

A NOVEL GALERKIN METHOD FOR SOLVING PDES ON THE SPHERE USING HIGHLY LOCALIZED KERNEL BASES

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ABSTRACT. The main goal of this paper is to introduce a novel meshless kernel Galerkin method for numerically solving partial differential equations on the sphere. Specifically, we will use this method to treat the partial differential equation for stationary heat conduction on \mathbb{S}^2 , in an inhomogeneous, anisotropic medium. The Galerkin method used to do this employs spatially well-localized, “small footprint”, robust bases for the associated kernel space. The stiffness matrices arising in the problem have entries decaying exponentially fast away from the diagonal. Discretization is achieved by first zeroing out small entries, resulting in a sparse matrix, and then replacing the remaining entries by ones computed via a very efficient kernel quadrature formula for the sphere. Error estimates for the approximate Galerkin solution are also obtained.

1. INTRODUCTION

The main goal of this paper is to introduce and analyze a novel meshless kernel Galerkin method for numerically solving partial differential equations on the sphere. Specifically, we will use this method to treat the partial differential equation for stationary heat conduction on \mathbb{S}^2 , the unit sphere in \mathbb{R}^3 , in an inhomogeneous, anisotropic medium. The equation for this heat-flow is

$$(1.1) \quad Lu = -\operatorname{div}(\mathbf{a}\cdot\nabla u) + bu = f,$$

where div and ∇ are the divergence and gradient on \mathbb{S}^2 , \mathbf{a} is a rank 2 positive definite tensor on \mathbb{S}^2 , f is in the Sobolev space H_s , $s \geq 0$, and $b : \mathbb{S}^2 \rightarrow \mathbb{R}$ is C^∞ and strictly positive. The analysis includes error estimates when the exact stiffness matrix is used, and also when various quadrature-based discretizations of that matrix are employed.

The kernels that we employ here are surface splines,

$$\phi_m(x \cdot y) = (-1)^m (1 - x \cdot y)^{m-1} \log(1 - x \cdot y), \quad m \geq 2.$$

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These are conditionally positive definite spherical basis functions (SBF). Their reproducing kernel Hilbert spaces (native spaces) are equivalent to the Sobolev spaces $H_m \approx W_2^m(\mathbb{S}^2)$. The associated approximation spaces involve spans of $\{\phi_m((\cdot) \cdot \xi)\}_{\xi \in X}$, with X being a discrete, finite set of quasi-uniformly distributed centers or nodes, along with spherical harmonics of order m . We denote them by $V_{\phi_m, X}$.

It is well known that, under mild conditions on an SBF ϕ , the spaces $V_{\phi, X}$ have excellent approximation power [23]. This makes them an obvious choice for use in meshless methods for solving PDEs. Kernel Galerkin methods using radial basis functions (RBFs) were employed in [38] to theoretically treat elliptic partial differential equations on \mathbb{R}^n . On \mathbb{S}^2 , SBF Galerkin methods for $-\Delta u = f$ have been studied in [18, 19]. More recently, Le Gia et al. [20, 21] used collocation and multi-level SBF methods for the purpose.

There are some drawbacks to these methods. Although convergence to the solution was established in [20, 21] under fairly mild assumptions, the rates obtained were not optimal. At each level, the rescaling required to ensure stability precludes optimality. For Galerkin methods on \mathbb{S}^2 , there is an additional problem that arises in connection with discretization. Entries in the stiffness matrices have to be numerically computed via quadrature. For instance, the method used in [18] requires solving an optimization problem to find the weights involved. This is a computationally expensive process. Applying it to large numbers of nodes is problematic.

The new meshless Galerkin method that we present here overcomes these difficulties. There are two novel features of our Galerkin approach to numerically approximating solutions to (1.1). First, for the surface splines, Fuselier et al. [8] recently showed in the Lagrange basis for $V_{\phi_m, X}$ each Lagrange function $\chi_\xi(x)$ is highly localized spatially; indeed, $\chi_\xi(x)$ decays exponentially fast as x moves away from ξ . Moreover, when χ_ξ is expressed in the kernel basis for $V_{\phi_m, X}$, the coefficient $\alpha_{\xi, \eta}$ of $\phi_m(x \cdot \eta)$ also decays exponentially fast as η moves away from ξ . Thus χ_ξ has a “small footprint” in the kernel basis. These features make the Lagrange basis robust. It is for this basis that we will build our initial theory.

L_2 -error estimates for the Galerkin solutions constructed using the thin-plate splines are derived in section 6.1. Indeed, we derive the error estimates for Galerkin solutions constructed using an arbitrary SBF ϕ , restricted only by the condition that coefficients in its expansion in spherical harmonics satisfy (6.1). (These conditions hold for ϕ_m , with $\tau = m$.) If $f \in H_s(\mathbb{S}^2)$, the L^2 -error bounds derived in section 6.1 are $C\|f\|_s h_X^{s+2}$, for $0 \leq s \leq 2m - 2$, and $C\|f\|_{2m-2} h_X^{2m}$, for $s > 2m - 2$. These error estimates differ from those found in works cited above in two ways: they hold for operators of the form L in (1.1), not just Δ , and they apply even when s is fractional.

Constructing the χ_ξ 's requires all of the points in X . In this sense, the χ_ξ 's form a *global* Lagrange basis; finding them is computationally expensive. In [8] a *local* Lagrange basis was also introduced. Each basis element χ_ξ^{loc} is constructed using only $\mathcal{O}((\log N)^2)$ in the neighborhood of ξ , and it approximates χ_ξ well. These elements have very fast spatial decay, although not exponential. They have the advantage that computing them is fast and parallelizable.

The computational advantages of these local bases make them a good choice for the implementation of our meshless method. We use our initial theory for the global Lagrange basis to further develop this method when a local Lagrange basis is used.

The stiffness matrix A in the global Lagrange basis is the key to the whole method. Our technique relies on having high quality approximations to this matrix. A itself has a number of very attractive properties in the global basis: (1) The entries $A_{\xi,\eta}$ decay exponentially fast in the distance between ξ and η . Thus A is essentially sparse and, as we shall see, zeroing out small entries will provide a sparse approximation. (2) If $N_X = \text{card}(X)$, then the number of *non-negligible* entries in each row is $\mathcal{O}((\log N_X)^2)$. (3) The *condition number* of A is $\mathcal{O}(q_X^{-2})$, where q_X is half the separation distance for X .

Discretization of the stiffness matrix is essential to the Galerkin method. After zeroing out small entries, the remaining matrix is sparse, having $\mathcal{O}(N_X(\log N_X)^2)$ entries. These entries are integrals that have to be computed via quadrature. To do this, we use kernel-based quadrature formulas for the sphere [9, 15, 32], with our kernels being surface splines. The special bases available to us for the approximation spaces corresponding to the ϕ_m 's enable us to efficiently construct the quadrature formulas [9]. Unlike the quadrature formula used in [18], the weights are obtained by solving a *linear* system of equations. Finding the weights follows readily by applying the techniques of [8]. These quadrature formulas are accurate, optimally so in many cases, even in the presence of noise, and they are stable when the number of nodes, which we denote by Y , increases. Indeed, tests run with this method handled over two-thirds of a million nodes [9].

When the problem is discretized, we get the continuous error plus a quadrature error. The quadrature error comes from replacing the stiffness matrix A by A^Y , whose entries are computed from those of A via Theorem 7.5, gives a theoretical bound on $\|A - A^Y\|_2$. If we ignore logarithms and similar terms, this is $\|A - A^Y\|_2 \sim (N_X/N_Y)^M$, where ϕ_M , $2 \leq M \leq m$, is used to obtain weights in the quadrature formula. The L^2 -error estimate for the discretized Galerkin solution is given in Theorem 7.9.

There is related work for \mathbb{R}^n . As mentioned earlier, Wendland [38] explored RBF Galerkin for domains in \mathbb{R}^n . However, problems with discretizing the stiffness matrix entries were limiting factors in implementing the method. For \mathbb{S}^n , the quadrature methods discussed above avoid these problems. In [2], Bond et al. successfully employed an RBF Galerkin method, using an approach based on the one used here, in numerical experiments for a peridynamic model of a non-local heat equation (See [4] for a discussion of peridynamics.)

We now discuss an outline of the paper and its organization. In section 2, we discuss background information on quasi-uniform sets of centers, geometry, and Sobolev spaces. In section 3, we discuss the relevant approximation spaces for the Galerkin method that we introduce. Section 4 describes the major tools used here: the highly localized Lagrange bases we employ throughout the work (section 4.1) and the quadrature formulas essential for discretizing the stiffness matrix (section 4.2). Section 5 discusses general properties of (1.1), including regularity of weak solutions and a useful application of the ‘‘Nitsche trick’’ [27]. Section 6.1 gives Galerkin error estimates in the case where the approximation spaces are generated by SBFs satisfying only (6.1). The next section, section 6.2, is key. It sets forth the properties of the stiffness matrix in the Lagrange basis: its quasi-bandedness, good conditioning, decay of matrix elements away from the diagonal, and general robustness. Section 7 discusses aspects of discretizing the problem. Theorem 7.9 is the main result of the section; it contains the theoretical L^2 -bound on the difference between discretized Galerkin solution and the exact solution. In section 8,

we discuss reducing the computational expense of numerically finding the Galerkin solution to the problem. There are two aspects of this. The first is obtaining a truncated approximation \tilde{A}^Y to the discretized stiffness matrix A^Y . Each row in \tilde{A}^Y has $\mathcal{O}((\log N_X)^2)$ non-zero entries, as opposed to N_X in A^Y . The second is to replace the global Lagrange basis with a local one, which is much easier to numerically find. The error estimates from making these approximations are virtually unchanged. In section 9, the results of numerical experiments that we did are presented. In terms of rates of convergence, the numerical results were actually better than the theory predicted. Finally, we wish to mention a few new results discussed in the appendix. In section A, we establish a generalized version of the so-called “doubling trick”, which applies to SBF interpolation of functions twice as smooth as those in the native space of the SBF [10, 29]. The result, which is given in Theorem A.3, holds for functions smoother than those in the native space for ϕ , but not having “double” the smoothness. In addition, it applies for SBFs that are conditionally positive definite.

2. PRELIMINARIES

2.1. Geometry of sets of centers. Although \mathbb{S}^2 is the underlying space for the Galerkin methods treated in this paper, much of what we will discuss in the next few sections applies to \mathbb{S}^n . In view of this, we will work in \mathbb{S}^n .

Let $\text{dist}(x, y)$ be the geodesic (great circle) distance between two points on \mathbb{S}^n . We will let $X = \{x_j\}_{j=1}^N \subset \mathbb{S}^n$ be a set of N distinct points; we will call X a set of centers. We remark that, apart from, say the vertices of the Platonic solids in \mathbb{S}^2 and similar quantities in \mathbb{S}^n , $n > 2$, there are no uniformly distributed sets of points in \mathbb{S}^n . We can, however, obtain *quasi-uniform* sets of points. We will explain this below.

There are three geometrical quantities associated with X . The first is the *separation radius*, q_X . For every $x \in X$, the radius of the largest ball whose interior contains no other point of X is given by $\frac{1}{2}\text{dist}(x, X \setminus \{x\})$; q_X is defined to be the smallest of these radii:

$$q_X := \min_{x \in X} \frac{1}{2} \text{dist}(x, X \setminus \{x\}).$$

Obviously $2q_X$ is the minimum distance between any two points in X . The second and third are the *mesh norm*, h_X , and the *mesh ratio*, ρ_X . The mesh norm h_X is the radius of the largest ball in \mathbb{S}^n whose interior contains no point of X . It also can be characterized as the largest distance of any point in \mathbb{S}^n from X . The mesh ratio is the ratio of h_X to q_X :

$$h_X := \max_{x \in \mathbb{S}^n} \text{dist}(x, X) \quad \text{and} \quad \rho_X := h_X/q_X.$$

The mesh norm, which is also called the *fill distance*, measures how tightly packed the centers are in \mathbb{S}^n . The mesh ratio measures how uniformly the centers are placed. When it is close to 1, the distribution of the points in X is said to be *quasi-uniform*.

For $\rho \geq 1$, let $\mathcal{F}_\rho = \mathcal{F}_\rho(\mathbb{S}^n)$ be the family of all sets of centers X with $\rho_X \leq \rho$; we will say that the family \mathcal{F}_ρ is ρ -*uniform*. Unless confusion would arise, we will not indicate either ρ or \mathbb{S}^n , and just use \mathcal{F} to designate the family $\mathcal{F}_\rho(\mathbb{S}^n)$. The specific ρ or sphere \mathbb{S}^n will be clear from the context.

On \mathbb{S}^2 , there are three important quasi-uniform sets of nodes (centers): Fibonacci nodes, icosahedral nodes, quasi-minimum energy nodes. All three of these families of nodes are quite popular in applications; see, for example, [12, 22, 28, 34] for the icosahedral nodes, [17, 31, 35] for the Fibonacci nodes, and [5, 6, 30, 40] for the quasi-minimum energy nodes. Similar considerations apply to \mathbb{S}^n , $n > 2$.

2.2. Spherical harmonics and Sobolev spaces. The sphere \mathbb{S}^n is of course a Riemannian manifold with metric tensor g_{ij} and invariant measure

$$d\mu = \sqrt{\det(g_{ij})} dx^1 \cdots dx^n,$$

where x^1, x^2, \dots, x^n is a smooth set of local coordinates. For \mathbb{S}^2 , the metric tensor in spherical coordinates (θ, φ) , with θ being the colatitude and φ being the longitude, has the form

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}.$$

The metric tensor for \mathbb{S}^n also can be expressed in a similar set of coordinates. On any Riemannian manifold there are two important operators: the covariant derivative ∇ , powers of covariant derivatives ∇^k , which is an operator on tensors, and the Laplace-Beltrami operator $\Delta = -\nabla^* \nabla$. The covariant derivative operating on a function is the usual gradient, expressed appropriately. Other powers are tensor operators. For example, ∇^2 plays the role of a Hessian. In local coordinates, Δ has the form

$$(2.1) \quad \Delta u = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j} \frac{\partial}{\partial x^i} \sqrt{\det(g_{ij})} g^{ij} \frac{\partial u}{\partial x^j},$$

where $g^{ij} = (g_{ij})^{-1}$. For \mathbb{S}^2 , in spherical coordinates, the Laplace-Beltrami operator is given by

$$\Delta u = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2}.$$

We now turn to a discussion of spherical harmonics; the details may be found in [25]. Spherical harmonics are eigenfunctions of Δ . On \mathbb{S}^n , the eigenvalues of $-\Delta$ are $\lambda_\ell = \ell(\ell + n - 1)$. The eigenspace corresponding to λ_ℓ is degenerate, and has dimension

$$(2.2) \quad d_\ell = \begin{cases} 1, & \ell = 0, \\ \frac{(2\ell + n - 1)\Gamma(\ell + n - 1)}{\Gamma(\ell + 1)\Gamma(n)} \sim \ell^{n-1}, & \ell \geq 1. \end{cases}$$

We note that $d_\ell = \mathcal{O}(\ell^{n-1})$. The eigenfunctions corresponding to λ_ℓ are denoted by $Y_{\ell,k}$, where $k = 1, \dots, d_\ell$. We will use the real-valued versions of the spherical harmonics. The eigenspace of λ_ℓ will be denoted by \mathcal{H}_ℓ . The space of all spherical harmonics of order L or less will be denoted by $\Pi_L = \bigoplus_{\ell=0}^L \mathcal{H}_\ell$. In addition, we mention the well-known addition formula. Let $x, y \in \mathbb{S}^n$ and let $x \cdot y$ denote the usual dot product from \mathbb{R}^{n+1} . Then,

$$(2.3) \quad \sum_{k=1}^{d_\ell} Y_{\ell,k}(x) Y_{\ell,k}(y) = \frac{2\ell + n - 1}{(n - 1)\omega_n} P_\ell^{\frac{n-1}{2}}(x \cdot y),$$

where ω_n is the volume of \mathbb{S}^n , and $P_\ell^{\frac{n-1}{2}}$ is the degree ℓ ultraspherical polynomial of order $\frac{n-1}{2}$.

The spherical harmonics form a complete orthonormal set in $L^2(\mathbb{S}^n)$. Given f in $L^2(\mathbb{S}^n)$, we can expand f in the series $f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{d_{\ell}} \hat{f}_{\ell,k} Y_{\ell,k}$. As usual, given $f, g \in L^2$, we have $\langle f, g \rangle_{L^2} = \sum_{\ell=0}^{\infty} \sum_{k=1}^{d_{\ell}} \hat{f}_{\ell,k} \overline{\hat{g}_{\ell,k}}$.

In this paper we will work with fractional order Sobolev spaces defined in terms of Bessel potentials [33, 37]. The Sobolev space of order $\tau \geq 0$ is

$$H_{\tau} := \{f \in L^2 : \|f\|_{H_{\tau}} := \|(I - \Delta)^{\tau/2} f\|_{L^2} < \infty\}.$$

This is a Hilbert space in the inner product

$$(2.4) \quad \langle f, g \rangle_{H_{\tau}} = \langle (I - \Delta)^{\tau/2} f, (I - \Delta)^{\tau/2} g \rangle_{L^2} = \sum_{\ell=0}^{\infty} \sum_{k=1}^{d_{\ell}} (1 + \lambda_{\ell})^{\tau} \hat{f}_{\ell,k} \overline{\hat{g}_{\ell,k}}.$$

When $\tau = m$ is an integer, these spaces agree, up to norm equivalence, with $W_2^m(\mathbb{S}^n)$, which are defined in terms of covariant derivatives [1, 37].

3. SPHERICAL BASIS FUNCTIONS AND APPROXIMATION SPACES

In the following, we will be working on \mathbb{S}^n . We start with zonal functions. A continuous function $\phi : [-1, 1] \rightarrow \mathbb{R}$ is said to be *zonal* if it has the expansion,

$$(3.1) \quad \phi(t) := \sum_{\ell=0}^{\infty} \hat{\phi}_{\ell} \frac{2\ell + n - 1}{(n - 1)\omega_n} P_{\ell}^{\frac{n-1}{2}}(t),$$

where $P_{\ell}^{\frac{n-1}{2}}$ is a degree ℓ ultra spherical polynomial [36, §4.7]. Zonal functions give rise to kernels on the sphere in the following way. Let $t = x \cdot y$, $x, y \in \mathbb{S}^n$. Using the addition formula (2.3) in (3.1), we see that

$$(3.2) \quad \phi(x \cdot y) = \sum_{\ell=0}^{\infty} \hat{\phi}_{\ell} \sum_{k=1}^{d_{\ell}} Y_{\ell,k}(x) Y_{\ell,k}(y),$$

which is a kernel mapping $\mathbb{S}^n \times \mathbb{S}^n$ to \mathbb{R} .

SBFs are zonal functions having $\hat{\phi}_{\ell} > 0$ for all $\ell \geq 0$. They are strictly positive definite functions on \mathbb{S}^n . This means that the matrix $A = (\phi(\xi \cdot \eta))_{\xi, \eta \in X}$ is positive definite for every choice of X . Equivalently,

$$\sum_{\xi, \eta \in X} c_{\xi} c_{\eta} \phi(\xi \cdot \eta) > 0,$$

except when the c 's are all 0. A being positive definite allows us to interpolate arbitrary continuous functions (or data, for that matter) on \mathbb{S}^n , using functions from the *approximation space*

$$V_{\phi, X} := \text{span}\{\phi((\cdot) \cdot \xi) : \xi \in X\}.$$

This follows because the existence of $c = A^{-1} f|_X$ implies that

$$I_X f(x) = \sum_{\xi \in X} c_{\xi} \phi(x \cdot \xi)$$

interpolates f on X , and does so uniquely. The SBF ϕ is also a reproducing kernel for the Hilbert space

$$\mathcal{N} := \{f \in L^2 : \sum_{\ell, k} \hat{\phi}_{\ell}^{-1} |\hat{f}_{\ell, k}|^2 < \infty\}.$$

This space is often called the *native space* of ϕ ; it has the inner product

$$(3.3) \quad \langle f, g \rangle_{\mathcal{N}} = \sum_{\ell=0}^{\infty} \sum_{k=1}^{d_{\ell}} \hat{\phi}_{\ell}^{-1} \hat{f}_{\ell,k} \bar{g}_{\ell,k}.$$

The SBF ϕ_{τ} for which $\hat{\phi}_{\tau,\ell} = (1 + \lambda_{\ell})^{-\tau}$, with $\tau > n/2$, is especially important. The native space for ϕ_{τ} is the Sobolev space H_{τ} , since $\|f\|_{\mathcal{N}} = \|(I - \Delta)^{\tau/2} f\|_{L^2} = \|f\|_{H_{\tau}}$. Making use of this observation yields a fractional order “zeros lemma”, similar to integer order ones proved in [14, Appendix A]. This will be important in the sequel.

Lemma 3.1 (Zeros lemma). *Let $\sigma, \tau \in \mathbb{R}$ satisfy $\tau > n/2$, $0 \leq \sigma \leq \tau$. In addition, let $X \subset \mathbb{S}^n$ be quasi-uniform. If $u \in \mathcal{H}_{\tau}$ satisfies $u|_X = 0$, then, for h_X sufficiently small, we have*

$$\|u\|_{H_{\sigma}} \leq Ch_X^{\tau-\sigma} \|u\|_{H_{\tau}}.$$

Proof. Let $I_{X, \phi_{\tau}}$ be the interpolation operator corresponding to ϕ_{τ} . By [26, Theorem 5.5], we have that

$$\|u - I_{X, \phi_{\tau}} u\|_{H_{\sigma}} \leq Ch_X^{\tau-\sigma} \|u\|_{H_{\tau}}.$$

Note that $u|_X = 0$ implies that $I_{X, \phi_{\tau}} u \equiv 0$. Using this in the previous equation then yields the result. \square

The SBFs discussed above are all strictly positive definite. We will also need to make use of *conditionally* positive definite SBFs. These SBFs have the form given in (3.1), but the $\hat{\phi}_{\ell}$ ’s need only be positive for $\ell > L$. For $0 \leq \ell \leq L$, $\hat{\phi}_{\ell}$ is arbitrary. Conditionally positive definite SBFs are employed to interpolate scattered data, with the requirement that the interpolants reproduce Π_L , the space of spherical harmonics of degree L or less. (Other spaces are also possible.) For a conditionally positive definite SBF ϕ , the corresponding approximation space is defined to be

$$V_{\phi,L,X} := \left\{ \sum_{\xi \in X} a_{\xi} \phi((\cdot) \cdot \xi) : \sum_{\xi \in X} a_{\xi} p(\xi) = 0 \ \forall p \in \Pi_L \right\} + \Pi_L.$$

The interpolation operator that both interpolates continuous functions and reproduces Π_L is

$$(3.4) \quad I_{X,L} f = \sum_{\xi \in X} a_{\xi,L} \phi(x \cdot \xi) + p_{X,L}, \quad p_{X,L} \in \Pi_L, \quad \text{and} \quad \sum_{\xi \in X} a_{\xi,L} p(\xi) = 0, \quad p \in \Pi_L.$$

The coefficients $a_{\xi,L}$ and the polynomial $p_{X,L}$ are determined by the requirements that the interpolation condition $I_{X,L} f|_X = f|_X$ hold and also that the coefficients satisfy the condition on the right above, provided X is unisolvent. Again, the interpolant is unique. There is also a semi-Hilbert space \mathcal{N} associated with ϕ . This is defined to be

$$\mathcal{N} := \{f \in L^2 : \sum_{\ell=L+1}^{\infty} \sum_{k=1}^{d_{\ell}} \hat{\phi}_{\ell}^{-1} |\hat{f}_{\ell,k}|^2 < \infty\}, \quad \langle f, g \rangle_{\mathcal{N}} = \sum_{\ell=L+1}^{\infty} \sum_{k=1}^{d_{\ell}} \hat{\phi}_{\ell}^{-1} \hat{f}_{\ell,k} \bar{g}_{\ell,k}.$$

In addition, we will need the following well-known fact, which we state without proof.

Proposition 3.2. *Suppose that $\tau > n/2$ and that ϕ is an SBF such that there are constants c, C and $L \in \mathbb{N}$ for which $c(1 + \lambda_{\ell})^{-\tau} \leq \hat{\phi}_{\ell} \leq C(1 + \lambda_{\ell})^{-\tau}$ holds either for all $\ell \geq 0$ or for all $\ell \geq L + 1$. If $\epsilon > 0$, then $\phi \in H_{2\tau - \frac{n}{2} - \epsilon}$.*

4. HIGHLY LOCALIZED “SMALL FOOTPRINT” BASES

Surface splines, or *polyharmonic kernels*, are special conditionally positive definite SBFs, and are a key ingredient in the kernel methods presented here. While they can be defined for any \mathbb{S}^n [14], we will restrict our attention to the case of \mathbb{S}^2 [8]. Their explicit forms are given below:

$$(4.1) \quad \left. \begin{aligned} \phi_m(t) &= (-1)^m (1-t)^{m-1} \log(1-t), \quad 1 < m \in \mathbb{N}, \\ \hat{\phi}_{m,\ell} &= C_m \frac{\Gamma(\ell-m+1)}{\Gamma(\ell+m+1)} \sim \ell^{-2m} \sim \lambda_\ell^{-m}, \quad \ell > m-1. \end{aligned} \right\}$$

Here $C_m = 2^{m+1} \pi \Gamma(m)^2$. These kernels are conditionally positive definite, and interpolation with them will reproduce Π_{m-1} . The space \mathcal{N} associated with ϕ_m is, up to norm equivalence, the Sobolev space $H_m(\mathbb{S}^2)$ modulo Π_{m-1} . Also, since $\hat{\phi}_{m,\ell} \sim \ell^{-2m}$, it is easy to show that $\phi_m \in H_{2m-1-\epsilon}(\mathbb{S}^2)$, $\epsilon > 0$. Furthermore, this implies that the approximation space for ϕ_m satisfies

$$V_{\phi_m, X} := V_{\phi_m, m-1, X} \subset H_{2m-1-\epsilon}(\mathbb{S}^2), \quad \forall \epsilon > 0.$$

4.1. Lagrange functions. We can form a basis for $V_{\phi_m, X}$ using *Lagrange functions* or *cardinal functions*. A Lagrange function χ_ξ is defined as the unique interpolant from $V_{\phi_m, X}$ that satisfies $\chi_\xi(\eta) = \delta_{\xi, \eta}$. Since $\chi_\xi \in V_{\phi_m, X}$, it has the form

$$(4.2) \quad \chi_\xi = \sum_{\zeta \in X} \alpha_{\xi, \zeta} \phi_m((\cdot) \cdot \zeta) + p_\xi, \quad p_\xi \in \Pi_{m-1}, \quad \sum_{\zeta \in X} \alpha_{\xi, \zeta} p(\zeta) = 0 \quad \forall p \in \Pi_{m-1}.$$

Interpolation using the χ_ξ 's is simple: If f is a continuous function on \mathbb{S}^2 , with $f|_X$ given, then $I_{\phi_m, X} f = \sum_{\xi \in X} f(\xi) \chi_\xi$.

There are two important properties of the Lagrange functions constructed from the ϕ_m 's. First, they are well localized in space. Indeed, $\chi_\xi(x)$ decays exponentially in $\text{dist}(x, \xi)$. Second, they have a small “footprint” in the set of basis elements. Again, the coefficients $\alpha_{\xi, \zeta}$ decay exponentially in $\text{dist}(\xi, \zeta)$. Each χ_ξ is effectively using only a small number of kernels from the set $\{\phi_m((\cdot) \cdot \xi) : \xi \in X\}$; i.e., χ_ξ has a small “footprint” in the set of kernels. The precise result is stated below:

Theorem 4.1 ([8, Theorem 5.3]). *Let $\rho > 0$ be a fixed mesh ratio and let $\mathbb{N} \ni m \geq 2$. There exist constants h^* , ν , c_1 , c_2 and C , depending only on m and ρ , so that if $h_X \leq h^*$, then χ_ξ given in (4.2) has these properties:*

$$(4.3) \quad |\chi_\xi(x)| \leq C \exp\left(-\nu \frac{\text{dist}(x, \xi)}{h_X}\right),$$

$$(4.4) \quad |\alpha_{\zeta, \xi}| \leq C q_X^{2-2m} \exp\left(-\nu \frac{\text{dist}(\xi, \zeta)}{h_X}\right),$$

$$(4.5) \quad c_1 q_X^{2/p} \|\boldsymbol{\beta}\|_{\ell_p(X)} \leq \left\| \sum_{\xi \in \Xi} \beta_\xi \chi_\xi \right\|_{L^p(\mathbb{S}^2)} \leq c_2 q_X^{2/p} \|\boldsymbol{\beta}\|_{\ell_p(X)}.$$

In addition to the various bounds above, we will also need a bound on $\nabla \chi_\xi$, the covariant derivative of χ_ξ . The lemma below will be needed to obtain this bound, as well as several others in the sequel.

Lemma 4.2. *Let $x \in \mathbb{S}^2$ be fixed. Then, there is a constant C that is independent of ν and X for which we have*

$$(4.6) \quad \sum_{\xi \in X} e^{-\frac{\nu}{h_X} \text{dist}(x, \xi)} < \frac{C\rho_X^2}{(1 - e^{-\nu})^2}.$$

In addition, if $B(x, r_0)$ is the ball of radius r_0 and center x , then

$$(4.7) \quad \sum_{\xi \in X \cap B(x, r_0)} e^{-\frac{\nu}{h_X} \text{dist}(x, \xi)} < C\rho_X^2 \frac{n_0 e^{-(n_0-1)\nu}}{(1 - e^{-\nu})^2}, \quad n_0 = \lceil r_0/h_X \rceil.$$

Proof. Divide the sphere into bands of width $\sim h_X$, center x , and outer radius $\sim nh_X$, $n \geq 1$. The sum then satisfies the inequality

$$\begin{aligned} \sum_{\xi \in X} e^{-\frac{\nu}{h_X} \text{dist}(x, \xi)} &= \sum_{n=1}^{n_{\max}} \sum_{x \in \text{band}_n \cap X} e^{-\frac{\nu}{h_X} d(x, \xi)} \\ &< \sum_{n=1}^{n_{\max}} \#(\text{band}_n \cap X) e^{-(n-1)\nu}, \end{aligned}$$

where $n_{\max} \sim \pi/h_X$. The area of band_n is $\sim nh_X^2$. Consequently, we have that cardinality $\#(\text{band}_n \cap X)$ is $\sim nh_X^2/q_X^2 = n\rho_X^2$. Using this in the equation above yields

$$\sum_{\xi \in X} e^{-\frac{\nu}{h_X} d(x, \xi)} < C\rho_X^2 \sum_{n=1}^{\infty} n e^{-(n-1)\nu}.$$

Summing the series on the right above yields (4.6). To obtain (4.7), we sum the series $\sum_{n=n_0}^{\infty} n e^{-(n-1)\nu}$ and use the fact that $n_0 - (n_0 - 1)e^{-\nu} < n_0$. \square

Theorem 4.3. *Adopt the notation of Theorem 4.1. There exists a constant $C = C(\rho, m)$ such that*

$$(4.8) \quad |\nabla \chi_{\xi}(x)| \leq Cq_X^{-1} e^{-\frac{\nu}{h_X} d(x, \xi)}.$$

In addition, $\|\nabla \chi_{\xi}\|_{L^\infty} \leq Cq_X^{-1}$. Finally,

$$(4.9) \quad \Lambda_1 := \max_{x \in \mathbb{S}^2} \sum_{\xi \in X} |\nabla \chi_{\xi}(x)| < C\rho_X^2 q_X^{-1} \frac{1}{(1 - e^{-\nu})^2}.$$

Proof. The Hölder estimate given in [14, Theorem 5.3], with $\epsilon = 1$, is

$$|\chi_{\xi}(x) - \chi_{\xi}(y)| \leq C \frac{d(x, y)}{q_X} e^{-\frac{\nu}{h_X} d(x, \xi)}.$$

Fixing x and dividing by $d(x, y)$ yields

$$\left| \frac{\chi_{\xi}(x) - \chi_{\xi}(y)}{d(x, y)} \right| \leq Cq_X^{-1} e^{-\frac{\nu}{h_X} d(x, \xi)}.$$

Let \hat{t} be a unit tangent vector based at x . Choose y to be a point along the geodesic starting at x with tangent \hat{t} . Then,

$$\lim_{d(x, y) \rightarrow 0} \left| \frac{\chi_{\xi}(x) - \chi_{\xi}(y)}{d(x, y)} \right| = |D_{\hat{t}}(\chi_{\xi})(x)| \leq Cq_X^{-1} e^{-\frac{\nu}{h_X} d(x, \xi)}.$$

This holds for every direction \hat{t} . Since $\max_{\hat{t}} |D_{\hat{t}}(\chi_{\xi})(x)| = |\nabla \chi_{\xi}(x)|$, the bound (4.8) follows immediately. Obviously, we also have $\|\nabla \chi_{\xi}\|_{L^\infty} \leq Cq_X^{-1}$. \square

Proposition 4.4. *Adopt the notation and assumptions of Theorem 4.1 and suppose that $a, b \in C^\infty$, $m \geq 2$ and $0 < \epsilon < 2m - 3$. Then, $b\chi_\xi\chi_\eta \in H_{2m-1-\epsilon} \cap L^\infty$. Moreover, for h_X sufficiently small, there exists $C = C(\rho, m)$ such that*

$$(4.10) \quad \|b\chi_\xi\chi_\eta\|_{H_{2m-1-\epsilon}} \leq Ch_X^{2+\epsilon-2m} \|b\|_{H_{2m}},$$

$$(4.11) \quad \|a\nabla\chi_\xi \cdot \nabla\chi_\eta\|_{H_{2m-\epsilon-2}} \leq Ch_X^{1+\epsilon-2m} \|a\|_{H_{2m}}.$$

Proof. In the proof below we will need the inequalities $2m-\epsilon-1 > 2$ and $2m-\epsilon-2 > 1$, which follows easily from $0 < \epsilon < 2m - 3$.

Theorem B.1 applies to $b\chi_\xi$, because $b \in C^\infty \subset H_{2m-1-\epsilon} \cap L^\infty$ and, by Theorem 4.1, $\chi_\xi \in H_{2m-1-\epsilon} \cap L^\infty$. Consequently, the three products $b\chi_\xi \chi_\eta$, $b\chi_\xi$ and $\chi_\xi\chi_\eta$ are in $H_{2m-1-\epsilon} \cap L^\infty$. A straightforward application of Theorem B.1, equation (B.1), to the various products then results in this bound:

$$\begin{aligned} \|b\chi_\xi\chi_\eta\|_{H_{2m-1-\epsilon}} &\leq C' \|b\|_{L^\infty} (\|\chi_\xi\|_{L^\infty} \|\chi_\eta\|_{H_{2m-1-\epsilon}} + \|\chi_\xi\|_{H_{2m-1-\epsilon}} \|\chi_\eta\|_{L^\infty}) \\ &\quad + C \|b\|_{H_{2m-1-\epsilon}} \|\chi_\xi\|_{L^\infty} \|\chi_\eta\|_{L^\infty}. \end{aligned}$$

By (4.5), with $p = \infty$, we have that both $\|\chi_\xi\|_{L^\infty}$ and $\|\chi_\eta\|_{L^\infty}$ are bounded by the constant c_2 , because the corresponding β 's have a single entry, 1. Moreover, since $2m - 1 - \epsilon > 1$, the Sobolev embedding theorem and a standard inclusion inequality imply that $\|b\|_{L^\infty} \leq C \|b\|_{H_{2m-1-\epsilon}} \leq C \|b\|_{H_{2m}}$. Inserting these in the previous inequality then yields

$$(4.12) \quad \|b\chi_\xi\chi_\eta\|_{H_{2m-1-\epsilon}} \leq Cc_2 \|b\|_{H_{2m}} (\|\chi_\xi\|_{H_{2m-1-\epsilon}} + \|\chi_\eta\|_{H_{2m-1-\epsilon}} + c_2).$$

We will now employ a Bernstein inequality¹ [23, Theorem 6.1] that holds for functions in $V_{\phi_m, X}$. The parameters in the theorem are $\beta = 2m$, from (4.1), $p = 2$, $n = 2$, $\gamma = 2m-1-\epsilon > 2$ and $g = \chi_\xi$. The theorem then implies that $\|\chi_\xi\|_{H_{2m-1-\epsilon}} \leq Cq_X^{1+\epsilon-2m} \|\chi_\xi\|_{L^2}$. Moreover, if we set $p = 2$ in (4.5), we also have both $\|\chi_\xi\|_{L^2}$ and $\|\chi_\eta\|_{L^2}$ bounded by c_2q_X . Thus, $\|\chi_\xi\|_{H_{2m-1-\epsilon}} \leq Cq_X^{2+\epsilon-2m}$. Combining the various bounds above we arrive at $\|b\chi_\xi\chi_\eta\|_{H_{2m-1-\epsilon}} \leq Cc_2 \|b\|_{H_{2m}} q_X^{2+\epsilon-2m} (2 + c_2q_X^{2m-2-\epsilon})$. Since $q_X \ll 1$ and $2m - 2 - \epsilon > 1$, we have $\|b\chi_\xi\chi_\eta\|_{H_{2m-1-\epsilon}} \leq Cc_2 \|b\|_{H_{2m}} q_X^{2+\epsilon-2m}$. From this, (4.10) follows on observing that $q_X \sim h_X$.

To obtain the second bound, note that, by Corollary B.3, the conditions on χ_ξ, χ_η imply that $\nabla\chi_\xi \cdot \nabla\chi_\eta \in H_{2m-\epsilon-2} \cap L^\infty$, since $2m - \epsilon - 2 > 1$, and that those on a are the ones used for b . Consequently,

$$\|a\nabla\chi_\xi \cdot \nabla\chi_\eta\|_{H_{2m-\epsilon-2}} \leq C \|a\|_{H_{2m}} (\|\nabla\chi_\xi \cdot \nabla\chi_\eta\|_{H_{2m-\epsilon-2}} + \|\nabla\chi_\xi \cdot \nabla\chi_\eta\|_{L^\infty}).$$

Since $2m - 2 - \epsilon > 1$, we may again apply the Sobolev embedding theorem to obtain $\|\nabla\chi_\xi \cdot \nabla\chi_\eta\|_{L^\infty} \leq C \|\nabla\chi_\xi \cdot \nabla\chi_\eta\|_{H_{2m-\epsilon-2}}$. Combining this with the previous inequality results in

$$(4.13) \quad \|a\nabla\chi_\xi \cdot \nabla\chi_\eta\|_{H_{2m-\epsilon-2}} \leq C \|a\|_{H_{2m}} \|\nabla\chi_\xi \cdot \nabla\chi_\eta\|_{H_{2m-\epsilon-2}}.$$

To estimate the norm on the right we will use Corollary B.3. This implies that

$$\|\nabla\chi_\xi \cdot \nabla\chi_\eta\|_{H_{2m-\epsilon-2}} \leq C (\|\chi_\xi\|_{H_{2m-1-\epsilon}} + \|\chi_\eta\|_{H_{2m-1-\epsilon}}) (\|\nabla\chi_\xi\|_{L^\infty} + \|\nabla\chi_\eta\|_{L^\infty}).$$

¹The precise version of the theorem holds for a positive definite SBF. However, it is easy to modify it so that it will hold for a conditionally positive definite SBF.

We may use Theorem 4.3 and the bounds on $\|\chi_\xi\|_{H_{2m-1-\epsilon}}$, $\|\chi_\eta\|_{H_{2m-1-\epsilon}}$ found above to obtain this:

$$(4.14) \quad \|\nabla\chi_\xi \cdot \nabla\chi_\eta\|_{H_{2m-\epsilon-2}} \leq Ch_X^{2+\epsilon-2m}q_X^{-1} \leq Ch_X^{1+\epsilon-2m}.$$

Finally, using the bound from (4.14) in (4.13) yields (4.11). □

4.2. Quadrature formulas. Numerically computing the integrals that arise in any Galerkin method ultimately requires a quadrature formula. In the setting of a sphere and other homogeneous manifolds, kernel quadrature formulas [9, 15, 32] have been developed and analyzed. Let f be continuous and consider the surface spline ϕ_m given in (4.1). In addition, let Y be a quasi-uniform set of points on \mathbb{S}^2 , which may be different from X . The quantities q_Y , h_Y , and ρ_Y , and cardinality N_Y have their usual meanings. Using ϕ_m , form the Lagrange functions $\tilde{\chi}_\zeta$, $\zeta \in Y$ corresponding to Y and the interpolant $I_Y f = \sum_{\zeta \in Y} f(\zeta)\tilde{\chi}_\zeta$. The quadrature formula is obtained integrating $I_Y f$:

$$Q_Y(f) = \int_{\mathbb{S}^2} I_Y f(x) d\mu(x) = \sum_{\zeta \in Y} f(\zeta)w_\zeta, \quad w_\zeta := \int_{\mathbb{S}^2} \tilde{\chi}_\zeta(x) d\mu(x).$$

We point out that a few of the weights w_ζ can be near zero or become slightly negative in the case of arbitrary Y . This is usually *not* the case for most quasi-uniform sets Y . (See the discussion in [9, Section 2.2.1; 32].) In fact, not only are the weights positive for most sets, but they also satisfy the lower bound

$$(4.15) \quad w_\zeta \geq Ch_Y^2.$$

In the rest of our discussion, we will assume that (4.15) holds. The only situation where this assumption comes into play will be in stability considerations of the discretized version of the stiffness matrix.

Positive or not, the weights all satisfy an *upper* bound; namely,

$$(4.16) \quad |w_\zeta| \leq Ch_Y^2.$$

Since $w_\zeta = \int_{\mathbb{S}^2} \tilde{\chi}_\zeta(x) d\mu(x)$, we have that $|w_\zeta| \leq \|\tilde{\chi}_\zeta\|_{L_1(\mathbb{S}^2)}$. To estimate the right side, use (4.5), with $p = 1$, $\beta_\zeta = 1$ and all of the other β 's equal to 0. This gives us $\|\beta\|_{\ell_1} = 1$, and so $\|\tilde{\chi}_\zeta\|_{L_1(\mathbb{S}^2)} \leq c_2 q_Y^2 \leq c_2 \rho_Y^{-2} h_Y^2$.

The salient feature of this quadrature formula is that the weights can be obtained by solving a linear system of equations that is stable and, while not sparse, has entries that decay rapidly as they move away from the diagonal. For $m = 2$, weights for a set Y having 600,000 points were easily computed [9, section 5].

Error estimates for the quadrature formula Q_Y have been derived for functions in various integer valued Sobolev spaces. However, we will need stronger results. We begin with the proposition below, which holds on \mathbb{S}^n , $n \geq 2$. Consider a (conditionally) positive definite SBF ϕ that satisfies $c(1+\lambda_\ell)^{-\tau} \leq \hat{\phi}_\ell \leq C(1+\lambda_\ell)^{-\tau}$ for all $\ell \geq L+1$. The novel feature of this result is that it uses a new version of the “doubling trick”, which is established in Theorem A.3, to obtain higher convergence rates for functions smoother than ones in the native space of ϕ . The result is this:

Proposition 4.5. *Let $\tau > n/2$, $2\tau \geq \mu > n/2$, and $f \in H_\mu$. If h_Y is sufficiently small, then*

$$(4.17) \quad \left| \int_{\mathbb{S}^n} f(x) d\mu - Q_Y(f) \right| \leq Ch_Y^\mu \|f\|_{H_\mu}.$$

Proof. Note that $|\int_{\mathbb{S}^n} f d\mu - Q_Y(f)| \leq \int_{\mathbb{S}^n} |f - I_Y f| d\mu \leq \omega_n^{1/2} \|f - I_Y f\|_{L^2}$, where ω_n is the volume of \mathbb{S}^n . In addition, by Theorem A.3, with $\beta = 0$, we have $\|f - I_Y f\|_{L^2} \leq Ch_Y^\mu \|f\|_{H_\mu}$. Combining the two inequalities yields (4.17). \square

5. WEAK AND STRONG SOLUTIONS TO $Lu = f$

In this section we will lay out the properties, assumptions and various aspects of weak and strong solutions to (1.1). In local coordinates on \mathbb{S}^2 , this equation has the form,

$$(5.1) \quad Lu = -\frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j} \frac{\partial}{\partial x^i} \sqrt{\det(g_{ij})} a^{ij}(x) \frac{\partial u}{\partial x^j} + b(x)u = f.$$

Here, g_{ij} is the covariant form of the standard metric tensor g on \mathbb{S}^2 ; as usual, $g^{ij} = [g_{ij}]^{-1}$ are the contravariant components of g . The a^{ij} 's are contravariant components of a C^∞ , symmetric rank 2 tensor a that is positive definite in the sense that there exist positive constants c_1, c_2 such that

$$(5.2) \quad c_1 \sum_{i,j} g^{ij}(x) v_i v_j \leq \sum_{i,j} a^{ij}(x) v_i v_j \leq c_2 \sum_{i,j} g^{ij}(x) v_i v_j$$

holds for all vectors v in the tangent space at $x \in \mathbb{S}^2$. The function $b(x)$ is C^∞ . In addition, we assume that there are constants b_1, b_2 such that, for all $x \in \mathbb{S}^2$, $0 < b_1 \leq b(x) \leq b_2$. We note that in the case that $a = g$, this reduces to the case $Lu = -\Delta u + bu$.

With L as given in (5.1) and $f \in L^2(\mathbb{S}^2)$, we can place $Lu = f$ into weak form by multiplying by $v \in H_1$ and integrating by parts to arrive at

$$(5.3) \quad \langle u, v \rangle_a := \int_{\mathbb{S}^2} \left(\sum_{i,j=1}^2 a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} + buv \right) d\mu = \int_{\mathbb{S}^2} f v d\mu := \ell(v).$$

By (5.2) and the assumptions on $b(x)$, the bilinear form $\langle \cdot, \cdot \rangle_a$ satisfies

$$(5.4) \quad M_1 \|u\|_{H_1}^2 = (c_1 + b_1) \langle u, u \rangle_{H_1} \leq \underbrace{\langle u, u \rangle_a}_{\|u\|_a^2} \leq (c_2 + b_2) \langle u, u \rangle_{H_1} = M_2 \|u\|_{H_1}^2.$$

A straightforward application of the Lax-Milgram theorem, together with $\ell(v)$ being a bounded linear functional on H_1 , then yields the following result:

Proposition 5.1. *The bilinear form $\langle \cdot, \cdot \rangle_a$ is coercive and bounded on H_1 and defines an inner product on H_1 , with the norms $\| \cdot \|_a$ and $\| \cdot \|_{H_1}$ being equivalent. In addition, for $f \in L^2$, there is a unique $u \in H_1$ such that (5.3) is satisfied; that is, $u \in H_1$ weakly solves $Lu = f$. Finally, $\|u\|_{L^2} \leq \|f\|_{L^2}$.*

We now turn to the regularity of the weak solution to $Lu = f$. A priori estimates of the general type needed here may be found in the survey article by Mikhalets and Murach [24, Theorem 6.6], along with the references. They are, however, given for pseudo-differential operators. A simpler approach is to use the local regularity theorems in [7, pgs. 261-269], which apply to open sets in \mathbb{R}^n , and so to coordinate patches on \mathbb{S}^n . Since the sphere is compact, they apply globally to $Lu = f$, and so we have the (standard) regularity result that we will use in the sequel.

Proposition 5.2. *Let L be as described above. If u is a distributional solution to $Lu = f$, where $f \in H_s$, $0 \leq s$, $s \in \mathbb{R}$, then $u \in H_{s+2}$. In addition, for any $t < s + 1$ there is a constant $C_t > 0$ such that $\|u\|_{H_{s+2}} \leq C_t(\|Lu\|_{H_s} + \|u\|_{H_t})$ and $\|u\|_{H_{s+2}} \leq C\|Lu\|_{H_s}$ all hold.*

Proof. Let $\{(\mathfrak{U}_\alpha, \phi_\alpha)\}$ be an atlas for \mathbb{S}^n . Since \mathbb{S}^n is a compact, C^∞ manifold, we may cover \mathbb{S}^n with a finite number of \mathfrak{U}_α 's. When $\phi_\alpha(\mathfrak{U}_\alpha) \cap \phi_\beta(\mathfrak{U}_\beta) \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1}$ is C^∞ . In a fixed chart $(\mathfrak{U}_\alpha, \phi_\alpha)$, the points $p \in \mathfrak{U}_\alpha$ are parametrized by $p = \phi_\alpha^{-1}(x)$, where $x = (x^1, \dots, x^d) \in U_\alpha = \phi_\alpha(\mathfrak{U}_\alpha)$. In addition, we choose the atlas so that in local coordinates the operator L is strongly elliptic.

Choose $\{\rho_\alpha\}$ to be a C^∞ partition of unity subordinate to $\{(\mathfrak{U}_\alpha, \phi_\alpha)\}$. A distribution on \mathbb{S}^n is a continuous linear functional on $D(\mathbb{S}^n) := C^\infty(\mathbb{S}^n)$. As usual, we denote the set of distributions by $D'(\mathbb{S}^n)$. Distributions may be characterized in terms of local charts and a subordinate partition of unity: $u \in D'(\mathbb{S}^n)$ if and only if $\rho_\alpha u \circ \phi_\alpha^{-1}$ is compactly supported in $U_\alpha \subseteq \mathbb{R}^n$ [16, §1.8]. Next, because of the properties of a partition of unity, $\rho_\alpha \circ \phi_\alpha^{-1}$ will be 1 in a neighborhood Ω_0 of some point x_0 in U_α . It follows that in local \mathfrak{U}_α -coordinates on Ω_0 , we have $L\rho_\alpha u \circ \phi_\alpha^{-1} = Lu \circ \phi_\alpha^{-1} = f \circ \phi_\alpha^{-1} \in H_{s+2}^{loc}(\Omega_0)$. By [7, Theorem 6.30], we then have $u \circ \phi_\alpha^{-1}|_{\Omega_0} \in H_{s+2}^{loc}(\Omega_0)$. If we use a different partition of unity $\{\rho'_\alpha\}$, we can find another point $x_1 \in U_\alpha$ and a corresponding neighborhood Ω_1 for which $u \circ \phi_\alpha^{-1}|_{\Omega_1} \in H_{s+2}^{loc}(\Omega_1)$ holds. Continuing in this way will yield $u \circ \phi_\alpha^{-1} \in H_{s+2}^{loc}(U_\alpha)$ and $Lu \circ \phi_\alpha^{-1} = f \circ \phi_\alpha^{-1}$ on U_α . Since $\rho_\alpha \circ \phi_\alpha^{-1}$ is C^∞ , with support within U_α , $\rho_\alpha u \circ \phi_\alpha^{-1}$ also has its support within U_α . Thus, $u \circ \phi_\alpha^{-1}$ being in $H_{s+2}^{loc}(U_\alpha)$ implies that $\rho_\alpha u \circ \phi_\alpha^{-1} \in H_{s+2}(U_\alpha)$. Obviously $\rho_\alpha u \circ \phi_\alpha^{-1}$ may be extended to all of \mathbb{R}^n by setting it equal to 0 outside its support in U_α . Let u_α be this extension; note that u_α is in $H_{s+2}(\mathbb{R}^n)$. Since there are only finitely many α 's, $\sum_\alpha \|u_\alpha\|_{H_{s+2}(\mathbb{R}^n)} < \infty$. By [11, Proposition, p. 35], we have, up to equivalent norms, that $\|u\|_{H_{s+2}(\mathbb{S}^n)} = \sum_\alpha \|u_\alpha\|_{H_{s+2}(\mathbb{R}^n)} < \infty$; consequently, $u \in H_{s+2}(\mathbb{S}^n)$.

We now turn to the inequalities. The support of u_α is contained in \mathfrak{U}_α , where L is strongly elliptic. By [7, Corollary 6.27], $\Omega = \mathfrak{U}_\alpha$, we have, with $s \rightarrow s + 2$ and $t < s - 1 \rightarrow t < s + 1$,

$$\|u_\alpha\|_{H_{s+2}(\mathbb{R}^n)} \leq C_t(\|L(u_\alpha)\|_{H_s(\mathbb{R}^n)} + \|u_\alpha\|_{H_t(\mathbb{R}^n)}).$$

The last part of the argument in step 2 of [7, Theorem 6.25] starts with $L(u_\alpha) = L(\rho_\alpha u) = \rho_\alpha Lu + [L, \rho_\alpha]u$, in local coordinates; the commutator term is a differential operator having order lower than L . Repeating the rest of the arguments in [7, Theorem 6.25] and [7, Corollary 6.27], we arrive at

$$\|u_\alpha\|_{H_{s+2}(\mathbb{R}^n)} \leq C_t(\|\rho_\alpha Lu\|_{H_s(\mathbb{R}^n)} + \|u_\alpha\|_{H_t(\mathbb{R}^n)}).$$

Summing over α then yields $\|u\|_{H_{s+2}(\mathbb{S}^n)} \leq C_t(\|Lu\|_{H_s(\mathbb{S}^n)} + \|u\|_{H_t(\mathbb{S}^n)})$, which in the notation $\|\cdot\|_s = \|\cdot\|_{H_s(\mathbb{S}^n)}$ is the first inequality. To obtain the second inequality, start by setting $t = 0$ in the first. Also note that, from (5.3), we have $\min(b)\|u\|_{L^2}^2 \leq \|u\|_a^2 = \langle Lu, u \rangle_{L^2} \leq \|Lu\|_{L^2}\|u\|_{L^2}$. Dividing by $\|u\|_{L^2}$, we obtain $\|u\|_{L^2} \leq C\|Lu\|_{L^2}$, since $\min(b) > 0$. Thus $\|u\|_{s+2} \leq C(\|Lu\|_{H_s} + \|Lu\|_{L^2})$. The inequality that we want then follows from the observation that $\|Lu\|_{L^2} \leq \|Lu\|_{H_s}$. \square

Remark 5.3. The proof above holds for any C^∞ compact Riemannian manifold \mathbb{M} and any strongly elliptic differential operator on \mathbb{M} .

We close this section with a corollary to the regularity result above for L defined in (1.1). The corollary forms the basis of the ‘‘Nitsche trick’’ [27] that we will use later.

Corollary 5.4. *Let V be a closed subspace of H_1 and let P_V be the orthogonal projection of V onto H_1 , relative to the inner product $\langle \cdot, \cdot \rangle_a$. If $u \in H_1$ and $Lw = u - P_V u$, then*

$$(5.5) \quad \|u - P_V u\|_{L^2}^2 \leq \|w - P_V w\|_a \|u - P_V u\|_a.$$

Proof. Since we have $Lw = u - P_V u$, the regularity result above implies that $w \in H_3$. Integrating by parts in $\langle w, u - P_V u \rangle_a$ yields

$$\langle w, u - P_V u \rangle_a = \langle Lw, u - P_V u \rangle_{L^2} = \langle u - P_V u, u - P_V u \rangle_{L^2} = \|u - P_V u\|_{L^2}^2.$$

Next, note that $P_V w$ is in V , and so $P_V w$ is orthogonal to $u - P_V u$, relative to $\langle \cdot, \cdot \rangle_a$. Consequently, $\langle w, u - P_V u \rangle_a = \langle w - P_V w, u - P_V u \rangle_a$. It follows that $\|u - P_V u\|_{L^2}^2 = \langle w - P_V w, u - P_V u \rangle_a$. Applying Schwarz’s inequality then yields (5.5). \square

6. GALERKIN APPROXIMATION FOR $Lu = f$

6.1. Error estimates. We will use spaces of spherical basis functions to obtain approximate solutions to $Lu = f$; specifically, the $V_{\phi, X}$ ’s and the $V_{\phi, L, X}$ ’s defined earlier. Let ϕ be an SBF on \mathbb{S}^2 that is positive definite or conditionally positive definite. For $\tau > 1$, we will make the assumption that the Fourier-Legendre coefficients of ϕ satisfy

$$(6.1) \quad c(1 + \lambda_\ell)^{-\tau} \leq \hat{\phi}_\ell \leq C(1 + \lambda_\ell)^{-\tau}, \quad \forall \ell \geq L + 1,$$

where c and C are positive constants and L is the highest order special harmonic reproduced by interpolation from $V_{\phi, L, X}$. In later sections, when we will be concerned with the stability of discretizing the problem, we will restrict the SBFs to the thin-plate splines, because they have highly localized bases. For obtaining error estimates, this is unnecessary, since the estimates depend only on the space and not the basis employed.

Let $P_X := P_{V_X}$ be the orthogonal projection of H_1 onto the finite dimensional space V_X , in the $\langle \cdot, \cdot \rangle_a$ inner product, and let I_X be the interpolation operator associated with V_X . Since $\|u - P_X u\|_a = \min_{v \in V_X} \|u - v\|_a$, we have that

$$(6.2) \quad \|u - P_X u\|_a \leq \|u - I_X u\|_a \leq C \|u - I_X u\|_{H_1},$$

where the last inequality follows from the equivalence of the norms $\|\cdot\|_a$ and $\|\cdot\|_{H_1}$. The same reasoning applies to the solution w to $Lw = u - P_X u$, so $\|w - P_X w\|_a \leq C \|w - I_X w\|_{H_1}$. Combining these estimates with the one from Corollary 5.4 then yields this:

$$(6.3) \quad \|u - P_X u\|_{L^2}^2 \leq C \|w - I_X w\|_{H_1} \|u - I_X u\|_{H_1}.$$

The regularity results in Proposition 5.2 imply that if $f \in H_s$, then the solution u to $Lu = f$ is in H_{s+2} . Moreover, if $\tau > 1$, then the projection $P_X u$ exists and is in $H_{\tau+\alpha}$, for any $\alpha < \tau - 1$. Thus, $u - P_X u$ belongs to H_σ , $\sigma := \min(s + 2, \tau + \alpha)$. Applying the elliptic regularity result to $Lw = u - P_X u$ then gives us $w \in H_{\sigma+2}$.

We are mainly interested in the case of \mathbb{S}^2 – i.e., $n = 2$. For that case, we have the following lemma, which will be needed to obtain error estimates.

Lemma 6.1. *Let $n = 2$. In the notation used above,*

$$(6.4) \quad \|w - I_X w\|_{H_1} \leq Ch^2 \|u - I_X u\|_{H_1}.$$

Proof. The solution to $Lw = u - P_X u$ is in $w \in H_{\sigma+2}$, where $\sigma := \min(s+2, \tau + \alpha)$, $\alpha < \tau - n/2 = \tau - 1$, and $\tau > n/2 = 1$. It follows that $\sigma > 1$ and so w is in $H_{\sigma+2} \subset H_3$. Applying Theorem A.3 then yields

$$(6.5) \quad \|w - I_X w\|_{H_1} \leq Ch^2 \|w\|_{H_3}.$$

Furthermore, by Proposition 5.2, $\|w\|_{H_3} \leq C \|Lw\|_{H_1} = \|u - P_X u\|_{H_1}$. Since the usual norm for H_1 is equivalent to the $\|\cdot\|_a$ norm, we have $\|w\|_{H_3} \leq C \|u - P_X u\|_a$. Then, by this inequality and (6.2), we see that $\|w\|_{H_3} \leq C \|u - I_X u\|_{H_1}$. Combining this with (6.5) gives us (6.4). \square

Theorem 6.2. *Let $n = 2$ and $Lu = f$, $f \in H_s$, $s \geq 0$. In the notation used above,*

$$(6.6) \quad \|u - P_X u\|_{L^2} \leq \begin{cases} Ch^{s+2} \|u\|_{H_{s+2}} \leq Ch^{s+2} \|f\|_{H_s}, & \text{if } s \leq 2\tau - 2, \\ Ch^{2\tau} \|u\|_{H_{2\tau}} \leq Ch^{2\tau} \|f\|_{H_{2\tau-2}}, & \text{if } 2\tau - 2 < s. \end{cases}$$

Proof. By Theorem A.3, we have that

$$(6.7) \quad \|u - I_X u\|_{H_1} \leq \begin{cases} Ch^{s+1} \|u\|_{H_{s+2}}, & \text{if } s \leq 2\tau - 2, \\ Ch^{2\tau-1} \|u\|_{H_{2\tau}}, & \text{if } 2\tau - 2 < s. \end{cases}$$

The estimate in terms of the Sobolev norms of u then follows from (6.3) and (6.4). Proposition 5.2 implies that $\|u\|_{s+2} \leq C \|Lu\|_s$. Since $f = Lu$, $\|u\|_{s+2} \leq C \|f\|_s$. When $s > 2\tau - 2$, then the previous argument applies, with $s + 2$ being replaced by 2τ . \square

6.2. The stiffness matrix in the Lagrange basis. The error estimates obtained above are, as we noted, independent of the basis used for the Galerkin approximation space. However, to actually solve for the Galerkin approximation, we must pick a suitable ϕ for which there is a good basis for $V_{\phi,L,X}$, one that results in a numerically robust method for finding the Galerkin solution. We will show that a surface spline ϕ_m and the corresponding Lagrange basis $\{\chi_\xi : \xi \in X\}$ will provide the required robustness.

Let $m \geq 2$ and set $V_{m,X} = V_{\phi_m, m-1, X}$. Take the basis for $V_{m,X}$ to be $\{\chi_\xi : \xi \in X\}$. The Galerkin approximation to the solution $Lu = f$ is the orthogonal projection $u_X := P_{m,X} u$ of u onto $V_{m,X}$, in the $\langle \cdot, \cdot \rangle_a$ inner product. If $u_X = \sum_{\xi \in X} \alpha_\xi \chi_\xi$, then, from the weak form of $Lu = f$ and the usual normal equations, we obtain the *stiffness matrix*:

$$(6.8) \quad A\alpha = \tilde{f}, \text{ where } A_{\xi,\eta} = \langle \chi_\xi, \chi_\eta \rangle_a, \alpha = (\alpha_\xi), \tilde{f} = (\langle f, \chi_\xi \rangle_{L^2}).$$

Eventually, we will discretize the problem by using quadrature methods to approximate A . For now, we will restrict our attention to A .

6.2.1. Stability of A . We want to estimate $\kappa_2(A)$, the condition number for A . We begin with the observation that A is a real, self-adjoint matrix. It is also a Gram matrix for the linearly independent set, $\{\chi_\xi : \xi \in X\}$, and is therefore positive definite as well. Consequently, $\kappa_2(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$.

We will begin by estimating $\lambda_{\min}(A)$. First of all, the operator L is self-adjoint and positive definite. Standard variational methods then imply that

$$(6.9) \quad \min_{v \in H_1} \langle v, v \rangle_a = \lambda_{\min}(L), \quad \|v\|_{L^2} = 1.$$

Let $v = \sum_{\xi \in X} \alpha_\xi \chi_\xi$. Consider the quadratic form $\alpha^T A \alpha = \langle v, v \rangle_a$. By (6.9), we have that

$$\alpha^T A \alpha \geq \lambda_{\min}(L) \|v\|_{L^2}^2.$$

Next, from (4.5), with $p = 2$, we have $\|v\|_{L^2} = \|\sum_{\xi \in X} \alpha_\xi \chi_\xi\|_{L^2} \geq c_1 q_X \|\alpha\|_{\ell_2(X)}$. From the inequality above, we then have

$$\alpha^T A \alpha \geq \lambda_{\min}(L) \|v\|_{L^2}^2 \geq c_1^2 q_X^2 \lambda_{\min}(L) \|\alpha\|_{\ell_2(X)}^2,$$

which holds for all $\alpha \in \mathbb{R}^{|X|}$. Hence, we have that

$$(6.10) \quad \lambda_{\min}(A) \geq c_1^2 q_X^2 \lambda_{\min}(L).$$

Estimating $\lambda_{\max}(A)$ requires the Bernstein inequality from [23, Theorem 6.1]. In the case at hand, the theorem cited² holds with $n = 2$, $p = 2$, $0 < \gamma < 2m - 1$, and $g = v$. For every $v \in V_{m,X}$ and every $0 < \gamma < 2m - 1$, we have that there is a constant $C > 0$, where $C = C(m, \rho)$, such that

$$(6.11) \quad \|g\|_{H_\gamma} \leq C q_X^{-\gamma} \|g\|_{L^2}.$$

Since $\alpha^T A \alpha = \langle v, v \rangle_a$ and, by (5.4), $\langle v, v \rangle_a \leq M_2 \|v\|_{H_1}^2$, then, from (6.11) ($\gamma = 1$) and (4.5) ($p = 2, n = 2$), we have that $\alpha^T A \alpha \leq C q_X^{-2} \|v\|_{L^2}^2 \leq C c_2 \|\alpha\|_{\ell_2(X)}^2$. Consequently,

$$(6.12) \quad \lambda_{\max}(A) \leq C, \quad C = C(m, \rho, \|\cdot\|_a).$$

Combining (6.12) and (6.10) results in the following:

Theorem 6.3. *Let A be the stiffness matrix in the basis $\{\chi_\xi : \xi \in X\}$ for $V_{m,X}$. If q_X is sufficiently small and ρ is fixed, then there is a constant $C = C(m, \rho, \|\cdot\|_a)$ for which the condition number $\kappa_2(A)$ satisfies $\kappa_2(A) \leq C q_X^{-2}$.*

6.2.2. Exponential decay of the entries of A . At this point, we turn to the behavior of the entries $A_{\xi,\eta}$ in A . What we will see is that the entries in A decay exponentially in $\text{dist}(\xi, \eta)$, making A nearly sparse. In section 8, we will use this decay to construct a sparse discretization for A . Establishing decay requires the following lemma.

Lemma 6.4. *Let f, g be in $C^1(\mathbb{S}^2)$, and let $\nabla f, \nabla g$ be their covariant derivatives. Then, there is a constant C such that for all $x \in \mathbb{S}^2$ we have*

$$(6.13) \quad \left| \sum_{i,j} a^{ij}(x) \nabla_i f(x) \nabla_j g(x) \right| \leq C |\nabla f(x)| |\nabla g(x)|.$$

Proof. The matrix a^{ij} is positive definite, so we may use it as an inner product. Schwarz's inequality applied to this inner product implies that

$$\left| \sum_{i,j} a^{ij} \nabla_i f \nabla_j h \right| \leq \left(\sum_{i,j} a^{ij} \nabla_i f \nabla_j f \right)^{1/2} \left(\sum_{i,j} a^{ij} \nabla_i g \nabla_j g \right)^{1/2}.$$

By (5.2), we have that $\sum_{i,j} a^{ij} \nabla_i f \nabla_j f \leq \sum_{i,j} g^{ij} \nabla_i f \nabla_j f = |\nabla f|^2$. This also holds for g as well. Applying these inequalities to the previous one yields (6.13). \square

Proposition 6.5. *For h_X sufficiently small and ν as in Theorem 4.1,*

$$(6.14) \quad |A_{\xi,\eta}| \leq C h_X^{-2} e^{-\frac{\nu}{h_X} \text{dist}(\xi,\eta)}.$$

²The precise version of the theorem holds for a positive definite SBF. However, it is easy to modify it so that it will hold for a conditionally positive definite SBF.

Proof. Since we have $A_{\xi,\eta} = \langle \chi_\xi, \chi_\eta \rangle_a$, we have that

$$|A_{\xi,\eta}| \leq \int_{\mathbb{S}^2} \left| \sum_{i,j} a^{ij} \nabla_i \chi_\xi \nabla_j \chi_\eta + b(x) \chi_\xi \chi_\eta \right| d\mu(x).$$

By Lemma 6.4 and the boundedness of b , we have that

$$\left| \sum_{i,j} a^{ij} \nabla_i \chi_\xi \nabla_j \chi_\eta + b(x) \chi_\xi \chi_\eta \right| \leq C |\nabla \chi_\xi| |\nabla \chi_\eta| + \|b\|_{L^\infty} |\chi_\xi| |\chi_\eta|.$$

Moreover, using (4.8) and (4.3), we see that

$$C |\nabla \chi_\xi| |\nabla \chi_\eta| + \|b\|_{L^\infty} |\chi_\xi| |\chi_\eta| \leq (C_1 q_X^{-2} + C_2) \exp \left(-\nu \frac{\text{dist}(x, \xi) + \text{dist}(x, \eta)}{h_X} \right).$$

Because $\text{dist}(\cdot, \cdot)$ is a metric on \mathbb{S}^2 , we may use the triangle inequality: $\text{dist}(x, \xi) + \text{dist}(x, \eta) \geq \text{dist}(\xi, \eta)$. In addition, h_X small implies that $q_X = h_X / \rho_X$ is also small. Combining these facts yields, uniformly in x , this inequality:

$$\left| \sum_{i,j} a^{ij} \nabla_i (\chi_\xi) \nabla_j (\chi_\eta) + b(x) \chi_\xi \chi_\eta \right| \leq C h_X^{-2} e^{-\frac{\nu}{h_X} \text{dist}(\xi, \eta)}.$$

Integrating both sides above then establishes (6.14). □

7. THE DISCRETIZED GALERKIN SOLUTION

The discretized solution to the Galerkin problem is obtained simply by replacing the stiffness matrix A from the original problem with a discretized version, which is obtained via quadrature, and then solving as usual. In order to carry out a complete analysis of this method, in the sequel we will restrict the tensor a^{ij} to have the form $a^{ij} = a g^{ij}$, where $a \in C^\infty(\mathbb{S}^2)$.

7.1. Discretizing the stiffness matrix. We now turn to the task of discretizing the stiffness matrix. Our approach is to approximate the (ξ, η) entry $A_{\xi,\eta} = \int_{\mathbb{S}^2} (a \nabla \chi_\xi \cdot \nabla \chi_\eta + b \chi_\xi \chi_\eta) d\mu$ by means of the quadrature formulas discussed in section 4.2. In doing so, we will allow for the surface spline used in the Galerkin method, ϕ_m , to differ from the one used in the quadrature formula. We will denote the latter by ϕ_M , with $M \geq 2$.

7.1.1. Discretization error for the stiffness matrix. To discretize the stiffness matrix, we will employ a set of nodes Y that is chosen independently of X . In general, Y will be much larger than X and need not contain X as a subset. That said, the discretization of $A_{\xi,\eta} = \int_{\mathbb{S}^2} (a \nabla \chi_\xi \cdot \nabla \chi_\eta + b \chi_\xi \chi_\eta) d\mu$ is $Q_Y(a \nabla \chi_\xi \cdot \nabla \chi_\eta + b \chi_\xi \chi_\eta)$. In explicit form, this is given by

$$(7.1) \quad A_{\xi,\eta}^Y := \sum_{\zeta \in Y} (a \nabla \chi_\xi \cdot \nabla \chi_\eta + b \chi_\xi \chi_\eta) \Big|_\zeta w_\zeta.$$

We will need the following lemma to obtain bounds on the error $|A_{\xi,\eta} - A_{\xi,\eta}^Y|$:

Lemma 7.1. *Suppose that $m > M$ are positive integers and that $\epsilon > 0$. Let $\sigma = m - M$. If $M \geq \sigma + 1$ and $\epsilon < 2\sigma - 1$, then $\epsilon < m - 2$ and $2m - 1 - \epsilon > 2M$.*

Proof. We will begin by showing the first inequality. By assumption, $\epsilon < 2\sigma - 1$, so $\epsilon < \sigma + \sigma + 1 - 2 \leq \sigma + M - 2 = m - 2$. To get the second, note that $2m - \epsilon - 1 = 2M + 2\sigma - 1 - \epsilon$. Since $2\sigma - 1 - \epsilon > 0$, we have that $2m - \epsilon - 1 > 2M$. □

Corollary 7.2. *Suppose that $a(x), b(x) \in C^\infty(\mathbb{S}^2)$, and adopt the notation and assumptions from Lemma 7.1. In addition, let $\{\chi_\xi: \xi \in X\}$ and $\{\tilde{\chi}_\xi: \xi \in Y\}$ be the Lagrange bases for $V_{m,X} := V_{\phi_m, X}$ and for $V_{M,Y} := V_{\phi_M, Y}$, respectively. Then, if $\delta := 2\sigma - 1 - \epsilon > 0$, we have:*

$$(7.2) \quad |A_{\xi, \eta} - A_{\xi, \eta}^Y| \leq C(h_Y/h_X)^{2M} h_X^{-\delta}, \quad C = C(\|a\|_{H_{2m}}, \|b\|_{H_{2m}}),$$

where $A_{\xi, \eta}^Y = Q_Y(a \nabla \chi_\xi \cdot \nabla \chi_\eta + b \chi_\xi \chi_\eta)$. Moreover, $\delta \leq m - 2 - \epsilon$, so it may be made as small as we wish by taking ϵ close to $m - 2$.

Proof. By Proposition 4.4, $b \chi_\xi \chi_\eta$ is in $H_{2m-1-\epsilon}$ for all $0 < \epsilon < m - 2$. By Lemma 7.1, if $M \geq \sigma + 1$ and $\epsilon < 2\sigma - 1$, then we have both $2m - 1 - \epsilon > 2M$ and $\epsilon < m - 2$. It follows that we may use (4.17) with $\tau = M$ and $\mu = 2M < 2m - 1 - \epsilon = 2M + 2\sigma - 1 - \epsilon$. From the bound in (4.10), and from $\|b \chi_\xi \chi_\eta\|_{H_{2M}} \leq \|b \chi_\xi \chi_\eta\|_{H_{2m-1-\epsilon}}$, we see that

$$(7.3) \quad \begin{aligned} \left| \int_{\mathbb{S}^2} b \chi_\xi \chi_\eta d\mu - Q_Y(b \chi_\xi \chi_\eta) \right| &\leq C h_Y^{2M} \|b \chi_\xi \chi_\eta\|_{H_{2m-1-\epsilon}} \\ &\leq C h_Y^{2M} h_X^{1+\epsilon-(2\sigma-1)-2M} \|b\|_{H_{2m}} \\ &\leq C (h_Y/h_X)^{2M} h_X^{1-\delta} \|b\|_{H_{2m}}. \end{aligned}$$

Using the same argument, but with the bounds from (4.11) instead of (4.10), we have

$$(7.4) \quad \left| \int_{\mathbb{S}^2} a \nabla \chi_\xi \cdot \nabla \chi_\eta d\mu - Q_Y(a \nabla \chi_\xi \cdot \nabla \chi_\eta) \right| \leq C (h_Y/h_X)^{2M} h_X^{-\delta} \|a\|_{H_{2m}}.$$

If we combine (7.3) and (7.4) and note that $h_X < \pi$, we obtain (7.2). To prove the statement concerning δ , observe that $\delta = \sigma - 2 - \epsilon + (\sigma + 1) \leq m - M - 2 + \epsilon + M = m - 2 - \epsilon$. □

Remark 7.3. If $M \geq m$ and $0 < \epsilon < m - 2$, then, obviously, $1 < m + 1 < 2m - \epsilon - 1 < 2M$. Thus, we may use (4.17) with $\tau = M$ and $\mu = 2m - 1 - \epsilon$. The same arguments employed above then imply that the error estimate in (7.2) becomes

$$(7.5) \quad |A_{\xi, \eta} - A_{\xi, \eta}^Y| \leq C \max(\|a\|_{H_{2m}}, \|b\|_{H_{2m}}) (h_Y/h_X)^{2m-1-\epsilon}.$$

The right side in this inequality depends on m and ϵ , but not on M . It follows that there is no advantage in choosing $M > m$.

One important fact is that $A_{\xi, \eta}^Y$ decays in $\text{dist}(\xi, \eta)$ in the same way as $A_{\xi, \eta}$. We will establish this below. Before carrying out the proof, we mention that, although we use the assumption (4.15) in our proof, it is not necessary to do so. We also wish to point out that the denominator of the fraction in the exponent is h_X , and not h_Y , as one would first suppose.

Proposition 7.4. *The discretized entry $A_{\xi, \eta}^Y$ satisfies the bound*

$$|A_{\xi, \eta}^Y| \leq C h_X^{-2} \exp\left(-\frac{\nu}{h_X} \text{dist}(\xi, \eta)\right).$$

Proof. The same argument used to establish (6.14) yields

$$|A_{\xi, \eta}^Y| \leq C h_X^{-2} \left(\sum_{\zeta \in Y} |w_\zeta| \right) e^{-\frac{\nu}{h_X} \text{dist}(\xi, \eta)}.$$

By our assumption that the weights are positive, we have that $\sum_{\zeta \in Y} |w_\zeta| = \sum_{\zeta \in Y} w_\zeta = 4\pi$, from which the result is immediate. \square

The theorem below gives us the desired bound on the error $\|A - A^Y\|_2$ that is made in using quadrature to compute the entries in the stiffness matrix.

Theorem 7.5. *Let $m > M \geq 2$ and let ϕ_M be the surface spline used for quadrature. If $0 < \delta < 2(m - M) - 1$, then, for h_X and h_Y sufficiently small, we have*

$$(7.6) \quad \|A - A^Y\|_2 \leq C(\log(h_Y))^2 (h_Y/h_X)^{2M} h_X^{-\delta}.$$

Proof. Recall that, for a self-adjoint matrix C , $\|C\|_2 \leq \|C\|_1 = \|C\|_\infty$. Applying this to the self-adjoint matrix $A - A^Y$ yields

$$\|A - A^Y\|_2 \leq \max_{\eta \in X} \left(\sum_{\xi \in X} |A_{\xi,\eta} - A_{\xi,\eta}^Y| \right).$$

The task is now to bound the sums on the right above. Let $r_0 > h_x$ and let $B(\eta, r_0)$ be the ball with center η and radius r_0 . We may break up the sum over X into a sum over centers inside $B(\eta, r_0)$ and those in $B(\eta, r_0)^c$:

$$\sum_{\xi \in X} |A_{\xi,\eta} - A_{\xi,\eta}^Y| = \sum_{\xi \in X \cap B(\eta, r_0)} |A_{\xi,\eta} - A_{\xi,\eta}^Y| + \sum_{\xi \in X \cap B(\eta, r_0)^c} |A_{\xi,\eta} - A_{\xi,\eta}^Y|.$$

By Propositions 6.5 and 7.4, we have that $|A_{\xi,\eta} - A_{\xi,\eta}^Y| \leq Ch_X^{-2} e^{-\frac{\nu}{h_X} \text{dist}(\xi,\eta)}$. Using this and (4.7) yields

$$(7.7) \quad \begin{aligned} \sum_{\xi \in X \cap B(\eta, r_0)^c} |A_{\xi,\eta} - A_{\xi,\eta}^Y| &\leq Ch_X^{-2} \sum_{\xi \in X \cap B(\eta, r_0)^c} e^{-\frac{\nu}{h_X} \text{dist}(\xi,\eta)} \\ &< C\rho_X^2 h_X^{-2} \frac{e^{-(n_0-1)\nu} n_0}{(1 - e^{-\nu})^2}, \quad n_0 = \lceil r_0/h_X \rceil. \end{aligned}$$

The set of remaining centers is $X \cap B(\eta, r_0)$, whose cardinality may be bounded by $\text{vol}(B(\eta, r_0))/\text{vol}(B(\eta, q_x)) \sim r_0^2/q_x^2 = \rho_X^2 (r_0/h_X)^2 < \rho_X^2 n_0^2$. From this fact and the uniform estimate on $|A_{\xi,\eta} - A_{\xi,\eta}^Y|$ in (7.2), we see that

$$(7.8) \quad \sum_{\xi \in X \cap B(\eta, r_0)} |A_{\xi,\eta} - A_{\xi,\eta}^Y| \leq C\rho_X^2 n_0^2 (h_Y/h_X)^{2M} h_X^{-\delta}.$$

Choose a constant K_0 so that $K_0\nu > 2M$, where ν is as in Theorem 4.1, and pick $r_0 = K_0 h_X |\log(h_Y)|$ so that $n_0 \sim K_0 |\log(h_Y)|$. (The h_Y is *not* a mistake.) The bounds in (7.7) and (7.2) are then, respectively, $Ch_X^{-2} h_Y^{K_0\nu} |\log(h_Y)|$ and $C(\log(h_Y))^2 (h_Y/h_X)^{2M} h_X^{-\delta}$. Note that $h_X^{-2} h_Y^{K_0\nu} \leq h_Y^{2M} h_X^{-2M} \leq (h_Y/h_X)^{2M}$. Since h_Y and h_X are small and $\delta > 0$, we also have that $|\log(h_Y)| \leq (\log(h_Y))^2 h_X^{-\delta}$. Combining the various bounds above results in

$$\sum_{\xi \in X} |A_{\xi,\eta} - A_{\xi,\eta}^Y| \leq C(\log(h_Y))^2 (h_Y/h_X)^{2M} h_X^{-\delta}.$$

This holding uniformly in η immediately implies (7.6). \square

7.1.2. *Stability of the discretized stiffness matrix.* We now turn to the question of how stable, numerically, the discretized stiffness matrix A^Y is. Answering this question requires the following lemma, which relates certain L^2 norms. We will need the two lemmas below.

Lemma 7.6. *Let X, Y be quasi-uniform, with $\rho_X, \rho_Y \leq \rho$ and suppose that $u \in V_X$ and $I_Y u$ is the interpolant of u relative to the kernel ϕ_m , $m \geq 2$ and its associated space V_Y . Then, there exists a constant $C(\rho)$ such that $\frac{1}{2}\|u\|_{L^2} \leq \|I_Y u\|_{L^2} \leq \frac{3}{2}\|u\|_{L^2}$, provided $h_Y \leq C(\rho)q_X$.*

Proof. Let $\tilde{u} = I_Y u$. Since $\|u\|_{L^2} - \|u - \tilde{u}\|_{L^2} \leq \|\tilde{u}\|_{L^2} \leq \|u\|_{L^2} + \|u - \tilde{u}\|_{L^2}$, we need only find $C(\rho)$ such that $\|u - \tilde{u}\|_{L^2} \leq \frac{1}{2}\|u\|_{L^2}$. From [9, Theorem 4.6], we have that

$$\|u - \tilde{u}\|_{L^2} = \|u - I_Y u\|_{L^2} \leq C_1 h_Y^2 \|u\|_{W_2^3},$$

since $u \in W_2^3 \subset W_2^2$ and $m \geq \mu = 2$. The constant C_1 only depends on Y through ρ . We now apply the Bernstein inequality³ from [23, Theorem 6.1]. First, $u \in V_X$, the space associated with ϕ_3 . This is essentially the Green's function associated with $\beta = 6$ in the Bernstein inequality. In addition, we may take $\gamma = 2$. Thus, we have

$$\|u\|_{W_2^2} \approx \|u\|_{H_2} \leq C_2 q_X^{-2} \|u\|_{L^2}, \text{ where } C_2 = C_2(\rho).$$

Combining this with the previous inequality yields $\|u - \tilde{u}\|_{L^2} \leq C_1 C_2 (h_Y/q_X)^2 \|u\|_{L^2}$. The result immediately follows on choosing $C(\rho) \leq (2C_1 C_2)^{-1/2}$. \square

Theorem 7.7. *Let $a, b \in C^\infty$ satisfy $a(x) \geq a_0 > 0$ and $b(x) \geq b_0 > 0$. Then $\lambda_{\min}(A^Y) \geq C q_X^2$, provided $h_Y \leq C(\rho)q_X$.*

Proof. Let $u \in V_X$, and so, $u = \sum_{\xi \in X} u(\xi)\chi_\xi$ and $\nabla u = \sum_{\xi \in X} u(\xi)\nabla\chi_\xi$. Moreover, $u|_X$ is an arbitrary vector in \mathbb{R}^{N_X} . It follows that

$$\begin{aligned} (u|_X)^T A^Y u|_X &= \sum_{\zeta \in Y} (a(\zeta)\nabla u(\zeta) \cdot \nabla u(\zeta) + b(\zeta)u(\zeta)^2)w_\zeta \\ &\geq b_0 \sum_{\zeta \in Y} u(\zeta)^2 w_\zeta \geq C b_0 N_Y^{-1} \sum_{\zeta \in Y} u(\zeta)^2 \text{ (by (4.15))} \\ &\geq C b_0 h_Y^2 \|u|_Y\|_{\ell^2}^2. \end{aligned}$$

Let $\tilde{u} = I_Y u = \sum_{\zeta \in Y} u(\zeta)\tilde{\chi}_\zeta$, which is the interpolant of u relative to V_Y , the space associated with ϕ_m and Y . Of course, since \tilde{u} is the interpolant of u on Y , $\tilde{u}|_Y = u|_Y$. By Corollary 3.11, (3.1) and (3.3) in [13], $\|u|_Y\|_{\ell^2} = \|\tilde{u}|_Y\|_{\ell^2} \geq C q_Y^{-1} \|\tilde{u}\|_{L^2}$. In addition, Lemma 7.6 implies that $\|\tilde{u}\|_{L^2} = \|I_Y u\|_{L^2} \geq \frac{1}{2}\|u\|_{L^2}$. Again applying the results from [13] then yields $\|u\|_{L^2} \geq C q_X \|u|_X\|_{\ell^2}$. Consequently, $\|u|_Y\|_{\ell^2} \geq C q_Y^{-1} q_X \|u|_X\|_{\ell^2}$. Combining this with the lower bound on the quadratic form $(u|_X)^T A^Y u|_X$ then gives us $(u|_X)^T A^Y u|_X \geq C b_0 (h_Y/q_Y)^2 q_X^2 \|u|_X\|_{\ell^2}^2 = C b_0 \rho^2 q_X^2 \|u|_X\|_{\ell^2}^2$, so $\lambda_{\min}(A^Y) \geq C q_X^2$. \square

Corollary 7.8. *If h_Y is chosen so that $(\log(h_Y))^2 (h_Y/h_X)^{2M} h_X^{-\delta} \leq C$, then the condition number $\kappa_2(A^Y)$ satisfies $\kappa_2(A^Y) \leq C q_X^{-2}$.*

³The theorem actually requires the SBFs involved to be strictly positive definite. However, by a simple adaptation of the argument used to prove [9, Theorem 4.6], one can establish the result needed here.

Proof. Because $\lambda_{\max}(A^Y) = \|A^Y\|_2 \leq \|A\|_2 + \|A - A^Y\|_2 = \lambda_{\max}(A) + \|A - A^Y\|_2$, we have, from (6.12), (7.6) and the condition on h_Y , that $\lambda_{\max}(A^Y) \leq C$. Applying Theorem 7.7 then yields the result. \square

7.2. Error estimates for the discretized Galerkin solution. Let \tilde{f} be the column vector⁴ defined in (6.8) and let $u_h := u_{h_X} = \sum_{\xi} \alpha_{\xi} \chi_{\xi}$ be the Galerkin approximation to the solution u of $Lu = f$. The coefficient vector α is given by $\alpha = A^{-1}\tilde{f}$. The discretized solution $u_h^Y := u_{h_X}^Y$ is obtained by replacing the stiffness matrix A by A^Y in the problem. The solution that results is $u_h^Y = \sum_{\xi} \alpha_{\xi}^Y \chi_{\xi}$, where $\alpha^Y = (A^Y)^{-1}\tilde{f}$.

Our goal is to analyze the L^2 -error between u and u_h^Y . The triangle inequality implies that $\|u - u_h^Y\|_{L^2} \leq \|u - u_h\|_{L^2} + \|u_h - u_h^Y\|_{L^2}$. We can estimate $\|u - u_h\|_{L^2}$ using (6.7):

$$(7.9) \quad \|u - u_h\|_{L^2} \leq Ch^{s+2}\|f\|_{H_s}.$$

We also have, by (4.5) and $A^{-1} - (A^Y)^{-1} = (A^Y)^{-1}(A^Y - A)A^{-1}$, that

$$\begin{aligned} \|u_h - u_h^Y\|_{L^2} &= \|\sum_{\xi} (\alpha_{\xi} - \alpha_{\xi}^Y) \chi_{\xi}\|_{L^2} \\ &\leq c_2 q_X \|\alpha - \alpha^Y\|_{\ell^2} \\ &\leq c_2 q_X \|(A^Y)^{-1}\| \|A^Y - A\| \underbrace{\|A^{-1}\tilde{f}\|}_{\alpha} \|\ell^2. \end{aligned}$$

Using (4.5) again, we have $\|\alpha\|_{\ell^2} \leq c_1^{-1} q_X^{-1} \|\sum_{\xi} \alpha_{\xi} \chi_{\xi}\|_{L^2} = c_1^{-1} q_X^{-1} \|u_h\|_{L^2}$. In addition, from Theorem 7.7, $\|(A^Y)^{-1}\| \leq C q_X^{-2}$. Combining these inequalities results in

$$\|u_h - u_h^Y\|_{L^2} \leq C q_X^{-2} \|A^Y - A\| \|u_h\|_{L^2}.$$

Because $\|u - u_h\|_{L^2} \leq Ch^{s+2}\|f\|_{H_s}$, we have $\|u_h\|_{L^2} \leq \|u\|_{L^2} + Ch_X^{s+2}\|f\|_{H_s}$. Moreover, by Proposition 5.1, $\|u\|_{L^2} \leq C\|f\|_{L^2} \leq C\|f\|_{H_s}$. Thus, for $h_X \sim q_X$ small,

$$\|u_h - u_h^Y\|_{L^2} \leq C q_X^{-2} \|A^Y - A\| (\|f\|_{H_s} + Ch_X^{s+2}\|f\|_{H_s}) \leq C q_X^{-2} \|A^Y - A\| \|f\|_{H_s}.$$

From this and (7.9), it follows that

$$(7.10) \quad \|u - u_h^Y\|_{L^2} \leq C(h_X^{s+2} + q_X^{-2} \|A^Y - A\|_2) \|f\|_{H_s}.$$

The above discussion together with Theorem 7.5 yields (7.11) below. Note that the second term in (7.11) measures the quadrature error. Ideally, the ‘‘fine set’’ Y can be chosen so that the second term is comparable to the optimal error $O(h_X^{s+2})$.

Theorem 7.9. *Let f, s be as in Theorem 6.2 and let m, M, δ be as in Theorem 7.5. Then, for h_X and h_Y sufficiently small, we have*

$$(7.11) \quad \|u - u_h^Y\|_{L^2} \leq C \left(h_X^{s+2} + \frac{(\log(h_Y)h_Y^M)^2}{h_X^{2M+2+\delta}} \right) \|f\|_{H_s}.$$

⁴We assume that \tilde{f} is known exactly here. In practice, it can be computed very accurately via quadrature. See section 9.

8. SPARSE APPROXIMATION AND LOCAL LAGRANGE FUNCTIONS

This section discusses reducing the computational expense of numerically finding the Galerkin solution to the problem. There are two aspects of this. The first is obtaining a truncated approximation \tilde{A}^Y to the discretized stiffness matrix A^Y . Each row in \tilde{A}^Y has $\mathcal{O}((\log(N_X))^2)$ non-zero entries, as opposed to N_X in A^Y . The second is to replace the global Lagrange basis with a local one. As mentioned in the introduction, computing this basis requires inverting a number small matrices, a task that is parallelizable. The error estimates from making these approximations are virtually unchanged.

8.1. Sparse approximation. So far, we have not addressed the question of how well a *sparse* approximation to the stiffness matrix would perform. Suppose that, in A^Y , we discard all entries $A_{\xi,\eta}^Y$ that satisfy $\text{dist}(\xi, \eta) \geq Kh_X |\log h_X|$, where $K\nu > 2$. Let the matrix we get in this way be \tilde{A}^Y , where

$$(8.1) \quad \tilde{A}_{\xi,\eta}^Y := \begin{cases} 0, & \text{dist}(\xi, \eta) > Kh_X |\log h_X|, \\ A_{\xi,\eta}^Y, & \text{dist}(\xi, \eta) \leq Kh_X |\log h_X|. \end{cases}$$

The matrix $\tilde{A}_{\xi,\eta}^Y$ is symmetric. The number of non-zero elements in each row is approximately the ratio of the areas of caps having radii $Kh_X |\log h_X|$ and h_X , respectively. If we make use of this and of the fact that, since X is quasi-uniform, $h_X \sim N_X^{-1/2}$, then we see that

$$(8.2) \quad \#\{\text{row } \eta\} \approx \frac{(Kh_X |\log h_X|)^2}{h_X^2} = K^2 (\log(h_X))^2 \approx \frac{1}{4} K^2 (\log(N_X))^2,$$

as opposed to N_X for A^Y itself.

Proposition 8.1. *Let $K\nu > 2$ and \tilde{A}^Y be defined by (8.1). Then,*

$$(8.3) \quad \|A^Y - \tilde{A}^Y\|_2 \leq \frac{2CKe^{-\nu}}{(1 - e^{-\nu})^2} h_X^{K\nu-2} |\log(h_X)|.$$

Proof. We will follow the proof of Theorem 7.5. Because A^Y and \tilde{A}^Y are symmetric, the norm $\|A^Y - \tilde{A}^Y\|_2$ satisfies the bound

$$\|A^Y - \tilde{A}^Y\|_2 \leq \|A^Y - \tilde{A}^Y\|_1 = \|A^Y - \tilde{A}^Y\|_\infty = \max_{\eta \in X} (\sum_{\xi \in X} |A_{\xi,\eta}^Y - \tilde{A}_{\xi,\eta}^Y|).$$

We again want to estimate each term in the sums above. Let B_η be the ball centered at η and having radius $r_h = Kh_X |\log(h_X)|$. From Proposition 7.4 and the definition of \tilde{A}^Y , we have that

$$\begin{aligned} \sum_{\xi \in X} |A_{\xi,\eta}^Y - \tilde{A}_{\xi,\eta}^Y| &= \sum_{\xi \in X \cap B_\eta^c} |A_{\xi,\eta}^Y| \\ &\leq Ch_X^{-2} \sum_{\xi \in X \cap B_\eta^c} e^{-\frac{\nu}{h_X} \text{dist}(\xi, \eta)}. \end{aligned}$$

Next, divide $B_\eta^{\mathbb{C}}$ into bands of width $\sim h_X$, the n^{th} band being a distance approximately $r_h + nh_X$ from η . Repeating the derivation of (7.7), mutatis mutandis, we obtain

$$\begin{aligned} \sum_{\xi \in X} |A_{\xi, \eta}^Y - \tilde{A}_{\xi, \eta}^Y| &\leq Ch_X^{-2} e^{-\nu K |\log(h_X)|} \sum_{n=1}^{\infty} (K |\log(h_X)| + n) e^{-\nu n} \\ &\leq \frac{2CK e^{-\nu}}{(1 - e^{-\nu})^2} h_X^{K\nu-2} |\log(h_X)|. \end{aligned}$$

Combining the inequalities above yields (8.3). □

8.2. Local Lagrange functions. The result above quantifies the error made in zeroing out the entries $A_{\xi, \eta}^Y$ corresponding to all ξ, η such that $\text{dist}(\xi, \eta) > Kh_X |\log(h_X)|$. To obtain the rest of the entries we still need to use (7.1); this entails finding the χ_ξ 's, which are global in the sense that they require all of the points in X for their computation.

There is a way around this. In [8], Fuselier et al. introduced a basis for $V_{\phi_m, X}$ composed of *local* Lagrange functions, $\{\chi_\xi^{loc} : \xi \in X\}$. These basis functions are simply Lagrange functions for the points in X that lie in a ball of radius $Kh_X |\log(h_X)|$ about $\xi \in X$, where K is appropriately chosen. A detailed description of their construction and properties is given in [8, §6.3]. We will simply list what we need here, in the theorem below.

Theorem 8.2 ([8, Theorem 6.5]). *Let the notation and assumptions of Theorem 4.1 hold; define $\Upsilon_\xi := X \cap B(\xi, Kh_X |\log(h_X)|)$, where $K > 0$ is as above. There exists⁵ $\mu = \mu(m)$ such that for K chosen to satisfy $J := K\nu - 4m + 2 - 2\mu > 0$ these hold:*

$$(8.4) \quad \begin{aligned} \|\chi_\xi^{loc} - \chi_\xi\|_{L^\infty} &\leq C h_X^J, \\ |\chi_\xi^{loc}(x)| &\leq C(1 + \text{dist}(x, \xi)/h_X)^{-J}. \end{aligned}$$

Furthermore, when $J > 2$, the set $\{\chi_\xi^{loc}\}$ is L^p stable: there are $C_1, C_2 > 0$ for which

$$C_1 q_X^{2/p} \|\beta\|_{\ell^p(X)} \leq \|\sum_{\xi \in X} \beta_X \chi_\xi^{loc}\|_{L^p(\mathbb{S}^2)} \leq C_2 q_X^{2/p} \|\beta\|_{\ell^p(X)}.$$

Local Lagrange functions may be expanded in global ones. Because χ_ξ^{loc} is a Lagrange function for Υ_ξ , it satisfies $\chi_\xi^{loc}(\eta) = \delta_{\xi, \eta}$, for $\eta \in \Upsilon_\xi$. Of course, we also have $\chi_\xi(\eta) = \delta_{\xi, \eta}$, for all $\eta \in X$. Expanding $\chi_\xi^{loc}(x)$ in terms of the basis $\{\chi_\eta\}_{\eta \in X}$ results in

$$(8.5) \quad \chi_\xi^{loc}(x) = \chi_\xi(x) + \sum_{\eta \notin \Upsilon_\xi} \chi_\xi^{loc}(\eta) \chi_\eta(x) = \chi_\xi(x) + \sum_{\eta \notin \Upsilon_\xi} (\chi_\xi^{loc}(\eta) - \chi_\xi(\eta)) \chi_\eta(x),$$

since, for $\xi \neq \eta \in X$, $\chi_\xi(\eta) = \delta_{\xi, \eta} = 0$. Taking the covariant derivative in the equation above yields

$$(8.6) \quad \nabla \chi_\xi^{loc}(x) = \nabla \chi_\xi(x) + \sum_{\eta \notin \Upsilon_\xi} \chi_\xi^{loc}(\eta) \nabla \chi_\eta(x) = \nabla \chi_\xi(x) + \sum_{\eta \notin \Upsilon_\xi} (\chi_\xi^{loc}(\eta) - \chi_\xi(\eta)) \nabla \chi_\eta(x).$$

From (8.6) and (8.4), it easily follows that

$$|\nabla \chi_\xi^{loc}(x) - \nabla \chi_\xi(x)| \leq \|\chi_\xi^{loc} - \chi_\xi\|_{L^\infty} \sum_{\eta \notin \Upsilon_\xi} |\nabla \chi_\eta(x)| \leq Ch^J \sum_{\eta \in X} |\nabla \chi_\eta(x)|.$$

⁵One may take $\mu = \iota$, where ι is constructed in [8, Lemma 6.4].

Applying (4.9) to the rightmost inequality then yields the following result:

Lemma 8.3. *If $J > 1$, then $\|\nabla\chi_\xi^{loc} - \nabla\chi_\xi\|_{L^\infty} \leq C\rho_X^2 h_X^{J-1}$.*

The result we are aiming at is estimating the error made in replacing the *exact* Lagrange functions by the *local* Lagrange functions in computing $A_{\xi,\eta}^Y$. Specifically, define

$$(8.7) \quad A_{loc,\xi,\eta}^Y := \sum_{\zeta \in Y} (a\nabla\chi_\xi^{loc} \cdot \nabla\chi_\eta^{loc} + b\chi_\xi^{loc}\chi_\eta^{loc})|_\zeta w_\zeta.$$

We want to estimate $|A_{\xi,\eta}^Y - A_{loc,\xi,\eta}^Y|$. This we do in the proposition below.

Proposition 8.4. *Let $A_{\xi,\eta}^Y$ be given by (7.1) and let $A_{loc,\xi,\eta}^Y$ be as above. Then, for h_X sufficiently small and $J > 2$,*

$$(8.8) \quad |A_{\xi,\eta}^Y - A_{loc,\xi,\eta}^Y| \leq Ch_X^{J-2}$$

holds uniformly for $\xi, \eta \in X$ and the set Y .

Proof. Note that, at $x \in \mathbb{S}^2$, we have

$$\begin{aligned} |\nabla\chi_\xi^{loc} \cdot \nabla\chi_\eta^{loc} - \nabla\chi_\xi \cdot \nabla\chi_\eta| &\leq |\nabla\chi_\xi^{loc} - \nabla\chi_\xi| |\nabla\chi_\eta| \\ &\quad + |\nabla\chi_\eta^{loc} - \nabla\chi_\eta| |\nabla\chi_\xi| + |\nabla\chi_\xi^{loc} - \nabla\chi_\xi| |\nabla\chi_\eta^{loc} - \nabla\chi_\eta|. \end{aligned}$$

By this inequality and Theorem 4.3, we see that

$$\begin{aligned} |\nabla\chi_\xi^{loc} \cdot \nabla\chi_\eta^{loc} - \nabla\chi_\xi \cdot \nabla\chi_\eta| &\leq C_1 h_X^{J-1} q_X^{-1} + C_2 h_X^{2J-2} \\ &= C_1 \rho_X h_X^{J-2} + C_2 h_X^{2J-2} \sim C \rho_X h_X^{J-2}. \end{aligned}$$

A similar calculation yields $|\chi_\xi^{loc}\chi_\eta^{loc} - \chi_\xi\chi_\eta| \leq Ch_X^J$. From this, the previous inequality, and h_X being small, we have that $|A_{\xi,\eta}^Y - A_{loc,\xi,\eta}^Y| \leq Ch_X^{J-2} \sum_{\zeta \in Y} w_\zeta = 4\pi Ch_X^{J-2} \sim Ch_X^{J-2}$. \square

Distance estimates. We have already dealt with a bound on $\|A^Y - \tilde{A}^Y\|_2$ in Proposition 8.1. We are really only interested in the ‘‘chopped’’ version of A_{loc}^Y – i.e., \tilde{A}_{loc}^Y , which is defined analogously to \tilde{A}^Y in (8.1).

Proposition 8.5. *For h_X sufficiently small and $J > 2$, we have that*

$$(8.9) \quad \|\tilde{A}_{loc}^Y - \tilde{A}^Y\|_2 < CK^2(\log(h_X))^2 h_X^{J-2}.$$

Proof. As before, we have $\|\tilde{A}_{loc}^Y - \tilde{A}^Y\|_2 \leq \max_{\eta \in X} (\sum_{\xi \in X} |\tilde{A}_{\xi,\eta}^Y - \tilde{A}_{loc,\xi,\eta}^Y|)$. For fixed η , all terms with $d(\xi, \eta) > Kh_X |\log(h_X)|$ are 0. It follows that

$$\sum_{\xi \in X} |\tilde{A}_{\xi,\eta}^Y - \tilde{A}_{loc,\xi,\eta}^Y| = \sum_{\xi \in B(\eta, r_h) \cap X} |A_{\xi,\eta}^Y - A_{loc,\xi,\eta}^Y|, \quad r_h = Kh_X |\log(h_X)|.$$

From (8.8), the difference in the right sum is uniformly bounded by Ch_X^{J-2} . Consequently, applying (8.2) then yields

$$\begin{aligned} \|\tilde{A}_{loc}^Y - \tilde{A}^Y\|_2 &\leq \max_{\eta \in X} (\sum_{\xi \in X} |\tilde{A}_{\xi,\eta}^Y - \tilde{A}_{loc,\xi,\eta}^Y|) \\ &\leq Ch_X^{J-2} \max_{\eta \in X} \#\{\text{row } \eta\} < CK^2(\log(h_X))^2 h_X^{J-2}, \end{aligned}$$

which is (8.9). \square

Corollary 8.6. *Assume that the hypotheses of Theorem 7.5, Proposition 8.1 and Proposition 8.5 hold. Then,*

$$\begin{cases} \|A - \tilde{A}^Y\|_2 \leq C((\log(h_Y))^2(h_Y/h_X)^{2M}h_X^{-\delta} + |\log(h_X)|h_X^{K\nu-2}), \\ \|A - \tilde{A}_{loc}^Y\|_2 \leq C((\log(h_Y))^2(h_Y/h_X)^{2M}h_X^{-\delta} + (\log(h_X))^2h_X^{J-2}). \end{cases}$$

Proof. The first result follows from first applying the triangle inequality and the three distance estimates from Theorem 7.5, Proposition 8.1 and Proposition 8.5 to $\|A - \tilde{A}^Y\|_2$. Establishing the second is done in a similar way, employing the additional fact that $h_X^{K\nu-2}|\log(h_X)| < (\log(h_X))^2h_X^{J-2}$, since $K\nu - 2 > J - 2$. \square

Stability. The matrices \tilde{A}^Y and \tilde{A}_{loc}^Y both have roughly the same stability properties as A and A^Y . We will establish them in Theorem 8.8 below. To do this, we will need the following elementary result from linear algebra, which we state without proof.

Lemma 8.7. *Let S and T be Hermitian $n \times n$ matrices and let S be positive definite. If there exists $0 < \varepsilon < 1$ such that $\|S - T\|_2 \leq \varepsilon\lambda_{\min}(S)$, then T is positive definite, and, in addition, these hold:*

$$\begin{aligned} (1 - \varepsilon)\lambda_{\min}(S) &\leq \lambda_{\min}(T) \leq (1 + \varepsilon)\lambda_{\min}(S), \\ \frac{1 - \varepsilon}{1 + \varepsilon}\kappa_2(S) &\leq \kappa_2(T) \leq \frac{1 + \varepsilon}{1 - \varepsilon}\kappa_2(S). \end{aligned}$$

Theorem 8.8. *Suppose that the conditions of Theorem 7.7 and Corollary 7.8 are satisfied and that J is as in Theorem 8.2. If $J > 4$, then both \tilde{A}^Y and \tilde{A}_{loc}^Y are positive definite, have $\lambda_{\min}(\tilde{A}^Y) \sim \lambda_{\min}(\tilde{A}_{loc}^Y) \sim q_X^2$, and also have $\kappa_2(\tilde{A}^Y) \sim \kappa_2(\tilde{A}_{loc}^Y) \sim q_X^{-2}$.*

Proof. Note that $J > 4$ implies that $K\nu > 4$, so that, as long as q_X is small, (8.3) holds, and so, using $h_X = \rho_X q_X$ and applying Theorem 7.7, we have

$$\|A^Y - \tilde{A}^Y\|_2 \leq Cq_X^{K\nu-4}|\log(q_X)|q_X^2 \leq \underbrace{Cq_X^{K\nu-4}|\log(q_X)|}_{\varepsilon_1}\lambda_{\min}(A^Y) = \varepsilon_1\lambda_{\min}(A^Y).$$

Since $K\nu - 4 > 0$, we may choose q_X so small that $\varepsilon_1 < 1$. Lemma 8.7 then implies the results stated for \tilde{A}^Y . Using this result and (8.9), we get

$$\|\tilde{A}^Y - \tilde{A}_{loc}^Y\|_2 \leq \underbrace{Cq_X^{J-4}(\log(q_X))^2}_{\varepsilon_2}\lambda_{\min}(\tilde{A}^Y) = \varepsilon_2\lambda_{\min}(\tilde{A}^Y).$$

Because $J > 4$, we may choose q_X sufficiently small so that $\varepsilon_2 < 1$. Applying Lemma 8.7 then yields the result for \tilde{A}_{loc}^Y . \square

Sparse and sparse local Lagrange Galerkin error estimates. We conclude by giving errors for the L^2 -Galerkin approximations to u , \tilde{u}_h^Y and $\tilde{u}_{loc,h}^Y$, which are obtained by discretizing with the chopped matrices \tilde{A}^Y and \tilde{A}_{loc}^Y . The estimates below are

gotten in very nearly the same way as the one in Theorem 7.9. The only change is that $\|(A^Y)^{-1}\| \|A^Y - A\| = \lambda_{\min}(A^Y)^{-1} \|A^Y - A\|$ gets replaced by

$$\lambda_{\min}(\tilde{A}^Y)^{-1} \|A - \tilde{A}^Y\|_2 \sim q_X^{-2} \|A - \tilde{A}^Y\|_2$$

in the first instance, and by

$$\lambda_{\min}(\tilde{A}_{loc}^Y)^{-1} \|A - \tilde{A}_{loc}^Y\|_2 \sim q_X^{-2} \|A - \tilde{A}_{loc}^Y\|_2$$

in the second.

Theorem 8.9. *Let J and K be as defined in Theorem 8.2. If the conditions of Theorem 7.7 and Corollary 7.8 are satisfied, then*

(8.10)

$$\|u - \tilde{u}_h^Y\|_{L^2} \leq C \left(h_X^{s+2} + \frac{(\log(h_Y)h_Y^M)^2}{h_X^{2M+2+\delta}} + |\log(h_X)|h_X^{K\nu-4} \right) \|f\|_{H_s}, \quad K\nu > 4,$$

(8.11)

$$\|u - \tilde{u}_{loc,h}^Y\|_{L^2} \leq C \left(h_X^{s+2} + \frac{(\log(h_Y)h_Y^M)^2}{h_X^{2M+2+\delta}} + (\log(h_X))^2 h_X^{J-4} \right) \|f\|_{H_s}, \quad J > 4.$$

9. IMPLEMENTATION AND NUMERICAL EXPERIMENTS

This section discusses the practical aspects of implementation and presents numerical experiments. We discuss the construction of point sets for the discrete approximation space and quadrature points, the assembly of the stiffness matrix, and the assembly of the right hand side vector.

The numerical scheme requires two sets of points on the sphere. The coarse set X is used to build a basis for the approximation space $V_{\phi_m, X}$ discussed in section 4. This space is spanned by the Lagrange functions $\{\chi_\xi\}_{\xi \in X}$ defined in (4.2), which have this form:

$$\chi_\xi(x) = \sum_{\eta \in X} \alpha_{\eta, \xi} \phi_m(x \cdot \eta) + \sum_{\ell=0}^{m-1} \sum_{k=1}^{2\ell+1} \beta_{\ell, k, \xi} Y_{\ell, k}(x),$$

$$\sum_{\eta} \alpha_{\eta, \xi} Y_{\ell, k}(\eta) = 0, \quad 0 \leq \ell \leq m-1, \quad 1 \leq k \leq 2\ell+1.$$

The $\alpha_{\xi, \eta}$ and $\beta_{\ell, k, \xi}$ coefficients must be computed for each $\xi \in X$. They are determined by $\chi_\xi(\zeta) = \delta_{\xi, \zeta}$ and the second set of equations above. Solving for them is not very difficult, even though each Lagrange function requires all of the points in X for its computation. However, for large data sets, there is a very efficient, parallelizable way to numerically compute local Lagrange functions. These functions, which were introduced in [8], require relatively few nodes from X and give very good approximations for the χ_ξ 's.

The second, finer set Y is used, in the quadrature formulas, to discretize entries in the stiffness matrix (7.1). By Theorem 7.5, the mesh norm of the set Y determines the error in the discrete stiffness matrix and should be chosen appropriately to obtain a desired accuracy in numerically approximating it. The quadrature weights $\{w_\zeta\}_{\zeta \in Y}$ satisfy

$$\int_{\mathbb{S}^2} \tilde{\chi}_\zeta(x) d\mu(x) = w_\zeta,$$

where $\tilde{\chi}_\zeta$ is the Lagrange function centered at $\zeta \in Y$ for the kernel ϕ_M . The kernel ϕ_M need not be the same kernel as the one used in the construction of the approximation space V_X , which was ϕ_m . The weights can be computed efficiently by solving a single linear system that can be preconditioned by the local Lagrange functions; see [9] for details. Solving the system with Generalized Minimum Residual method (GMRES) and the local Lagrange preconditioner requires few iterations. Experiments performed in [9] demonstrate that the number of iterations required seems to be independent of the number of points in Y .

The stiffness matrix assembly requires computing the quadrature nodes Y and quadrature weights $\{w_\zeta\}_{\zeta \in Y}$ and the coefficients $\{\alpha_{\eta,\xi}\}_{\xi,\eta \in X}$ and $\{\beta_{\ell,\xi}\}_{\xi \in X}$. We recall the discrete stiffness matrix entries found via quadrature:

$$A_{\xi,\eta}^Y = \sum_{\zeta \in Y} (a \nabla \chi_\xi \cdot \nabla \chi_\eta + b \chi_\xi \chi_\eta) |_\zeta w_\zeta.$$

We provide some details of the computation of $\nabla \chi_\xi \cdot \nabla \chi_\eta(\zeta)$. We expand the Lagrange functions in terms of the surface splines ϕ_m (denoted ϕ) as $\chi_\xi(\zeta) = \sum_{\tau \in X} \alpha_{\tau,\xi} \phi(\zeta, \tau) + p_\xi(\zeta)$ and $\chi_\eta(\zeta) = \sum_{\gamma \in X} \alpha_{\gamma,\eta} \phi(\zeta, \gamma) + p_\eta(\zeta)$. Let $x = \sin(\theta) \cos(\varphi)$, $y = \sin(\theta) \sin(\varphi)$, $z = \cos(\theta)$ where $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. On \mathbb{S}^2 , $\nabla f = \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{\sin(\theta)} \frac{\partial f}{\partial \varphi} \hat{\varphi}$. Let $\phi'(\zeta, \tau) = (m-1)(1-\zeta \cdot \tau)^{m-2} \log(1-\zeta \cdot \tau) + \zeta \cdot \tau - 1$. Let $\tau = (\tau_x, \tau_y, \tau_z)$ in Cartesian coordinates. We note that $\frac{\partial \phi(\cdot, \tau)}{\partial x} |_\zeta = \phi'(\zeta, \tau) \tau_x$, and similarly for the y and z partial derivatives. Evaluating the covariant derivative of the restricted surface spline in Cartesian coordinates then yields

$$\begin{aligned} \nabla \phi(\cdot, \tau) |_\zeta &= \phi'(\zeta, \tau) \left((1 - \zeta_x^2) \tau_x - \zeta_x \zeta_y \tau_y - \zeta_x \zeta_z \tau_z \right) \hat{i} \\ &\quad + (-\zeta_x \zeta_y \tau_x + (1 - \zeta_y^2) \tau_y - \zeta_x \zeta_y \tau_z) \hat{j} \\ &\quad + (-\zeta_x \zeta_z \tau_x - \zeta_y \zeta_z \tau_z + (1 - \zeta_z^2) \tau_z) \hat{k}. \end{aligned}$$

The evaluation of $\nabla \chi_\xi \cdot \nabla \chi_\eta(\zeta)$ then reduces to

(9.1)

$$\nabla \chi_\xi(\zeta) \cdot \nabla \chi_\eta(\zeta) = \left(\sum_{\tau \in X} \alpha_{\tau,\xi} \nabla \phi(\zeta, \tau) + \nabla p_\xi(\zeta) \right) \cdot \left(\sum_{\gamma \in X} \alpha_{\gamma,\eta} \nabla \phi(\zeta, \gamma) + \nabla p_\eta(\zeta) \right).$$

One final comment. In section 7.2, we noted that the column vector \tilde{f} , where $\tilde{f}_\xi = \langle f, \chi_\xi \rangle$, had to be numerically computed via quadrature, which is what was done in the numerical experiments described below, where $\tilde{f}_\xi \doteq \sum_{\zeta \in Y} f(\zeta) \chi_\xi(\zeta) w_\zeta$.

9.1. Numerical experiments. In this section, we discuss numerical results of various experiments that explore the computational properties of the Galerkin method. We consider different differential operators, explore the effects of the quadrature node density on the L^2 -error of the discrete solution, and compute condition numbers for the discrete stiffness matrix. We also demonstrate that local Lagrange functions, as discussed in [8], provide a computationally less expensive approximation space and yield comparable error and condition numbers as the approximation space generated by the Lagrange functions. We choose the spherical basis function $\phi_3(t) = (1-t)^2 \log(1-t)$ to construct the approximation space and $\phi_2(t) = (1-t) \log(1-t)$ for the quadrature weights. We use the minimum energy points for the centers X used in the approximation space $V_{\phi_3, X}$. For the quadrature nodes, we use the icosahedral nodes and quasi-minimum energy points. These

points are available for download; see [39]. For each experiment, the L^2 -error is computed by evaluating the discrete solution on a set of evaluation points E and applying the Lagrange function quadrature rule. The set E is 62,500 quasi-minimum energy points, which is used for each experiment independent of X and Y . Let N_X and N_Y denote the number of points in X and Y respectively. We approximate h_Y by $\frac{1}{\sqrt{N_Y}}$.

We first consider the problem $-\Delta u + u = f$ with $u = \exp(\cos(\theta))$ and $f = \exp(\cos(\theta))(\cos^2(\theta) + 2z \cos(\theta))$. In the second and third columns of Table 1 we display the relative L^2 -errors of the discrete solution for two separate experiments. To obtain the discrete stiffness matrix, we first fixed 961 centers for X and varied the number of quadrature points used in Y . The quadrature points are icosahedral nodes with between 2,562 points to 92,162 points. We theoretically expect the L^2 -error to be $\mathcal{O}(|\log(h_Y)|^2 h_Y^4)$. In fact, the numerically observed error is $\mathcal{O}(|\log(h_Y)|^2 h_Y^{5.2})$. The experiment was repeated with $N_X = 3,721$ minimum energy nodes and using the same Y . This time, ignoring the $N_Y = 2,562$ outlier, $|\log(h_Y)| h_Y^{5.5}$ is observed, indicating that improvement in the theoretical errors rate is possible. The Lagrange basis was used for these two sets.

Next, we treated the problem $-\operatorname{div}(\mathbf{a} \cdot \nabla u) + u = f$ for the case in which $\mathbf{a} = a(\theta, \phi) \mathbf{g}$, where \mathbf{g} is the metric tensor for \mathbb{S}^2 and $a(\theta, \phi) = 1 - \frac{1}{2} \cos(\theta)$. We again chose $u = \exp(\cos(\theta))$, which results in the right hand side being

$$f = \left(-\frac{1}{2}(\cos^3(\theta) + \cos^2(\theta) - 5 \cos(\theta) + 1) + 1 \right) \exp(\cos(\theta)).$$

We also consider the possibility of using a local Lagrange basis to discretize the PDE. In this case, the approximation space is $V_X = \operatorname{span}\{\chi_\xi^{loc} : \xi \in X\}$, where the χ_ξ^{loc} functions are constructed using only kernels $\phi(\cdot, \eta)$ such that $\operatorname{dist}(\xi, \eta) \leq 7h_X |\log(h_X)|$. See [8] for a detailed description of the theoretical properties of this basis. The χ_ξ^{loc} 's may be constructed in parallel by solving a small linear system. This reduces computational complexity associated with assembling the $\alpha_{\xi, \eta}$ coefficients. By appropriately tuning the number of kernels used per Lagrange function, the local Lagrange function can be made to satisfy $\|\chi_\xi - \chi_\xi^{loc}\|_{L^\infty} \sim h_X^{2m}$, where m is the smoothness of the kernel ϕ . For the anisotropic problem, the fifth

TABLE 1. Both $-\Delta u + u = f$ and $-\operatorname{div}(\mathbf{a} \cdot \nabla u) + u = f$ were numerically solved using minimum energy point sets for X and icosahedral point sets for Y . The L^2 -error for all cases was $\mathcal{O}(|\log(h_Y)|^2 h_Y^{5+})$. Here, $h_Y = N_Y^{-1/2}$. For the first equation, a Lagrange basis was used, and, for the second, a *local* Lagrange basis.

$-\Delta u + u = f$ Lagrange Basis			$-\operatorname{div}(\mathbf{a} \cdot \nabla u) + u = f$ Local Lagrange Basis		
N_Y	$N_X = 961$	$N_X = 3721$	N_Y	$N_X = 961$	$N_X = 3721$
2,562	7.86e-5	2.19e-2	2,500	8.00e-5	2.10e-2
10,242	2.22e-6	3.76e-5	10,000	2.46e-6	3.23e-5
23,042	3.34e-7	3.83e-6	22,500	3.02e-7	4.78e-6
40,962	8.96e-8	9.32e-7	40,000	7.80e-8	1.04e-6
92,162	1.50e-8	1.27e-7	90,000	1.10e-8	1.49e-7

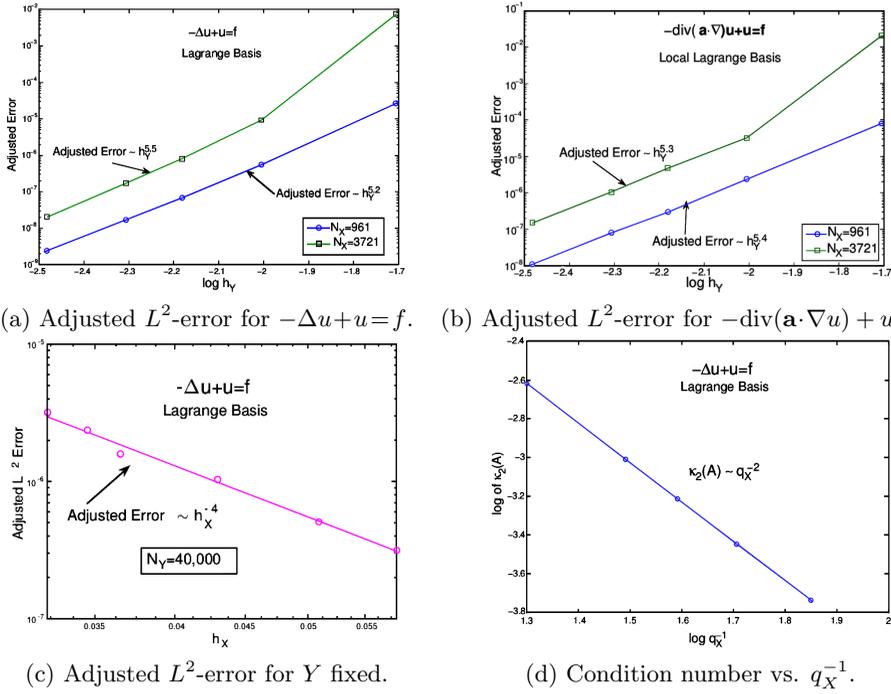


FIGURE 1. In (a) and (b), semi-log plots of the errors (adjusted by removing log factors) for $-\Delta u + u = f$ and $-\operatorname{div}(\mathbf{a} \cdot \nabla u) + u = f$ are shown. The minimum energy points were used for X and icosahedral points were used for Y . In (c), a loglog plot of the L^2 -error vs. h_X is plotted. For this experiment, the number of quadrature points is fixed and the number of centers used for the approximation space varies. In (d), the log of the condition number for the stiffness matrix for $-\Delta u + u = f$ is plotted.

and sixth columns in Table 1 display the results of the experiment using the local Lagrange basis. For $N_X = 961$, each local Lagrange function is constructed using about 423 centers and for $N_X = 3,721$, each local Lagrange function is constructed using around 776 centers, where the number of centers used per kernel is chosen to be all centers with distance at most $7h_X |\log(h_X)|$ from the center. The computed L^2 -errors from using the local basis versus the full basis are negligible, confirming the results in section 8.2. Since the local bases offer comparable L^2 -error while being computationally simpler, they offer no drawbacks when compared to the full basis and certainly are a good choice for doing the discretization step. The results of the two experiments are plotted in Figure 1(a) and Figure 1(b).

A third experiment was conducted keeping Y with fixed and varying X . The result is displayed in Figure 1(c). In this experiment, the error *increases* with decreasing h_X . This is counter intuitive, but in complete agreement with the theory. What this illustrates is that the dominant term in the L^2 -error comes from quadrature. This is no surprise and is a well-known phenomenon in Galerkin methods.

The condition number of the discrete stiffness matrix is dependent primarily on the separation radius of the centers, q_X . We theoretically predicted the condition

number to be $\mathcal{O}(q_X^{-2})$, which we validated numerically. See Figure 1(d). In addition, the theory predicts that changing the *quadrature nodes* should not significantly alter the condition number of the stiffness matrix. Again, this result was validated.

APPENDIX A. INTERPOLATION ERRORS AND THE “DOUBLING TRICK”

In this section we will discuss interpolation errors for spherical basis functions. Previous work on error estimates concentrated on interpolating functions *not* smooth enough to be in the reproducing Hilbert space \mathcal{N} corresponding to an SBF ϕ .

We will also need error estimates for interpolating functions *smoother* than those in \mathcal{N} . Results of this kind have been developed by Schaback [29] for positive definite functions on \mathbb{R}^n and on manifolds. In addition, Fuselier and Wright [10, Proposition 11] give a thorough treatment of the topic. For SBFs, the main result is that if \mathcal{N} is equivalent to Sobolev space H_τ , $\tau > n/2$, then, for functions in $H_{2\tau}$, the error rate is double the one obtained for functions in \mathcal{N} . This result is known as the “doubling trick”.

Throughout this section we will assume that an SBF ϕ has coefficients $\hat{\phi}_\ell$ that satisfy the following condition. There are constants c, C and $L \in \mathbb{N}$ such that

$$(A.1) \quad c(1 + \lambda_\ell)^{-\tau} \leq \hat{\phi}_\ell \leq C(1 + \lambda_\ell)^{-\tau},$$

holds either for all $\ell \geq 0$ or for all $\ell \geq L + 1$. Here $\lambda_\ell = \ell(\ell + n - 1)$ is an eigenvalue of $-\Delta_{\mathbb{S}^n}$.

A.1. Positive definite SBFs. In this section, we will deal with positive definite SBFs, so $\hat{\phi}_\ell > 0$ for all ℓ . The proposition below is a statement of the “doubling trick” in the case where $f \in H_{\tau+\alpha}$, $0 \leq \alpha \leq \tau$. We follow this up with a general result combining the doubling trick with estimates from [26, Theorem 5.5]. We separate the two so that the doubling trick itself is clearly stated.

Proposition A.1. *Let $\alpha, \beta, \tau \in \mathbb{R}$, with $\tau > n/2$ and $\alpha, \beta \in [0, \tau]$. Suppose that $f \in H_{\tau+\alpha}(\mathbb{S}^n)$ and that (A.1) holds for all $\ell \geq 0$. If h_X is sufficiently small, then*

$$(A.2) \quad \|f - I_X f\|_{H_\beta} \leq Ch_X^{\tau+\alpha-\beta} \|f\|_{H_{\tau+\alpha}}.$$

Proof. We will first deal with the $\beta = \tau$ case. The interpolant $I_X f$ being the projection of f onto V_X in the native space \mathcal{N} implies that $\langle f - I_X f, v \rangle_{\mathcal{N}} = 0$ for all $v \in V_X$. Consequently, we have $\langle f - I_X f, I_X f \rangle_{\mathcal{N}} = 0$ and so $\|f - I_X f\|_{\mathcal{N}}^2 = \langle f - I_X f, f \rangle_{\mathcal{N}}$. Let $g := f - I_X f$. The previous equation then takes the form $\|g\|_{\mathcal{N}}^2 = \langle g, f \rangle_{\mathcal{N}}$. From (3.3) and the bounds on $\hat{\phi}_\ell$, we see that

$$\begin{aligned} \|g\|_{\mathcal{N}}^2 &= \langle g, f \rangle_{\mathcal{N}} = \sum_{l=0}^{\infty} \sum_{m=1}^{d_\ell} (\hat{\phi}_\ell)^{-1} \hat{f}_{lm} \overline{\hat{g}_{lm}} \\ &\leq C \sum_{l=0}^{\infty} \sum_{m=1}^{d_\ell} (1 + \lambda_\ell)^{(\tau+\alpha)/2} |\hat{f}_{lm}| (1 + \lambda_\ell)^{(\tau-\alpha)/2} |\hat{g}_{lm}| \\ &\leq C \left(\sum_{\ell,m} (1 + \lambda_\ell)^{\tau+\alpha} |\hat{f}_{lm}|^2 \right)^{\frac{1}{2}} \left(\sum_{\ell,m} (1 + \lambda_\ell)^{\tau-\alpha} |\hat{g}_{lm}|^2 \right)^{\frac{1}{2}} \\ (A.3) \quad &= C \|f\|_{H_{\tau+\alpha}} \|g\|_{H_{\tau-\alpha}}. \end{aligned}$$

Applying Lemma 3.1 to $g = f - I_X f$ gives us $\|g\|_{H_{\tau-\alpha}} \leq Ch_X^\alpha \|g\|_{H_\tau}$. Combining this and (A.3) then yields

$$\|g\|_{\mathcal{N}}^2 \leq Ch_X^\alpha \|f\|_{H_{\tau+\alpha}} \|g\|_{H_\tau}.$$

In addition, the conditions on $\hat{\phi}_\ell$ imply that $\|g\|_{\mathcal{N}}^2 \geq c\|g\|_{H_\tau}^2$, and so

$$c\|g\|_{H_\tau}^2 \leq Ch_X^\alpha \|f\|_{H_{\tau+\alpha}} \|g\|_{H_\tau}.$$

Dividing both sides above by $\|g\|_{H_\tau}$ and replacing g by $g = f - I_X f$ then yields the $\beta = \tau$ case. If $0 < \beta \leq \tau$, Lemma 3.1 implies that $\|f - I_X f\|_{H_\beta} \leq Ch^{\tau-\beta} \|f - I_X f\|_{H_\tau}$. Since we have already shown that $\|f - I_X f\|_{H_\tau} \leq Ch_X^\alpha \|f\|_{H_{\tau+\alpha}}$, we have $\|f - I_X f\|_{H_\beta} \leq Ch^{\tau-\beta} \|f - I_X f\|_{H_\tau} \leq Ch^{\tau+\alpha-\beta} \|f\|_{H_{\tau+\alpha}}$. \square

We want to combine this with the result found in [26, Theorem 5.5], which deals with estimates for $f \in H_\mu$, $\tau \geq \mu > n/2$, to get the following general result.

Theorem A.2. *Let $\beta, \mu, \tau \in \mathbb{R}$, with $\tau > n/2$, $n/2 < \mu \leq 2\tau$, and $\beta \leq \min(\mu, \tau)$. Suppose that $f \in H_\mu(\mathbb{S}^n)$ and that (A.1) holds for all $\ell \geq 0$. If h_X is sufficiently small, then*

$$(A.4) \quad \|f - I_X f\|_{H_\beta} \leq Ch_X^{\mu-\beta} \|f\|_{H_\mu}.$$

Proof. If $\mu \geq \tau$, the result then follows from (A.2), with $\alpha = \mu - \tau$. If $\mu \leq \tau$, then (A.4) follows [26, Theorem 5.5]. \square

A.2. Conditionally positive definite SBFs. The SBFs dealt with above are all strictly positive definite. We will also need to obtain interpolation estimates for the conditionally positive definite SBFs discussed in section 3. Recall that for these SBFs, the $\hat{\phi}_\ell$'s need only be positive for $\ell > L$. For $0 \leq \ell \leq L$, the $\hat{\phi}_\ell$'s can be arbitrary. The interpolation operator for a conditionally positive definite SBF ϕ that reproduces Π_L is given in (3.4). Note that the coefficients $a_{\xi,L}$ and the polynomial $p_{X,L}$ are determined by the requirements that the interpolation condition $I_{X,L} f|_X = f|_X$ hold and also that the coefficients satisfy the condition on the right above. This condition also implies that changing the $\hat{\phi}_\ell$'s, with $0 \leq \ell \leq L$, will *not* change $I_{X,L} f$, because

$$\sum_{\xi \in X} a_{\xi,L} \sum_{\ell=0}^L \hat{\phi}_\ell Y_{\ell,m}(x) Y_{\ell,m}(\xi) = \sum_{\ell=0}^L \hat{\phi}_\ell Y_{\ell,m}(x) \underbrace{\sum_{\xi \in X} a_{\xi,L} Y_{\ell,m}(\xi)}_0 = 0.$$

The same reasoning further gives us that the terms $\sum_{\xi \in X} a_{\xi,L} \phi(x \cdot \xi)$ and $p_{X,L}$ are orthogonal. Thus, letting P_{Π_L} be the orthogonal projection onto Π_L , we have

$$(A.5) \quad P_{\Pi_L} I_{X,L} f = p_{X,L}.$$

These remarks above allow us to assume that $\hat{\phi}_\ell = 1$ for $0 \leq \ell \leq L$, with no loss of generality. We will thus make this assumption. Doing so turns ϕ into a strictly positive definite SBF and, consequently, makes possible forming the standard SBF interpolant $I_X f(x) = \sum_{\xi \in X} a_\xi \phi(x \cdot \xi)$, with the a_ξ 's determined by $I_X f|_X = f|_X$.

The two interpolants $I_X f$ and $I_{X,L} f$ are related in several ways. First of all, the difference of the two is given by

$$I_X f - I_{X,L} f = \sum_{\xi \in X} (a_\xi - a_{\xi,L}) \phi((\cdot) \cdot \xi) - p_{X,L}.$$

Since $(I_X f - I_{X,L} f)|_X = 0$, $\sum_{\xi \in X} (a_{\xi,L} - a_\xi) \phi((\cdot) \cdot \xi)$ interpolates $p_{X,L}$. Or, put another way, $\sum_{\xi \in X} (a_\xi - a_{\xi,L}) \phi((\cdot) \cdot \xi) = I_X p_{X,L}$. Rewriting the equation above using this fact yields

$$(A.6) \quad I_X f - I_{X,L} f = I_X p_{X,L} - p_{X,L}.$$

Theorem A.3. *Let $\beta, \mu, \tau \in \mathbb{R}$, with $\tau > n/2$, $n/2 < \mu \leq 2\tau$, and $\beta \leq \min(\mu, \tau)$. Suppose that $f \in H_\mu(\mathbb{S}^n)$ and that (A.1) holds for all $\ell \geq L+1$. If h_X is sufficiently small, then*

$$(A.7) \quad \|f - I_{X,L} f\|_{H_\beta} \leq Ch_X^{\mu-\beta} \|f\|_{H_\mu}.$$

Proof. From (A.6) and Theorem A.2 we see that

$$(A.8) \quad \begin{aligned} \|I_{X,L} f - f\|_{H_\beta} &\leq \|I_X f - f\|_{H_\beta} + \|I_X p_{X,L} - p_{X,L}\|_{H_\beta} \\ &\leq Ch_X^{\mu-\beta} \|f\|_{H_\mu} + \|I_X p_{X,L} - p_{X,L}\|_{H_\beta}. \end{aligned}$$

Because $p_{X,L}$ is a degree L polynomial, it is analytic, so of course it is in H_μ . It follows from Theorem A.2 that $\|I_X p_{X,L} - p_{X,L}\|_{H_\beta} \leq Ch_X^{\mu-\beta} \|p_{X,L}\|_{H_\mu}$. Furthermore, because $p_{X,L} \in \Pi_L$,

$$\lambda_L^{-(\mu-\beta)/2} \|p_{X,L}\|_{H_\mu} \leq \|p_{X,L}\|_{H_\beta} \leq \lambda_L^{(\mu-\beta)/2} \|p_{X,L}\|_{H_\mu}.$$

Consequently, $\|I_X p_{X,L} - p_{X,L}\|_{H_\beta} \leq Ch_X^{\mu-\beta} \|p_{X,L}\|_{H_\beta}$. Rewriting (A.5) as $\mathbb{P}_{\Pi_L}(I_{X,L} f - f) + \mathbb{P}_{\Pi_L} f = p_{X,L}$, taking the H_β norm, and using $\|f\|_{H_\beta} \leq \|f\|_{H_\mu}$, we obtain

$$\|p_{X,L}\|_{H_\beta} = \|\mathbb{P}_{\Pi_L}(I_{X,L} f - f) + \mathbb{P}_{\Pi_L} f\|_{H_\beta} \leq \|I_{X,L} f - f\|_{H_\beta} + \|f\|_{H_\mu},$$

from which it follows that

$$(A.9) \quad \|I_X p_{X,L} - p_{X,L}\|_{H_\beta} \leq Ch_X^{\mu-\beta} (\|I_{X,L} f - f\|_{H_\beta} + \|f\|_{H_\mu}).$$

From (A.8) and the previous inequality, we have

$$\|I_{X,L} f - f\|_{H_\beta} \leq Ch_X^{\mu-\beta} \|f\|_{H_\mu} + Ch_X^{\mu-\beta} \|I_{X,L} f - f\|_{H_\beta}.$$

Choose h_X so small that $Ch_X^\alpha < \frac{1}{2}$ yields

$$\|I_{X,L} f - f\|_{H_\tau} \leq Ch_X^\alpha \|f\|_{H_{\tau+\alpha}} + \frac{1}{2} \|I_{X,L} f - f\|_{H_\beta}.$$

Subtracting $\frac{1}{2} \|I_{X,L} f - f\|_{H_\beta}$ from both sides and manipulating the result gives us (A.7). \square

APPENDIX B. SOBOLEV SPACE ALGEBRAS

In addition to the estimates on interpolation error estimates derived above, we will need to deal with bounds on Sobolev norms of products of functions. Fortunately, Coulhon et al. [3] have established the requisite results. We will state these results for \mathbb{S}^n , in our notation, and only for the “p=2” cases. Before we state these results, we point out that $L_\tau^2 = H_\tau$ and that the norm $\|f\|_{\tau,2} = \|(-\Delta)^{\tau/2} f\|_{L^2} + \|f\|_{L^2}$, which is defined in [3, p. 286], is equivalent to $\|f\|_{H_\tau}$.

Theorem B.1 ([3, Theorem 27]). *Let f, g be in $H_\tau \cap L^\infty$, where $\tau \in [0, \infty)$. Then, $fg \in H_\tau \cap L^\infty$ and there exists $C > 0$ such that*

$$(B.1) \quad \|fg\|_{H_\tau} \leq C(\|f\|_{L^\infty} \|g\|_{H_\tau} + \|g\|_{L^\infty} \|f\|_{H_\tau}).$$

Proof. We just need to verify that the conditions in [3, Theorem 27] are satisfied. The parameters in [3, Theorem 27] connect with ours this way: $\tau := \alpha$, $p_1 = q_2 = \infty$, $p_2 = q_1 = 2$. If we take $\mathbb{M} = \mathbb{S}^n$, then all the conditions imposed on the manifold, including that of bounded geometry and positive injectivity radius, are satisfied. Thus, (B.1) holds for $\tau \in [0, \infty)$. \square

Using Banach space interpolation methods, Coulhon et al.⁶ showed that the following holds:

Proposition B.2 ([3, p. 334]). *Let $\tau > 0$. Then f is in $H_{\tau+1}$ if and only if f and $|\nabla f|$ are in H_τ . In addition, $\|f\|_{H_{\tau+1}} \sim \|f\|_{H_\tau} + \| |\nabla f| \|_{H_\tau}$.*

Corollary B.3. *Let $f, g \in H_{\tau+1}$ and suppose that $f, g, |\nabla f|, |\nabla g| \in L^\infty$. Then $\nabla f \cdot \nabla g \in H_\tau \cap L^\infty$ and*

$$\|\nabla f \cdot \nabla g\|_{H_\tau} \leq C(\|f\|_{H_{\tau+1}} + \|g\|_{H_{\tau+1}})(\| |\nabla f| \|_{L^\infty} + \| |\nabla g| \|_{L^\infty}).$$

Proof. We will first prove the result for $f = g$. By Proposition B.2, $|\nabla f| \in H_\tau$ and $\| |\nabla f| \|_{H_\tau} \leq C(\|f\|_{H_{\tau+1}} + \|f\|_{H_\tau}) \leq C\|f\|_{H_{\tau+1}}$. Moreover, by Theorem B.1, $|\nabla f|^2 \in H_\tau$. Also, since $|\nabla f| \in L^\infty$, $|\nabla f|^2 \in L^\infty$. The remarks above and (B.1) then imply that

$$\| |\nabla f|^2 \|_{H_\tau} \leq C\| |\nabla f| \|_{H_\tau} \| |\nabla f| \|_{L^\infty} \leq C\|f\|_{H_{\tau+1}} \| |\nabla f| \|_{L^\infty}.$$

The result for $f = g$ then follows immediately. For the general case, just use $|\nabla(f+g)|^2 - |\nabla(f-g)|^2 = 4\nabla f \cdot \nabla g$ and apply the result for the $f = g$ case. \square

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⁶As stated, the theorem [3, Theorem 30], which was employed in the interpolation process, requires that the manifold be unbounded. However, examining the result as stated in [37, Theorem 7.4.5] does not make this assumption, and so the interpolation result holds for compact manifolds as well.

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