DETERMINANTAL REPRESENTATIONS OF HYPERBOLIC CURVES VIA POLYNOMIAL HOMOTOPY CONTINUATION

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ABSTRACT. A smooth curve of degree d in the real projective plane is hyperbolic if its ovals are maximally nested, i.e., its real points contain $\lfloor \frac{d}{2} \rfloor$ nested ovals. By the Helton-Vinnikov theorem, any such curve admits a definite symmetric determinantal representation. We use polynomial homotopy continuation to compute such representations numerically. Our method works by lifting paths from the space of hyperbolic polynomials to a branched cover in the space of pairs of symmetric matrices.

INTRODUCTION

Let $p \in \mathbb{C}[t, x, y]$ be a homogeneous polynomial of degree $d \geqslant 1$. A *(linear symmetric) determinantal representation* of p is an expression

$$p = \det(tM_1 + xM_2 + yM_3),$$

where M_1, M_2, M_3 are complex symmetric matrices of size $d \times d$. Determinantal representations of plane curves are a classical topic of algebraic geometry. Existence for smooth curves of arbitrary degree was first proved by Dixon in 1902 [3]. For an exposition in modern language, see Beauville [2].

Real determinantal representations of real curves have only been studied systematically much later in the work of Dubrovin [4] and Vinnikov [15]. Of particular interest here are the *definite representations*, where some linear combination of the matrices M_1, M_2, M_3 is positive definite. By a celebrated result due to Helton and Vinnikov [9], these correspond exactly to the *hyperbolic curves*, whose real points consist of maximally nested ovals in the real projective plane.

The Helton-Vinnikov theorem (previously known as the Lax Conjecture) has attracted attention in connection with semidefinite programming, since it characterizes the boundary of those convex subsets of the real plane that can be described by linear matrix inequalities. See Vinnikov [17] for an excellent survey.

While the Helton-Vinnikov theorem ensures the existence of a definite determinantal representation for any hyperbolic curve, finding such a representation for a given polynomial p remains a difficult computational problem. With a suitable choice of coordinates, we can restrict to representations of the form

$$p = \det(tI_d + xD + yR)$$

where I_d is the identity matrix, D is a real diagonal and R a real symmetric matrix. The hyperbolicity of p is reflected in the fact that for any point $(u, v) \in \mathbb{R}^2$, all

roots of the univariate polynomial $p(t, u, v) \in \mathbb{R}[t]$ are real. Given such p, the computational task of finding the unknown entries of D and R leads, in general, to a zero-dimensional system of polynomial equations. However, as d grows, this direct approach quickly becomes infeasible in practice. This, as well as symbolic methods and an alternative approach via theta functions based on the proof of the Helton-Vinnikov theorem, have been investigated in [13]. As far as actual computations are concerned, $d = 6$ was the largest degree for which computations terminated in reasonable time.

We present here a more sophisticated numerical approach, implemented with NAG4M2: the *NumericalAlgebraicGeometry* package [10] for Macaulay2 [7]. We consider the branched cover of the space of homogeneous polynomials by pairs of matrices (D, R), with D diagonal and R symmetric, via the determinantal map

$$(D, R) \mapsto \det(tI_d + xD + yR).$$

We use known results on the number of equivalence classes of complex determinantal representations to show that the determinantal map is unramified over the set of smooth hyperbolic polynomials (Theorem 1.4). We then use the fact that this set is path-connected. In fact, an explicit path connecting any hyperbolic polynomial to a certain fixed polynomial was constructed by Nuij in [12], which we refer to as the N-path. Our algorithm works by constructing a lifting of the N-path to the covering space. The advantage over an application of a blackbox homotopy continuation solver to the zero-dimensional system of equations is that we need to track a *single* path instead of as many paths as there are complex solutions. We also recover the Helton-Vinnikov theorem from these topological considerations and the count of equivalence classes of complex representations.

Since the singular locus has codimension at least 2 inside the set of strictly hyperbolic polynomials, the N-path avoids singularities for almost all starting polynomials (Proposition 2.5). In the unlikely event that the N-path goes through the singular locus, it is possible to perturb the starting point and obtain an approximate determinantal representation. We also provide an algorithm that produces a complex determinantal representation via the complexification of the N-path.

We modify the approach of Nuij to introduce a *randomized N-path*, which depends on a choice of random linear forms. For a given starting polynomial p, we conjecture (Conjecture 2.8) that the randomized N-path avoids the singular locus with probability 1. This also results in a better practical complexity of the computation than the original N-path (Remark 4.1).

Our proof-of-concept implementation is written in the top-level interpreted language of Macaulay2 and, by default, uses standard double floating point precision. Even with these limitations we can compute small examples in reasonable time (see the example in §4 for $d = 6$): we are able to finish examples with $d \leq 10$ within one day. With arbitrary precision arithmetic and speeding up the numerical evaluation procedure, we see no obstacles to computing robustly for d in double digits using the present-day hardware.

We note that the developed method constructs an *intrinsically real* homotopy to find real solutions to a (specially structured) polynomial system. The only other intrinsically real homotopy known to us is introduced for Khovanskii-Rolle continuation in [1].

1. Hyperbolic and determinantal polynomials

We consider real or complex homogeneous polynomials of degree $d \geqslant 1$ in $n + 1$ variables (t, x), $x = x_1, \ldots, x_n$. Let

$$\mathcal{F} = \big\{ p \in \mathbb{C}[t, x] \mid p \text{ is homogeneous of total degree } d \text{ and } p(1, 0, \ldots, 0) = 1 \big\},$$
$$\mathcal{F}_\mathbb{R} = \mathcal{F} \cap \mathbb{R}[t, x].$$

A polynomial $p \in \mathcal{F}_\mathbb{R}$ is called *hyperbolic* if all roots of the univariate polynomial $p(t, u) \in \mathbb{R}[t]$ are real, for all $u \in \mathbb{R}^n$. It is called *strictly hyperbolic* if all these roots are distinct, for all $u \in \mathbb{R}^n$, $u \neq 0$. Write

$$\mathcal{H} = \big\{ p \in \mathcal{F}_\mathbb{R} \mid p \text{ is hyperbolic} \big\}.$$

Proposition 1.1.

(1) *The interior* $\mathrm{int}(\mathcal{H})$ *of* $\mathcal{H}$ *is the set of strictly hyperbolic polynomials and* $\mathcal{H}$ *is the closure of* $\mathrm{int}(\mathcal{H})$ *in* $\mathcal{F}_\mathbb{R}$.

(2) *The set* $\mathrm{int}(\mathcal{H})$ *is contractible and path-connected (hence so is* $\mathcal{H}$).

(3) *A polynomial* $f \in \mathcal{H}$ *is strictly hyperbolic if and only if the projective variety* $\mathcal{V}_\mathbb{C}(f)$ *defined by* f *has no real singular points.*

(4) *Let* $\mathcal{H}^\circ$ *be the set of hyperbolic polynomials* $p \in \mathcal{H}$ *for which* $\mathcal{V}_\mathbb{C}(p)$ *is smooth. Then* $\mathrm{int}(\mathcal{H}) \setminus \mathcal{H}^\circ$ *has codimension at least* 2 *in* $\mathcal{F}_\mathbb{R}$.

Proof. (1) and (2) are proved by Nuij [12] (see also Section 2 below). (3) is proved in [14, Lemma 2.4]. (4) follows from the fact that the elements of $\mathrm{int}(\mathcal{H})$ have no real singularities, while complex singularities must come in conjugate pairs. $\qquad\square$

For the remainder of this section, we restrict to the case $n = 2$ (plane projective curves) and use (x, y) instead of (x_1, x_2).

Remark 1.2. If $n = 2$ and $d \leqslant 3$, then $\mathcal{H}^\circ = \mathrm{int}(\mathcal{H})$, i.e., every strictly hyperbolic curve of degree at most 3 is smooth. This is simply because a real plane curve of degree at most 3 cannot have any non-real singularities.

When $d \geqslant 4$, a strictly hyperbolic curve may still have complex singularities. For example, let $p = 1/19(19t^4 - 31x^2t^2 - 86y^2t^2 + 9x^4 + 41x^2y^2 + 39y^4)$. One can check that p is hyperbolic and that the projective plane curve defined by p has no real singularities, hence p is strictly hyperbolic. However, $(1 : \pm 2 : \pm i)$ are two pairs of complex-conjugate singular points of $\mathcal{V}_\mathbb{C}(p)$. Thus, $p \in \mathrm{int}(\mathcal{H}) \setminus \mathcal{H}^\circ$.

We will use the notation

$$\mathcal{M} = \big\{ (D, R) \in \big(\mathrm{Sym}_d(\mathbb{C}) \big)^2 \mid D \text{ is diagonal} \big\},$$
$$\mathcal{M}_\mathbb{R} = \mathcal{M} \cap (\mathrm{Sym}(\mathbb{R}))^2.$$

Note that, since $n = 2$, we have $\dim_\mathbb{C} \mathcal{F} = \dim_\mathbb{R} \mathcal{F}_\mathbb{R} = \dim_\mathbb{R} \mathcal{M}_\mathbb{R} = \dim_\mathbb{C} \mathcal{M} = \frac{d(d+3)}{2}$. We study the map

$$\Phi \colon \begin{cases} \mathcal{M} & \to & \mathcal{F} \\ (D, R) & \mapsto & \det(tI_d + xD + yR) \end{cases}$$

and its restriction to $\mathcal{M}_\mathbb{R}$.

The image of $\mathcal{M}_\mathbb{R}$ under Φ is contained in $\mathcal{H}$. It is also not hard to show that it is closed (see [14, Lemma 3.4]). Our first goal is to find a connected open subset U of $\mathcal{H}$ such that the restriction of Φ to $\Phi^{-1}(U)$ is smooth.

For fixed $p \in \mathcal{H}$, the group $\mathrm{SL}_d(\mathbb{C}) \times \{\pm 1\}$ acts on the determinantal representations $p = \det(tM_1 + xM_2 + yM_3)$ via symmetric equivalence. In other words, any $A \in \mathrm{SL}_d(\mathbb{C}) \times \{\pm 1\}$ gives a new representation $p = \det(tAM_1A^T + xAM_2A^T + yAM_3A^T)$. When we restrict to the normalized representations we are considering, we have an action on pairs $(D, R) \in \Phi^{-1}(p)$ by those elements $A \in \mathrm{SL}_d(\mathbb{C}) \times \{\pm 1\}$ for which $AA^T = I_d$ (i.e., $A \in \mathrm{O}_d(\mathbb{C})$) and ADA^T is diagonal.

Theorem 1.3. *For $n = 2$, any $p \in \mathcal{F}$ has only finitely many complex representations $p = \det(tI_d + xD + yR)$ up to symmetric equivalence. If the curve $\mathcal{V}_{\mathbb{C}}(p)$ is smooth, the number of equivalence classes is precisely $2^{g-1} \cdot (2^g + 1)$, where $g = \binom{d-1}{2}$ is the genus of $\mathcal{V}_{\mathbb{C}}(p)$.*

Proof. For smooth curves, the equivalence classes of symmetric determinantal representations are in canonical bijection with ineffective even theta characteristics; see [13, Thm. 2.1] and the references given there. $\square$

Theorem 1.4. *The set $\mathcal{H}^{\circ}$ of smooth hyperbolic polynomials in three variables is an open, dense, path-connected subset of $\mathcal{H}$, and each fiber of Φ over a point of $\mathcal{H}^{\circ}$ consists of exactly $2^{g-1} \cdot (2^g + 1) \cdot 2^{d-1} \cdot d!$ distinct points.*

Proof. The statements about the topology of $\mathcal{H}^{\circ}$ follow immediately from Proposition 1.1. Let $p \in \mathrm{int}(\mathcal{H})$ and let $(D, R) \in \Phi^{-1}(p)$, which means $p = \det(tI_d + xD + yR)$. The diagonal entries of D are the zeros of $p(t, -1, 0)$. Since p is strictly hyperbolic, these zeros are real and distinct. So D is a real diagonal matrix with distinct entries. It follows then that the centralizer of D in $\mathrm{O}_d(\mathbb{C})$ consists precisely of the 2^d diagonal matrices with entries ± 1. Let S be such a matrix with $S \neq \pm I_d$. We want to identify the set of symmetric matrices R that commute with S. Up to permutation, we may assume that the first k diagonal entries of S are equal to -1 and the remaining $d - k$ are equal to 1. It follows then that any R with $SR = RS$ must have $r_{ij} = r_{ji} = 0$ if $i > k \geqslant j$, so that R is block-diagonal. For such R to show up in a pair $(D, R) \in \Phi^{-1}(p)$, the polynomial p must be reducible. In particular, if $p \in \mathcal{H}^{\circ}$, there is no such S commuting with R. It follows then that $\{SRS \mid S \text{ diagonal with } S^2 = I_d\}$ has 2^{d-1} distinct elements. Permuting the distinct diagonal entries of D gives $d!$ possible choices of D. This, combined with the count of equivalence classes in the preceding theorem, completes the proof. $\square$

Corollary 1.5. *The restriction of Φ to $\Phi^{-1}(\mathcal{H}^{\circ})$ is smooth.*

Proof. The restriction of Φ to $\Phi^{-1}(\mathcal{H}^{\circ})$ is a polynomial map with finite fibers that is unramified over $\mathcal{H}^{\circ}$, since the cardinality of the fiber does not change. Hence it is smooth (see, for example, Hartshorne [8, III.10]). $\square$

We sketch an argument for deducing the Helton-Vinnikov theorem from Theorem 1.4.

Corollary 1.6 (Helton-Vinnikov theorem). *Every hyperbolic polynomial $p \in \mathcal{H}$ in three variables admits a determinantal representation $p = \det(tI_d + xD + yR)$ with D diagonal and R real symmetric.*

Proof. Since all fibers of Φ over $\mathcal{H}^{\circ}$ have the same cardinality and $\mathcal{H}^{\circ}$ is path-connected, the number of real points in each fiber must also be constant over $\mathcal{H}^{\circ}$. That number cannot be zero, since there exist fibers with real points. (This amounts to showing that for each $d \geqslant 1$ there exists a real pair (D, R) such that

$p = \det(tI_d + xD + yR)$ defines a smooth curve. This can, for example, be deduced with the help of Bertini's theorem.) It follows that $\mathcal{H}^\circ$ is contained in $\Phi(\mathcal{M}_\mathbb{R})$. On the other hand, $\Phi(\mathcal{M}_\mathbb{R})$ is closed in $\mathcal{F}_\mathbb{R}$ by [14, Lemma 3.4] and contained in $\mathcal{H}$, hence $\Phi(\mathcal{M}_\mathbb{R}) = \mathcal{H}$. □

Remark 1.7. The number of equivalence classes of real definite representations of a hyperbolic curve is in fact also known, namely it is 2^g. See [13] and references to [16] given there. We conclude that $\Phi^{-1}(p) \cap \mathcal{M}_\mathbb{R}$ consists of $2^g \cdot 2^{d-1} \cdot d!$ distinct points for every $p \in \mathcal{H}^\circ$.

Note also that, even if p is hyperbolic, it will typically admit real determinantal representations $p = \det(tM_1 + xM_2 + yM_3)$ that are not definite, i.e., are not equivalent to such a representation with $M_1 = I_d$ and M_2, M_3 real. Such representations do not reflect the hyperbolicity of p.

2. The Nuij path

In order to use homotopy continuation methods for numerical computations, we need an explicit path connecting any two given points in the space $\mathcal{H}$ of hyperbolic polynomials.

2.1. **Original N-path.** Following Nuij [12], we consider the following operators on polynomials $\mathcal{F}_\mathbb{R} \subset \mathbb{R}[t, x] = \mathbb{R}[t, x_1, \ldots, x_n]$:

$$T_s^\ell \colon p \mapsto p + s\ell\frac{\partial p}{\partial t} \quad (\ell \in \mathbb{R}[x] \text{ a linear form}),$$
$$G_s \colon p \mapsto p(t, sx),$$
$$F_s = (T_s^{x_1})^d \cdots (T_s^{x_n})^d,$$
$$N_s = F_{1-s}G_s,$$

where $s \in \mathbb{R}$ is a parameter. For fixed s, all of these are linear operators on $\mathbb{R}[t, x]$ taking the affine-linear subspace $\mathcal{F}_\mathbb{R}$ to itself. Clearly, G_s preserves hyperbolicity for any $s \in \mathbb{R}$, and $G_0(p) = t^d$ for all $p \in \mathcal{F}_\mathbb{R}$. The operator F_s is used to "smoothen" the polynomials along the path $s \mapsto G_s(p)$. The exact statement is the following.

Proposition 2.1 (Nuij [12]). *For $s \geqslant 0$, the operators T_s^ℓ preserve hyperbolicity. Moreover, for $p \in \mathcal{H}$, we have $N_s(p) \in \mathrm{int}(\mathcal{H})$ for all $s \in [0, 1)$, with $N_0(p) \in \mathrm{int}(\mathcal{H})$ not depending on p and $N_1(p) = p$.* □

For $p \in \mathcal{F}_\mathbb{R}$, we call $[0, 1] \ni s \mapsto N_s(p)$ the *N-path* of p.

Remark 2.2. The N-path defines the contraction that appeared in Proposition 1.1(2): for all $p \in \mathcal{H}$, the N-path leads to $N_0(p) = F_1G_0(p) = F_1(t^d) = N_0(t^d)$.

However, it does not define a *strong* deformation retract as $N_s(t^d) = t^d$ holds only at the end of the N-path.

In order to ensure smoothness of the map $\Phi \colon \mathcal{M} \to \mathcal{H}$ along an N-path, we would like to ensure that the N-path stays inside the set $\mathcal{H}^\circ$ of smooth hyperbolic polynomials and thus away from the ramification locus, by the discussion in the preceding section.

Example 2.3. If $n = 2$ and $d \leqslant 3$, we know that $\mathcal{H}^\circ = \mathrm{int}(\mathcal{H})$ (Remark 1.2). For $n = d = 2$, we verify by explicit computation that the N-path stays inside the strictly hyperbolic conics, which in this case is just equivalent to irreducibility.

Let $D = \mathrm{Diag}(d_1, d_2)$ and $R = \begin{bmatrix} r_{(1,1)} & r_{(1,2)} \\ r_{(1,2)} & r_{(2,2)} \end{bmatrix}$. The quadric

$$N_s(\Phi(D, R)) = Ax^2 + Bxy + Cy^2 + Dxt + Eyt + Ft^2$$

is not contained in $\mathcal{H}^\circ$ if and only if it factors, which happens if and only if

$$
\begin{aligned}
4 \, \mathrm{Disc}(N_s(\Phi(D, R))) &= 4 \det \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \\
&= 2s^2(s-1)^2(d_1 - d_2)^2 \\
&\quad + 2s^2(s-1)^2 (r_{11} - r_{22})^2 \\
&\quad + s^4 r_{12}^2 (d_1 - d_2)^2 \\
&\quad + 8r_{12}^2 s^2 (1-s)^2 \\
&\quad + 16(s-1)^4 \\
&= 0.
\end{aligned}
$$

The sum-of-squares representation was produced using the intuition obtained by a numerical sum-of-squares decomposition delivered by YALMIP and further exact symbolic computations in MATHEMATICA (see `Nuij-d2.mathematica` [11]). The components of the sum-of-squares decomposition above vanish simultaneously only when $s = 1$ (this proves the proposition) and either $r_{12} = 0$ or $d_2 = d_1$.

Unfortunately, for $d \geqslant 4$ it is no longer true that the N-path always stays inside $\mathcal{H}^\circ$, due to the existence of strictly hyperbolic polynomials with complex singularities.

Example 2.4. Consider again the polynomial

$$p = 1/19(19t^4 - 31x^2t^2 - 86y^2t^2 + 9x^4 + 41x^2y^2 + 39y^4),$$

which is contained in $\mathrm{int}(\mathcal{H})$ but not in $\mathcal{H}^\circ$ (cf. Remark 1.2). One can verify through direct computation that the polynomial

$$
\begin{aligned}
r = \frac{1}{124659} \Big(&124659t^4 - 221616t^3(x + y) - 324t^2 - \big(205x^2 - 912xy + 1580y^2\big) \\
&+ 1440t \big(98x^3 + 41 - x^2y + 316xy^2 + 373y^3\big) \\
&+ 40 \big(1099x^4 - 1568x^3 - y + 5540x^2y^2 - 5968xy^3 + 5849y^4\big) \Big)
\end{aligned}
$$

is hyperbolic and smooth, i.e., $r \in H^\circ$, but $N_{1/9}(r) = p$, so that the N-path for r is not fully contained in $\mathcal{H}^\circ$.

However, one can still attempt to avoid the ramification points in the following manner.

Proposition 2.5. *The N-path $N_s(p)$, with parameter s varied along a piecewise-linear path $[0, c] \cup [c, 1) \subset \mathbb{C}$, does not meet the ramification locus of Φ, for almost all $c \in \mathbb{C}$. (More precisely, this holds for any c taken in the complement of some proper real algebraic subset of $\mathbb{C} \simeq \mathbb{R}^2$.)*

Proof. This immediately follows from the fact that the ramification locus is a proper complex subvariety of $\mathcal{F}$ and therefore has real codimension at least 2. $\square$

Remark 2.6. Using a random path as described in the proposition may result in a non-real determinantal representation; indeed, a non-real path for s is not guaranteed to result in a real point in the fiber $\Phi^{-1}(N_0(p))$ when a real point in $\Phi^{-1}(N_1(p))$ is taken.

One may ask for the probability of obtaining a real determinantal representation at the end of the path described in Proposition 2.5. (For a more precise question, one may pick c on a unit circle with a uniform distribution.) While this probability is clearly non-zero, deriving an explicit lower bound seems to be a very hard problem. A naive intuition suggests that the probability can be estimated as a ratio of the number of real representations (Remark 1.7) to the total count of complex representations (Theorem 1.4). The experiments with quartic hyperbolic curves, however, suggest that the probability is much higher.

On the other hand, the set of polynomials for which the N-path avoids the ramification locus altogether is dense, as the following proposition shows.

Proposition 2.7. *The set of strictly hyperbolic polynomials p such that $N_s(p) \in \mathcal{H}^\circ$ for all $s \in (0, 1]$ is dense in $\mathcal{H}$. (More precisely, its complement is a semialgebraic subset of positive codimension.)*

Proof. Consider the map

$$N\colon \begin{cases} \mathcal{F}_\mathbb{R} \times \mathbb{R} & \to & \mathcal{F}_\mathbb{R} \\ (p, s) & \mapsto & N_s(p), \end{cases}$$

and put $\mathcal{R} = \mathrm{int}(\mathcal{H}) \setminus \mathcal{H}^\circ \subset \mathcal{F}_\mathbb{R}$. We want to show that the projection of the semialgebraic set $N^{-1}(\mathcal{R}) \subset \mathcal{F}_\mathbb{R} \times \mathbb{R}$ onto $\mathcal{F}_\mathbb{R}$ has codimension at least 1.

The linear operator $N_s = (T_{1-s}^{x_1})^d \cdots (T_{1-s}^{x_n})^d G_s$ is bijective for all $s \neq 0$. For $p = -sx_k(\partial p/\partial t)$ is only possible for $p = 0$, since $sx_k(\partial p/\partial t)$ has strictly lower degree in t than p, so $T_s^{x_k}$ has trivial kernel. Furthermore, G_s is bijective for $s \neq 0$, hence so is N_s. This implies that all fibers $N^{-1}(p)$ of N have dimension at most 1 (except if $p = N_0(p)$ is the fixed endpoint of the N-path). In particular, all fibers of N over $\mathcal{R}$ are at most 1-dimensional, and since $\mathcal{R}$ has codimension 2 in $\mathcal{F}_\mathbb{R}$, this implies that $N^{-1}(\mathcal{R})$ also has codimension at least 2 in $\mathcal{F}_\mathbb{R} \times \mathbb{R}$, so that the projection of $N^{-1}(\mathcal{R})$ onto $\mathcal{F}_\mathbb{R}$ has codimension at least 1, as claimed. □

In principle, this proposition can be used as follows: Suppose $p \in \mathcal{H}$ is such that the N-path intersects the ramification locus. If $p \in \mathrm{int}(\mathcal{H})$, we can apply the algorithm to small random perturbations of p, which will avoid the ramification locus with probability 1. If p is a boundary point of $\mathcal{H}$, i.e., if p is not strictly hyperbolic, we can first replace p by $N_{1-\varepsilon}(p)$ for small $\varepsilon > 0$, which is strictly hyperbolic, and then perturb further if necessary.

On the other hand, we do not know whether the fixed endpoint $N_0(p) = F_1(t^d)$ is contained in $\mathcal{H}^\circ$ in all degrees d, although the direct computation in our experiments suggests that this is the case. (See also Theorem 2.10 below.)

2.2. **Randomized N-path.** The following modification of Nuij's construction has proved itself useful in computations. Let $e = \max\{d, n\}$ and choose a sequence $L = (\ell_1, \ldots, \ell_e) \in \mathbb{R}[x]^e$ of e linear forms. The *randomized N-path* N_s^L given by L is defined by

$$F_s^L = T_s^{\ell_1} T_s^{\ell_2} \cdots T_s^{\ell_e},$$
$$N_s^L = F_{1-s}^L G_s,$$

with T, G as before. Thus the randomized N-path only involves $\max\{d, n\}$ differential operators rather than dn. In practice, replacing the N-path by the randomized N-path has worked very well (cf. Remark 4.1).

The product of operators in F_s^L can be expanded explicitly, namely

$$F_s^L(p) = \sum_{k=0}^{e} s^k \sigma_k(\ell_1, \ldots, \ell_e) \frac{\partial}{\partial t^k} p$$

for all $p \in \mathcal{F}_{\mathbb{R}}$, where $\sigma_k(y_1, \ldots, y_e)$ denotes the elementary symmetric polynomial of degree k in the variables $y_1, \ldots, y_e$.

Conjecture 2.8. *Let $e \geq \{d, n\}$.*
  (1) *For a given $p \in \mathcal{H}$, the set of L such that $N_s^L(p) \in \mathcal{H}^\circ$ for $s \in [0, 1)$ is dense in $\mathbb{R}[x]^e$.*
  (2) *For a general choice of the linear forms $L \in \mathbb{R}[x]^e$, the set of polynomials p such that $N_s^L(p) \in \mathcal{H}^\circ$ for all $s \in [0, 1]$ is dense in $\mathcal{H}$.*

Note that e is chosen minimally in the sense that if $e < d$ or $e < n$, the fixed endpoint $N_0^L(p) = F_1^L(t^d)$ is no longer smooth or even strictly hyperbolic.

We will show that if $n = 2$, then at least that endpoint lies in $\mathcal{H}^\circ$ for a generic choice of L. The proof relies on Bertini's theorem in the following form.

Theorem 2.9 (Bertini's theorem, extended form; [6, Thm. 4.1])**.** *On an arbitrary ambient variety, if a linear system has no fixed components, then the general member has no singular points outside of the base locus of the system and of the singular locus of the ambient variety.*

For the proof, see also [5, Thm. 6.6.2].

Theorem 2.10. *Let $n = 2$. For all $d \geqslant 2$, the plane projective curve $\mathcal{V}_{\mathbb{C}}(F_1^L(t^d))$ is smooth for a generic choice of $L \in \mathbb{C}[x, y]_1^d$.*

Proof. We first show the following: Let $q \in \mathbb{C}[t, x, y]$ be homogeneous and monic in t with $\mathcal{V}_{\mathbb{C}}(q)$ smooth, $q \neq t$. Let $\ell \in \mathbb{C}[x, y]_1$ and k a positive integer, then $T_1^\ell(t^k q) = t^k q + \ell(kt^{k-1}q + t^k q') = t^{k-1}((t + k\ell)q + t\ell q')$, where $q' = \partial q/\partial t$. Put

$$r_\ell = (t + k\ell)q + t\ell q'.$$

We claim that, for generic ℓ, the variety $\mathcal{V}_{\mathbb{C}}(r_\ell)$ is smooth. To see this, consider $R = (t+u)q+tvq' \in \mathbb{C}[t, x, y, u, v]$. We find $\partial R/\partial u = q$ and $\partial R/\partial v = tq'$, hence the singular locus of the variety $\mathcal{V}_{\mathbb{C}}(R)$ in $\mathbb{P}^4$ is contained in $\mathcal{V}_{\mathbb{C}}(q) \cap \mathcal{V}_{\mathbb{C}}(tq')$. Consider the linear series on $\mathcal{V}_{\mathbb{C}}(R)$ defined by $u = k\ell$, $v = \ell$, $\ell \in \mathbb{C}[x, y]_1$. It is basepoint-free (in particular without fixed components), since the only basepoint on $\mathbb{P}^4$ is $(1 : 0 : 0 : 0 : 0)$, and that is not a point on $\mathcal{V}_{\mathbb{C}}(R)$. By Bertini's theorem as stated above, the variety $\mathcal{V}_{\mathbb{C}}(r_\ell)$ has no singular points outside the singular locus of $\mathcal{V}_{\mathbb{C}}(R)$ for generic $\ell \in \mathbb{C}[x, y]_1$. Thus we are left with showing that, for any point $P \in \mathcal{V}_{\mathbb{C}}(q) \cap \mathcal{V}_{\mathbb{C}}(tq')$, we have $(\nabla r_\ell)(P) \neq 0$ for generic $\ell \in \mathbb{C}[x, y]_1$. Since $\mathcal{V}_{\mathbb{C}}(q)$ is smooth by assumption, q and tq' are coprime in $\mathbb{C}[t, x, y]$, hence the intersection $\mathcal{V}_{\mathbb{C}}(q) \cap \mathcal{V}_{\mathbb{C}}(tq')$ in $\mathbb{P}^2$ is finite. If P is any of these intersection points, suppose first that $q'(P) = 0$, which implies either $(\partial q/\partial x)(P) \neq 0$ or $(\partial q/\partial y)(P) \neq 0$, since $\mathcal{V}_{\mathbb{C}}(q)$ is smooth. Suppose $a = (\partial q/\partial x)(P) \neq 0$ and put $b = (\partial q'/\partial x)(P)$, then

$$(\partial r_\ell/\partial x)(P) = (t(P) + k\ell(P))a + t(P)\ell(P)b$$
$$= (ka + bt(P))\ell(P) + at(P).$$

If $ka \neq -bt(P)$, there is at most one value $\ell(P)$ for which $(\partial r_\ell/\partial x)(P) = 0$. Otherwise, if $ka = -bt(P)$, then $at(P) \neq 0$, so $(\partial r_\ell/\partial x)(P) \neq 0$. The case $(\partial q/\partial x)(P) = 0$ and $(\partial q/\partial y)(P) \neq 0$ is analogous. Finally, if $q'(P) \neq 0$, then we must have $t(P) = 0$, hence $(\partial r_\ell/\partial t)(P) = (k + 1)\ell(P)q'(P)$ is non-zero, provided that $\ell(P) \neq 0$.

Thus we have shown that $\mathcal{V}_\mathbb{C}(r_\ell)$ is smooth for generic ℓ. To prove the original claim, let $L \in \mathbb{C}[x, y]_1^d$ and consider

$$F_1^L(t^d) = T_1^{\ell_1} \cdots T_1^{\ell_d}(t^d).$$

Applying the above, with $k = d$ and $q = 1$, shows that $T_1^{\ell_d}(t^d)$ is of the form $t^{d-1}q$, and $\mathcal{V}_\mathbb{C}(q)$ is smooth for generic ℓ_d. The claim now follows by induction. $\qquad\square$

3. Algorithm and implementation

Given a hyperbolic polynomial $p \in \mathcal{H}$, the N-path $N_s(p)$ connects $p = N_1(p)$ with $p_0 = N_0(p)$ which does not depend on p. This suggests the following algorithm to compute a determinantal representation $(D_p, R_p) \in \mathcal{M}_\mathbb{R}$ for p:

(1) Pick $(D_q, R_q) \in \mathcal{M}_\mathbb{R}$ giving a strictly hyperbolic polynomial $q = \Phi(D_q, R_q)$. Track the homotopy path $N_s(q)$ from $s = 1$ with the *start* solution (D_q, R_q) to $s = 0$ producing the *target* solution (D_{p_0}, R_{p_0}). Then $p_0 = \Phi(D_{p_0}, R_{p_0})$.
(2) Track the homotopy path $N_s(p)$ from $s = p0$ with the start solution (D_{p_0}, R_{p_0}) to $s = 1$ to obtain (D_p, R_p) such that $p = \Phi(D_p, R_p)$.

In principle, the first step only has to be performed once in each degree d. In what follows we describe two ways to set up a polynomial homotopy continuation for the pullback of an N-path.

3.1. N-path in the monomial basis.
One way is to take the coefficients of the polynomial $\Phi(D, R) - N_s(p) \in \mathbb{C}[D, R, s][t, x, y]$ with respect to the monomial basis of $\mathcal{F}$. This gives a family of square (#equations=#unknowns) systems of polynomial equations in $\mathbb{C}[D, R]$ parametrized by s.

Then this family is passed to a homotopy continuation software package (we use NAG4M2 [10]). As long as $s \in \mathbb{C}$ follows a path that ensures that $N_s(p)$ stays in $\mathcal{H}° \subset \mathcal{H}$, there are no singularities on the homotopy path except, perhaps, at the target system (see the discussion in Section 2).

The bottleneck of this approach is the expansion of the determinant in the expression $\Phi(D, R)$ and evaluation of its (t, x, y)-coefficients: it takes $\Theta(d!)$ operations and results in an expression with $\Theta(d!)$ terms. This limits us to $d \leq 5$ in the current implementation of this approach.

3.2. N-path with respect to a dual basis.
While it may seem that picking a basis of $\mathcal{F}$ different from the monomial one does not bring any advantage, it turns out to be crucial for practical computation in case of larger d.

We fix a *dual basis* in $\mathcal{F}^*$ consisting of $m = \dim \mathcal{F}$ evaluations e_i at general points $(t_i, x_i, y_i) \in \mathbb{C}^3$, for $i = 1, \ldots, m$. The current implementation generates the points with coordinates on the unit circle in $\mathbb{C}$ at random.

Now the family of polynomial systems to consider is

$$h_i = e_i(\Phi(D, R) - N_s(p)) \in \mathbb{C}[D, R, s], \ i = 1, \ldots, m.$$

Since $e_i(\Phi(D, R)) = \det(It_i + Dx_i + Ry_i)$, the evaluation of h_i and its partial derivatives costs $O((\dim \mathcal{F})^3) = O(d^6)$.

Evaluation of the (unexpanded) expression $\Phi(D, R)$ and its partial derivatives is much faster than expanding it in the monomial basis. The latter costs $\Theta(d!)$ in the worst case and, in addition, numerical tracking procedures would still need to evaluate the large expanded expression and its partial derivatives.

We modified the NAG4M2 implementation of evaluation circuits, which can be written as straight-line programs, to include taking a determinant as an atomic operation.

4. EXAMPLE

The last improvement in the implementation allows us to compute examples for larger d. With an implementation of the homotopy tracking in arbitrary precision arithmetic, we see no obstacles to computing determinantal representations for d in double digits.

To give an example, we choose the sextic

$$p = -36x^6 - 157x^4y^2 - 20x^3y^3 - 109x^2y^4 + 246xy^5 - 92y^6 - 12x^3y^2t + 90x^2y^3t$$
$$+ 10xy^4t + 76y^5t + 49x^4t^2 + 156x^2y^2t^2 - 16xy^3t^2 + 132y^4t^2 + 12xy^2t^3$$
$$- 14y^3t^3 - 14x^2t^4 - 27y^2t^4 + t^6.$$

The polynomial p is hyperbolic, since $p = \Phi(D, R)$ with

$$(4.1) \qquad D = \mathrm{Diag}(-3, -2, -1, 1, 2, 3), \quad R = \begin{bmatrix} 0 & 1 & -1 & 1 & 2 & 1 \\ 1 & 0 & -1 & -2 & 1 & -1 \\ -1 & -1 & 0 & 1 & 2 & 1 \\ 1 & -2 & 1 & 0 & -1 & 1 \\ 2 & 1 & 2 & -1 & 0 & -2 \\ 1 & -1 & 1 & 1 & -2 & 0 \end{bmatrix}.$$

Assuming this pair (D, R) is not known, let us describe the application of our algorithm to recover a determinantal representation of p; one can reproduce the following results by running lines in `showcase.m2` [11]. First, taking an arbitrary pair (D_q, R_q) and tracking the N-path $N_s(q)$ from the strictly hyperbolic polynomial $q = \Phi(D_q, R_q) = N_1(q)$ to the fixed polynomial $p_0 = N_0(q)$, we get

$$D_{p_0} = \mathrm{Diag}(.222847, 1.18893, 2.99274, 5.77514, 9.83747, 15.9829),$$

$$R_{p_0} = \begin{bmatrix} 6 & 2.51352 & 1.19571 & 4.04309 & 1.42786 & -1.98597 \\ 2.51352 & 6 & 3.08656 & .468873 & 2.38468 & 1.05948 \\ 1.19571 & 3.08656 & 6 & .785785 & 4.66027 & 2.29433 \\ 4.04309 & .468873 & .785785 & 6 & 1.6226 & .933245 \\ 1.42786 & 2.38468 & 4.66027 & 1.6226 & 6 & 3.50198 \\ -1.98597 & 1.05948 & 2.29433 & .933245 & 3.50198 & 6 \end{bmatrix}.$$

Tracking the N-path $N_s(p)$ from $p_0 = N_0(p) = N_0(q)$ to $p = N_1(p)$, we obtain

$$D' = \mathrm{Diag}(-3, -2, -1, 1, 2, 3),$$

$$R' = \begin{bmatrix} 0 & .596508 & -1.43241 & 2.00316 & 1.10471 & -.725394 \\ .596508 & 0 & .739773 & 1.79407 & .0604427 & -1.60948 \\ -1.43241 & .739773 & 0 & 1.56816 & 1.66137 & -.165953 \\ 2.00316 & 1.79407 & 1.56816 & 0 & .839374 & 2.00885 \\ 1.10471 & .0604427 & 1.66137 & .839374 & 0 & 1.57679 \\ -.725394 & -1.60948 & -.165953 & 2.00885 & 1.57679 & 0 \end{bmatrix},$$

which is an alternative determinantal representation of p. While we returned to the same point p in the base of the cover Φ, the route taken has led us to a different sheet than the sheet of the fiber point $(D, R) \in \Phi^{-1}(p)$ in (4.1) used to construct this example.

With the default settings of NAG4M2, the homotopy tracking algorithm takes 28 steps on the first path and 15 steps on the second. We were not able to find a determinantal representation for this example trying to solve the system $p = \Phi(D, R)$ directly. This is in line with what is reported in [13]: the largest examples that the general solvers could compute with this naïve strategy are in degree $d = 5$.

Remark 4.1. The following is a table of experiments that can be reproduced using the examples posted at [11]. Note that the paths produced by the original and randomized strategies for the same problem are *different paths* and some random choices are made even in the algorithm that follows the original (not randomized) N-path; see §3.2.

d	$m = \dim \mathcal{F}$	randomized?	precision(bits)	#steps	time(seconds)
6	27	*no*	53	22	1161
6	27	*yes*	53	24	1326
6	27	*no*	53	fail	
6	27	*no*	100	38	1870
6	27	*yes*	53	39	2098
7	35	*no*	53	fail	
7	35	*no*	100	42	9273
7	35	*yes*	53	37	7315
8	44	*no*	53	27	18173
8	44	*yes*	53	22	12091
9	54	*no*	53	fail	
9	54	*no*	100	43	60692
9	54	*yes*	53	38	43410
10	65	*no*	53	fail	
10	65	*no*	100	fail	
10	65	*yes*	53	36	163744

Our empirical conclusion is that using the randomized N-path instead of the original N-path allows computing determinantal representations for larger examples than the original N-path.

The main underlying reason that may explain the observed lower average practical complexity, in our opinion, is that $\deg_s N_s^L(p) = d$ while $\deg_s N_s(p) = 2d$, where $d = \deg p$.

One other reason for a path with lower degree in s having better properties on average is better average conditioning; one may observe that higher precision was needed to finish with the original strategy for several examples. The same increase in precision did not help in our example for $d = 10$.

That said, condition numbers for the fixed end ($s = 0$ in Proposition 2.1) of the original N-path can be better than the corresponding end of the random N-path: the latter depends on the random choices and can be significantly worse if unlucky. We provide two examples for $d = 6$ (separated by a dashed line): if the original strategy does not fail, then it is even slightly faster than the randomized one. The

former is faster near one end of the path $(s = 0)$, while the later is faster near the other end $(s = 1)$.

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