1+m^2\right)\left(1-6m^2+m^4\right)p},\\
d_{12} &= 2(1+p^2),\\
d_{13} &= -\frac{p^2+m^8p^2-2m^4(1-21p^2+p^4)+m^2(1-14p^2+p^4)+m^6(1-14p^2+p^4)}{m(-1+m^2)(1-6m^2+m^4)p},\\
d_{23} &= \frac{p^2+4mp^2+4m^3p^2-4m^5p^2-4m^7p^2+m^8p^2-m^2N_1-m^6N_1+2m^4N_2}{m\left(-1+m^2\right)\left(1+m^2\right)\left(-1-2m+m^2\right)p},
\end{aligned}$$

where $p, m \in \mathbb{Q}$ and $N_1 = 1-6p^2+p^4$, $N_2 = 1+p^2+p^4$.

A *Mathematica*® program [6], developed by [5], returns the vector coordinates for the embedding of a parallelepiped in $\mathbb{R}^3$ given its edge lengths and a diagonal length of each face. For the parameterization described in equation (2.4), the embedding in 3-space is

$$\begin{aligned}
\vec{u} &= \{2\left(-1+p^2\right), 0, 0\},\\
\vec{v} &= \{0, 4p, 0\},\\
\vec{w} &= \{0, j, \frac{A\sqrt{h}}{2m(-1+m^2)(1-6m^2+m^4)p}\},
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
j(p,m) = {} & \frac{1}{2m(-1+m^2)(1-5m^2-5m^4+m^6)^2p^3}\\
& \times (p^4-8m^2p^4-8m^{14}p^4+m^{16}p^4\\
& + m^8(-6+24p^2+98p^4+24p^6-6p^8)-m^4(1-4p^2+10p^4-4p^6+p^8)\\
& - m^{12}(1-4p^2+10p^4-4p^6+p^8)+4m^6(1-4p^2+24p^4-4p^6+p^8)\\
& + 4m^{10}(1-4p^2+24p^4-4p^6+p^8)),
\end{aligned}$$

$$
\begin{aligned}
h(p,m) = {} & ((3p^8 + 3m^{32}p^8 - 24m^6p^4(3 + 45p^2 + 334p^4 + 45p^6 + 3p^8) \\
& - 24m^{26}p^4(3 + 45p^2 + 334p^4 + 45p^6 + 3p^8) + 2m^4p^4(3 + 76p^2 \\
& + 630p^4 + 76p^6 + 3p^8) + 2m^{28}p^4(3 + 76p^2 + 630p^4 + 76p^6 + 3p^8) \\
& - 8m^2(p^6 + 12p^8 + p^{10}) - 8m^{30}(p^6 + 12p^8 + p^{10}) + m^{16}(-70 + 560p^2 \\
& - 640p^4 + 32496p^6 + 291614p^8 + 32496p^{10} - 640p^{12} + 560p^{14} - 70p^{16}) \\
& + m^8(-1 + 8p^2 + 176p^4 + 3048p^6 + 21398p^8 + 3048p^{10} + 176p^{12} \\
& + 8p^{14} - p^{16}) + m^{24}(-1 + 8p^2 + 176p^4 + 3048p^6 + 21398p^8 + 3048p^{10} \\
& + 176p^{12} + 8p^{14} - p^{16}) + 8m^{10}(1 - 8p^2 + 55p^4 + 99p^6 + 844p^8 + 99p^{10} \\
& + 55p^{12} - 8p^{14} + p^{16}) + 8m^{22}(1 - 8p^2 + 55p^4 + 99p^6 + 844p^8 + 99p^{10} \\
& + 55p^{12} - 8p^{14} + p^{16}) + 8m^{14}(7 - 56p^2 + 178p^4 - 411p^6 + 6874p^8 \\
& - 411p^{10} + 178p^{12} - 56p^{14} + 7p^{16}) + 8m^{18}(7 - 56p^2 + 178p^4 - 411p^6 \\
& + 6874p^8 - 411p^{10} + 178p^{12} - 56p^{14} + 7p^{16}) - 2m^{12}(14 - 112p^2 \\
& + 827p^4 + 7932p^6 + 61898p^8 + 7932p^{10} + 827p^{12} - 112p^{14} + 14p^{16}) \\
& - 2m^{20}(14 - 112p^2 + 827p^4 + 7932p^6 + 61898p^8 + 7932p^{10} + 827p^{12} \\
& - 112p^{14} + 14p^{16}))/((1 + m^2)^4(1 - 6m^2 + m^4)^2p^4A^2))
\end{aligned}
$$

and

$$A(p,m) = (p^2 + m^8p^2 - m^2(1 + 10p^2 + p^4) - m^6(1 + 10p^2 + p^4) + 2m^4(1 + 17p^2 + p^4))^2).$$

We now discuss the restrictions on p and m required for this embedding to produce realizable rational parallelepipeds.

Lemma 2.2. *If p and m are rational values such that $p \neq 0, m \neq 0, \pm 1$, and $A \neq 0$, then the coordinates in equations* (2.5) *will all be well defined.*

Proof. The terms in the denominators of the aforementioned vectors will be zero precisely when $p = 0, m = 0, m = \pm 1, A = 0$, as well as for several irrational values of m. $\square$

The plot of when $A = 0$ can be seen as the black curve in the white region of the plot in Figure 2 at the end of this section.

Lemma 2.3. *Given h as defined in equations* (2.5) *and rational values for p and m for which the hypotheses of Lemma* 2.2 *are satisfied, if $h > 0$ then the parameterization described in equations* 2.5 *will yield three vectors in $\mathbb{R}^3$.*

Proof. Assuming p and m are rational and the terms are well defined, it is clear to see that the majority of the coordinates will be real. The only coordinate which is not guaranteed to be real is the third coordinate of $\vec{w}$. We can see, however, that if the expression under the square root in this coordinate, h, were positive, then the coordinate would be real. Thus, if $h > 0$, then the vectors defined by the parameterized edge and diagonal lengths would all be in $\mathbb{R}^3$. $\square$

Figure 2 at the end of this section displays the region in which $h > 0$ as the black shaded region. We can see that the curve described in Lemma 2.2 representing when terms of the vectors may be undefined does not coincide with this shaded region.

Lemma 2.4. *If the hypotheses of Lemmas* 2.2 *and* 2.3 *are satisfied and* $p \neq \pm 1$*, then the vectors* $\vec{u}, \vec{v}$*, and* $\vec{w}$ *will not be coplanar.*

Proof. We observe that if the first coordinate of $\vec{u}$, the second of $\vec{v}$, and the third of $\vec{w}$ are not equal to zero, then the three vectors will not be coplanar. For $\vec{u}$ to be nontrivial, we need $p \neq \pm 1$. For $\vec{v}$ to be nontrivial, we need $p \neq 0$. For the third coordinate of $\vec{w}$ to be nonzero, we need $h > 0$ and $A \neq 0$. Thus by satisfying the hypotheses of Lemmas 2.2 and 2.3 and the additional condition $p \neq \pm 1$, we ensure that $\vec{u}, \vec{v}$, and $\vec{w}$ will not be coplanar. □

Theorem 2.5. *For rational* p *and* m *that satisfy* $p \neq 0, \pm 1$*,* $m \neq 0, \pm 1$*,* $A(p,m) \neq 0$*, and* $h(p,m) > 0$*, the vectors parameterized in equations* (2.6) *will form a realizable rational parallelepiped.*

Proof. Let p and m be parameters that satisfy the hypotheses. It is known that any three vectors in 3-space that are not coplanar form a parallelepiped. Lemmas 2.2 and 2.3 guarantee the vectors to be well defined and real, and Lemma 2.4 guarantees them to be not coplanar. Therefore, p and m form a realizable parallelepiped. Direct computation [6] verifies that under these conditions the vectors guarantee rational body diagonals, face diagonals, and edge lengths. Thus, in fact, they form a realizable rational parallelepiped. □

Our final parameterization of a family of rational parallelepipeds is given by $\vec{u}, \vec{v}$, and $\vec{w}$ defined as above in equation (2.5) satifying the above hypotheses and with edge, face diagonal, and body diagonal lengths given by

$$\begin{gathered} \|\vec{u}\|, \|\vec{v}\|, \|\vec{w}\|, \|\vec{u}+\vec{v}\|, \|\vec{u}-\vec{v}\|, \|\vec{u}+\vec{w}\|, \|\vec{u}-\vec{w}\|, \|\vec{v}+\vec{w}\|, \|\vec{v}-\vec{w}\|, \\ \|\vec{u}+\vec{v}+\vec{w}\|, \|\vec{u}+\vec{v}-\vec{w}\|, \|\vec{u}-\vec{v}+\vec{w}\|, \text{ and, } \|\vec{u}-\vec{v}-\vec{w}\|. \end{gathered} \tag{2.6}$$

Further, we note that the lengths of the sides and face diagonals are precisely the absolute values of e_1, e_2, e_3, d_{12}, d_{13}, and d_{23} given in equations (2.4).

2.3. Infinitely many dissimilar realizable perfect parallelepipeds.

To show that our family does in fact contain infinitely many dissimilar realizable perfect parallelepipeds, we first find parameters that result in a single realizable perfect parallelepiped.

Theorem 2.6. *For the particular values* $p = 6$ *and* $m = \frac{1}{5}$*, the parameterization in equations* (2.6) *yields a realizable perfect parallelepiped.*

Proof. Direct computation [6] verifies that

$\vec{u} = \langle 70, 0, 0 \rangle$,

$\vec{v} = \langle 0, 24, 0 \rangle$, and

$\vec{w} = \langle 0, \frac{5443248}{244205}, \frac{1728\sqrt{5369118}}{244205} \rangle$.

Direct computation [6] verifies that

$$\begin{gathered} \|\vec{u}\| = 70, \|\vec{v}\| = 24, \|\vec{w}\| = \frac{2352}{85}, \|\vec{u}+\vec{v}\| = 74, \|\vec{u}-\vec{v}\| = 74, \|\vec{u}+\vec{w}\| = \frac{6398}{85}, \\ \|\vec{u}-\vec{w}\| = \frac{6398}{85}, \|\vec{v}+\vec{w}\| = \frac{3192}{65}, \|\vec{v}-\vec{w}\| = \frac{18216}{1105}, \|\vec{u}+\vec{v}+\vec{w}\| = \frac{5558}{65}, \\ \|\vec{u}+\vec{v}-\vec{w}\| = \frac{79466}{1105}, \|\vec{u}-\vec{v}+\vec{w}\| = \frac{79466}{1105}, \quad \text{and} \quad \|\vec{u}-\vec{v}-\vec{w}\| = \frac{5558}{65}. \end{gathered}$$

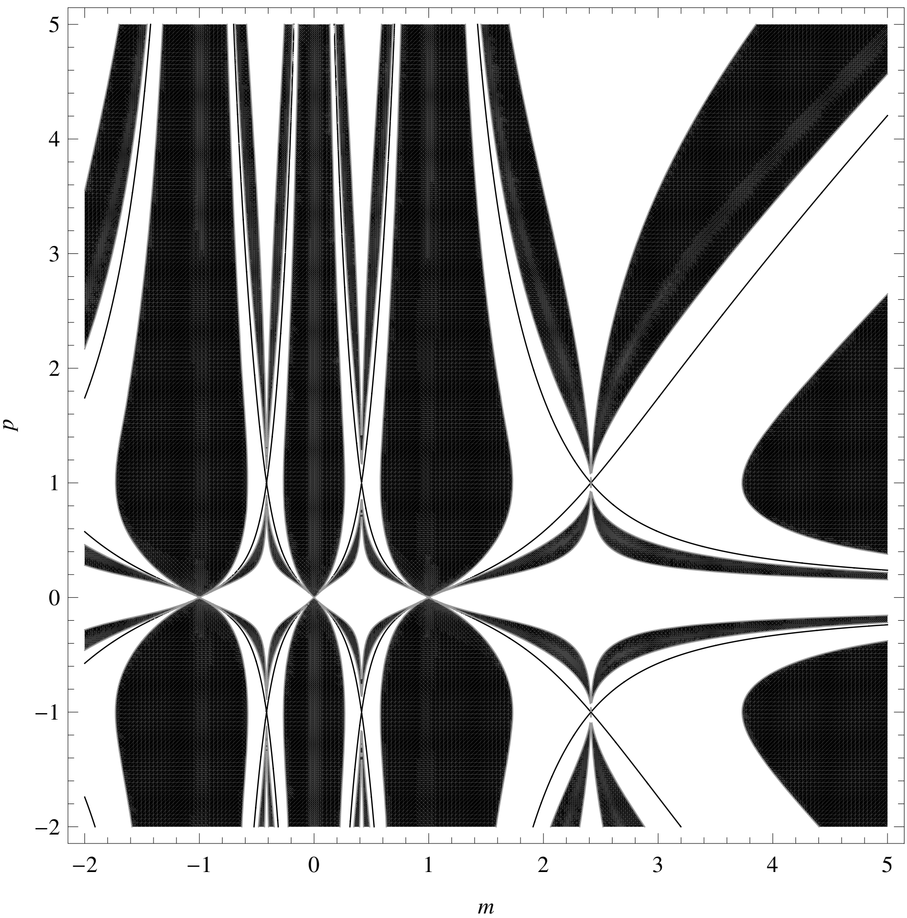

FIGURE 2. Contour plot of regions described in Lemmas 2.2 and 2.3

By multiplying each of these values by the LCM of their denominators, we obtain a perfect parallelepiped. □

To show that this leads to infinitely many dissimilar perfect parallelepipeds, we fix m at $\frac{1}{5}$ and argue that by varying p, we achieve the desired result.

Theorem 2.7. *There are infinitely many dissimilar perfect parallelepipeds in a neighborhood of the particular perfect parallelepiped obtained when* $p = 6$ *and* $m = \frac{1}{5}$.

Proof. Let $p = 6$ and $m = \frac{1}{5}$. By Theorem 2.6, we know this gives a realizable rational parallelepiped. Clearly, based on the embeddings shown in the proof of Theorem 2.6 the function h, as defined in equations (2.5), is positive for $p = 6$ and $m = \frac{1}{5}$. Since $h(p, \frac{1}{5})$ is a rational function in p and defined when $p = 6$, then it will be continuous at $p = 6$. Similarly, the function A, as defined in equations (2.5), is nonzero for $p = 6$ and $m = \frac{1}{5}$. $A(p, \frac{1}{5})$ is also a rational function in p, defined when $p = 6$, so it will be continuous at $p = 6$. Thus, when $m = \frac{1}{5}$, there is a neighborhood about $p = 6$ containing infinitely many rationals for which Theorem 2.5 is satisfied, and the parameterization produces a rational parallelepiped. To show that we in fact have infinitely many dissimilar rational parallelepipeds in this neighborhood, we consider the angle between edges $\vec{v}$ and $\vec{w}$. Since the other two edges are always at right angles, two parallelepipeds with different acute angles between $\vec{v}$ and $\vec{w}$ will be necessarily dissimilar from one another. By direct computation [6] the cosine of the desired angle for the particular value $p = 6$ is $c_{23} = \frac{113401}{140777}$. The derivative of c_{23}, with respect to p, at that particular value of p is $\frac{77834125}{165553752}$. Since the derivative is defined at $p = 6$, the cosine is continuous and nonconstant in a neighborhood of 6. We conclude that the angle θ_{23} takes on infinitely many acute values in a neighborhood of 6; hence, there are infinitely many dissimilar rational parallelepipeds in such a neighborhood. By clearing denominators, the examples remain dissimilar, and we obtain infinitely many dissimilar perfect parallelepipeds. □

We will see in Section 5 that we can choose rational p and m such that the angle between $\vec{v}$ and $\vec{w}$ is arbitrarily close to 90°.

3. Symmetries in parameters

The parameterization presented in equations (2.6) is valid for any rational p and m provided that the hypotheses of Theorem 2.5 hold. However, a significantly smaller domain is required to parameterize all dissimilar examples. Suppose parameters p, m and p', m' parameterize perfect parallelepipeds with lengths $|e_1|$, $|e_2|$, $|e_3|$, $|d_{12}|$, $|d_{13}|$, $|d_{23}|$, and $|e_1'|$, $|e_2'|$, $|e_3'|$, $|d_{12}'|$, $|d_{13}'|$, $|d_{23}'|$, respectively. Then the resulting parallelepipeds are similar if there exists some scaling factor s such that

$$\begin{gathered} |e_1| = s|e_1'|,\ |e_2| = s|e_2'|,\ |e_3| = s|e_3'|, \\ |d_{12}| = s|d_{12}'|,\ |d_{13}| = s|d_{13}'|, \text{ and } |d_{23}| = s|d_{23}'|. \end{gathered} \tag{3.1}$$

Other cases of similarity are possible; however, we restrict ourselves to this case, as it is the most frequent.

In the following lemmas, we will use the function notation $e_i(p, m)$ to denote the value of e_i when parameters p and m are used; likewise for $d_{ij}(p, m)$. In addition, for real numbers a and b, we let $(a,\, b)$ be the set of all rational x satisfying $a < x < b$.

Lemma 3.1. *Every parallelepiped obtained by the vectors parameterized in equations* (2.6) *with parameter values $p \in (-\infty,\, 0)$ and $m \in \mathbb{Q}$ is similar to a parallelepiped with parameter values $p' \in (0,\, \infty)$ and m.*

Proof. Let $p \in (-\infty,\, 0)$ and $m \in \mathbb{Q}$. Now let $p' = -p$, and observe that $p' \in (0,\, \infty)$. Direct computation [6] verifies that $e_1(p,\, m) = e_1(p',\, m)$, $e_2(p,\, m) = -e_2(p',\, m)$, $e_3(p,\, m) = -e_3(p',\, m)$, $d_{12}(p,\, m) = d_{12}(p',\, m)$, $d_{13}(p,\, m) = -d_{13}(p',\, m)$, and $d_{23}(p,\, m) = -d_{23}(p',\, m)$. Thus for the scaling factor of $s = 1$, equations (3.1) hold, and we conclude that the parallelepipeds are similar. □

Since no realizable parallelepiped has $p = 0$, we obtain all dissimilar parallelepipeds by restricting ourselves to positive p values.

Lemma 3.2. *Every parallelepiped obtained by the vectors parameterized in equations* (2.6) *with parameter values $p \in (1,\, \infty)$ and $m \in \mathbb{Q}$ is similar to a parallelepiped with parameter values $p' \in (0,\, 1)$ and m.*

Proof. Let $p \in (1,\, \infty)$ and $m \in \mathbb{Q}$. Now let $p' = \frac{1}{p}$, and observe that $p' \in (0,\, 1)$. Direct computation [6] verifies that $e_1(p,\, m) = -p^2 e_1(p',\, m)$, $e_2(p,\, m) = p^2 e_2(p',\, m)$,

$e_3(p, m) = p^2 e_3(p', m)$, $d_{12}(p, m) = p^2 d_{12}(p', m)$, $d_{13}(p, m) = p^2 d_{13}(p', m)$, and $d_{23}(p, m) = p^2 d_{23}(p', m)$. Thus for the scaling factor of $s = p^2$, equations (3.1) hold, and we conclude that the parallelepipeds are similar. $\square$

Again since no realizable parallelepiped has $p = 1$, we are safe to restrict ourselves to p values between 0 and 1. We have analogous lemmas for m.

Lemma 3.3. *Every parallelepiped obtained by the vectors parameterized in equations* (2.6) *with parameter values* $p \in (0, 1)$ *and* $m \in (-\infty, -1) \cup (1, \infty)$ *is similar to a parallelepiped with parameter values* $p \in (0, 1)$ *and* $m' \in (-1, 1)$.

Proof. Let $p \in (0, 1)$ and $m \in (-\infty, -1) \cup (1, \infty)$. Now let $m' = \frac{-1}{m}$, and observe that $m' \in (-1, 1)$. Direct computation [6] verifies that $e_1(p, m) = e_1(p, m')$, $e_2(p, m) = e_2(p, m')$, $e_3(p, m) = e_3(p, m')$, $d_{12}(p, m) = d_{12}(p, m')$, $d_{13}(p, m) = d_{13}(p, m')$, and $d_{23}(p, m) = -d_{23}(p, m')$. Thus for the scaling factor of $s = 1$, equations (3.1) hold, and we conclude that the parallelepipeds are similar. $\square$

Lemma 3.4. *Every parallelepiped obtained by the vectors parameterized in equations* (2.6) *with parameter values* $p \in (0, 1)$ *and* $m \in (\sqrt{2} - 1, 1)$ *is similar to a parallelepiped with parameter values* $p \in (0, 1)$ *and* $m' \in (0, \sqrt{2} - 1)$.

Proof. Let $p \in (0, 1)$ and $m \in (\sqrt{2} - 1, 1)$. Now let $m' = \frac{1-m}{1+m}$, and observe that $m' \in (0, \sqrt{2} - 1)$. Direct computation [6] verifies that $e_1(p, m) = e_1(p, m')$, $e_2(p, m) = e_2(p, m')$, $e_3(p, m) = -e_3(p, m')$, $d_{12}(p, m) = d_{12}(p, m')$, $d_{13}(p, m) = -d_{13}(p, m')$, and $d_{23}(p, m) = d_{23}(p, m')$. Thus for the scaling factor of $s = 1$, equations (3.1) hold, and we conclude that the parallelepipeds are similar. $\square$

Lemma 3.5. *Every parallelepiped obtained by the vectors parameterized in equations* (2.6) *with parameter values* $p \in (0, 1)$ *and* $m \in (-1, 1 - \sqrt{2})$ *is similar to a parallelepiped with parameter values* $p \in (0, 1)$ *and* $m' \in (1 - \sqrt{2}, 0)$.

Proof. Let $p \in (0, 1)$ and $m \in (-1, 1 - \sqrt{2})$. Now let $m' = \frac{1+m}{m-1}$, and observe that $m' \in (1 - \sqrt{2}, 0)$. Direct computation [6] verifies that $e_1(p, m) = e_1(p, m')$, $e_2(p, m) = e_2(p, m')$, $e_3(p, m) = -e_3(p, m')$, $d_{12}(p, m) = d_{12}(p, m')$, $d_{13}(p, m) = -d_{13}(p, m')$, and $d_{23}(p, m) = -d_{23}(p, m')$. Thus for the scaling factor of $s = 1$, equations (3.1) hold, and we conclude that the parallelepipeds are similar. $\square$

These lemmas culminate in the following result:

Theorem 3.6. *Any rational parallelepiped obtained by the vectors parameterized in equations* (2.6) *is similar to a rational parallelepiped obtained by rational parameters* $0 < p < 1$ *and* $1 - \sqrt{2} < m < \sqrt{2} - 1$.

Proof. The theorem follows from Lemmas 3.1–3.5 and the fact that no parallelepiped obtained by the vectors parameterized in equations (2.6) has $p = \pm 1$ or $p = 0$. $\square$

For example, direct computation [6] verifies that using $p = \frac{1}{6}$ and $m = \frac{1}{5}$ give e_i and d_{ij} that differ from those in Theorem 2.3 by a factor of ± 36. The embedding in this case is given by $u = \left\{-\frac{35}{18}, 0, 0\right\}$, $v = \left\{0, \frac{2}{3}, 0\right\}$, and $w = \left\{0, \frac{453604}{732615}, \frac{48\sqrt{5369118}}{244205}\right\}$, which involves a negative entry. However, upon clearing the denominators in both examples, identical primitive perfect parallelepipeds appear with $\|\vec{u}\| = 38675$, $\|\vec{v}\| = 13260$, $\|\vec{w}\| = 15288$, $\|\vec{u} \pm \vec{v}\| = 40885$, $\|\vec{u} \pm \vec{w}\| = 41587$, $\|\vec{v} + \vec{w}\| = 27132$, $\|\vec{v} - \vec{w}\| = 9108$, $\|\vec{u} \pm (\vec{v} + \vec{w})\| = 47243$, $\|\vec{u} \pm (\vec{v} - \vec{w})\| = 39733$.

4. Methodology

Here we will discuss our methodology in searching for the infinite family of perfect parallelepipeds we found. Using the initial parameterizations described in equations (2.2) we ran brute force computer searches. These searches quickly found the four examples of perfect parallelepipeds with two nonparallel rectangular faces which Sawyer and Reiter had discovered [5]. The searches then yielded an expansive data set for us to work with.

In examining the data, we noticed certain values of m appeared more prevalently than others and thus decided to run searches based upon those fixed values of m. Once we had a sufficient amount of data from such a search, we created a scatterplot comparing the values of p_1 to e_1 for each of the rational parallelepipeds our search produced. Depicted in Figure 3 is the scatterplot for the case of $m = \frac{1}{5}$. Note that each of the points on the plot represents a rational parallelepiped obtained through our searches.

In attempts to define e_1 in terms of p_1, we examined the scatterplot for collections of points that fell along a single curve. We took samplings of points that appeared to be related and then fit a function curve to those points. In several cases we were able to obtain simple functions giving e_1 in terms of p_1. For these cases we then were able to plot the corresponding s versus p_1 values and similarly fit curves to the points, obtaining simple equations giving s in terms of p_1. From these equations of e_1 and s, the lengths of the remaining sides could be determined, and they guaranteed rational body diagonal lengths. This resulted in several equations of curves that hit strings of points in the scatterplot. We then compared these equations and generalized them by introducing a new variable, k, thus leading to a parameterization of $e_1 = 4kp_1 + 2p_1^2$ and $s = \frac{119}{30}(k^2 + kp_1)$. Repeating this process

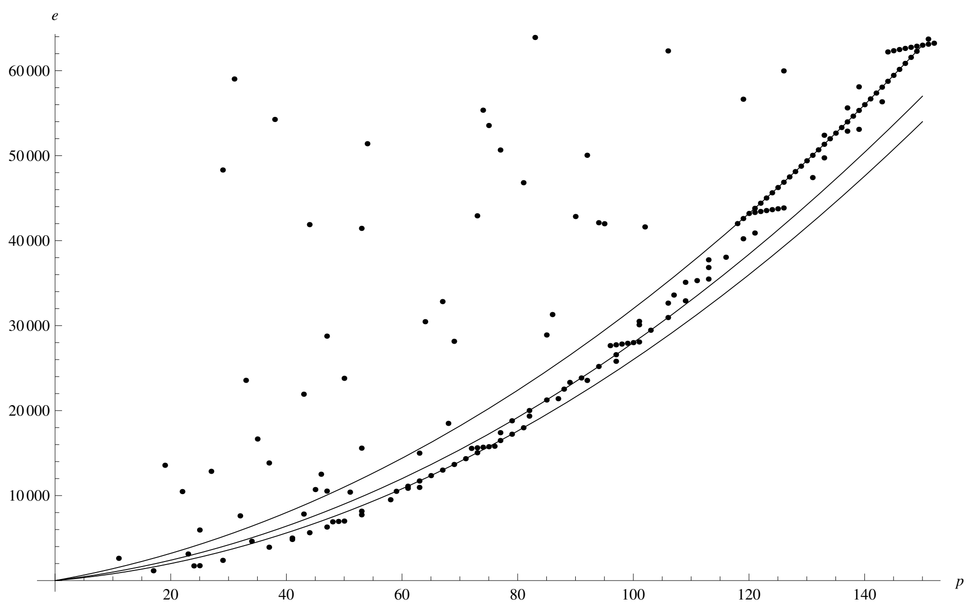

FIGURE 3. Scatterplot of rational parallelepipeds when $m = \frac{1}{5}$

for various fixed values of m, we obtained several equation sets parameterizing families for fixed m values. Specifically we saw

$$m = \tfrac{1}{5}\colon\ e_1 = 4kp_1 + 2p_1^2 \text{ and } s = \tfrac{119}{30}(k^2 + kp_1),$$
$$m = \tfrac{1}{4}\colon\ e_1 = 4kp_1 + 2p_1^2 \text{ and } s = \tfrac{161}{60}(k^2 + kp_1),$$
$$m = \tfrac{2}{5}\colon\ e_1 = 4kp_1 + 2p_1^2 \text{ and } s = \tfrac{41}{210}(k^2 + kp_1).$$

We recognized the similarity between each of the cases and sought a way to generalize the unique factor in s for any valid m value. We discovered that we could generalize to

$$e_1 = 4kp_1 + 2p_1^2 \text{ and } s = \tfrac{(1-6m^2+m^4)}{m(-1+m^2)}(k^2 + kp_1).$$

We observed that this parameterization led to all of the equations for the other edge and diagonal lengths of the shape being homogeneous in p_1 and k, which tells us that the dissimilarity of the resulting parallelepipeds is solely dependent on the ratio of the values of p_1 and k. Thus, by allowing p_1 and k to be rational values, it suffices to set $k = 1$ in our equations. Additionally, we see that by allowing the substitution of $p_1 = p - 1$, our parameterization becomes symmetric about zero. This leaves us simply with our current parameterization of

$$e_1 = 2(p^2 - 1) \text{ and } s = \tfrac{(1-6m^2+m^4)p}{m(-1+m^2)}.$$

Recall our initial parameterization of Pythagorean triples as seen in equations (2.1). We have also explored the different possible combinations of how the shared edge of the two Pythagorean triples, that is e_1, could be represented in each of the two triples. Note that the two legs of a Pythagorean triple can be represented as $2\alpha\beta$ and $\alpha^2 - \beta^2$. In the family from equations (2.6), e_1 appears in the form of $2\alpha\beta$ in both of the triples. We have also found infinite families for which it appears as $\alpha^2 - \beta^2$ in both of the triples, and as $2\alpha\beta$ in one, and as $\alpha^2 - \beta^2$ in the other. After further examination, however, it was discovered that these families were algebraic transformations of the original, and thus not unique. We checked over a thousand perfect parallelepipeds with two nonparallel rectangular faces that we discovered using the searches described above, and they all can be produced using equations (2.6).

Conjecture 4.1. *Up to scaling, every rational parallelepipid with at least* 2 *non-parallel rectangular faces may be found using suitable parameters in equations* (2.6).

5. Notes on the perfect cuboid

The infinite family of perfect parallelepipeds we have obtained can also be viewed as an infinite family of nearly perfect cuboids. The parallelepipeds in this family are but one right angle from being cuboids. Through our parameterizations we will be able to give a condition, which if met, would result in the discovery of a perfect cuboid.

Recall the embedding in $\mathbb{R}^3$ defined in equations (2.5) and note that the three vectors are mutually orthogonal if $j(p, m) = 0$. This leads to the following result.

Theorem 5.1. *There exist rational p and m such that the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ yield a perfect rational cuboid if and only if there exist rational p and m with $0 < p < 1$ and $0 < m < \sqrt{2} - 1$ such that $j(p, m) = 0$.*

Proof. First suppose that there exist $0 < p < 1$ and $0 < m < \sqrt{2}-1$, both rational, such that $j(p,m) = 0$. We need to check that the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ satisfy the hypotheses of Theorem 2.5. Clearly, we have $p \neq 0$, $p \neq \pm 1$, $m \neq 0$, and $m \neq \pm 1$. Solving $j(p,m) = 0$ for p yields eight solutions for p in terms of m. Four of these solutions guarantee p to be negative and thus are disregarded. For each of the four nonnegative solutions, we verify that $A(p,m)$ is nonzero for all choices of m. Direct computation [6] verifies that in all cases $A(p,m) = 0$ implies that m is irrational. We conclude $A(p,m)$ is nonzero.

Further, direct computation [6] verifies that for each of the four aforementioned solutions, $h(p,m)$ evaluates to 4 for all rational m. Thus we conclude that $h > 0$, and by Theorem 2.5 the parameters p and m result in a rational parallelepiped. Since the y coordinate of $\vec{w}$ is zero, we conclude that the vectors $\vec{v}$ and $\vec{w}$ are orthogonal. Thus the rational parallelepiped has three nonparallel rectangular faces and is a perfect rational cuboid.

Now suppose that there exist rational p and m such that $\vec{u}$, $\vec{v}$, and $\vec{w}$ yield a perfect rational cuboid. By Theorem 3.6, without loss of genrality, we may assume $0 < p < 1$ and $1-\sqrt{2} < m < \sqrt{2}-1$. The vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are mutually orthogonal; hence the y coordinate of $\vec{w}$ must be zero, thus $j(p,m) = 0$. It remains to show that this cuboid is similar to a perfect rational cuboid with $0 < p' < 1$ and $0 < m' < \sqrt{2}-1$. If $0 < m < \sqrt{2}-1$, we are done. We know $m \neq 0$. So we may assume $1-\sqrt{2} < m < 0$. As above, solving $j(p,m) = 0$ yields four viable expressions for p in terms of m. In each case, direct computation verifies that the parallelepiped generated by p and m is similar to the parallelepiped generated by p and $-m$. Thus we let $p' = p$ and $m' = -m$ and note that p' and m' are in the desired intervals and generate a perfect rational cuboid. ∎

Conjecture 5.2. *There are no rational choices of p and m for which $j(p,m) = 0$.*

While four viable solutions for p in terms of m are produced when solving $j(p,m) = 0$, a contour plot reveals that only one of them attains values in the desired region $0 < p < 1$ and $0 < m < \sqrt{2}-1$. The contour plot is given in Figure 4. The key equation represented by the white curve in the black region is

$$p = \frac{\sqrt{-\frac{-1+14m^4+2m^6-m^8+\sqrt{G}+m^2\left(2+\sqrt{G}\right)}{m^2(-1+m^2)^2}}}{\sqrt{2}}, \tag{5.1}$$

where $G(m) = 1 - 6m^2 - 17m^4 + 108m^6 - 17m^8 - 6m^{10} + m^{12}$.

The plot suggests that $\vec{u}$, $\vec{v}$, and $\vec{w}$ yield a perfect rational cuboid if and only if equation (5.1) has a solution for rational p and m, $0 < m < \sqrt{2}-1$.

In addition, since the cosine of the angle between $\vec{v}$ and $\vec{w}$ is continuous where it is defined, we have the following result:

Theorem 5.3. *We can obtain rational parallelepipeds such that the angle between $\vec{v}$ and $\vec{w}$ is arbitrarily close to $90°$.*

Proof. We obtain these parallelepipeds by picking rational p and m in the valid (black) region that are sufficiently close the white curve. ∎

For example, direct computation [6] shows that the nonrectangular face in the parallelepiped corresponding to $p = \frac{1}{6}$, $m = \frac{1}{5}$ has an angle of approximately 36.34 degrees. If we put $m = \frac{1}{5}$ into equation (5.1) and compute a five term

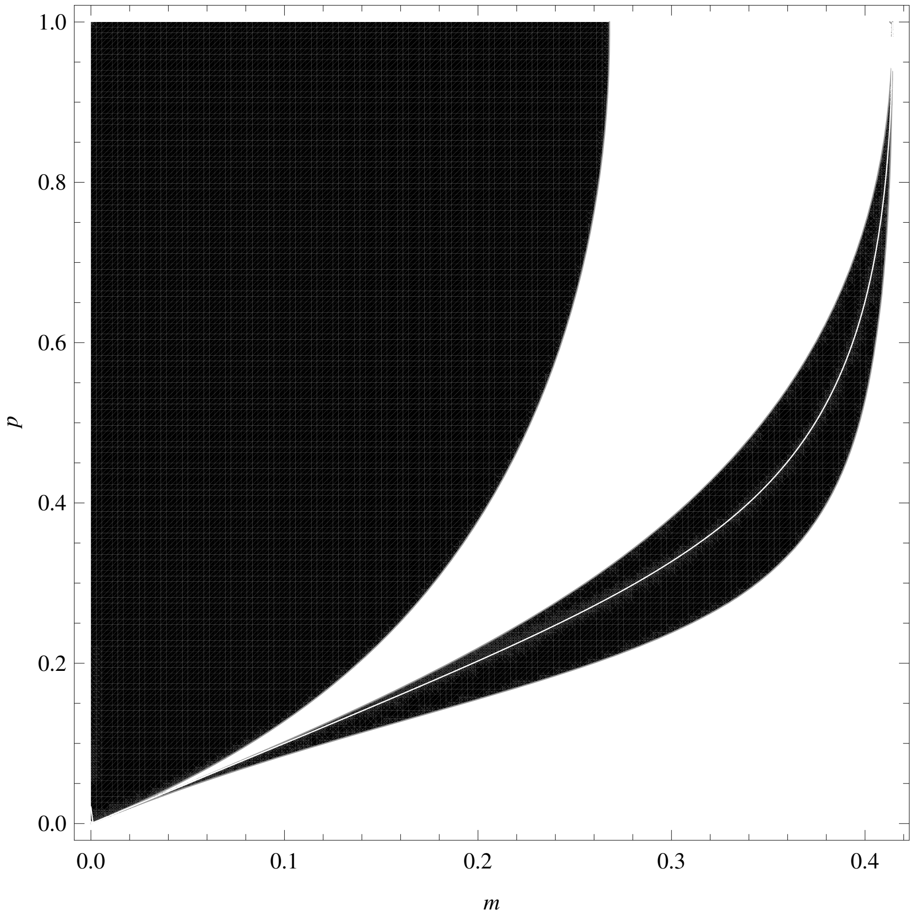

FIGURE 4. The black region represents the parameter space that yields realizable parallelepipeds. The thin white curve, given by equation (5.1), represents the parameters that yield cuboids.

simple continued fraction expansion for the algebraic value p, we obtain the rational approximation $\frac{43}{212}$. Using that approximation produces a rational parallelepiped with nonrectangluar face having an angle of approximately 89.98 degrees. Using a 100 term continued fraction expansion and resulting rational approximation gives an angle that is 90 degrees to 87 decimal places.

Acknowledgments

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