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# LINEAR REGRESSION MDP SCHEME FOR DISCRETE BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS UNDER GENERAL CONDITIONS

EMMANUEL GOBET AND PLAMEN TURKEDJIEV

Abstract. We design a numerical scheme for solving the Multi-step Forward Dynamic Programming (MDP) equation arising from the time-discretization of backward stochastic differential equations. The generator is assumed to be locally Lipschitz, which includes some cases of quadratic drivers. When the large sequence of conditional expectations is computed using empirical least-squares regressions, under general conditions we establish an upper bound error as the average, rather than the sum, of local regression errors only, suggesting that our error estimation is tight. Despite the nested regression problems, the interdependency errors are justified to be at most of the order of the statistical regression errors (up to logarithmic factor). Finally, we optimize the algorithm parameters, depending on the dimension and on the smoothness of value functions, in the limit as the time mesh size goes to zero and we compute the complexity needed to achieve a given accuracy. Numerical experiments are presented illustrating theoretical convergence estimates.

## 1. Introduction

**Framework.** Let $T > 0$ be a fixed terminal time and $W$ a $q$-dimensional ($q \geq 1$) Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual hypotheses; the filtration may be larger than that generated by $W$. Let $\pi := \{0 = t_0 < \ldots < t_N = T\}$ be a time-grid for the interval $[0, T]$, whose $(i+1)$-th time-step $t_{i+1} - t_i$ is denoted by $\Delta_i$, whose mesh size is defined by $|\pi| := \max_{0\leq i<N} \Delta_i \leq T$, and the $(i + 1)$-th Brownian motion increment $W_{t_{i+1}} - W_{t_i}$ is defined by $\Delta W_i$. The conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_{t_i}]$ is denoted by $\mathbb{E}_i[\cdot]$.

In this paper, we study an algorithm to approximate the solution to a discrete time backward stochastic differential equation (BSDE) in the form of the Multi-step

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Forward Dynamic Programming (MDP for short) equation given by

(1.1)
$$\left.\begin{array}{rcl} Y_i & = & \mathbb{E}_i[\xi + \sum_{k=i}^{N-1} f_k(Y_{k+1}, Z_k)\Delta_k], \\ \Delta_i Z_i & = & \mathbb{E}_i[(\xi + \sum_{k=i+1}^{N-1} f_k(Y_{k+1}, Z_k)\Delta_k)\Delta W_i^\top] \end{array}\right\} \quad \text{for } i \in \{0, \ldots, N-1\},$$

where the so called terminal condition $\xi$ is a given $\mathcal{F}_T$-measurable random variable in $\mathbf{L}_2$ and, for each $i$, the so-called driver $(\omega, y, z) \mapsto f_i(y, z)$ is $\mathcal{F}_{t_i} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^q)$-measurable. The operator $^\top$ denotes the vector transpose. Our results are shown under quite general conditions on the driver $f$ and the terminal condition $\xi$, which allows us, for instance, to treat the challenging case of quadratic drivers in a suitable Markovian setting (see Section 2.2). Because in general the conditional expectations in (1.1) are not known in closed form, one has to approximate them in order to have a fully implementable scheme. We follow the *empirical least-squares regression approach* presented in [27] and estimate the global error that this method incurs in the approximation of $(Y, Z)$. We call the resulting algorithm *LSMDP*. Let us briefly and informally recall this approach. Under rather general conditions (see Section 2), the solution of the MDP equation takes the form

$$(Y_i, Z_i) := (y_i(X_i), z_i(X_i)) \qquad \text{for } i \in \{0, \ldots, N-1\} \tag{1.2}$$

for some (unknown but deterministic) measurable functions $y_i(\cdot), z_i(\cdot)$ and a $d$-dimensional explanatory process $X := (X_i)_{0 \le i \le N}$. Each conditional expectation $\mathbb{E}_i(\cdot)$ can be viewed as a solution of a least-squares problem in $\mathbf{L}_2(\mathcal{F}_{t_i}, \mathbb{P})$, so the functions $y_i(\cdot)$ and $z_i(\cdot)$ are approximated by solving this problem on a finite-dimensional subspace. The approximations are computed using empirical least-squares regression using simulations of the paths of the explanatory variable $X$. Our main result (Theorem 4.11) is a full analysis of the quadratic error for the LSMDP. Informally speaking, the quadratic error can be decomposed into three different contributions:

$$\begin{aligned} \text{quadratic error } \le & \text{ approximation error } + \text{ statistical error} \\ & + \text{ interdependency error }. \end{aligned} \tag{1.3}$$

Using these error estimates, we optimize the algorithm parameters depending on the number of time points $N$ and consider the asymptotics $N \to +\infty$. This is detailed in Section 4.4, where the complexity required for a given theoretical accuracy is also computed. The approximation error depends on the choice of the finite-dimensional space and its magnitude usually depends on the smoothness of the solutions $y_i(\cdot)$ and $z_i(\cdot)$. We demonstrate how higher orders of smoothness of the value functions $y_i$ and $z_i$ defined in (1.2) lead to improvements in the quadratic error vs. computational work trade-off as $N \to +\infty$. The statistical error is due to the finite number of Monte Carlo simulations and it is usually increasing with respect to the dimensions of the approximation spaces. Thus, a careful balance between approximation and statistical errors has to be drawn and a curse of dimensionality usually appears. Non-asymptotic quadratic error estimates are fundamental to achieve the optimal trade-off. Finally, the interdependency error arises because of the interdependency between regression problems and is inherited from the non-linear driver term in the structure of the DP equation. We prove that in our algorithm, the interdependency error is of the same magnitude of the statistical error (up to a log factor). This is a very important improvement compared to previous works [27, 29]. In this sense, the estimates of Theorem 4.11 are tight. In Section 5, we provide numerical experiments that confirm the rate of convergence predicted, for a multidimensional quadratic BSDE with the Hölder continuous terminal condition.

The estimates of Theorem 4.11 are obtained by exploiting stability inequalities (Section 3) for discrete BSDEs: our MDP scheme leads to better weighted norms compared to the usual One step forward DP scheme (ODP for short)

$$Y_N = \xi, \qquad \left.\begin{array}{rcl} Y_i &=& \mathbb{E}_i\left(Y_{i+1} + f_i(Y_{i+1}, Z_i)\Delta_i\right), \\ \Delta_i Z_i &=& \mathbb{E}_i\left(Y_{i+1}\Delta W_i^\top\right), \end{array}\right\} \text{ for } i \in \{0, \dots, N-1\} \tag{1.4}$$

of [27], and thus better error estimates: the quadratic error is the average of local error terms, rather than the sum, which also means that the result is (in a sense) tight. Unlike [27, 29] which use a single cloud of simulations for the whole scheme, the LSMDP uses a different, independent cloud of simulations to approximate the conditional expectation for each time point. Resimulation is also used in the work of [7]. It is a standard technique in non-parametric statistics to split the data into independent samples (one for learning, one for testing). While increasing the a priori simulation cost, this reduces large errors due to interdependencies between the regressions at the different time-points, leading to a theoretically more efficient scheme. In practice, it is often observed that one-cloud schemes obtain similar rates of convergence to multi-cloud schemes, suggesting that the resimulation of the cloud is wasteful. Whether this is generally the case, however, is an open question, because the tools of non-parametric statistics currently available imply one-cloud schemes incur a much greater interdependency error. This is observed in [18] for the ODP scheme; for the MDP scheme, one may compare the results of this paper with the results of Chapter 2 in [34]. There is an additional, practical argument in favour of the resimulation approach: storing the simulated values is by far the highest memory expense because the number of simulations $M$ must be much larger than the approximation space dimension $K$ (to avoid overfitting). Thus, roughly speaking, we need to store $(d+q) \times M \times N$ values (for Markov chain and Brownian motion) if we do not resimulate data, and only $(2d+q+1) \times M$ with resimulation (1 for the Markov Chain sample at current time, 1 for the Markov chain over the running time, 1 for the the Brownian motion at the current time, and 1 for the sum of driver and terminal condition).

The use of an empirical regression scheme is supported by two further very powerful features: first, since we use distribution-free tools [24, Chapters 10-12], the estimates of Theorem 4.11 are model robust and, as such, can be applied to very general stochastic models because we make very few assumptions on the explanatory process $X$; see Section 2. Presumably, the constants obtained are too conservative, but the convergence rates are optimal. Second, the method of regression requires only independent simulations of $X$ and the Brownian increment, and no knowledge of the distribution of $X$; in this sense, the algorithm is a black-box mapping model simulations into value functions. This enables the use of data-driven methods such as, for example, the purely historical technique of [30].

**Applications.** Equation (1.1) appears naturally when approximating a continuous-time BSDE by a discrete-time process along the time grid $\pi$. Continuous time BSDEs have a multiplicity of applications in the theory of mathematical finance, stochastic optimal control, and partial differential equations (e.g. KPP equations, reaction-diffusion equations). Over the last ten years, the various numerical methods developed have been based on Lipschitz continuity in $(y,z)$ with uniform Lipschitz constant for the driver $f_i$ and $\xi = \Phi(X_{i_0},\dots,X_{i_L})$ for some Lipschitz continuous function $\Phi$ [1,7,8,11,17,19,27,28]. Very few papers [9,15,20,26,32] have handled greater generality. These papers, however, only deal with the discretization error, and, to the best of our knowledge, there are no works that consider the global error analysis of the full numerical scheme in higher generality. In this paper, we treat the global error analysis under several important relaxations to the traditional assumptions of numerical schemes for BSDEs. We do not treat the discretization error in this paper, however, and refer readers to [35] for this aspect, under the relaxed regularity assumptions of the current paper, of the error analysis. First, the terminal condition may satisfy substantially weaker regularity constraints without impacting the error estimates. This is in contrast to [27], where Lipschitz continuity is necessary. Second, the driver may satisfy a weaker, local Lipschitz continuity condition, where the Lipschitz constant of $f_i$ may depend on the time $t_i$; see ($\mathbf{A_F}$**-i**) in Section 2. This allows our results, combined with a simple truncation procedure that is presented in Section 2.2, to be applied to the important class of quadratic BSDEs with bounded, Hölder continuous terminal conditions *without incurring any additional error due to the truncation.* This is a very important feature, because truncation can have a very severe impact on numerical scheme. The methods of [9,26,32] use a truncation procedure which incurs error even at the discretization level. Third, the driver may satisfy a weaker, local bound at $(y,z) = (0,0)$ (see ($\mathbf{A_F}$**-ii**) in Section 2). Consequently, our results can be applied to a particular proxy technique presented in Section 2.2. Finally, the full error analysis is performed for possibly non-uniform time grid $\pi$; see ($\mathbf{A_F}$**-iii**) in Section 2. Indeed, to reduce the discretization error for BSDE with irregular terminal conditions $\xi = \Phi(X_T)$, it has been recently proposed in [20] to choose non-uniform grids which are consistent with our assumption ($\mathbf{A_F}$**-iii**). Similar non-uniform grids are obtained for path dependent $\xi$ in [15].

**Comparison to other works.** We would like to mention further two works that are relevant for comparison with this work, but which will not receive so much attention later in the paper as [27].

- We remark that equation (1.1) is inspired by the algorithm of [3], an algorithm based on Picard iterations and implicit equations that we do not use here. The use of explicit equations leads to different challenges in the computation of error estimates, particularly in obtaining a priori estimates (Section 3). Avoiding Picard iterations greatly simplifies the analysis error, because there is no supremum over the Picard iterations in the approximation error (compare with [29]).
- We have designed an alternative algorithm based on Malliavin weights [22]. Although this algorithm is potentially more efficient than the one presented in this paper, it suffers from a narrower scope of applicability. Indeed, for the approximation of continuous time BSDE, the $Z$ part of the continuous-time solution must satisfy a Malliavin integration-by-parts representation [34]. This may not be valid in the degenerate setting or in the setting with jumps. The scheme of the current paper is simpler to implement since we do not need to simulate the Malliavin weights.

**Organization of the paper.** In Section 2, we state our working assumptions and give several examples to show how these assumptions are useful for approximating a wide variety of continuous-time BSDEs. In Section 3, we establish stability estimates for discrete BSDEs, and apply them to derive tight pointwise and $\mathbf{L}_2$-estimates for $(Y, Z)$. In Section 4, we define the LSMDP scheme. The global error is stated in Theorem 4.11. The rest of the section is devoted to proofs and to a discussion related to algorithm complexity. In Section 5, we present numerical experiments in which the algorithm is applied to a quadratic BSDE with Hölder continuous driver in several dimensions, and show that the rate of convergence predicted by our main Theorem 4.11 holds for experiments. We note that, unlike many papers, these experiments concern the full quadratic error of the algorithm, rather than the error in the approximation of $Y$ only at time 0. Some intermediate results are detailed in the Appendix.

**Further notation.**

- $|x|$ stands for the Euclidean norm of the vector $x$, $\mathrm{Tr}(A)$ denotes the trace of the matrix $A$.
- $|U|_{\mathbf{L}_p} = (\mathbb{E}|U|^p)^{\frac{1}{p}}$ stands for the $\mathbf{L}_p(\mathbb{P})$-norm ($p \geq 1$) of a random variable $U$. To indicate that $U$ is additionally measurable w.r.t. the $\sigma$-algebra $\mathcal{Q}$, we may write $U \in \mathbf{L}_p(\mathcal{Q}, \mathbb{P})$.
- For a $q(\geq 1)$-dimensional process $U = (U_i)_{0\leq i\leq N}$, its $l$-th component is denoted by $U_l = (U_{l,i})_{0\leq i\leq N}$.
- For any finite $L > 0$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define the truncation function $\mathcal{T}_L(x) := (-L \vee x_1 \wedge L, \ldots, -L \vee x_n \wedge L)$.
- For finite $x > 0$, $\log(x)$ is the natural logarithm of $x$.

## 2. Standing assumptions and applications

In this section, we give the standing assumptions for this paper and we outline several examples to demonstrate how these more general assumptions lead to wider applicability of our results to practical continuous-time BSDE problems than is, to the best of our knowledge, currently available.

### 2.1. Standing assumptions.

The standing assumptions are separated into two parts: the first set consists of the *minimal* assumptions that are in force throughout the entirety of the paper, and the second set consists of the *Markovian* assumptions in force throughout Section 4 (or otherwise where stated). The minimal assumptions are:

($\mathbf{A}_\xi$) $\xi$ is in $\mathbf{L}_2(\mathcal{F}_T, \mathbb{P})$,

($\mathbf{A_F}$) i) $(\omega, y, z) \mapsto f_i(y, z)$ is $\mathcal{F}_{t_i} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^q)$-measurable for every $i < N$, and there exist deterministic parameters $\theta_L \in (0, 1]$ and $L_f \in [0, +\infty)$ such that

(2.1)
$$|f_i(y, z) - f_i(y', z')| \leq \frac{L_f}{(T - t_i)^{(1-\theta_L)/2}}(|y - y'| + |z - z'|) \qquad \forall\, i \in \{0, \ldots, N - 1\},$$

for any $(y, y', z, z') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^q$,

ii) there exist deterministic parameters $\theta_c \in (0, 1]$ and $C_f \in [0, +\infty)$ such that

$$|f_i(0, 0)| \leq \frac{C_f}{(T - t_i)^{1-\theta_c}}, \quad \forall\, i \in \{0, \ldots, N - 1\}, \tag{2.2}$$

iii) the time-grids $\pi := \{0 = t_0 < \ldots < t_N = T\}$ are such that

$$C_\pi = \sup_{k<N} \frac{\Delta_k}{(T - t_k)^{1-\theta_L}} \to 0 \quad \text{as } N \to +\infty, \tag{2.3}$$

$$\limsup_{N\to\infty} R_\pi < +\infty, \quad \text{where } R_\pi = \sup_{0\le k\le N-2} \frac{\Delta_k}{\Delta_{k+1}}. \tag{2.4}$$

Under $(\mathbf{A}_\xi)$ and $(\mathbf{A_F}\text{-}\mathbf{i}\text{-}\mathbf{ii})$, it is straightforward to check from (1.4) that $(Y_i)_{0\le i\le N}$ and $(Z_i)_{0\le i<N}$ are well defined and belong to $\mathbf{L}_2$ (see Proposition 3.2 for tight estimates). Note that taking $\theta_L = \theta_c = 1$ in $(\mathbf{A_F}\text{-}\mathbf{i}\text{-}\mathbf{ii})$ reduces the assumptions to the usual globally Lipschitz driver setting.

When analyzing the influence of using simulations in Section 4, we reinforce the basic assumptions with the following set of *Markovian* assumptions:

$(\mathbf{A_X})$ $X$ is a Markov chain in $\mathbb{R}^d$ $(1 \le d < +\infty)$ adapted to $(\mathcal{F}_{t_i})_i$. Moreover, for every $(i, j) \in \{0, \ldots, N\}^2$ with $i < j$, there exists a function $V_j^i : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ which is $\mathcal{G}_i \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, for some sub-$\sigma$-algebra $\mathcal{G}_i \subset \mathcal{F}_T$ independent of $\mathcal{F}_{t_i}$, such that $X_j = V_j^i(X_i)$.

$(\mathbf{A}'_\xi)$ i) $\xi$ is a bounded $\mathcal{F}_T$-measurable random variable:

$$C_\xi := \mathbb{P} - \operatorname*{ess\,sup}_\omega |\xi(\omega)| < +\infty.$$

ii) $\xi$ is of form $\xi = \Phi(X_N)$ for a bounded, measurable function $\Phi$.

$(\mathbf{A}'_\mathbf{F})$ For every $i < N$, the driver is of the form $f_i(y, z) = f_i(X_i, y, z)$ where $(x, y, z) \mapsto f_i(x, y, z)$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^q)$-measurable and satisfies $(\mathbf{A_F})$.

These yield a Markov representation for solutions of the discrete BSDEs: for all $k < N$, there exist measurable, deterministic functions $y_k : \mathbb{R}^d \to \mathbb{R}$ and $z_k : \mathbb{R}^d \to \mathbb{R}^q$ such that $Y_k = y_k(X_k)$ and $Z_k = z_k(X_k)$ hold almost surely (see Subsection 4.1). We emphasize that we do not make any further assumptions on $X$—no non-degeneracy condition, no specific distributions, etc; our error estimates are *model-free* in this sense. This lends flexibility and robustness to the empirical least-squares regression scheme: it is a black box mapping model simulations into value functions.

At first glance, the boundedness assumption $(\mathbf{A}'_\xi\text{-}\mathbf{i})$ appears to be a serious restriction of our scheme. The *raison d'être* of $(\mathbf{A}'_\xi\text{-}\mathbf{i})$ is to derive robust estimates for the global error (see Theorem 4.11) using the tools of non-parametric regression [24]. On the other hand, $\xi_n = -n \vee \xi \wedge n$ $(n \ge 0)$ defines a sequence of bounded approximations of $\xi$ and by $\mathbf{L}_2$-stability results on continuous-time BSDEs (see [14, Proposition 2.1] for instance), the truncation error converges to 0 as $n \to +\infty$. Since in our global error estimates we keep track on the dependence on $C_\xi$, it would be a priori possible to let this upper bound go appropriately quickly to infinity, while maintaining the convergence rate of the scheme (up to log terms).

## 2.2. Applications to motivate $(\mathbf{A_F})$.

► **Assumptions $(\mathbf{A_F}\text{-}\mathbf{i}\text{-}\mathbf{ii})$.** We outline two canonical examples, both related to a $\mathbb{R}^d$-valued Brownian diffusion process $(X_t)_{0\le t\le T}$ (which could include jumps as in [27]) with infinitesimal generator $\mathcal{L}$. Assume that $q = d$, that the coefficients of $X$ are smooth and bounded and that its diffusion coefficient $\sigma(t, x)$ satisfies a uniform ellipticity condition. The typical continuous-time BSDE at hand is of the form

$$Y_t = \Phi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

Under various conditions on $\Phi, f$ and on the approximation of $X_{t_i}$ by $X_i$, the discrete time process $(Y_i, Z_i)_{0\le i<N}$ generated by (1.1) (or equivalently (1.4)) converges to $(Y, Z)$, in suitable $\mathbf{L}_2$-spaces, as the mesh size $|\pi|$ goes to 0. See [7,19,20,26,32,36] among others. More precisely, the convergence under the current local Lipschitz assumptions $(\mathbf{A_F})$ was recently analysed in [35].

*Approximating Quadratic BSDEs.* Consider a quadratic growth driver satisfying

$$|f(t,x,y,z)| \le c\ (1 + |y| + |z|^2),$$
$$|f(t,x,y,z) - f(t,x,y',z')| \le c\ (1 + |z| + |z'|)(|y-y'| + |z-z'|)$$

for any $(t,x,y,y',z,z') \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and for a given constant $c \ge 0$. Quadratic BSDEs have important applications in mathematical finance, for example, in utility optimization in incomplete markets [13, 25, 33]. Assume additionally that the terminal function $\Phi$ is Hölder continuous and bounded. Then, [12, Theorem 2.1] yields that there exist constants $\theta \in (0,1]$ and $C_u \in \mathbb{R}^+$ such that $(T-t)^{(1-\theta)/2}|Z_t| \le C_u$ for all $t \in [0,T)$ almost surely. Now, using the truncation function $\mathcal{T}_\cdot(.)$, set $\varphi_t : \zeta \in \mathbb{R}^d \mapsto \varphi_t(\zeta) := \mathcal{T}_{C_u(T-t)^{(\theta-1)/2}}(\zeta)$ and define a new driver $\bar f(t,x,y,z) := f\big(t,x,y,\varphi_t(z)\big)$. Observe that $\bar f(t, X_t, Y_t, Z_t) = f(t, X_t, Y_t, Z_t)$ almost surely. Therefore, the BSDE with terminal condition $\Phi(X_T)$ and driver $f(t, X_t, y, z)$ admits the same solution as the BSDE with terminal condition $\Phi(X_T)$ and driver $\bar f(t, X_t, y, z)$ due to uniqueness of solutions, so it is equivalent to solve the BSDE with driver $f$ or $\bar f$. This implies that this technique incurs zero error due to the truncation. Notice also that $\varphi_t(\cdot)$ is 1-Lipschitz continuous and bounded by $C_u\sqrt{d}(T-t)^{-(1-\theta)/2}$, hence $f_i(y,z) := \bar f(t_i, X_{t_i}, y, z)$ satisfies $(\mathbf{A_F}$-**i**-**ii**$)$ with $C_f = c$, $\theta_c = 1$, $L_f = c(T^{(1-\theta)/2} + 2\sqrt{d}C_u)$, $\theta_L = \theta$. It is possible to show that the exponent $\theta$ equals the Hölder coefficient of $\Phi$; see [35], [34]. In contrast, it may be difficult to obtain an explicit value for the constant $C_u$ and therefore the constant $L_f$; see for example [31]. On the other hand, one may replace this constant by $\widetilde{C}_u\sqrt{\log(e + \log N)}$, where $N$ is the number of time-points in the time-grid and where $\tilde{C}_u$ is an estimate of $C_u$; for sufficiently large $N$, this will dominate $L_f$. The effect of adding this dependence on $N$ into the Lipschitz constant will deteriorate the rate of convergence only by a factor of $(e + \log N)^{\bar c \widetilde{C}_u}$, because the constants obtained in Theorem 4.11 depend on $L_f$ at worst through the factor $e^{\bar c L_f^2}$.

*Using proxys.* This is a technique that may improve the numerical efficiency of BSDE algorithms given a priori knowledge. Consider a standard Lipschitz driver $f$. Assume that we know by some a priori information that the solution $(Y_t, Z_t)_t$ is close to $(v(t, X_t), \nabla v(t, X_t)\sigma(t, X_t))_t$, where $v$ is the explicit solution to a linear parabolic equation $\partial_t v(t,x) + \tilde{\mathcal{L}}v(t,x) + \tilde f(t,x) = 0$; the diffusion process associated to $\tilde{\mathcal{L}}$, the terminal condition and the driver may have changed to produce an analytical solution. $v$ is called a *proxy* in [6]. This is a standard method in PDE theory. It is then natural to numerically compute the residual $(Y_t^0, Z_t^0) := (Y_t - v(t, X_t), Z_t - \nabla v(t, X_t)\sigma(t, X_t))$, which solves a BSDE with terminal function $\Phi(.) - v(T, .)$ and driver

$$f^0(t,x,y,z) := f(t, x, y + v(t,x), z + \nabla v(t,x)\sigma(t,x)) - \tilde f(t,x) + (\mathcal{L} - \tilde{\mathcal{L}})v(t,x).$$

The new driver $f^0$ is uniformly Lipschitz w.r.t. $y$ and $z$, so ($\mathbf{A_F}$-**i**) is satisfied with $\theta_L = 1$. If $v(T,\cdot)$ is $\theta$-Hölder continuous ($\theta \in (0,1]$), then usual PDE estimates on the parabolic operator $\tilde{\mathcal{L}}$ give $(T-t)^{(\frac{k-\theta}{2})_+}|D_x^k v(t,x)| \le C_v$ ($k=0,1,2$), from which ($\mathbf{A_F}$-**ii**) is derived with $\theta_c = \theta/2$. In fact, for $\tilde{\mathcal{L}} = \mathcal{L}$, $v(T,.) = \Phi(.)$ and $\tilde{f} = 0$, one has the improvement $\theta_c = (1+\theta)/2$.

To conclude this example, we mention that in the case $\tilde{\mathcal{L}} = \mathcal{L}$, $v(T,.) = \Phi(.)$ and $\tilde{f} = 0$, it is proved in [20] that the $\mathbf{L}_2$-time-regularity of $(Y^0, Z^0)$ is usually more well-behaved than that of $(Y,Z)$, implying that the discretization error from the DP equation for $(Y^0, Z^0)$ is smaller.

▶ Assumption ($\mathbf{A_F}$-**iii**). This assumption is used to derive stability results for discrete BSDEs (see Proposition 3.2) and for the numerical schemes (see Theorem 4.11) as the number $N$ of grid-times becomes large. This condition is satisfied, for example, by the grid suggested in [20] (i.e., $t_k = T - T(1-k/N)^{1/\theta_\pi}$ with $\theta_\pi \in (0,1]$); see [21, Lemma A.1]. In that article, it was shown that this class of time-grid is in some sense optimal for the time-discretization of the continuous BSDE, a result extended to the weaker conditions used in this paper in [35], so the Assumption ($\mathbf{A_F}$-**iii**) is not restrictive in this respect.

## 3. General a priori estimates

In this section, our effort concentrates on establishing $\mathbf{L}_2$ estimates that will be uniform as $N \to +\infty$. These results will be crucial for Section 4. The methods of proof are somewhat standard but due to our non-globally Lipschitz assumptions, there are technicalities that are worth mentioning.

The first two results of this section, Lemma 3.1 and Proposition 3.2, determine stability estimates for a class of BSDEs whose drivers satisfy rather weaker conditions than afforded by ($\mathbf{A_F}$). These weaker assumptions are critical for the analysis of BSDEs with sample-dependant drivers in Section 4. We consider two discrete BSDEs, $(Y_{1,i}, Z_{1,i})_{0\le i\le N}$ and $(Y_{2,i}, Z_{2,i})_{0\le i\le N}$, given by

$$\begin{aligned} Y_{j,i} &= \mathbb{E}_i\left[\xi_j + \textstyle\sum_{k=i}^{N-1} f_{j,k}(Y_{j,k+1}, Z_{j,k})\Delta_k\right], \\ \Delta_i Z_{j,i} &= \mathbb{E}_i\left[(\xi_j + \textstyle\sum_{k=i+1}^{N-1} f_{j,k}(Y_{j,k+1}, Z_{j,k})\Delta_k)\Delta W_i^\top\right], \end{aligned}$$

for $i \in \{0,\ldots,N-1\}$, $j \in \{1,2\}$, and we aim to study the differences

$$\Delta Y_i = Y_{1,i} - Y_{2,i}, \qquad \Delta Z_i = Z_{1,i} - Z_{2,i}.$$

For this, we define

$$\Delta f_i = f_{1,i}(Y_{1,i+1}, Z_{1,i}) - f_{2,i}(Y_{1,i+1}, Z_{1,i}), \qquad \Delta\xi = \xi_1 - \xi_2.$$

The assumptions on the drivers $f_{j,i}(y,z)$ differ from ($\mathbf{A_F}$) and, before continuing, we briefly outline how. First, the driver $f_{2,i}(y,z)$ is Lipschitz continuous w.r.t. $(y,z)$ and the dependence of their Lipschitz constant w.r.t. $i$ is general. Second, there are no Lipschitz continuity assumptions on $f_{1,i}(y,z)$; however, we assume that each $f_{1,i}(Y_{1,i+1}, Z_{1,i})$ are in $\mathbf{L}_2(\mathcal{F}_T)$, so that $Y_{1,i}$ and $Z_{1,i}$ are also square integrable for any $i$. Finally, we do not insist that the drivers be adapted, which will be needed in the setting of sample-dependant drivers.

Using the tower property of conditional expectations, observe that the MDP and ODP definitions of the discrete BSDEs coincide, i.e.,

$$\tag{3.1} \begin{cases} Y_{j,N} = \xi_j, & Y_{j,i} = \mathbb{E}_i\left[Y_{j,i+1} + f_{j,i}(Y_{j,i+1}, Z_{j,i})\Delta_i\right], \\ & \Delta_i Z_{j,i} = \mathbb{E}_i\left[Y_{j,i+1}\Delta W_i^\top\right], \end{cases}$$

for $i \in \{0, \dots, N-1\}$, $j \in \{1, 2\}$. The first stability result, the lemma below, is an intermediate result used repeatedly later in the proof of Proposition 3.2:

**Lemma 3.1** (Local estimates)**.** *For $j \in \{1, 2\}$, assume that $\xi_j$ is in $\mathbf{L}_2(\mathcal{F}_T)$. Moreover, for each $i \in \{0, \dots, N-1\}$, assume that $f_{1,i}(Y_{1,i+1}, Z_{1,i})$ is in $\mathbf{L}_2(\mathcal{F}_T)$ and $f_{2,i}(y, z)$ is Lipschitz continuous w.r.t. $y$ and $z$, with a finite Lipschitz constant $L_{f_{2,i}} \geq 0$. Then, for any $\Delta_i \leq T$ and $\gamma_i > 0$ satisfying $6q(\Delta_i + \frac{1}{\gamma_i})L_{f_{2,i}}^2 \leq 1$, it follows that*

$$\tag{3.2} |\Delta Y_i|^2 \leq (1 + (\gamma_i + \frac{1}{2})\Delta_i)\mathbb{E}_i(|\Delta Y_{i+1}|^2) + 3(\Delta_i + \frac{1}{\gamma_i})\Delta_i\mathbb{E}_i[|\Delta f_i|^2].$$

*Proof.* <u>*Preliminary estimates for $\Delta Z_i$.*</u> Since the Brownian increment $\Delta W_i$ is conditionally centered, it follows that

$$\Delta_i \Delta Z_i = \mathbb{E}_i\left[(\Delta Y_{i+1} - \mathbb{E}_i[\Delta Y_{i+1}])\Delta W_i^\top\right].$$

By the Cauchy-Schwarz inequality, $|\mathbb{E}_i\left[(\Delta Y_{i+1} - \mathbb{E}_i[\Delta Y_{i+1}])\Delta W_i^\top\right]|^2$ is bounded above by $q\Delta_i\Big(\mathbb{E}_i[(\Delta Y_{i+1})^2] - (\mathbb{E}_i[\Delta Y_{i+1}])^2\Big)$, whence

$$\tag{3.3} \Delta_i|\Delta Z_i|^2 \leq q\Big(\mathbb{E}_i[(\Delta Y_{i+1})^2] - (\mathbb{E}_i[\Delta Y_{i+1}])^2\Big).$$

<u>*Estimates for $\Delta Y_i$.*</u> We have $\Delta Y_i = \mathbb{E}_i\Delta Y_{i+1} + \Delta_i\mathbb{E}_i[\Delta f_i] + \Delta_i\mathbb{E}_i[f_{2,i}(Y_{1,i+1}, Z_{1,i}) - f_{2,i}(Y_{2,i+1}, Z_{2,i})]$. Combining Young's inequality, $(a+b)^2 \leq (1+\gamma_i\Delta_i)a^2 + (1 + \frac{1}{\gamma_i\Delta_i})b^2$ for any $(a, b) \in \mathbb{R}^2$, and the Lipschitz property of $f_{2,i}$ and (3.3), one deduces that

$$\begin{aligned} \text{(3.4)}\quad (\Delta Y_i)^2 &\leq (1+\gamma_i\Delta_i)(\mathbb{E}_i[\Delta Y_{i+1}])^2 \\ &\quad + 3(\Delta_i + \frac{1}{\gamma_i})\Delta_i\left[\mathbb{E}_i[|\Delta f_i|^2] + L_{f_{2,i}}^2\mathbb{E}_i[(\Delta Y_{i+1})^2] + L_{f_{2,i}}^2|\Delta Z_i|^2\right] \\ &\leq \left(1 + \gamma_i\Delta_i - 3qL_{f_{2,i}}^2(\Delta_i + \frac{1}{\gamma_i})\right)(\mathbb{E}_i[\Delta Y_{i+1}])^2 + 3(\Delta_i + \frac{1}{\gamma_i})\Delta_i\mathbb{E}_i[|\Delta f_i|^2] \\ \text{(3.5)}\quad &\quad + \left[3(\Delta_i + \frac{1}{\gamma_i})\Delta_i L_{f_{2,i}}^2 + 3qL_{f_{2,i}}^2(\Delta_i + \frac{1}{\gamma_i})\right]\mathbb{E}_i[(\Delta Y_{i+1})^2]. \end{aligned}$$

The assumptions on $\gamma_i$ and $\Delta_i$ ensure that $1 + \gamma_i\Delta_i - 3qL_{f_{2,i}}^2(\Delta_i + \frac{1}{\gamma_i}) \geq 0$ for any $\Delta_i$, whence, applying Jensen's inequality to the terms in $(\mathbb{E}_i\Delta Y_{i+1})^2$ in (3.5), it follows that

$$\begin{aligned} (\Delta Y_i)^2 &\leq \left(1 + \gamma_i\Delta_i - 3qL_{f_{2,i}}^2(\Delta_i + \frac{1}{\gamma_i})\right)\mathbb{E}_i[(\Delta Y_{i+1})^2] + 3(\Delta_i + \frac{1}{\gamma_i})\Delta_i\mathbb{E}_i(|\Delta f_i|^2) \\ &\quad + \left[3(\Delta_i + \frac{1}{\gamma_i})\Delta_i L_{f_{2,i}}^2 + 3qL_{f_{2,i}}^2(\Delta_i + \frac{1}{\gamma_i})\right]\mathbb{E}_i[(\Delta Y_{i+1})^2] \\ &= \left(1 + \gamma_i\Delta_i + 3(\Delta_i + \frac{1}{\gamma_i})\Delta_i L_{f_{2,i}}^2\right)\mathbb{E}_i[(\Delta Y_{i+1})^2] + 3(\Delta_i + \frac{1}{\gamma_i})\Delta_i\mathbb{E}_i(|\Delta f_i|^2). \end{aligned}$$

Finally, using that $3(\Delta_i + \frac{1}{\gamma_i})\Delta_i L_{f_{2,i}}^2 \leq \frac{\Delta_i}{2}$ completes the proof of (3.2). ∎

We now come to the main stability result, which will be used extensively in the error analysis of Section 4:

**Proposition 3.2** (Global pointwise estimates). *For $j \in \{1,2\}$, assume that $\xi_j$ is in $\mathbf{L}_2(\mathcal{F}_T)$. Moreover, for each $i \in \{0,\dots,N-1\}$, assume that $f_{1,i}(Y_{1,i+1}, Z_{1,i})$ is in $\mathbf{L}_2(\mathcal{F}_T)$ and $f_{2,i}(y,z)$ is Lipschitz continuous w.r.t. $y$ and $z$, with a finite Lipschitz constant $L_{f_{2,i}} \geq 0$. Then, for any time-grid $\pi$ and $\gamma \in (0,+\infty)^N$ satisfying $6q(\Delta_k + \frac{1}{\gamma_k})L^2_{f_{2,k}} \leq 1$ for all $k \leq N-1$, we have, for $0 \leq i \leq N$,*

$$
\begin{aligned}
&|\Delta Y_i|^2\Gamma_i + \sum_{k=i}^{N-1} \Delta_k \mathbb{E}_i(|\Delta Z_k|^2)\Gamma_k \\
&\qquad \leq C_{(3.6)}\Big(\Gamma_N \mathbb{E}_i[|\Delta\xi|^2] + 3\sum_{k=i}^{N-1}\big(\frac{1}{\gamma_k} + \Delta_k\big)\Delta_k \mathbb{E}_i[|\Delta f_k|^2]\Gamma_k\Big), \qquad (3.6)
\end{aligned}
$$

*where $\Gamma_i := \prod_{k=0}^{i-1}(1+\gamma_k\Delta_k)$ and $C_{(3.6)} := 2q + (1+T)e^{T/2}$.*

Note that, whenever necessary, the above *conditional* pointwise estimates can be easily turned into uniform $\mathbf{L}_2$-estimates:

$$
\begin{aligned}
&\sup_{i\leq k\leq N} \mathbb{E}(|\Delta Y_k|^2)\Gamma_k + \sum_{k=i}^{N-1} \Delta_k \mathbb{E}(|\Delta Z_k|^2)\Gamma_k \\
&\qquad \leq C_{(3.6)}\Big(\Gamma_N \mathbb{E}[|\Delta\xi|^2] + 3\sum_{k=i}^{N-1}\big(\frac{1}{\gamma_k} + \Delta_k\big)\Delta_k \mathbb{E}[|\Delta f_k|^2]\Gamma_k\Big).
\end{aligned}
$$

*Proof of Proposition* 3.2. Multiplying both sides of equation (3.2) by

$$\lambda_i := (1 + (\gamma_{i-1} + \frac{1}{2})\Delta_{i-1})\lambda_{i-1}, \quad \text{where } \lambda_0 := 1,$$

one obtains $|\Delta Y_i|^2\lambda_i \leq \mathbb{E}_i[|\Delta Y_{i+1}|^2]\lambda_{i+1} + 3(\Delta_i + 1/\gamma_i)\Delta_i[|\Delta f_i|^2]\lambda_i$; summing both sides of this inequality between $i$ to $N-1$ and taking the conditional expectation $\mathbb{E}_i[\cdot]$, one deduces that

$$(\Delta Y_i)^2\lambda_i \leq \lambda_N \mathbb{E}_i(|\Delta\xi|^2) + 3\sum_{k=i}^{N-1}(\frac{1}{\gamma_k} + \Delta_k)\Delta_k \mathbb{E}_i[|\Delta f_k|^2]\lambda_k.$$

From the simple inequality $\Gamma_i \leq \lambda_i = \exp\big(\sum_{k=0}^{i}\log(1+(\gamma_k+\frac{1}{2})\Delta_k)\big) \leq e^{T/2}\Gamma_i$, it follows that, for all $i \in \{0,\dots,N\}$,

$$(\Delta Y_i)^2\Gamma_i \leq e^{T/2}\Gamma_N \mathbb{E}_i[|\Delta\xi|^2] + 3e^{T/2}\sum_{k=i}^{N-1}(\frac{1}{\gamma_k} + \Delta_k)\Delta_k \mathbb{E}_i[|\Delta f_k|^2]\Gamma_k. \qquad (3.7)$$

*Final estimates for $\Delta Z_i$.* From (3.3), $\sum_{k=i}^{N-1}\Delta_k \mathbb{E}_i[|\Delta Z_k|^2]\Gamma_k$ is bounded above by

$$
\begin{aligned}
&\sum_{k=i}^{N-1} q\Gamma_{k+1}\big(\mathbb{E}_i[(\Delta Y_{k+1})^2] - \mathbb{E}_i[(\mathbb{E}_k\Delta Y_{k+1})^2]\big) \\
&\qquad \leq q\Gamma_N \mathbb{E}_i[|\Delta\xi|^2] + \sum_{k=i+1}^{N-1} q\Gamma_k\Big(\mathbb{E}_i[(\Delta Y_k)^2] - (1+\gamma_k\Delta_k)\mathbb{E}_i[(\mathbb{E}_k\Delta Y_{k+1})^2]\Big)
\end{aligned}
$$

and, from (3.4), $\mathbb{E}_i[(\Delta Y_k)^2] - (1+\gamma_k\Delta_k)\mathbb{E}_i[(\mathbb{E}_k[\Delta Y_{k+1}])^2]$ is bounded above by

$$3(\frac{1}{\gamma_k}+\Delta_k)\Delta_k\Big[\mathbb{E}_i[|\Delta f_k|^2] + L^2_{f_{2,k}}\mathbb{E}_i[(\Delta Y_{k+1})^2] + L^2_{f_{2,k}}\mathbb{E}_i[|\Delta Z_k|^2]\Big]$$

whence it follows that $\sum_{k=i}^{N-1}\Delta_k\mathbb{E}_i[|\Delta Z_k|^2]\Gamma_k$ is bounded above by

$$q\Gamma_N\mathbb{E}_i(\Delta\xi^2) + 3\sum_{k=i+1}^{N-1} q(\frac{1}{\gamma_k}+\Delta_k)\Delta_k L^2_{f_{2,k}}\mathbb{E}_i(|\Delta Z_k|^2)\Gamma_k$$
$$+3\sum_{k=i+1}^{N-1} q(\frac{1}{\gamma_k}+\Delta_k)\Delta_k\mathbb{E}_i(|\Delta f_k|^2)\Gamma_k + 3\sum_{k=i+1}^{N-1} q(\frac{1}{\gamma_k}+\Delta_k)\Delta_k L^2_{f_{2,k}}\mathbb{E}_i[(\Delta Y_{k+1})^2]\Gamma_k.$$

Using the assumptions on $\gamma_k$ and $\Delta_k$ of the proposition statement, it follows that $\sum_{k=i}^{N-1}\Delta_k\mathbb{E}_i[|\Delta Z_k|^2]\Gamma_k$ is bounded above by

$$2q\Gamma_N\mathbb{E}_i[|\Delta\xi|^2] + 6\sum_{k=i+1}^{N-1} q(\frac{1}{\gamma_k}+\Delta_k)\Delta_k\mathbb{E}_i[|\Delta f_k|^2]\Gamma_k + \sum_{k=i+1}^{N-1}\Delta_k\mathbb{E}_i[(\Delta Y_{k+1})^2]\Gamma_k$$
$$\le (2q+Te^{T/2})\Gamma_N\mathbb{E}_i[|\Delta\xi|^2] + (6q+3Te^{T/2})\sum_{k=i+1}^{N-1}(\frac{1}{\gamma_k}+\Delta_k)\Delta_k\mathbb{E}_i[|\Delta f_k|^2]\Gamma_k,$$

where the estimate (3.7) on $\Delta Y$ is used in the last inequality. □

In the final result of this section, Proposition 3.3 below, we return to the discrete BSDE $(Y_i, Z_i)_{0\le i\le N}$ defined in (1.1) and show that, when the terminal condition is bounded, almost sure absolute bounds on $Y_i$ and $Z_i$ are available for all $i \in \{0,\dots,N-1\}$. The a priori estimates of Proposition 3.2 are vital in the proof. Note that assumption $(\mathbf{A_F})$ is automatically in force. These bounds will be critical for determining model-free estimates with the non-parametric tools [24, Chapters 10-12] in Section 4.

**Proposition 3.3** (*a.s.* upper bounds)**.** *Assume* $(\mathbf{A}'_\xi$-$\mathbf{i})$. *For any* $\pi$ *with* $N$ *large enough (such that* $C_\pi L_f^2 \le \frac{1}{12q}$*, see* $(\mathbf{A_F}$-$\mathbf{iii})$*), the following almost sure bounds on* $Y_i$ *and* $Z_i$ *apply*

$$|Y_i| \le C_{(3.8)}\Big(C_\xi + \sqrt{\frac{T^{\theta_L\wedge\theta_c}(T-t_i)^{2\theta_c-\theta_L\wedge\theta_c}}{4q(2\theta_c-\theta_L\wedge\theta_c)}}C_f\Big) \tag{3.8}$$
$$\le C_y := C_{(3.8)}\Big(C_\xi + \frac{T^{\theta_c}}{\sqrt{4q(2\theta_c-\theta_L\wedge\theta_c)}}C_f\Big),$$

$$|Z_{l,i}| \le \frac{C_{(3.8)}}{\sqrt{\Delta_i}}\Big(C_\xi + \sqrt{\frac{T^{\theta_L\wedge\theta_c}(T-t_i)^{2\theta_c-\theta_L\wedge\theta_c}}{4q(2\theta_c-\theta_L\wedge\theta_c)}}C_f\Big) \tag{3.9}$$
$$\le C_{z,i} := \frac{C_y}{\sqrt{\Delta_i}},$$

*for* $0\le i\le N-1$ *and* $C_{(3.8)} = \exp\big(\frac{T}{4} + \frac{6q(1\vee L_f^2)}{\theta_L\wedge\theta_c}(T_L^\theta\vee 1)\big)$.

Observe that $C_y$ and $C_{(3.8)}$ are uniform in $i$, and that they remain bounded as $L_f$ and $T$ go to 0 (as we naturally expect). Moreover, if the terminal condition is

0, so that $C_\xi = 0$, we obtain the simplification

$$|Y_i| \le C_{(3.10)}(T-t_i)^{\theta_c - \frac{1}{2}\theta_L\wedge\theta_c}, \qquad |Z_{l,i}| \le \frac{C_{(3.10)}}{\sqrt{\Delta_i}}(T-t_i)^{\theta_c-\frac{1}{2}\theta_L\wedge\theta_c}. \tag{3.10}$$

where $C_{(3.10)} := \frac{C_{(3.8)}T^{\frac{1}{2}\theta_L\wedge\theta_c}}{\sqrt{4q(2\theta_c-\theta_L\wedge\theta_c)}}$. Therefore, we see that $Y_i$ and $Z_i$ get smaller as $i$ tends to $N$. This observation is useful in particular for the proxy method of Section 2.2, in the setting where $\tilde{\mathcal{L}} = \mathcal{L}$, $v(T,.) = \Phi(.)$ and $\tilde{f} = 0$; indeed, tighter bounds on $Y$ and $Z$ improve the constants of the numerical convergence (but presumably not the rates).

*Proof.* Let $(Y_{1,i}, Z_{1,i})_{0\le i\le N} := (0,0)_{0\le i\le N}$, the discrete BSDE with 0 terminal condition and 0 driver, and $(Y_{2,i}, Z_{2,i})_{0\le i\le N} := (Y_i, Z_i)_{0\le i\le N}$, the discrete BSDE given by (1.1). The terminal condition and driver of $(Y_{j,i}, Z_{j,i})_{0\le i\le N}$ (for $j\in\{1,2\}$, respectively) satisfy the conditions of Proposition 3.2. In order to apply the a priori estimates from this proposition, it is sufficient from (**$\mathbf{A_F}$-iii**) to take $N$ large enough so that $C_\pi L_f^2 \le \frac{1}{12q}$, whence $6q\Delta_i\frac{L_f^2}{(T-t_i)^{1-\theta_L}} \le 1/2$ for each $i$, and to find a $\gamma_i > 0$ (explicited below) such that $6qL_f^2(T-t_i)^{\theta_L-1}/\gamma_i \le 1/2$ for each $i\in\{0,\dots,N-1\}$. Then, it follows from (3.7) and bound of $f_k(0,0)$ given in (**$\mathbf{A_F}$-ii**) that

$$\begin{aligned}
(\Delta Y_i)^2 &\le e^{T/2}\Big(\Gamma_N\mathbb{E}_i(\xi^2) + 3\sum_{k=i}^{N-1}\frac{(1+\gamma_k\Delta_k)}{\gamma_k}\Gamma_k\Delta_k\mathbb{E}_i[|f_k(0,0)|^2]\Big)\\
&\le e^{T/2}\Gamma_N\big(C_\xi^2 + 3C_f^2\sum_{k=i}^{N-1}\frac{\Delta_k}{\gamma_k(T-t_k)^{2(1-\theta_c)}}\big), \qquad (3.11)
\end{aligned}$$

where $\Gamma_i$ is defined by $\prod_{k=0}^{i-1}(1+\gamma_k\Delta_k)$ for each $i\in\{0,\dots,N\}$. The appropriate $\gamma_i$'s are determined constructively: setting

$$\gamma_i := 12q\frac{(1\vee T^{-\theta_L})(1\vee L_f^2)}{(T-t_i)^{1-\theta_L}}\big(\frac{T}{T-t_i}\big)^{\theta_L-\theta_L\wedge\theta_c} \ge \frac{12qL_f^2}{(T-t_i)^{1-\theta_L}}, \quad \text{for } i\in\{0,\dots,N-1\},$$

it follows that $6q\frac{1}{\gamma_i}\frac{L_f^2}{(T-t_i)^{1-\theta_L}} \le 1/2$ for every $i$, as required. It only remains to compute upper bounds for $\Gamma_N$. Observing that $2\theta_c - \theta_L\wedge\theta_c \ge \theta_c > 0$, it follows that

$$\begin{aligned}
\Gamma_N &\le \exp\big(\sum_{k=0}^{N-1}\gamma_k\Delta_k\big) \le \exp\big(12q(1\vee T^{-\theta_L})(1\vee L_f^2)T^{\theta_L-\theta_L\wedge\theta_c}\int_0^T(T-t)^{\theta_L\wedge\theta_c-1}dt\big)\\
&= \exp\big(\frac{12q(1\vee L_f^2)}{\theta_L\wedge\theta_c}(T_L^\theta\vee 1)\big), \qquad (3.12)
\end{aligned}$$

$$\begin{aligned}
\sum_{k=i}^{N-1}\frac{\Delta_k}{\gamma_k(T-t_k)^{2(1-\theta_c)}} &= \sum_{k=i}^{N-1}\frac{\Delta_k(1\wedge T^{\theta_L})T^{\theta_L\wedge\theta_c-\theta_L}}{12q(1\vee L_f^2)(T-t_k)^{(1-2\theta_c+\theta_L\wedge\theta_c)}}\\
&\le \frac{T^{\theta_L\wedge\theta_c}(T-t_i)^{2\theta_c-\theta_L\wedge\theta_c}}{12q(2\theta_c-\theta_L\wedge\theta_c)} \le \frac{T^{2\theta_c}}{12q(2\theta_c-\theta_L\wedge\theta_c)}. \qquad (3.13)
\end{aligned}$$

Substituting (3.12) and (3.13) into (3.11), we obtain the required upper bounds (3.8) on $Y$. The bound on $Z_i$ is clear from the Cauchy-Schwarz inequality and the bound on $Y_{i+1}$. ☐

## 4. Monte Carlo regression scheme: the LSMDP algorithm

In this section, we present the LSMDP scheme: the conditional expectations in (1.1) are approximated using linear least-squares regression on simulated data to provide a fully implementable algorithm. A full error analysis is undertaken in Section 4.3 and the algorithm complexity is discussed in Section 4.4. Finally, Section 4.5 is devoted to the proofs. Throughout this section, the Markovian assumptions $(\mathbf{A_X})$, $(\mathbf{A}'_\xi)$ and $(\mathbf{A}'_\mathbf{F})$ are always in force.

4.1. **Preliminaries.** Due to the Markovian assumptions, there are measurable, deterministic (but unknown) functions $y_i(\cdot):\mathbb{R}^d\to\mathbb{R}$ and $z_i(\cdot):\mathbb{R}^d\to\mathbb{R}^q$ for each $i\in\{0,\dots,N-1\}$ such that the solution $(Y_i,Z_i)_{0\le i\le N}$ of the discrete BSDE (1.1) is given by

$$(Y_i,Z_i):=\big(y_i(X_i),z_i(X_i)\big). \tag{4.1}$$

This is a similar result to [4, Theorem 3.1], but without Picard iterations. It is shown by mathematical induction using assumption $(\mathbf{A_X})$ and the following corollary of the Monotone Class theorem.

**Lemma 4.1.** *Suppose that $\mathcal{G}$ and $\mathcal{H}$ are independent sub-$\sigma$-algebras of $\mathcal{F}$. For $l\ge 1$, let $F:\Omega\times\mathbb{R}^d\to\mathbb{R}^l$ be bounded and $\mathcal{G}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable, and $U:\Omega\to\mathbb{R}^d$ be $\mathcal{H}$-measurable. Then, $\mathbb{E}[F(U)|\mathcal{H}]=j(U)$ where $j(h)=\mathbb{E}[F(h)]$ for all $h\in\mathbb{R}^d$.*

We have used assumption $(\mathbf{A_X})$—i.e., that $X_j=V_j^i(X_i)$ for all $j>i$—in order to apply Lemma 4.1 for each $i\in\{0,\dots,N-1\}$ with $\mathcal{H}=\sigma(X_i)$, $\mathcal{G}=\sigma(\Delta W_i)\vee\mathcal{G}_i$, $U=X_i$, and

$$F(x)=F_{Y,i}(x):=\Phi\big(V_N^i(x)\big)+\sum_{k=i}^N f_k\big(V_k^i(x),y_{k+1}(V_{k+1}^i(x)),z_k(V_k^i(x))\big)\Delta_k \ \text{ for } y_i(\cdot),$$

and $F(x)=F_{Z,i}(x):=\dfrac{1}{\Delta_i}F_{Y,i+1}(x)\Delta W_i^\top$ for$z_i(\cdot)$.

**Definition 4.2** (Linear least-squares regression)**.** For $l,l'\ge 1$ and for probability spaces $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and $(\mathbb{R}^l,\mathcal{B}(\mathbb{R}^l),\nu)$, let $S$ be a $\tilde{\mathcal{F}}\otimes\mathcal{B}(\mathbb{R}^l)$-measurable $\mathbb{R}^{l'}$-valued function such that $S(\omega,\cdot)\in\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\nu)$ for $\tilde{\mathbb{P}}$-a.e. $\omega\in\tilde\Omega$, and $\mathcal{K}$ a linear vector subspace of $\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\nu)$ spanned by deterministic $\mathbb{R}^{l'}$-valued functions $\{p_k(.),k\ge 1\}$. The least squares approximation of $S$ in the space $\mathcal{K}$ with respect to $\nu$ is the ($\tilde{\mathbb{P}}\times\nu$-a.e.) unique, $\tilde{\mathcal{F}}\otimes\mathcal{B}(\mathbb{R}^l)$-measurable function $S^\star$ given by

$$S^\star(\omega,\cdot)=\arg\inf_{\phi\in\mathcal{K}}\int|\phi(x)-S(\omega,x)|^2\nu(dx). \tag{4.2}$$

We say that $S^\star$ solves $\mathbf{OLS}(S,\mathcal{K},\nu)$.

On the other hand, suppose that $\nu_M=\frac{1}{M}\sum_{m=1}^M\delta_{\mathcal{X}^{(m)}}$ is a discrete probability measure on $(\mathbb{R}^l,\mathcal{B}(\mathbb{R}^l))$, where $\delta_x$ is the Dirac measure on $x$ and $\mathcal{X}^{(1)},\dots,\mathcal{X}^{(M)}:\tilde\Omega\to\mathbb{R}^l$ are i.i.d. random variables. For an $\tilde{\mathcal{F}}\otimes\mathcal{B}(\mathbb{R}^l)$-measurable $\mathbb{R}^{l'}$-valued function $S$ such that $\big|S\big(\omega,\mathcal{X}^{(m)}(\omega)\big)\big|<\infty$ for any $m$ and $\tilde{\mathbb{P}}$-a.e. $\omega\in\tilde\Omega$, the least squares approximation of $S$ in the space $\mathcal{K}$ with respect to $\nu_M$ is the ($\tilde{\mathbb{P}}$-a.e.) unique, $\tilde{\mathcal{F}}\otimes\mathcal{B}(\mathbb{R}^l)$–measurable function $S^\star$ given by

$$S^\star(\omega,\cdot)=\arg\inf_{\phi\in\mathcal{K}}\frac{1}{M}\sum_{m=1}^M|\phi\big(\mathcal{X}^{(m)}(\omega)\big)-S\big(\omega,\mathcal{X}^{(m)}(\omega)\big)|^2. \tag{4.3}$$

We say that $S^\star$ solves $\mathbf{OLS}(S,\mathcal{K},\nu_M)$.

From (4.1), the MDP equation (1.1) can be reformulated in terms of Definition 4.2: taking for $\mathcal{K}_i^{(l')}$ any dense subset in the $\mathbb{R}^{l'}$-valued functions belonging to $\mathbf{L}_2(\mathcal{B}(\mathbb{R}^d),\mathbb{P}\circ(X_i)^{-1})$, for each $i\in\{0,\dots,N-1\}$,

$$\text{(4.4)}\qquad \begin{cases} y_i(\cdot) \text{ solves } \mathbf{OLS}(\ S_{Y,i}(\underline{\mathbf{x}})\ ,\ \mathcal{K}_i^{(1)}\ ,\ \nu_i\ ),\\ \qquad \text{where } S_{Y,i}(\underline{\mathbf{x}}) := \Phi(x_N) + \displaystyle\sum_{k=i}^{N-1} f_k\big(x_k, y_{k+1}(x_{k+1}), z_k(x_k)\big)\Delta_k,\\ z_i(\cdot) \text{ solves } \mathbf{OLS}(\ S_{Z,i}(w,\underline{\mathbf{x}})\ ,\ \mathcal{K}_i^{(q)}\ ,\ \nu_i\ ),\\ \qquad \text{where } S_{Z,i}(w,\underline{\mathbf{x}}) := \dfrac{1}{\Delta_i} S_{Y,i+1}(\underline{\mathbf{x}})\ w^\top\\ \text{and}\quad \nu_i := \mathbb{P}\circ(\Delta W_i, X_i,\dots,X_N)^{-1}, \end{cases}$$

where $w\in\mathbb{R}^q$ and $\underline{\mathbf{x}} := (x_0,\dots,x_N)\in(\mathbb{R}^d)^{N+1}$. However, the least-squares regressions in (4.4) encounter the computational problems that $\mathbf{L}_2(\mathcal{B}(\mathbb{R}^d),\mathbb{P}\circ(X_i)^{-1})$ may be infinite-dimensional and that one has to compute (4.2) using the law of $(\Delta W_i, X_i,\dots,X_N)$, which may be impossible. Therefore, the functions $y_i(\cdot)$ and $z_i(\cdot)$ are to be approximated on finite-dimensional function spaces with the sample-based empirical version of the law, as described in the next section.

4.2. **Algorithm.** In order to avoid the possible infinite dimensionality, we will regress on a predetermined finite-dimensional vector space.

**Definition 4.3** (Finite dimensional approximation spaces)**.** For $i\in\{0,\dots,N-1\}$, we are given two finite functional linear spaces of dimension $K_{Y,i}$ and $K_{Z,i}$:

$$\begin{cases} \mathcal{K}_{Y,i} := \operatorname{span}\{p_{Y,i}^{(1)},\dots,p_{Y,i}^{(K_{Y,i})}\},\ \text{for } p_{Y,i}^{(k)}:\mathbb{R}^d\to\mathbb{R} \text{ s.t. } \mathbb{E}[|p_{Y,i}^{(k)}(X_i)|^2]<+\infty,\\ \mathcal{K}_{Z,i} := \operatorname{span}\{p_{Z,i}^{(1)},\dots,p_{Z,i}^{(K_{Z,i})}\},\ \text{for } p_{Z,i}^{(k)}:\mathbb{R}^d\to(\mathbb{R}^q)^\top \text{ s.t. } \mathbb{E}[|p_{Z,i}^{(k)}(X_i)|^2]<+\infty. \end{cases}$$

The function $y_i(\cdot)$ will be approximated in the linear space $\mathcal{K}_{Y,i}$, whereas the function $z_i(\cdot)$ will be approximated in $\mathcal{K}_{Z,i}$. We define the squared quadratic approximation errors

$$T_{1,i}^Y := \inf_{\phi\in\mathcal{K}_{Y,i}} \mathbb{E}\Big[|\phi(X_i)-y_i(X_i)|^2\Big],\qquad T_{1,i}^Z := \inf_{\phi\in\mathcal{K}_{Z,i}} \mathbb{E}\Big[|\phi(X_i)-z_i(X_i)|^2\Big],$$

which are the best quadratic errors using the basis functions. In order to avoid the problem of explicit computations using the law of $(\Delta W_i, X_i,\dots,X_N)$, we will regress using the *empirical* measure, rather than the law. In order to define the empirical measure, we need to generate *simulations* of $(\Delta W_i, X_i,\dots,X_N)$.

**Definition 4.4** (Simulations and empirical measures)**.** For $i\in\{0,\dots,N-1\}$, generate $M_i\ge 1$ independent copies $\mathcal{C}_i := \{(\Delta W_i^{(i,m)}, X^{(i,m)})\ :\ m=1,\dots,M_i\}$ of $(\Delta W_i, X)$: $\mathcal{C}_i$ forms a *cloud of simulations* used for the regression at time $i$. Furthermore, we assume that the clouds of simulations $(\mathcal{C}_i : 0\le i<N)$ are independently generated. All these random variables are defined on a probability space $(\Omega^{(M)},\mathcal{F}^{(M)},\mathbb{P}^{(M)})$. Denote by $\nu_{i,M}$ the empirical probability measure of the $\mathcal{C}_i$-simulations, i.e.,

$$\nu_{i,M} = \frac{1}{M_i}\sum_{m=1}^{M_i} \delta_{(\Delta W_i^{(i,m)}, X_i^{(i,m)},\dots,X_N^{(i,m)})}.$$

Denoting by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space supporting $(\Delta W, X)$, which serves as a generic element for the clouds of simulations, the full probability space used to analyze our algorithm is the product space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) = (\Omega, \mathcal{F}, \mathbb{P}) \otimes (\Omega^{(M)}, \mathcal{F}^{(M)}, \mathbb{P}^{(M)})$.

*Remark* 4.5. The reader will observe that the simulations $X_j^{(i,m)}$ for $j < i$ are not used in the algorithm; however, we keep them in the notation for simplicity.

Allowing time-dependency in the number of simulations $M_i$ is coherent with our setting of time-dependent local Lipschitz driver. Without loss of generality, up to the generation of extra simulations, we assume $M_i \geq K_{Y,i} \vee K_{Z,i}$ for all $i \in \{0, \dots, N-1\}$.

By a slight abuse of notation, we write $\mathbb{P}$ (resp. $\mathbb{E}$) to mean $\bar{\mathbb{P}}$ (resp. $\bar{\mathbb{E}}$) from now on. We now come to the definition of the LSMDP algorithm.

**Definition 4.6** (LSMDP algorithm)**.** Recall the linear spaces $\mathcal{K}_{Y,i}$ and $\mathcal{K}_{Z,i}$ from Definition 4.3, the empirical measures $\{\nu_{i,M} : i = 0, \dots, N-1\}$ from Definition 4.4, the almost sure bounds from Proposition 3.3 and the truncation function $\mathcal{T}_L$ (notation in the Introduction).

Set $y_N^{(M)}(\cdot) := \Phi(\cdot)$.

For each $i = N-1, N-2, \dots, 0$, set the random functions $y_i^{(M)}(\cdot)$ and $z_i^{(M)}(\cdot)$ recursively as follows: define $y_i^{(M)}(\cdot) := \mathcal{T}_{C_y}\big(\psi_{Y,i}^{(M)}(\cdot)\big)$ and $z_i^{(M)}(\cdot) = \mathcal{T}_{C_{z,i}}\big(\psi_{Z,i}^{(M)}(\cdot)\big)$, where

$$
\begin{cases}
\psi_{Y,i}^{(M)}(\cdot) \quad \text{solves} \quad \mathbf{OLS}(S_{Y,i}^{(M)}(\underline{\mathbf{x}}) \ , \ \mathcal{K}_{Y,i} \ , \ \nu_{i,M}) \\
\qquad \text{for} \quad S_{Y,i}^{(M)}(\underline{\mathbf{x}}) := \Phi(x_N) + \displaystyle\sum_{k=i}^{N-1} f_k\big(x_k, y_{k+1}^{(M)}(x_{k+1}), z_k^{(M)}(x_k)\big)\Delta_k, \\
\psi_{Z,i}^{(M)}(\cdot) \quad \text{solves} \quad \mathbf{OLS}(\ S_{Z,i}^{(M)}(w, \underline{\mathbf{x}}) \ , \ \mathcal{K}_{Z,i} \ , \ \nu_{i,M}) \\
\qquad \text{for} \quad S_{Z,i}^{(M)}(w, \underline{\mathbf{x}}) := \dfrac{1}{\Delta_i} S_{Y,i+1}^{(M)}(\underline{\mathbf{x}}) \ w^\top,
\end{cases}
\tag{4.5}
$$

for every $w \in \mathbb{R}^q$, $\underline{\mathbf{x}} = (x_0, \dots, x_N) \in (\mathbb{R}^d)^{N+1}$.

Like the MDP, the LSMDP is computed recursively: first, $y_N^{(M)}(\cdot)$ is used to compute $z_{N-1}^{(M)}(\cdot)$, then $y_N^{(M)}(\cdot)$ and $z_{N-1}^{(M)}(\cdot)$ are both used to compute $y_{N-1}^{(M)}(\cdot)$, and so on. In practice, the empirical least squares regressions are computed using a Singular Value Decomposition (see [23, Section 5.5]). So long as it is possible to generate the simulations $(\Delta W_i^{(i,m)}, X^{(i,m)})_{i,m}$, it is possible to perform the LSMDP algorithm given in Definition 4.6. In this sense, the algorithm is fully implementable.

To conclude this section, we present the following result, which will feature frequently in Section 4.3 below, on almost sure bounds.

**Lemma 4.7.** *Recall the constants $C_y$ of Proposition* 3.3*. For $S_{Y,i}^{(M)}(\cdot)$ defined in* (4.5)*,*

$$
\sup_{0 \leq i \leq N} \sup_{\underline{\mathbf{x}} \in (\mathbb{R}^d)^{N+1}} |S_{Y,i}^{(M)}(\underline{\mathbf{x}})| \leq C_{4.7} := C_\xi + L_f C_y T^{\frac{\theta_L}{2}} \Big[\frac{2\sqrt{T}}{1+\theta_L} + \frac{\sqrt{q}\sqrt{N}}{\sqrt{\theta_L}}\Big] + C_f \frac{T^{\theta_c}}{\theta_c}.
$$

*Proof.* From ($\mathbf{A}'_\xi$-**i**), ($\mathbf{A_F}$-**i-ii**), and Proposition 3.3, we readily obtain

$$
\begin{aligned}
|S^{(M)}_{Y,i}(\underline{\mathbf{x}})| &\le C_\xi + \sum_{i=0}^{N-1}\Big[\frac{L_f}{(T-t_i)^{\frac{1-\theta_L}{2}}}(C_y+\sqrt{q}\frac{C_y}{\sqrt{\Delta_i}})\Delta_i + \frac{C_f}{(T-t_i)^{1-\theta_c}}\Delta_i\Big]\\
&\le C_\xi + L_f C_y\Big[\frac{T^{(1+\theta_L)/2}}{(1+\theta_L)/2} + \sqrt{q}\sqrt{N}\Big(\sum_{i=0}^{N-1}\big(\frac{\sqrt{\Delta_i}}{(T-t_i)^{\frac{1-\theta_L}{2}}}\big)^2\Big)^{1/2}\Big] + C_f\frac{T^{\theta_c}}{\theta_c}
\end{aligned}
$$

for any $\underline{\mathbf{x}} \in (\mathbb{R}^d)^{N+1}$ and $i \in \{0,\dots,N\}$, so the announced upper bound follows. □

4.3. **Error analysis.** We will estimate the error of the LSMDP approximation $\big(y_i^{(M)}(X_i), z_i^{(M)}(X_i)\big)_{0\le i\le N}$ of the discrete BSDE $\big(y_i(X_i), z_i(X_i)\big)_{0\le i\le N}$ in a $\mathbf{L}_2$-norm. Since both the functions $y_i^{(M)}(.), z_i^{(M)}(.)$ and its arguments $X_i$ are random, we shall define more precisely the norms chosen to quantify the error.

**Definition 4.8.** Let $\varphi : \Omega^{(M)} \times \mathbb{R}^d \to \mathbb{R}$ or $\mathbb{R}^q$ be $\mathcal{F}^{(M)} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. For each $i \in \{0,\dots,N-1\}$, define the random norms

$$
\|\varphi\|^2_{i,\infty} := \int_{\mathbb{R}^d} |\varphi(x)|^2\, \mathbb{P}\circ X_i^{-1}(dx), \qquad \|\varphi\|^2_{i,M} := \frac{1}{M_i}\sum_{m=1}^{M_i} |\varphi(X_i^{(i,m)})|^2.
$$

We are concerned with finding an upper bound for the error terms

$$
\bar{\mathcal{E}}(Y,M,i) := \mathbb{E}\Big[\|y_i^{(M)}(\cdot) - y_i(\cdot)\|^2_{i,\infty}\Big], \quad \bar{\mathcal{E}}(Z,M,i) := \mathbb{E}\Big[\|z_i^{(M)}(\cdot) - z_i(\cdot)\|^2_{i,\infty}\Big].
$$

Actually, in the error analysis of empirical regressions, the most natural error norms are related to empirical measures, i.e.,

$$
\mathcal{E}(Y,M,i) := \mathbb{E}\Big[\|y_i^{(M)}(\cdot) - y_i(\cdot)\|^2_{i,M}\Big], \quad \mathcal{E}(Z,M,i) := \mathbb{E}\Big[\|z_i^{(M)}(\cdot) - z_i(\cdot)\|^2_{i,M}\Big].
$$

In order to relate error terms $\bar{\mathcal{E}}(\cdot,M,i)$ and $\mathcal{E}(\cdot,M,i)$, we use simple inequalities

$$
\begin{aligned}
&\bar{\mathcal{E}}(Y,M,i) \le 2\mathcal{E}(Y,M,i)\\
&\qquad + \mathbb{E}\Big[\big(\|y_i^{(M)}(\cdot) - y_i(\cdot)\|^2_{i,\infty} - 2\|y_i^{(M)}(\cdot) - y_i(\cdot)\|^2_{i,M}\big)_+\Big], \qquad (4.6)\\
&\bar{\mathcal{E}}(Z,M,i) \le 2\mathcal{E}(Z,M,i)\\
&\qquad + \sum_{l=1}^{q}\mathbb{E}\Big[\big(\|z_{l,i}^{(M)}(\cdot) - z_{l,i}(\cdot)\|^2_{i,\infty} - 2\|z_{l,i}^{(M)}(\cdot) - z_{l,i}(\cdot)\|^2_{i,M}\big)_+\Big], \qquad (4.7)
\end{aligned}
$$

so that we are in a position to apply the following proposition which estimates sample deviation uniformly on the function spaces; we defer to Appendix B for the proof.

**Proposition 4.9.** *For finite $B > 0$, let $\mathcal{G} := \{\psi\big(\mathcal{T}_B\phi(\cdot) - \eta(\cdot)\big) : \phi \in \mathcal{K}\}$, where $\psi : \mathbb{R} \to [0,\infty)$ is Lipschitz continuous with $\psi(0) = 0$ and Lipschitz constant $L_\psi$, $\eta : \mathbb{R}^d \to [-B,B]$, and $\mathcal{K}$ is a finite $K$-dimensional vector space of functions. Then, for $\mathcal{X}^{(1)},\dots,\mathcal{X}^{(M)}$ i.i.d. random variables distributed as $\mathcal{X}$, we have*

$$
\mathbb{E}\Big[\sup_{g\in\mathcal{G}}\Big(\int_{\mathbb{R}^d} g(x)\mathbb{P}\circ\mathcal{X}^{-1}(dx) - \frac{2}{M}\sum_{m=1}^{M} g(\mathcal{X}^{(m)})\Big)_+\Big] \le \frac{507(K+1)BL_\psi\log(3M)}{M}.
$$

Recalling the constant $C_y$ from Proposition 3.3, we apply Proposition 4.9 with $B = C_y$ (resp. $C_{z,i} = C_y/\sqrt{\Delta_i}$), $\psi(x) = \big(|x| \wedge (2C_y)\big)^2$ (resp. $\big(|x| \wedge (2C_{z,i})\big)^2$—whence $L_\psi = 4C_y$ (resp. $4C_{z,i}$)—and $\mathcal{K} = \mathcal{K}_{Y,i}$ (resp. $\mathcal{K}_{Z,i}$), so that, from (4.6) and (4.7) we obtain the following preliminary result.

**Proposition 4.10.** *For each $i \in \{0, \dots, N-1\}$, we have*

$$\bar{\mathcal{E}}(Y, M, i) \le 2\mathcal{E}(Y, M, i) + \frac{2028(K_{Y,i} + 1)C_y^2 \log(3M_i)}{M_i},$$
$$\bar{\mathcal{E}}(Z, M, i) \le 2\mathcal{E}(Z, M, i) + \frac{2028(K_{Z,i} + 1)qC_y^2 \log(3M_i)}{\Delta_i M_i}.$$

Therefore, to control $\bar{\mathcal{E}}(Y, M, i)$ and $\bar{\mathcal{E}}(Z, M, i)$, it is sufficient to select sufficiently large $M_i$ to balance $K_{Y,i}$ and $K_{Z,i}$, and to control the error terms $\mathcal{E}(Y, M, i)$ and $\mathcal{E}(Z, M, i)$. We now state the main result of this paper.

**Theorem 4.11** (Error for the LSMDP scheme)**.** *Recall the constants $C_y$ from Proposition 3.3 and $C_{4.7}$ from Lemma 4.7. For each $k \in \{0, \dots, N-1\}$, define*

$$\begin{aligned}\mathcal{E}(k) := T^Y_{1,k} + T^Z_{1,k} + C^2_{4.7}\Big(\frac{3K_{Y,k}}{M_k} + 2q\frac{K_{Z,k}}{\Delta_k M_k}\Big)\\ + 800\Big((K_{Y,k} + 1) + \frac{(K_{Z,k} + 1)q}{\Delta_k}\Big)C_y^2\frac{\log(3M_k)}{M_k}.\end{aligned}$$

*For any $\pi$ such that $C_\pi L_f^2(R_\pi \vee 1) \le (192C_{(3.6)}(1+T))^{-1}$, we have, for all $0 \le i \le N-1$, that*

$$\mathbb{E}\Big[\|y_i^{(M)}(\cdot) - y_i(\cdot)\|^2_{i,M}\Big] \le T^Y_{1,i} + \frac{3C^2_{4.7}K_{Y,i}}{M_i} + C_{4.11}\sum_{k=i}^{N-1}\mathcal{E}(k)\Delta_k, \tag{4.8}$$

$$\sum_{k=i}^{N-1}\mathbb{E}\Big[\|z_k^{(M)}(\cdot) - z_k(\cdot)\|^2_{k,M}\Big]\Delta_k \le C_{4.11}\sum_{k=i}^{N-1}\mathcal{E}(k)\Delta_k, \tag{4.9}$$

*where $C_{4.11} = 8\exp\big(192(R_\pi \vee 1)C_{(3.6)}(1+T)T^{\theta_L}L_f^2/\theta_L\big)$.*

The proof of Theorem 4.11 is given in Section 4.5. As announced in (1.3), the global error estimates are a time-average of three different, local error contributions $\mathcal{E}(k)$ along the DP equation: the terms $T^Y_{1,k}$ and $T^Z_{1,k}$ are the best (squared quadratic) approximation errors using the basis functions, the terms with factor $C_{4.7}$ come from statistical errors, and last terms (with factor $C_y$) are interdependency terms due to nested regression problems. In our algorithm, these interdependency terms deteriorate the statistical error terms only by a $\log(M)$ factor; this is a very significant improvement compared to existing results. As a consequence, the theoretical rate of convergence is much better, even under our general conditions.

Since interdependency errors have a small impact, the global LSMDP error writes as an average of regression errors only, therefore it is hard to think of possible improvements to the convergence rates. In this sense, we believe that these error estimates are optimal. However, constants could be improved, for instance, by incorporating variance reduction techniques (see [2, 34]). In relation with the latter, the martingale basis method of Bender-Steiner [5] and the Wiener chaos decomposition of Briand-Labart [8] provide alternative algorithms to reduce the statistical errors by taking advantage (whenever possible) of appropriate basis functions for which conditional expectations are explicitly computed.

4.4. **Algorithm complexity.** As usual in empirical regression theory, the parameters of the algorithm play contradictory roles: the higher the dimension of the approximation spaces, the lower the squared quadratic approximation error $T^{\cdot}_{1,\cdot}$ but the larger the statistical errors in Theorem 4.11; the higher the number of simulations, the lower the statistical and interdependancy errors, but the more computational work to be done. Thus, it is essential to optimize these parameters by means of an error vs. computational work (*complexity*) analysis. For simplicity, we assume that the time-grid is uniform: $\Delta_i = T/N$. In the following, $c$ is a positive constant that does not depend on $N$ and may change from line to line; $c$ is assumed to be large enough for the arguments to be consistent.

► *Smooth Markovian functions.* Assume that the continuous-time limit of the value functions $(y_i(.), z_i(.))_i$ are $(u(t,.), v(t,.))_t$ and that these functions are, respectively, of class $C_b^{\kappa+1+\eta}$ and $C_b^{\kappa+\eta}$ in space uniformly in time for some $\kappa \in \mathbb{N}$ and $\eta \in (0,1]$, i.e., $u$ and $v$ are uniformly bounded and $\kappa+1$ (resp. $\kappa$)-continuously space-differentiable with bounded derivatives, and the $\kappa+1$ (resp. $\kappa$)-th derivatives are uniformly $\eta$-Hölder continuous in space. This qualitative information is related to semi-linear PDE estimates: they hold under standard conditions on the driver and the terminal function; see for instance [10, 12].

Assume furthermore that the discretization errors between $(y_i(.), z_i(.))$ and $(u(t_i,.), v(t_i,.))$ are uniformly bounded by $cN^{-\theta_{\text{conv}}}$ for some $\theta_{\text{conv}} > 0$: for Lipschitz data, $\theta_{\text{conv}} = \frac{1}{2}$ (see [7, 19, 36]) and for smoother data $\theta_{\text{conv}} = 1$ (see [16, Theorems 7 and 8]). Thus, the squared quadratic approximation errors are bounded as follows:

$$T^Y_{1,i} \le 2 \inf_{\phi \in \mathcal{K}_{Y,i}} \mathbb{E}\Big[|\phi(X_i) - u(t_i, X_i)|^2\Big] + \frac{2c^2}{N^{2\theta_{\text{conv}}}},$$

$$T^Z_{1,i} \le 2 \inf_{\phi \in \mathcal{K}_{Z,i}} \mathbb{E}\Big[|\phi(X_i) - v(t_i, X_i)|^2\Big] + \frac{2c^2}{N^{2\theta_{\text{conv}}}}.$$

We now aim at choosing parameters so that the local error terms $\mathcal{E}(i)$ are of order $N^{-2\theta_{\text{conv}}}$ (up to logarithm factor) uniformly in $i$.

► *Approximation spaces.* For the basis functions, we take local polynomials defined on disjoint hypercubes $(\mathcal{H}_n)_n$ with edge length $\delta_y$ (for $y$) and $\delta_z$ (for $z$). The union of these hypercubes is of the form $[-R, R]^d$ for each component $y, z_1, \ldots, z_q$. The degree of local polynomials is $\kappa+1$ for $y$, and $\kappa$ for $z$. $\mathcal{P}_l$ stands for the set of polynomials of degree less than or equal to $l$.

► *Approximation error.* The best squared quadratic approximation error $\inf_{\phi \in \mathcal{K}_{Y,i}} \mathbb{E}[|\phi(X_i) - u(t_i, X_i)|^2]$ is equal to

$$\begin{aligned}
&\mathbb{E}\Big[|u(t_i, X_i)|^2 \mathbf{1}_{|X_i|_\infty > R}\Big] + \sum_n \min_{\phi \in \mathcal{P}_{\kappa+1}} \mathbb{E}\Big[|\phi(t_i, X_i) - u(t_i, X_i)|^2 \mathbf{1}_{X_i \in \mathcal{H}_n}\Big] \\
&\qquad \le |u(t_i,.)|^2_\infty \mathbb{P}(|X_i|_\infty > R) + \sum_n c|u(t_i,.)|^2_{\kappa+1+\eta} (\delta_y^{\kappa+1+\eta})^2 \mathbb{P}(X_i \in \mathcal{H}_n) \\
&\qquad \le |u(t_i,.)|^2_\infty \mathbb{P}(|X_i|_\infty > R) + c|u(t_i,.)|^2_{\kappa+1+\eta} (\delta_y^{\kappa+1+\eta})^2
\end{aligned}$$

where we have used a Taylor expansion on each set $\mathcal{H}_n$ and taken the local polynomials to be equal to the first $\kappa+1$ terms of the expansion. Assume additionally that $X_i$ has exponential moments (uniformly in $i$), i.e., for some $\lambda > 0$, $\sup_{N \ge 1} \sup_{0 \le i \le N} \mathbb{E}(e^{\lambda |X_i|_\infty}) < +\infty$, so that the choice $R = 2\theta_{\text{conv}} \lambda^{-1} \log(N+1)$ is sufficient to ensure $\mathbb{P}(|X_i|_\infty > R) = O(N^{-2\theta_{\mathrm{conv}}})$. Hence, the choice $\delta_y = cN^{-\frac{\theta_{\mathrm{conv}}}{\kappa+1+\eta}}$ ensures that $T^Y_{1,i} = O(N^{-2\theta_{\mathrm{conv}}})$. With similar arguments for the $z$ components, we have to choose $\delta_z = cN^{-\frac{\theta_{\mathrm{conv}}}{\kappa+\eta}}$. Thus the sizes of the vector spaces are $K_{Y,i} = cN^{d\frac{\theta_{\mathrm{conv}}}{\kappa+1+\eta}}\log^d(N+1)$ and $K_{Z,i} = cN^{d\frac{\theta_{\mathrm{conv}}}{\kappa+\eta}}\log^d(N+1)$.

► *Statistical and interdependency error.* Because the continuous-time solution $(u, v)$ are bounded, we can improve the uniform upper bound for $|S^{(M)}_{Y,i}(\underline{\mathbf{x}})|$ and $|S_{Y,i}(\underline{\mathbf{x}})|$, replacing $C_{4.7}$ by a constant independent of $N$. Then, to ensure that the term of factor of $C^2_{4.7}$ in $\mathcal{E}(i)$ is of order $O(N^{-2\theta_{\mathrm{conv}}})$, it is enough to take $M_i = cN^{1+2\theta_{\mathrm{conv}}}K_{l,i} = cN^{1+2\theta_{\mathrm{conv}}+d\frac{\theta_{\mathrm{conv}}}{\kappa+\eta}}\log^d(N+1)$. With this choice of $M_i = M$, the last contribution (interdependency error) in $\mathcal{E}(i)$ is $O(N^{-2\theta_{\mathrm{conv}}}\log(N))$.

► *Complexity analysis for the LSMDP scheme.* By averaging the above error contributions, we obtain from Theorem 4.11 that $\sup_{i\le N}\Big(\mathbb{E}\Big[\|y^{(M)}_i(\cdot) - y_i(\cdot)\|^2_{i,M}\Big] + \sum_{k=i}^{N-1}\mathbb{E}\Big[\|z^{(M)}_k(\cdot) - z_k(\cdot)\|^2_{k,M}\Big]\Delta_k\Big) = O\Big(\frac{\log(N)}{N^{2\theta_{\mathrm{conv}}}}\Big)$. Since the hypercubes are disjoint, the final computational cost $\mathcal{C}$ (counting the elementary operations) is of order $MN^2$, that is,

$$\mathcal{C} = cN^{3+2\theta_{\mathrm{conv}}+d\frac{\theta_{\mathrm{conv}}}{\kappa+\eta}}\log^d(N+1).$$

Equivalently, the global error, as a function of complexity and ignoring log factors, is

$$N^{-\theta_{\mathrm{conv}}} \le c\,\mathcal{C}^{\frac{-\theta_{\mathrm{conv}}}{3+2\theta_{\mathrm{conv}}+d\frac{\theta_{\mathrm{conv}}}{\kappa+\eta}}} = c\,\mathcal{C}^{\frac{-1}{2(1+\frac{3}{2\theta_{\mathrm{conv}}}+\frac{d}{2(\kappa+\eta)})}}.$$

This analysis shows that the smaller the parameter $\frac{3}{2\theta_{\mathrm{conv}}} + \frac{d}{2(\kappa+\eta)}$, the quicker the convergence. There are several numerically significant implications of this:

- the higher the smoothness of the solution, the better the convergence;
- the higher the dimension, the worse the convergence; this is the usual curse of dimensionality.
- the better the discretization error ($\theta_{\mathrm{conv}}$ large), the better the convergence. This motivates the development of high-order discretization schemes for BSDEs.

If we apply the same analysis to [27, Theorem 2] for $\kappa + \eta \ge 1$ and $\theta_{\mathrm{conv}} = 1/2$, we obtain that the error is of order $\mathcal{C}^{-\frac{1}{2(4+\frac{2d}{\kappa+1+\eta})}}$ for the ODP, which is significantly worse than the current LSMDP (at least in the theoretical framework given in this section). There is a version of the ODP scheme which also uses resampling and is detailed in [18]; therefore, it also avoids the high interdependency error, but additionally avoids high resampling cost, because it only needs to resample one time-point per regression. The error of this scheme is theoretically of order $\mathcal{C}^{-\frac{1}{2(4+\frac{d}{\kappa+1+\eta})}}$. This is still higher than the error of the LSMDP scheme so long as the smoothness parameters satisfy $\kappa + \eta > 1$, which constitutes every scenario under which the differentiability assumption is valid. This demonstrates that using the multi-step rather than the one-step algorithm may bring about improvement in theoretical efficiency, and that this improvement in efficiency is in fact related to the multi-step structure of the algorithm and not only due to resimulation.

To conclude this section, let us turn our attention to the proxy method of Section 2.2. So far, our analysis indicates that the use of proxies does not deteriorate the rate of convergence. We focus particularly on the setting $\tilde{\mathcal{L}} = \mathcal{L}$, $v(T,.) = \Phi(.)$ and $\tilde{f} = 0$. As we saw in (3.10), the a priori bounds of the remainder $(y_i(\cdot) - v(t_i,\cdot), z_i(\cdot) - \nabla_x v(t_i,\cdot)\sigma(t_i,\cdot))$ are smaller, which induce tighter error bounds due to improved constants and therefore better convergences. Unfortunately, convergence rates are not improved.

4.5. **Proof of Theorem 4.11.** The following proposition is a key tool in the proof of Theorem 4.11:

**Proposition 4.12.** *With the notation of Definition* 4.2, *suppose that $\mathcal{K}$ is finite-dimensional and spanned by the functions $\{p_1(.), \ldots, p_K(.)\}$. Let $S^\star$ solve* $\mathbf{OLS}(S,\mathcal{K},\nu)$ *(resp.* $\mathbf{OLS}(S,\mathcal{K},\nu_M)$*), according to* (4.2) *(resp.* (4.3)*). The following properties are satisfied:*

(i) *linearity: the mapping $S \mapsto S^\star$ is linear;*
(ii) *stability property: $\|S^\star\|_{\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\mu)} \le \|S\|_{\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\mu)}$, where $\mu = \nu$ (resp. $\mu = \nu_M$);*
(iii) *conditional expectation solution: in the case of the discrete probability measure $\nu_M$, assume additionally that the sub-$\sigma$-algebra $\mathcal{Q} \subset \tilde{\mathcal{F}}$ is such that $\big(p_j(\mathcal{X}^{(1)}), \ldots, p_j(\mathcal{X}^{(M)})\big)$ is $\mathcal{Q}$-measurable for every $j \in \{1,\ldots,K\}$. Setting $S_{\mathcal{Q}}(\mathcal{X}^{(m)}) := \tilde{\mathbb{E}}[S(\mathcal{X}^{(m)})|\mathcal{Q}]$ for each $m \in \{1,\ldots,M\}$, then $\tilde{\mathbb{E}}[S^\star|\mathcal{Q}]$ solves $\mathbf{OLS}\big(S_{\mathcal{Q}},\mathcal{K},\nu_M\big)$;*
(iv) *bounded conditional variance: in the case of the discrete probability measure $\nu_M$, suppose that $S(\omega,x)$ is $\mathcal{G}\otimes\mathcal{B}(\mathbb{R}^l)$-measurable, for $\mathcal{G} \subset \tilde{\mathcal{F}}$ independent of $\sigma(\mathcal{X}^{(1:M)})$, there exists a Borel measurable function $h : \mathbb{R}^l \to \mathcal{E}$, for some Euclidean space $\mathcal{E}$, such that the random variables $\{p_j(\mathcal{X}^{(m)}) : m = 1,\ldots,M,\ j = 1,\ldots,K\}$ are $\mathcal{H} := \sigma(h(\mathcal{X}^{(m)}) : m = 1,\ldots,M)$-measurable, and there is a finite constant $\sigma^2 \ge 0$ that uniformly bounds the conditional variances $\tilde{\mathbb{E}}\big[|S(\mathcal{X}^{(m)}) - \tilde{\mathbb{E}}(S(\mathcal{X}^{(m)})|\mathcal{G}\vee\mathcal{H})|^2 \mid \mathcal{G}\vee\mathcal{H}\big] \le \sigma^2$ $\tilde{\mathbb{P}}$-a.s. and for all $m \in \{1,\ldots,M\}$. Then*

$$\tilde{\mathbb{E}}\Big[\|S^\star(\cdot) - \tilde{\mathbb{E}}[S^\star(\cdot)|\mathcal{G}\vee\mathcal{H}]\|^2_{\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\nu_M)} \;\Big|\; \mathcal{G}\vee\mathcal{H}\Big] \le \sigma^2 K/M.$$

The proof is given in Appendix A. The significance of the Borel function $h$ in Proposition 4.12(iv) is to distinguish between the dependency on the sample path of the basis functions and the observations used for the least-squares regression: observe that the basis functions in Definition 4.3 depend only on the marginal $X_k^{(k,m)}$, whereas the observations in Definition 4.4 depend also on the marginals $X_j^{(k,m)}$ for $j$ greater than $k$ and the Brownian increment $\Delta W_k^{(k,m)}$, so the Borel function $h : \mathbb{R}^{q+d\times(N+1)} \to \mathbb{R}^d$ is the projection operator $h(w,\underline{\mathbf{x}}) = x_k$. We need to introduce some further notation and preliminary results before we commence the proof.

**Conditioning.** To deal with the simulations, we introduce the following notation.

**Definition 4.13.** Define the $\sigma$-algebras:

$$\mathcal{F}_i^{(*)} := \sigma(\mathcal{C}^{i+1}, \dots, \mathcal{C}^{N-1}), \qquad \mathcal{F}_i^{(M)} := \mathcal{F}_i^{(*)} \vee \sigma(X_i^{(i,m)} \ : \ 1 \leq m \leq M_i).$$

For every $i \in \{0, \dots, N-1\}$, let $\mathbb{E}_i^M[\cdot]$ (resp. $\mathbb{P}_i^M$) with respect to $\mathcal{F}_i^{(M)}$.

*Remark* 4.14. The $\sigma$-algebras in Definition 4.13 *do not* form filtrations. Recall the ambiant filtration $(\mathcal{F}_t)_{0\leq t\leq T}$: we will make use of the extended filtration $\{ \ \mathcal{F}^{(M)} \vee \mathcal{F}_{t_i} \ : \ i = 0, \dots, N\}$ below.

**Intermediate processes and local error terms.** For each $k \in \{0, \dots, N-1\}$, recall the functions $S_{Y,k}(\underline{\mathbf{x}})$ and $S_{Z,k}(w, \underline{\mathbf{x}})$ from (4.4), the linear spaces $\mathcal{K}_{Y,k}$ and $\mathcal{K}_{Z,k}$ from Definition 4.3, and the empirical measure $\nu_{k,M}$ from Definition 4.13, and set

$$\left.\begin{array}{lll} \psi_{Y,k}(\cdot) & \text{solves} & \textbf{OLS}( \ S_{Y,k}(\underline{\mathbf{x}}) \ , \ \mathcal{K}_{Y,k} \ , \ \nu_{k,M} \ ), \\ \psi_{Z,k}(\cdot) & \text{solves} & \textbf{OLS}( \ S_{Z,k}(w,\underline{\mathbf{x}}) \ , \ \mathcal{K}_{Z,k} \ , \ \nu_{k,M} \ ). \end{array}\right\} \tag{4.10}$$

Observing that $(\mathbb{E}_k^M[S_{Y,k}(X^{(k,m)})] \ , \ \mathbb{E}_k^M[S_{Z,k}(\Delta W_k^{(k,m)}, X^{(k,m)})]) = \big(y_k(X_k^{(k,m)}), z_k(X_k^{(k,m)})\big)$ for each $m \in \{1, \dots, M_k\}$ where $\big(y_k(\cdot), z_k(\cdot)\big)$ are the unknown functions defined in (4.1), we can apply Proposition 4.12(iii) to prove the following lemma.

**Lemma 4.15.** *For each* $k \in \{0, \dots, N-1\}$,

$$\left.\begin{array}{lll} \mathbb{E}_k^M[\psi_{Y,k}(\cdot)] & \textit{solves} & \textbf{OLS}( \ y_k(.) \ , \ \mathcal{K}_{Y,k} \ , \ \nu_{k,M} \ ), \\ \mathbb{E}_k^M[\psi_{Z,k}(\cdot)] & \textit{solves} & \textbf{OLS}( \ z_k(.) \ , \ \mathcal{K}_{Z,k} \ , \ \nu_{k,M} \ ). \end{array}\right\}$$

*In addition,*

$$T_{1,k}^{Y,M} := \mathbb{E}\big[\|\mathbb{E}_k^M[\psi_{Y,k}(\cdot)] - y_k(\cdot)\|_{k,M}^2\big] = \mathbb{E}\big[\inf_{\phi \in \mathcal{K}_{Y,k}} \|\phi(\cdot) - y_k(\cdot)\|_{k,M}^2\big] \leq T_{1,k}^{Y},$$

$$T_{1,k}^{Z,M} := \mathbb{E}\big[\|\mathbb{E}_k^M[\psi_{Z,k}(\cdot)] - z_k(\cdot)\|_{k,M}^2\big] = \mathbb{E}\big[\inf_{\phi \in \mathcal{K}_{Z,k}} \|\phi(\cdot) - z_k(\cdot)\|_{k,M}^2\big] \leq T_{1,k}^{Z}.$$

**Proof of Theorem 4.11.** From $\mathcal{T}_{C_y}(y_k) = y_k$ and the Lipschitz continuity of $\mathcal{T}_{C_y}$, it follows that $\mathbb{E}[\|y_k(\cdot) - y_k^{(M)}(\cdot)\|_{k,M}^2]$ is less than or equal to $\mathbb{E}[\|y_k(\cdot) - \psi_{Y,k}^{(M)}(\cdot)\|_{k,M}^2]$. Applying Pythagoras' theorem and taking expectations then yields

$$\mathbb{E}\|y_k(\cdot) - y_k^{(M)}(\cdot)\|_{k,M}^2 \leq T_{1,k}^{Y,M} + \mathbb{E}[\|\psi_{Y,k}^{(M)}(\cdot) - \mathbb{E}_k^M[\psi_{Y,k}(\cdot)]\|_{k,M}^2].$$

Because $S_{Y,k}^{(M)}(\cdot)$ depends on $z_k^{(M)}(\cdot)$ computed with the same cloud of simulations $\mathcal{C}_k$ as that used to define the OLS solution $\psi_{Y,k}^{(M)}(\cdot)$, it raises some interdependency issue that we solve by making a small perturbation to the intermediate processes as follows: for $\underline{\mathbf{x}} = (x_0, \dots, x_N) \in (\mathbb{R}^d)^{N+1}$, define

$$\begin{aligned} \tilde{S}_{Y,k}^{(M)}(\underline{\mathbf{x}}) := \Phi(x_N) &+ f_k\big(x_k, y_{k+1}^{(M)}(x_{k+1}), z_k(x_k)\big)\Delta_k \\ &+ \sum_{i=k+1}^{N-1} f_i\big(x_i, y_{i+1}^{(M)}(x_{i+1}), z_i^{(M)}(x_i)\big)\Delta_i, \end{aligned}$$

$$\tilde{\psi}_{Y,k}^{(M)}(\cdot) \quad \text{solves} \quad \textbf{OLS}(\tilde{S}_{Y,k}^{(M)}(\underline{\mathbf{x}}) \ , \ \mathcal{K}_{Y,k} \ , \ \nu_{k,M}).$$

We do not need this perturbation for the $Z$-component, because $S^{(M)}_{Z,k}(w,\underline{\mathbf{x}})$ depends only on the subsequent clouds of simulations $\{\mathcal{C}_j, j>k\}$. Then by Young's inequality, we have

$$\begin{aligned}\mathbb{E}\|y_k(\cdot)-y^{(M)}_k(\cdot)\|^2_{k,M} \le T^Y_{1,k} &+ 3\mathbb{E}[\|\mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot)-\psi_{Y,k}(\cdot)]\|^2_{k,M}]\\ &+ 3\mathbb{E}[\|\tilde{\psi}^{(M)}_{Y,k}(\cdot)-\mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot)]\|^2_{k,M}]\\ &+ 3\mathbb{E}[\|\tilde{\psi}^{(M)}_{Y,k}(\cdot)-\psi^{(M)}_{Y,k}(\cdot)\|^2_{k,M}].\end{aligned}\tag{4.11}$$

We handle each term separately.

▶ Term $\mathbb{E}[\|\mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot)-\psi_{Y,k}(\cdot)]\|^2_{k,M}]$. Set $\tilde{\xi}^*_{Y,k}(x) := \mathbb{E}(\tilde{S}^{(M)}_{Y,k}(\underline{\mathbf{X}})-S_{Y,k}(\underline{\mathbf{X}})|X_k = x, \mathcal{F}^{(M)})$. Recalling that $\tilde{S}^{(M)}_{Y,k}(\underline{\mathbf{x}})-S_{Y,k}(\underline{\mathbf{x}})$ is built only using the clouds $\{\mathcal{C}_j, j\ge k+1\}$, it follows from Lemma 4.1 that $\mathbb{E}^M_k[\tilde{S}^{(M)}_{Y,k}(X^{(k,m)})-S_{Y,k}(X^{(k,m)})]$ is equal to $\tilde{\xi}^*_{Y,k}(X^{(k,m)}_k)$ for every $m\in\{1,\dots,M_k\}$. Then, using Proposition 4.12(i)(iii), $\mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot)-\psi_{Y,k}(\cdot)]$ solves $\mathbf{OLS}(\tilde{\xi}^*_{Y,k}(\cdot),\ \mathcal{K}_{Y,k}\ ,\ \nu_{k,M})$. By Proposition 4.12(ii),

$$\mathbb{E}[\|\mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot)-\psi_{Y,k}(\cdot)]\|^2_{k,M}] \le \mathbb{E}[\|\tilde{\xi}^*_{Y,k}(\cdot)\|^2_{k,M}] = \mathbb{E}[(\tilde{\xi}^*_{Y,k}(X_k))^2].$$

Defining $\xi^*_{Y,k}(x) := \mathbb{E}(S^{(M)}_{Y,k}(\underline{\mathbf{X}})-S_{Y,k}(\underline{\mathbf{X}})|X_k = x,\mathcal{F}^{(M)})$, Young's inequality yields

$$\begin{aligned}&\mathbb{E}[(\tilde{\xi}^*_{Y,k}(X_k))^2]\\ &\le 2\mathbb{E}[(\tilde{S}^{(M)}_{Y,k}(\underline{\mathbf{X}})-S^{(M)}_{Y,k}(\underline{\mathbf{X}}))^2] + 2\mathbb{E}[(\xi^*_{Y,k}(X_k))^2]\\ &\le 2\mathbb{E}[|f_k(X_k, y^{(M)}_{k+1}(X_{k+1}), z^{(M)}_k(X_k)) - f_k(X_k, y^{(M)}_{k+1}(X_{k+1}), z_k(X_k))|^2]\Delta_k^2\\ &+ 2\mathbb{E}[(\xi^*_{Y,k}(X_k))^2]\\ &\le \frac{2L_f^2\Delta_k^2}{(T-t_k)^{1-\theta_L}}\mathbb{E}[|z^{(M)}_k(X_k)-z_k(X_k)|^2] + 2\mathbb{E}[(\xi^*_{Y,k}(X_k))^2].\end{aligned}$$

▶ Term $\mathbb{E}[\|\tilde{\psi}^{(M)}_{Y,k}(\cdot)-\mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot)]\|^2_{k,M}]$. Since $\tilde{S}^{(M)}_{Y,k}(.)$ depends only on the clouds $\{\mathcal{C}_j, j>k\}$ and is bounded above by $C_{4.7}$ (like $S^{(M)}_{Y,k}(.)$), it follows from Proposition 4.12(iv) that $\mathbb{E}[\|\tilde{\psi}^{(M)}_{Y,k}(\cdot)-\mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot)]\|^2_{k,M}]$ is bounded above by $C^2_{4.7}K_{Y,k}/M_k$. This contribution is interpreted as a statistical error term in regression theory.

▶ Term $\mathbb{E}[\|\tilde{\psi}^{(M)}_{Y,k}(\cdot)-\psi^{(M)}_{Y,k}(\cdot)\|^2_{k,M}]$. Using Proposition 4.12(i)(ii), it follows that $\|\tilde{\psi}^{(M)}_{Y,k}(\cdot)-\psi^{(M)}_{Y,k}(\cdot)\|^2_{k,M}$ is bounded above by $\|\tilde{S}^{(M)}_{Y,k}(\cdot)-S^{(M)}_{Y,k}(\cdot)\|^2_{k,M}$, which equals

$$\begin{aligned}&\frac{\Delta_k^2}{M_k}\sum_{m=1}^{M_k}|f_k(X^{(k,m)}_k, y^{(M)}_{k+1}(X^{(k,m)}_{k+1}), z^{(M)}_k(X^{(k,m)}_k))\\ &\qquad\qquad - f_k(X^{(k,m)}_k, y^{(M)}_{k+1}(X^{(k,m)}_{k+1}), z_k(X^{(k,m)}_k))|^2\\ &\le \frac{L_f^2\Delta_k^2\|z_k(\cdot)-z^{(M)}_k(\cdot)\|^2_{k,M}}{(T-t_k)^{1-\theta_L}}.\end{aligned}$$

Collecting the bounds on the three terms and substituting them into (4.11) and applying Proposition 4.10 yields

$$\begin{aligned}
\mathbb{E}\|y_k(\cdot) - y_k^{(M)}(\cdot)\|_{k,M}^2 &\le T_{1,k}^Y + 6\mathbb{E}[(\xi_{Y,k}^*(X_k))^2] + \frac{3C_{4.7}^2 K_{Y,k}}{M_k} \\
&\quad + \frac{3L_f^2 \Delta_k^2}{(T-t_k)^{1-\theta_L}}\{\mathbb{E}[\|z_k(\cdot) - z_k^{(M)}(\cdot)\|_{k,M}^2] + 2\bar{\mathcal{E}}(Z,M,k)\} \\
&\le T_{1,k}^Y + 6\mathbb{E}[(\xi_{Y,k}^*(X_k))^2] + \frac{3C_{4.7}^2 K_{Y,k}}{M_k} \\
&\quad + 3L_f^2 \Delta_k C_\pi \Big\{5\mathcal{E}(Z,M,k) + \frac{4056(K_{Z,k}+1)qC_y^2 \log(3M_k)}{\Delta_k M_k}\Big\}. \qquad (4.12)
\end{aligned}$$

Analogously to (4.11), one obtains the upper bound

$$\begin{aligned}
\mathbb{E}\|z_k(\cdot) - z_k^{(M)}(\cdot)\|_{k,M}^2 &\le T_{1,k}^Z + 2\mathbb{E}[\|\mathbb{E}_k^M[\psi_{Z,k}^{(M)}(\cdot) - \psi_{Z,k}(\cdot)]\|_{k,M}^2] \\
&\quad + 2\mathbb{E}[\|\psi_{Z,k}^{(M)}(\cdot) - \mathbb{E}_k^M[\psi_{Z,k}^{(M)}(\cdot)]\|_{k,M}^2].
\end{aligned}$$

Since $S_{Z,k}^{(M)}(.)$ depends only on the clouds $\{\mathcal{C}_j, j > k\}$ and its $\mathcal{F}_k^{(M)}$-conditional variance is bounded above by $qC_{4.7}^2/\Delta_k$, it follows from Proposition 4.12(iv) that $\mathbb{E}[\|\psi_{Z,k}^{(M)}(\cdot) - \mathbb{E}_k^M[\psi_{Z,k}^{(M)}(\cdot)]\|_{k,M}^2]$ is bounded above by $qC_{4.7}^2 K_{Z,k}/(\Delta_k M_k)$ (computations similar to those for (3.3)). Defining $\xi_{Z,k}^*(x) := \mathbb{E}(S_{Z,k}^{(M)}(\Delta W_k, \underline{\mathbf{X}}) - S_{Z,k}(\Delta W_k, \underline{\mathbf{X}}) | X_k = x, \mathcal{F}^{(M)})$, it follows that $\mathbb{E}_k^M[\psi_{Z,k}^{(M)}(\cdot) - \psi_{Z,k}(\cdot)]$ solves $\mathbf{OLS}(\xi_{Z,k}^*(\cdot),\ \mathcal{K}_{Z,k}\ ,\ \nu_{k,M})$. Therefore,

$$\mathbb{E}\|z_k(\cdot) - z_k^{(M)}(\cdot)\|_{k,M}^2 \le T_{1,k}^Z + 2\mathbb{E}[|\xi_{Z,k}^*(X_k)|^2] + \frac{2qC_{4.7}^2 K_{Z,k}}{\Delta_k M_k}. \qquad (4.13)$$

Observe that $(\xi_{Y,k}^*(X_k), \xi_{Z,k}^*(X_k))$ solves a discrete BSDE with terminal condition 0 and driver $f_{\xi^*,k}(y,z) := f_k(X_k, y_{k+1}^{(M)}(X_{k+1}), z_k^{(M)}(X_k)) - f_k(X_k, y_{k+1}(X_{k+1}), z_k(X_k))$. Combined with the local Lipschitz continuity of $f_k$ and a choice of $(\gamma_0, \ldots, \gamma_{N-1}) \in (0, +\infty)^N$ such that

$$96(R_\pi \vee 1)C_{(3.6)}(1+T)(\frac{1}{\gamma_k} + \Delta_k)\frac{L_f^2}{(T-t_k)^{1-\theta_L}} \le 1, \qquad (0 \le k < N), \qquad (4.14)$$

and setting the weights $\Gamma_k := \prod_{j=0}^{k-1}(1+\gamma_j\Delta_j)$, Proposition 3.2 (with $(Y_{1,k}, Z_{1,k}) = (\xi_{Y,k}^*(X_k), \xi_{Z,k}^*(X_k))$ and $(Y_2, Z_2) = (0,0)$) and Proposition 4.10 yield the bound

$$\sum_{k=i}^{N-1} \Delta_k \mathbb{E}[|\xi^*_{Y,k}(X_k)|^2]\Gamma_k + \sum_{k=i}^{N-1} \Delta_k \mathbb{E}[|\xi^*_{Z,k}(X_k)|^2]\Gamma_k$$
$$\le 6C_{(3.6)}(1+T) \sum_{k=i}^{N-1} \Delta_{k+1} \frac{\Delta_k}{\Delta_{k+1}} \frac{L_f^2(1/\gamma_k+\Delta_k)}{(T-t_k)^{1-\theta_L}} \mathbb{E}\big[|y_{k+1}(X_{k+1})-y^{(M)}_{k+1}(X_{k+1})|^2\big]\Gamma_k$$
$$+ 6C_{(3.6)}(1+T) \sum_{k=i}^{N-1} \Delta_k \frac{L_f^2(1/\gamma_k+\Delta_k)}{(T-t_k)^{1-\theta_L}} \mathbb{E}\big[|z_k(X_k) - z^{(M)}_k(X_k)|^2\big]\Gamma_k$$
$$\le \frac{1}{16} \sum_{k=i}^{N-1} \Delta_k \big(\bar{\mathcal{E}}(Y,M,k) + \bar{\mathcal{E}}(Z,M,k)\big)\Gamma_k$$
$$\le \frac{1}{8} \sum_{k=i}^{N-1} \Delta_k \big(\mathcal{E}(Y,M,k) + \mathcal{E}(Z,M,k)\big)\Gamma_k$$
$$+ 127 \sum_{k=i}^{N-1} \Delta_k \Gamma_k \Big\{(K_{Y,k}+1) + \frac{(K_{Z,k}+1)q}{\Delta_k}\Big\} \frac{C_y^2 \log(3M_k)}{M_k}. \tag{4.15}$$

Combining (4.15) with (4.12) and (4.13), it follows that

$$\sum_{k=i}^{N-1} \Delta_k \big(\mathcal{E}(Y,M,k) + \mathcal{E}(Z,M,k)\big)\Gamma_k \tag{4.16}$$
$$\le \sum_{k=i}^{N-1} \Delta_k \big(\widetilde{\mathcal{E}}(k) + 15 L_f^2 \Delta_k C_\pi \mathcal{E}(Z,M,k)\big)\Gamma_k$$
$$+ \frac{6}{8} \sum_{k=i}^{N-1} \Delta_k \big(\mathcal{E}(Z,M,k) + k\mathcal{E}(Z,M,k)\big)\Gamma_k$$

where we set

$$\widetilde{\mathcal{E}}(k) := T^Y_{1,k} + T^Z_{1,k} + C^2_{4.7}\Big(\frac{3K_{Y,k}}{M_k} + 2q\frac{K_{Z,k}}{\Delta_k M_k}\Big)$$
$$+ 3L_f^2 \Delta_k C_\pi \times 4056 (K_{Z,k}+1) q C_y^2 \frac{\log(3M_k)}{\Delta_k M_k}$$
$$+ 6 \times 127 \Big((K_{Y,k}+1) + \frac{(K_{Z,k}+1)q}{\Delta_k}\Big) C_y^2 \frac{\log(3M_k)}{M_k}.$$

Observe that $C_{(3.6)} \ge 3$, $3L_f^2 \Delta_k C_\pi \times 4056 \le \frac{3\times 4056}{192\times 3} \le 22$ using the constraint on $C_\pi$ given in the statement of Theorem 4.11; it readily implies that $\widetilde{\mathcal{E}}(k) \le \mathcal{E}(k)$. In addition, $15L_f^2 \Delta_k C_\pi \le \frac{15}{192\times 3} \le 1/8$ implies that the inequality (4.16) becomes

$$\sum_{k=i}^{N-1} \Delta_k \big(\mathcal{E}(Z,M,k) + \mathcal{E}(Z,M,k)\big)\Gamma_k$$
$$\le \sum_{k=i}^{N-1} \Delta_k \mathcal{E}(k)\Gamma_k + \frac{7}{8} \sum_{k=i}^{N-1} \Delta_k \big(\mathcal{E}(Z,M,k) + \mathcal{E}(Z,M,k)\big)\Gamma_k$$
$$\le 8 \sum_{k=i}^{N-1} \Delta_k \mathcal{E}(k)\Gamma_k. \tag{4.17}$$

The choice $\gamma_k = 192(R_\pi \vee 1)C_{(3.6)}(1+T)L_f^2/(T-t_k)^{1-\theta_L}$ makes (4.14) valid and then, $8\Gamma_k \le C_{4.11}$ (similarly to (3.12)) and this completes the proof of (4.9).

We now establish (4.8). Observe that Proposition 3.2 gives a direct estimate on $\mathbb{E}[|\xi^*_{Y,i}(X_i)|^2]\Gamma_i$: similarly to (4.15), one obtains (using previous inequalities)

$$\mathbb{E}[|\xi^*_{Y,i}(X_i)|^2]\Gamma_i \le \frac{1}{8}\sum_{k=i}^{N-1}\Delta_k\big(\mathcal{E}(Y,M,k)+\mathcal{E}(Z,M,k)\big)\Gamma_k + \frac{127}{800}\sum_{k=i}^{N-1}\Delta_k\mathcal{E}(k)\Gamma_k.$$

Plugging this inequality into (4.12) and using $L_f^2 C_\pi \le 1/(192\times 3)$ leads to

$$\begin{aligned}\mathcal{E}(Y,M,i)\Gamma_i \le T^Y_{1,i}\Gamma_i + \frac{3C^2_{4.7}K_{Y,i}}{M_i}\Gamma_i + \frac{6}{8}\sum_{k=i}^{N-1}\Delta_k\big(\mathcal{E}(Y,M,k)+\mathcal{E}(Z,M,k)\big)\Gamma_k \\ + \frac{6\times 127}{800}\sum_{k=i}^{N-1}\Delta_k\mathcal{E}(k)\Gamma_k + \frac{15}{576}\Delta_i\mathcal{E}(Z,M,i)\Gamma_i + \frac{3\times 4056}{576\times 800}\mathcal{E}(i)\Gamma_i.\end{aligned}$$

Using (4.17) leads to (4.8). □

## 5. Numerical experiments

We consider a BSDE with terminal condition $\Phi(x) := \sin(|x|^{2\alpha})$ for $\alpha \in (0,1/2]$; this is a Hölder continuous function with exponent $2\alpha$. The time horizon is $T = 0.2$, and the model of the explanatory process (forward SDE) is $X_t = W_t$. Setting $\psi(t,x) = \sin\big((T-t+|x|^2)^\alpha\big)$ and $\phi(t,x) = \cos\big((T-t+|x|^2)^\alpha\big)$, we define a driver which is quadratic in $z$ (but independent of $y$):

$$f(t,x,z) := |z|^2 - |\nabla\psi(t,x)|^2 - (\partial_t + \frac{1}{2}\Delta)\psi(t,x).$$

By Itô's formula, we verify that the analytic solution of this BSDE is $Y_t = \psi(t,W_t)$ and $Z_t = \nabla_x\psi(t,W_t) = 2\alpha W_t^\top \phi(t,W_t)(T-t+|W_t|^2)^{\alpha-1}$. To obtain a locally Lipschitz BSDE satisfying the condition $(\mathbf{A_F})$, we use the truncation procedure given in Section 2.2. Using the notation of that section, we take $\theta = 2\alpha$ and $C_u = 2\alpha$. Recalling the soft truncation function $\mathcal{T}_\cdot(\cdot)$, the driver $f(t,x,z)$ above is replaced by $\bar{f}(t,x,z) := f(t,x,\mathcal{T}_{C_u(T-t)^{(\theta-1)/2}}(z))$ in order to perform computations. In what follows, we take $\alpha = 0.4$.

We now choose numerical parameters to ensure that the squared quadratic error of the approximation converges at rate 1 with respect to the number of time-points $N$ in the time-grid. Note that the analysis in this section will be slightly different from that of Section 4.4, due to the use of a terminal condition that is *not* continuously differentiable. Therefore, we adapt the arguments of that section to the current context. To discretize the BSDE, we use a uniform time-grid, i.e., $\Delta_i = T/N$. We regress on piecewise-constant functions on hypercubes. We set the outer boundary of the hypercubes at $(-2\sqrt{t_i}, 2\sqrt{t_i})^q$ for each time-point $t_i$; in other words at $2\sigma$'s of the explanatory process. The cube width is fixed arbitrarily at $\delta = 4\sqrt{T}/k$ for a parameter $k$ running over positive integers. Then, $K_i = k_i^q$ for $k_i = \lceil 4\sqrt{t_i}/\delta\rceil$. The number of time points $N$ is chosen to be $N = c_0^{-2}k^2$ for an arbitrary constant $c_0 > 0$, which implies that $\delta = 4\sqrt{T}/(c_0\sqrt{N})$.

Let us check that the squared quadratic approximation error induced by this choice of basis is $O(N^{-1})$; we remind the reader that the squared quadratic approximation error is equal to

$$\max_{0\le i\le N-1}\sum_H \inf_{\alpha_H\in\mathbb{R}} \mathbb{E}[|y_i(W_{t_i})-\alpha_H|^2\mathbf{1}_{W_{t_i}\in H}]$$
$$+\sum_{i=0}^{N-1}\sum_H \inf_{\alpha_H\in(\mathbb{R}^q)^\top} \mathbb{E}[|z_i(W_{t_i})-\alpha_H|^2\mathbf{1}_{W_{t_i}\in H}]\Delta_i,$$

where the sum over $H$ indicates the sum over the hypercubes. We focus on the terms $z_i$, the analysis for $y_i$ being similar. Let $w_H$ be any point in a given hypercube $H$: in each of the above terms, we can take $\alpha_H = z_i(w_H)$, as this bounds the inf value from above. Then a first order Taylor expansion yields $z_i(W_{t_i}) - z_i(w_H) = \int_0^1 \nabla_x z_i(\lambda W_{t_i} + (1-\lambda)w_H)d\lambda](W_{t_i} - w_H)$, whence

$$\begin{aligned}
&\sum_H \inf_{\alpha_H\in(\mathbb{R}^q)^\top} \mathbb{E}[|z_i(W_{t_i})-\alpha_H|^2\mathbf{1}_{W_{t_i}\in H}]\\
&\le q\delta^2\int_0^1\sum_H \mathbb{E}[|\nabla_x z_i(\lambda W_{t_i}+(1-\lambda)w_H)|^2\mathbf{1}_{W_{t_i}\in H}]d\lambda\\
&=: q\delta^2\int_0^1\sum_H \eta_i(H,\lambda)d\lambda.
\end{aligned}$$

Since $\delta^2 = O(N^{-1})$, our strategy now is to show that

$$\sum_{i=0}^{N-1}\sum_H \eta_i(H,\lambda)\Delta_i = O(1), \qquad \text{uniformly in } \lambda\in[0,1], \tag{5.1}$$

which implies that the squared quadratic approximation error on $z$ is $O(N^{-1})$ as required.

Actually, a direct computation shows $|\nabla_x z_i(x)|^2 \le c_\psi^2(T-t_i+|x|^2)^{2\alpha-2}$ for some constant $c_\psi$. We discuss the estimate (5.1) according to two different cases.

*1st case: $t_i \le T/2$.* Then $|\nabla_x z_i(x)|^2 \le c_\psi^2(T/2)^{2\alpha-2}$, and therefore $\eta_i(H,\lambda) \le c_\psi^2(T/2)^{2\alpha-2}\mathbb{P}(W_{t_i}\in H)$. This implies $\sum_{i=0}^{N-1}\sum_H \eta_i(H,\lambda)\mathbf{1}_{t_i\le T/2}\Delta_i = O(1)$.

*2nd case: $t_i \ge T/2$.* Define $c_1 := 8\sqrt{q}/c_0$; the role of this constant will become apparent in the following computations. We split the expectation $\mathbb{E}[|\nabla_x z_i(\lambda W_{t_i} + (1-\lambda)w_H)|^2\mathbf{1}_{W_{t_i}\in H}]$ according to the three partioning events:

$$\{|W_{t_i}|\le c_1\sqrt{T-t_i}\},\qquad \{c_1\sqrt{T-t_i}<|W_{t_i}|\le c_1\sqrt{T}\},\qquad \{c_1\sqrt{T}<|W_{t_i}|\}.$$

In the first case, we simply use $|\nabla_x z_i(x)|^2 \le c_\psi^2(T-t_i)^{2\alpha-2}$. In the second case, by triangular inequality and $\delta \le 4\sqrt{T-t_i}/c_0$, observe that $|\lambda W_{t_i} + (1-\lambda)w_H| \ge |W_{t_i}| - \sqrt{q}\delta = |W_{t_i}|(1-\frac{\sqrt{q}\delta)}{|W_{t_i}|}) \ge |W_{t_i}|(1-\frac{4\sqrt{q}}{c_0c_1}) = |W_{t_i}|/2$, therefore $|\nabla_x z_i(\lambda W_{t_i} + (1-\lambda)w_H)|^2 \le c_\psi^2|W_{t_i}/2|^{4\alpha-4}$. In the third case, $|\nabla_x z_i(\lambda W_{t_i} + (1-\lambda)w_H)|^2 \le$

$c_\psi^2(c_1^2 T)^{2\alpha-2}$. The conclusion of the above partitioning is that we obtain (for $t_i \geq T/2$):

$$\sum_H \eta_i(H,\lambda) \leq c_\psi^2 (T-t_i)^{2\alpha-2}\mathbb{P}(|W_{t_i}| \leq c_1\sqrt{T-t_i})$$
$$+ c_\psi^2 \mathbb{E}\left(|W_{t_i}/2|^{4\alpha-4}\mathbf{1}_{c_1\sqrt{T-t_i}<|W_{t_i}|\leq c_1\sqrt{T}}\right)$$
$$+ c_\psi^2 (c_1^2 T)^{2\alpha-2}\mathbb{P}(|W_{t_i}| > c_1\sqrt{T}).$$

The above expectations and probabilities are bounded from above by changing variables to polar coordinates and taking advantage of the fact that the distribution of $W_{t_i}$ has a density bounded from above by $(\pi T)^{-q/2}$ (owing to $t_i \geq T/2$); this gives

$$\sum_H \eta_i(H,\lambda) \leq C(T-t_i)^{2\alpha-2+q/2} + C(T-t_i)^{2\alpha-2+q/2} + C$$

with a new constant $C$ (depending only on $\alpha$, $c_\psi$, $q$ and $T$). Observe that the condition $2\alpha - 2 + q/2 > -1$ is sufficient to complete the proof of (5.1). This condition is satisfied under our choice of $\alpha = 0.4$ for any dimension $q$, so we are done.

Finally, to ensure the correct rate of convergence in the statistical and interdependency errors, in view of Theorem 4.11 one must choose the number of simulations $M_i = O(N^2 K_i)$. For the tests reported in Figures 1-3 in dimension $q = 3, 5, 7$, we have taken $N = (0.5 \times k)^2$, $M_i = 0.4 N^2 K_i$. On these graphs, the squared quadratic error is plotted w.r.t. $N$ (both in log scales). The best linear approximation is also provided to estimate the convergence rate: this shows that the rate of convergence is close to $O(N^{-1})$ in all three cases, as predicted by our main result Theorem 4.11.

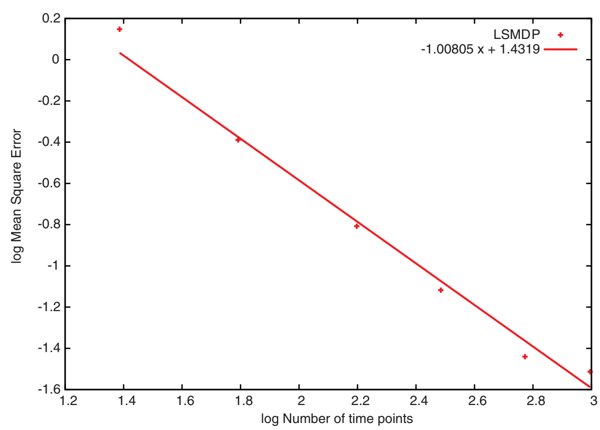


FIGURE 1. Log quadratic error versus $\log(N)$ in dimension $q = 3$.

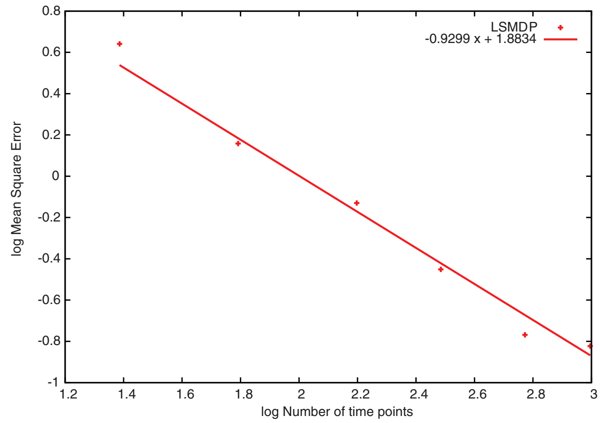


FIGURE 2. Log quadratic error versus $\log(N)$ in dimension $q = 5$.

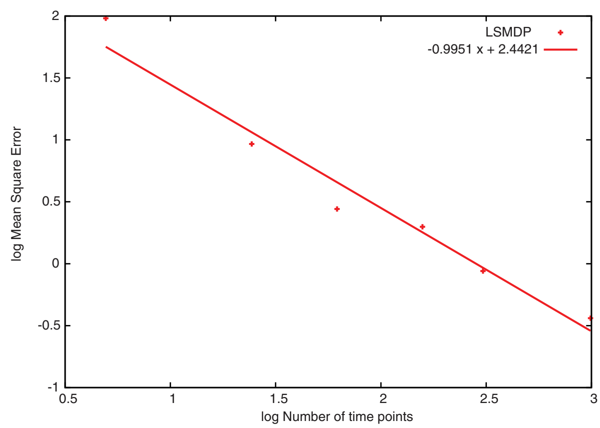


FIGURE 3. Log quadratic error versus $\log(N)$ in dimension $q = 7$.

## APPENDIX A. PROOF OF PROPOSITION 4.12

We prove the statements (i)-(ii) simultaneously for $\mu = \nu$ and $\mu = \nu_M$. Since $\mathcal{K}$ is finite-dimensional, there is an orthonormal (with respect to the norm $\|\cdot\|_{\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\mu)}$) basis $\{\tilde{p}_1, \ldots, \tilde{p}_{\tilde{K}}\}$ with $\tilde{K} \leq K$: if $\mu = \nu$, each $\tilde{p}_k$ is $\mathcal{B}(\mathbb{R}^l)$-measurable (deterministic) and if $\mu = \nu_M$, $\tilde{p}_k$ is $\tilde{\mathcal{F}} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable. Define by $\tilde{p}(\cdot)$ the $l' \times \tilde{K}$-matrix of functions $\big(\tilde{p}_1(\cdot), \ldots, \tilde{p}_{\tilde{K}}(\cdot)\big)$. Then, setting

$$\alpha^\star = \int_{\mathbb{R}^l} \tilde{p}^\top S \, d\mu \tag{A.1}$$

defined $\tilde{\mathbb{P}}$-a.s. as a $\tilde{K}$-dimensional (column) vector, we readily check the Pythagoras relation

$$\|\tilde{p}\,\alpha - S\|^2_{\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\mu)} = \|\tilde{p}\,\alpha^\star - S\|^2_{\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\mu)} + \|\tilde{p}\,(\alpha^\star - \alpha)\|^2_{\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\mu)}, \quad \tilde{\mathbb{P}}\text{-a.s.} \tag{A.2}$$

Then, clearly $S^\star(\omega,.) := \tilde{p}(\omega,.)\alpha^\star(\omega)$ solves $\mathbf{OLS}(S(\omega,\cdot),\mathcal{K},\mu)$, and is $\tilde{\mathbb{P}}\otimes\mu$-a.e. unique. In view of (A.1), the mapping $S \mapsto S^\star$ is linear. The stability property follows from (A.2) with $\alpha = 0$.

To prove the conditional expectation property (iii) under $\mu = \nu_M$, observe that the $\mathcal{Q}$-measurability of each $p_j(\mathcal{X}^{(m)})$ implies that of each $\tilde{p}_k(\mathcal{X}^{(m)})$. Hence, by taking the conditional expectation $\tilde{\mathbb{E}}[\cdot|\mathcal{Q}]$ in (A.1), it follows that

$$\tilde{\mathbb{E}}[\alpha^\star|\mathcal{Q}] = \tilde{\mathbb{E}}[\int_{\mathbb{R}^l}\tilde{p}^\top S\ d\nu_M|\mathcal{Q}] = \frac{1}{M}\sum_{m=1}^M \tilde{p}^\top(\mathcal{X}^{(m)})\ \tilde{\mathbb{E}}[S(\mathcal{X}^{(m)})|\mathcal{Q}] = \int_{\mathbb{R}^l}\tilde{p}^\top S_{\mathcal{Q}}\ d\nu_M.$$

The proof is then completed by combining the above and using $\tilde{\mathbb{E}}[S^\star(\cdot)|\mathcal{Q}] = \tilde{p}(.)\tilde{\mathbb{E}}[\alpha^\star|\mathcal{Q}]$.

To prove the bound on the conditional variance (iv), apply previous results with $\mathcal{Q} = \mathcal{G}\vee\mathcal{H}$ and write $S^\star(\mathcal{X}^{(m)}) - \tilde{\mathbb{E}}[S^\star(\mathcal{X}^{(m)})|\mathcal{G}\vee\mathcal{H}] = \tilde{p}(\mathcal{X}^{(m)})(\alpha^\star - \tilde{\mathbb{E}}[\alpha^\star|\mathcal{G}\vee\mathcal{H}])$. Using the orthogonality property of $\tilde{p}$ and the expression of $\alpha^\star$ as a summation over the sample yields

$$\begin{aligned}\|S^\star(\cdot) - \tilde{\mathbb{E}}[S^\star(\cdot)|\mathcal{G}\vee\mathcal{H}]\|^2_{\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\nu_M)} &= |\alpha^\star - \tilde{\mathbb{E}}[\alpha^\star|\mathcal{G}\vee\mathcal{H}]|^2\\ = \frac{1}{M^2}\sum_{m_1,m_2=1}^M \mathrm{Tr}\Big(\tilde{p}(\mathcal{X}^{(m_1)})\tilde{p}^\top(\mathcal{X}^{(m_2)})&(S(\mathcal{X}^{(m_2)})\\ - \tilde{\mathbb{E}}[S(\mathcal{X}^{(m_2)})|\mathcal{G}\vee\mathcal{H}])&(S(\mathcal{X}^{(m_1)}) - \tilde{\mathbb{E}}[S(\mathcal{X}^{(m_1)})|\mathcal{G}\vee\mathcal{H}])^\top\Big).\end{aligned}$$

The random variables $\{\mathcal{X}^{(1)},\dots,\mathcal{X}^{(M)}\}$ are independent, which implies that $\{S(\omega,\mathcal{X}^{(m)})\ : m = 1,\dots,M\}$ are independent conditionally on $\mathcal{G}\vee\mathcal{H}$. Thus, the conditional expectation $\tilde{\mathbb{E}}(.|\mathcal{G}\vee\mathcal{H})$ of the $(m_1,m_2)$-term above is 0 for $m_1 \neq m_2$, and so, by introducing the conditional covariance matrix $\Sigma^{(m)} := \tilde{\mathbb{E}}\big[(S(\mathcal{X}^{(m)}) - \tilde{\mathbb{E}}(S(\mathcal{X}^{(m)})|\mathcal{G}\vee\mathcal{H}))(S(\mathcal{X}^{(m)}) - \tilde{\mathbb{E}}(S(\mathcal{X}^{(m)})|\mathcal{G}\vee\mathcal{H}))^\top|\mathcal{G}\vee\mathcal{H}\big]$, it follows that

$$\begin{aligned}\tilde{\mathbb{E}}\Big[\|S^\star(\cdot) - \tilde{\mathbb{E}}[S^\star(\cdot)|\mathcal{G}\vee\mathcal{H}]\|^2_{\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\nu_M)}\ \big|\ \mathcal{G}\vee\mathcal{H}\Big] &= \frac{1}{M^2}\sum_{m=1}^M \mathrm{Tr}\Big([\tilde{p}\tilde{p}^\top](\mathcal{X}^{(m)})\Sigma^{(m)}\Big)\\ &\le \frac{1}{M^2}\sum_{m=1}^M \mathrm{Tr}\big([\tilde{p}\tilde{p}^\top](\mathcal{X}^{(m)})\big)\mathrm{Tr}(\Sigma^{(m)}),\end{aligned}$$

where we have used that $\mathrm{Tr}(AB) \le \mathrm{Tr}(A)\mathrm{Tr}(B)$ for any symmetric definite nonnegative matrices $A$ and $B$. The proof is completed by observing that $\mathrm{Tr}(\Sigma^{(m)}) \le \sigma^2$, $\frac{1}{M}\sum_{m=1}^M[\tilde{p}^\top\tilde{p}](\mathcal{X}^{(m)}) = \mathrm{Id}_{\mathbb{R}^{\tilde{K}}}$ and $\tilde{K} \le K$. □

## Appendix B. Upper bound of a deviation probability, uniform over a class of functions

We recall, for the benefit of the reader, the definition of a covering number given in [24, Definition 9.3(c)]. For a more detailed account on covering, see Chapter 9 in the above reference.

**Definition B.1.** If $\mathcal{G}$ is a class of functions from $\mathbb{R}^d$ to $\mathbb{R}$ and $x^{1:M} := \{x^{(1)},\dots, x^{(M)}\}$ are $M$ (possibly random) points of $\mathbb{R}^d$, an *$\varepsilon$-cover* ($\varepsilon > 0$) of $\mathcal{G}$ w.r.t. the $\mathbf{L}_p(p \ge 1)$-empirical norm $\|g\|_M := \big(\frac{1}{M}\sum_{m=1}^M |g(x^{(m)})|^p\big)^{\frac{1}{p}}$ is a finite collection of functions $g_1,\dots,g_n$ such that for any $g \in \mathcal{G}$, we can find a $j \in \{1,\dots,n\}$ such that

$\|g - g_j\|_M \le \varepsilon$. The smallest integer $n$ for which an $\varepsilon$-cover exists is called the *$\varepsilon$-covering number* and denoted by $\mathcal{N}_p(\varepsilon, \mathcal{G}, x^{1:M})$.

**Lemma B.2.** *Let $\mathcal{G}$ be a countable set of functions $g : \mathbb{R}^d \mapsto [0, B]$ with $B > 0$. Let $\mathcal{X}, \mathcal{X}^{(1)}, \dots, \mathcal{X}^{(M)}$ ($M \ge 1$) be i.i.d. $\mathbb{R}^d$-valued random variables. For any $\alpha > 0$ and $\varepsilon \in (0,1)$ one has*

$$\mathbb{P}\big(\sup_{g\in\mathcal{G}} \frac{\frac{1}{M}\sum_{m=1}^M g(\mathcal{X}^{(m)}) - \mathbb{E}[g(\mathcal{X})]}{\alpha + \frac{1}{M}\sum_{m=1}^M g(\mathcal{X}^{(m)}) + \mathbb{E}[g(\mathcal{X})]} > \varepsilon\big) \le 4\mathbb{E}\big(\mathcal{N}_1\big(\frac{\alpha\varepsilon}{5}, \mathcal{G}, \mathcal{X}^{1:M}\big)\big) \exp\big(-\frac{3\varepsilon^2\alpha M}{40B}\big),$$

$$\mathbb{P}\big(\sup_{g\in\mathcal{G}} \frac{\mathbb{E}[g(\mathcal{X})] - \frac{1}{M}\sum_{m=1}^M g(\mathcal{X}^{(m)})}{\alpha + \frac{1}{M}\sum_{m=1}^M g(\mathcal{X}^{(m)}) + \mathbb{E}[g(\mathcal{X})]} > \varepsilon\big) \le 4\mathbb{E}\big(\mathcal{N}_1\big(\frac{\alpha\varepsilon}{8}, \mathcal{G}, \mathcal{X}^{1:M}\big)\big) \exp\big(-\frac{6\varepsilon^2\alpha M}{169B}\big).$$

The first inequality is stated in [24, Theorem 11.6] for $B \ge 1$. The case $B \in (0,1)$ is obtained by a direct rescaling of the functions. The second inequality is proved in [21].

*Proof of Proposition* 4.9. Let

$$\mathcal{Z} := \sup_{g\in\mathcal{G}} \Big(\int_{\mathbb{R}^d} g(x)\mathbb{P}\circ\mathcal{X}^{-1}(dx) - \frac{2}{M}\sum_{m=1}^M g(\mathcal{X}^{(m)})\Big)_+.$$

We prove the result by taking advantage of the identity $\mathbb{E}[\mathcal{Z}] = \int_0^\infty \mathbb{P}(\mathcal{Z} > \varepsilon)d\varepsilon$. Using the equality

$$\mathbb{P}(\mathcal{Z} > \varepsilon) = \mathbb{P}\Big(\exists g \in \mathcal{G} : \frac{\int_{\mathbb{R}^d} g(x)\mathbb{P}\circ\mathcal{X}^{-1}(dx) - \frac{1}{M}\sum_{m=1}^M g(\mathcal{X}^{(m)})}{2\varepsilon + \int_{\mathbb{R}^d} g(x)\mathbb{P}\circ\mathcal{X}^{-1}(dx) + \frac{1}{M}\sum_{m=1}^M g(\mathcal{X}^{(m)})} > \frac{1}{3}\Big),$$

and that the elements of $\mathcal{G}$ take values in $[0, 2BL_\psi]$, it follows from Lemma B.2 that

$$\mathbb{P}(\mathcal{Z} > \varepsilon) \le 4\mathbb{E}\big[\mathcal{N}_1\big(\frac{\varepsilon}{12}, \mathcal{G}, \mathcal{X}^{1:M}\big)\big] \exp\Big(-\frac{2\varepsilon M}{507BL_\psi}\Big) \tag{B.1}$$

where $\mathcal{N}_1(\frac{\varepsilon}{12}, \mathcal{G}, \mathcal{X}^{1:M})$ is the $\frac{\varepsilon}{12}$-covering number of $\mathcal{G}$ with respect to the $\mathbf{L}_1$-empirical norm of $\{\mathcal{X}^{(1)}, \dots, \mathcal{X}^{(M)}\}$, as given in Definition B.1. Define by $\mathcal{T}_B\mathcal{K}$ the set of functions $\{\mathcal{T}_B\phi(\cdot) \ : \ \phi \in \mathcal{K}\}$. Since $|\psi\big(\phi_1(x) - \eta(x)\big) - \psi\big(\phi_2(x) - \eta(x)\big)| \le L_\psi|\phi_1(x) - \phi_2(x)|$ for all $x \in \mathbb{R}^d$ and all $(\phi_1, \phi_2)$, it follows that

$$\mathcal{N}_1\big(\frac{\varepsilon}{12}, \mathcal{G}, \mathcal{X}^{1:M}\big) \le \mathcal{N}_1\big(\frac{\varepsilon}{12L_\psi}, \mathcal{T}_B\mathcal{K}, \mathcal{X}^{1:M}\big).$$

Next, [24, Lemma 9.2, Theorem 9.4 and Theorem 9.5] gives that

$$\mathcal{N}_1\big(\frac{\varepsilon}{12L_\psi}, \mathcal{T}_B\mathcal{K}, \{\mathcal{X}_1, \dots, \mathcal{X}_M\}\big) \le 3\Big(\frac{48eBL_\psi}{\varepsilon}\log\big(\frac{72eBL_\psi}{\varepsilon}\big)\Big)^{K+1}$$

whenever $\varepsilon < 6BL_\psi$. Then the relation

$$\forall x \ge 12, \quad 2e\,x\log(3e\,x) \le 2e\,x[\log(36e) + \frac{3e\,x - 36e}{36e}] \le e\frac{1+\log(36)}{6}x^2 \le (\frac{3}{2}x)^2$$

combined with previous inequalities implies that

$$\mathbb{P}(\mathcal{Z} > \varepsilon) \leq 12\Big(\frac{36BL_\psi}{\varepsilon}\Big)^{2(K+1)} \exp\Big(-\frac{2\varepsilon M}{507BL_\psi}\Big) \qquad \text{whenever } \varepsilon \leq 2BL_\Psi. \tag{B.2}$$

On the other hand, $\mathbb{P}(\mathcal{Z} > \varepsilon) = 0$ for all $\varepsilon > 2BL_\Psi$. Setting $a = 36BL_\psi$, $b = 2/(507BL_\psi)$, it follows from (B.2) that

$$\mathbb{P}(\mathcal{Z} > \varepsilon) \leq 12(\frac{a}{\varepsilon})^{2(K+1)} \exp(-bM\varepsilon), \qquad \forall\, \varepsilon > 0.$$

Fix $\varepsilon_0$ to be some finite value (to be determined later) such that

$$\varepsilon_0 \geq \frac{a}{M(1+ab)}. \tag{B.3}$$

Using the fact that $\mathbb{E}[\mathcal{Z}]$ is equal to $\int_0^\infty \mathbb{P}(\mathcal{Z} > \varepsilon)d\varepsilon$, it follows that

$$\begin{aligned}
\mathbb{E}[\mathcal{Z}] &\leq \varepsilon_0 + \int_{\varepsilon_0}^\infty 12\Big(\frac{a}{\varepsilon}\Big)^{2(K+1)} \exp(-bM\varepsilon)d\varepsilon \\
&\leq \varepsilon_0 + \frac{12}{bM}\big(M(1+ab)\big)^{2(K+1)} \exp(-bM\varepsilon_0).
\end{aligned}$$

Now, taking $\varepsilon_0 = \frac{1}{bM}\log\Big(12\big((1+ab)M\big)^{2(K+1)}\Big)$ satisfies (B.3), because

$$\frac{1}{bM}\log\Big(12\big((1+ab)M\big)^{2(K+1)}\Big) \geq \frac{a}{M}\frac{\log(1+ab)}{ab} \geq \frac{a}{M}\frac{1}{1+ab}$$

using $\log(1+x) \geq x/(1+x)$ for all positive $x$. Moreover, this choice of $\varepsilon_0$ implies that

$$\begin{aligned}
\mathbb{E}[\mathcal{Z}] &\leq \frac{1}{bM}\Big(1 + \log(12) + 2(K+1)\log\big((1+ab)M\big)\Big) \\
&\leq \frac{2(K+1)}{bM}\log\Big((1+ab)\exp\big[\frac{1}{4}(1+\log(12))\big]M\Big)
\end{aligned}$$

which simplifies to the required result. □

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Centre de Mathématiques Appliquées, Ecole Polytechnique and CNRS, Route de Saclay, F 91128 Palaiseau cedex, France

*E-mail address*: `emmanuel.gobet@polytechnique.edu`

Centre de Mathématiques Appliquées, Ecole Polytechnique and CNRS, Route de Saclay, F 91128 Palaiseau cedex, France

*E-mail address*: `turkedjiev@cmap.polytechnique.fr`