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EXPONENTIAL INTEGRABILITY PROPERTIES OF NUMERICAL APPROXIMATION PROCESSES FOR NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

MARTIN HUTZENTHALER, ARNULF JENTZEN, AND XIAOJIE WANG

ABSTRACT. Exponential integrability properties of numerical approximations are a key tool for establishing positive rates of strong and numerically weak convergence for a large class of nonlinear stochastic differential equations. It turns out that well-known numerical approximation processes such as Euler-Maruyama approximations, linear-implicit Euler approximations, and some tamed Euler approximations from the literature rarely preserve exponential integrability properties of the exact solution. The main contribution of this article is to identify a class of stopped increment-tamed Euler approximations which preserve exponential integrability properties of the exact solution under minor additional assumptions on the involved functions.

1. INTRODUCTION

Let $T \in (0,\infty)$, $d, m \in \mathbb{N} = \{1, 2, \ldots\}$, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be locally Lipschitz continuous functions, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $W\colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, and let $X\colon [0,T] \times \Omega \to \mathbb{R}^d$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths satisfying that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = X_0 + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s. \tag{1}$$

The stochastic process X is thus a solution process of the stochastic differential equation (SDE) (1).

The goal of this paper is to identify numerical approximations $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, that converge in the strong sense to the exact solution of the SDE (1) and that *preserve exponential integrability properties* in the sense that for all suffciently regular functions $U\colon \mathbb{R}^d \to [0,\infty)$ with $\sup_{t\in[0,T]} \mathbb{E}\big[\exp(U(X_t))\big] < \infty$ it holds that $\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\big[\exp\big(U(Y_t^N)\big)\big] < \infty$. Our main motivation for this is that such exponential integrability properties are a key tool for establishing rates of strong and numerically weak convergence for a large class of nonlinear SDEs. To be more specific, strong convergence rates of approximations of a multi-dimensional

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The third author is the corresponding author.

SDE have, except in Dörsek [6] and except in [17], only been established if the coefficients of the SDE are globally monotone (see, e.g., (H2) in Prévôt and Röckner [29] for the global monotonicity assumption). Unfortunately, most of the nonlinear SDEs from the literature fail to satisfy the global monotonicity assumption (see, e.g., Section 4 in Cox et al. [3] for a list of examples). In Corollary 3.2 in Dörsek [6], strong convergence rates for spatial spectral Galerkin approximations of the solution of the vorticity formulation of two-dimensional stochastic Navier-Stokes equations have been established by exploiting *exponential integrability properties*. Moreover, the perturbation estimate in Theorem 1.2 in [17] implies in a general setting that suitable *exponential integrability properties* of a family of approximation processes are sufficient to establish *strong convergence rates*. This conditional result together with the *exponential integrability properties* established in this article then yields strong convergence rates for the numerical scheme proposed in this article (see (6) below) for several SDEs with nonglobally monotone coefficients (see Theorem 1.3 in [17] for details). In particular, to the best of our knowledge, the numerical scheme proposed in this article (see (6) below) is the first approximation method for which temporal strong convergence rates have been proved (see Theorem 1.3 in [17]) for at least one multi-dimensional SDE with nonglobally monotone coefficients (see Section 3.1 in [17] for a list of example SDEs for which temporal strong convergence rates for the numerical method (6) below have been proved). In addition, exponential integrability properties of numerical approximations are necessary for approximating expectations of exponentially growing test functions of the exact solution.

There are a number of SDEs in the literature that admit exponential integrability properties. We focus on Corollary 2.4 in Cox et al. [3] (see, for example, also Lemma 2.3 in Zhang [37]). Let $\rho \in [0, \infty)$, $U \in C^3(\mathbb{R}^d, [0, \infty))$, and $\bar{U} \in C(\mathbb{R}^d, \mathbb{R})$ satisfy for all $x \in \mathbb{R}^d$ that $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$, $\inf_{y \in \mathbb{R}^d} \bar{U}(y) > -\infty$, and

$$U'(x)\mu(x) + \tfrac{1}{2}\mathrm{tr}\Big(\sigma(x)\sigma(x)^*(\mathrm{Hess}\, U)(x)\Big) + \tfrac{1}{2}\left\|\sigma(x)^*(\nabla U)(x)\right\|^2 + \bar{U}(x) \leq \rho\, U(x). \tag{2}$$

Then Corollary 2.4 in Cox et al. [3] yields that for all $t \in [0, T]$ it holds that

$$\mathbb{E}\left[\exp\!\left(\tfrac{U(X_t)}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(X_s)}{e^{\rho s}}\, ds\right)\right] \leq \mathbb{E}\Big[e^{U(X_0)}\Big] \in (0, \infty). \tag{3}$$

Section 4 lists a selection of SDEs from the literature which satisfy condition (2). Further instructive exponential integrability results for solutions of SDEs can be found, e.g., in [1, 7, 8, 10, 12, 23, 37]. In light of inequality (3), the goal of this paper is, in particular, to identify numerical approximations that converge in the strong sense to the exact solution of the SDE (1) and that *preserve inequality* (3) in a suitable sense; see inequality (11) below.

It turns out that many well-known numerical methods for SDEs fail to preserve exponential integrability properties. For instance, in the special case where $d = m = 1$, where $X_0 = \xi \in \mathbb{R}$, and where for all $x \in \mathbb{R}$ it holds that $\mu(x) = -x^3$ and $\sigma(x) = 1$, the solution process X of the SDE (1) satisfies that for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = X_0 - \int_0^t (X_s)^3\, ds + W_t. \tag{4}$$

In that case, inequality (2) holds with $\rho = 0$, $\varepsilon \in (0, \frac{1}{2})$, $U = \big(\mathbb{R} \ni x \mapsto \varepsilon|x|^4 \in [0,\infty)\big)$, and $\bar{U} = \big(\mathbb{R} \ni x \mapsto 4\varepsilon\,(1-2\varepsilon)\,x^6 - 6\varepsilon x^2 \in \mathbb{R}\big)$; see Subsection 5.1 below. Thus, Corollary 2.4 in Cox et al. [3] implies for all $\varepsilon \in [0, \frac{1}{2}]$ that

$$\sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\varepsilon\,|X_t|^4 + \int_0^t 4\varepsilon\,(1-2\varepsilon)\,|X_s|^6 - 6\varepsilon\,|X_s|^2\,ds\bigg)\bigg] \leq \mathbb{E}\Big[e^{\varepsilon|X_0|^4}\Big] < \infty \tag{5}$$

and, in particular, for all $\varepsilon \in [0, \frac{1}{2})$ that $\sup_{t\in[0,T]} \mathbb{E}\big[\exp\big(\varepsilon|X_t|^4\big)\big] < \infty$. If $Y^N\colon [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, are the classical Euler-Maruyama approximations as defined in (181) below with $D_t = \mathbb{R}$, $t \in (0,T]$, then moments are finite but unbounded in the sense that for all $N \in \mathbb{N}$, $p \in (0,\infty)$ it holds that $\mathbb{E}\big[|Y_T^N|^p\big] < \infty$ and $\lim_{M\to\infty} \mathbb{E}\big[|Y_T^M|^p\big] = \infty$ (see Theorem 2.1 in [19] for the case $p \in [1,\infty)$ and Theorem 2.1 in [21]) whereas approximations of $\mathbb{E}\big[\exp\big(\delta|X_t|^4\big)\big]$, $\delta \in (0,\varepsilon)$, $t \in (0,T]$, are infinite in the sense that for all $N \in \mathbb{N}$, $p \in (0,\infty)$, $q \in (2,\infty)$ it holds that $\inf_{t\in(0,T]} \mathbb{E}\big[\exp(p|Y_t^N|^q)\big] = \infty$; see Lemma 5.1 below. Next, if $Y^N\colon [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, are the linear-implicit Euler approximations as defined in (184) below with $D_t = \mathbb{R}$, $t \in (0,T]$, then strong convergence holds in the sense that for all $p \in (0,\infty)$ it holds that $\lim_{N\to\infty} \mathbb{E}\big[|X_T - Y_T^N|^p\big] = 0$ whereas approximations of $\mathbb{E}\big[\exp\big(\delta|X_t|^4\big)\big]$, $\delta \in (0,\varepsilon)$, $t \in (0,T]$, are infinite in the sense that for all $N \in \mathbb{N}$, $p \in (0,\infty)$, $q \in (2,\infty)$ it holds that $\inf_{t\in(0,T]} \mathbb{E}\big[\exp(p|Y_t^N|^q)\big] = \infty$; see Lemma 5.2 below. Moreover, if $Y^N\colon [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, are tamed Euler approximations as defined in (194) or as in (199) below with $D_t = \mathbb{R}$, $t \in (0,T]$, then approximations of $\mathbb{E}\big[\exp\big(\delta|X_t|^4\big)\big]$, $\delta \in (0,\varepsilon)$, $t \in (0,T]$, are finite but unbounded in the sense that for all $N \in \mathbb{N}$, $p \in (0,\infty)$, $q \in (3,\infty)$ it holds that $\sup_{t\in[0,T]} \mathbb{E}\big[\exp(\varepsilon|Y_t^N|^4)\big] < \infty$ and $\lim_{M\to\infty} \inf_{t\in(0,T]} \mathbb{E}\big[\exp(p|Y_t^M|^q)\big] = \infty$; see Corollary 5.4 and Corollary 5.5 below. In the above sense, Euler-Maruyama approximations, linear-implicit Euler approximations, and tamed Euler approximations as defined in (194) or as in (199) are not suitable for approximating $\mathbb{E}\big[\exp\big(\delta|X_t|^4\big)\big]$, $\delta \in (0,\varepsilon)$, $t \in (0,T]$, in the numerical weak sense. Lemma 5.3 below also indicates that further numerical one-step approximation methods whose one-step increment function grows sufficiently fast as the discretization step size decreases are not suitable for approximating expectations of exponential functionals in the generality of Theorem 1.1 below.

There are many results in the literature which prove uniform boundedness of polynomial moments of numerical approximations of certain nonlinear SDEs with superlinearly growing coefficients; see, e.g., [1, 2, 5, 9, 11, 13–16, 18, 20, 25, 28, 30–32, 34–36]. To the best of our knowledge, the only reference on exponential integrability properties of appropriate numerical approximations for nonlinear SDEs is Bou-Rabee and Hairer [1]. More precisely, Lemma 3.6 in Bou-Rabee and Hairer [1] implies that there exists $\theta \in (0,\beta)$ such that $\sup_{h\in(0,1]} \mathbb{E}\big[\exp(\theta U(\bar{X}^h_{\lfloor 1/h\rfloor}))\big] < \infty$ where $\bar{X}^h\colon \{0,1,2,\dots\} \times \Omega \to \mathbb{R}^d$, $h \in (0,1]$, is a 'patched' version of the Metropolis-Adjusted Langevin Algorithm (MALA) for the overdamped Langevin dynamics (see Subsection 4.9 below) where the potential energy function $U \in C^4(\mathbb{R}^d,\mathbb{R})$ satisfies certain assumptions; see [1] for the details. In addition, Proposition 5.2 in Bou-Rabee and Hairer [1] provides an exponential one-step estimate for MALA.

In this article, we propose the following method to approximate the solution of the SDE (1) and to preserve inequality (3) in a suitable sense. Let $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be the mappings which satisfy that for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N-1\}$,

$t \in (\frac{nT}{N}, \frac{(n+1)T}{N}]$ that $Y_0^N = X_0$ and

$$
\begin{aligned}
Y_t^N &= Y_{\frac{nT}{N}}^N + \mathbb{1}_{\left\{\|Y_{nT/N}^N\| \leq \exp\left(|\ln(N/T)|^{1/2}\right)\right\}} \\
&\quad \cdot \left[\frac{\mu(Y_{nT/N}^N)(t-\frac{nT}{N}) + \sigma(Y_{nT/N}^N)(W_t - W_{nT/N})}{1 + \|\mu(Y_{nT/N}^N)(t-\frac{nT}{N}) + \sigma(Y_{nT/N}^N)(W_t - W_{nT/N})\|^2} \right].
\end{aligned}
\tag{6}
$$

This method differs from the classical Euler-Maruyama scheme in two aspects. First, the Euler-Maruyama increment is divided through by one plus the squared norm of the Euler-Maruyama increment. This ensures that the increments of the numerical method (6) are uniformly bounded. Second, the approximation paths with $N \in \mathbb{N}$ time discretizations are stopped after leaving the set $\{x \in \mathbb{R}^d \colon \|x\| \leq \exp\left(|\ln(N/T)|^{1/2}\right)\}$ where we choose the stopping levels mainly such that for all $p \in (0, \infty)$ it holds that $\lim_{N\to\infty} \exp\left(|\ln(N/T)|^{1/2}\right) N^{-p} = 0$. These a priori bounds give us control on certain rare events. In addition, observe that the numerical approximations $\{0, 1, \ldots, N\} \times \Omega \ni (n, \omega) \mapsto Y_{nT/N}^N(\omega) \in \mathbb{R}^d$, $N \in \mathbb{N}$, can be easily implemented recursively. In fact, this implementation requires only a few additional arithmetical operations in each recursion step compared to the classical Euler-Maruyama approximations. Theorem 1.1 below shows that the numerical approximations (6) preserve inequality (3) in a suitable sense under slightly stronger assumptions on μ, σ, U, and $\bar{U}$.

Theorem 1.1. *Assume the above setting, let* $p, c \in [1, \infty)$, *let* $\tau_N \colon \Omega \to [0, T]$, $N \in \mathbb{N}$, *be mappings, assume for all* $N \in \mathbb{N}$ *that* $\tau_N = \inf\left(\left\{t \in \{0, \frac{T}{N}, \frac{2T}{N}, \ldots, T\} : \|Y_t^N\| > \exp(|\ln(N/T)|^{1/2})\right\} \cup \{T\}\right)$, *and assume for all* $x, y \in \mathbb{R}^d$, $i \in \{1, 2, 3\}$ *that*

$$\|\mu(x)\| + \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d)} \leq c\,(1 + \|x\|^c)\,, \tag{7}$$

$$|\bar{U}(x) - \bar{U}(y)| \leq c\,(1 + \|x\|^c + \|y\|^c)\,\|x - y\|, \tag{8}$$

$$\|x\|^{1/c} \leq c\,(1 + U(x))\,, \tag{9}$$

$$\|U^{(i)}(x)\|_{L^{(i)}(\mathbb{R}^d, \mathbb{R})} \leq c\,(1 + U(x))^{\max\{1 - i/p, 0\}}\,. \tag{10}$$

Then it holds for all $r \in (0, \infty)$ *that* $\lim_{N\to\infty} \left(\sup_{t\in[0,T]} \mathbb{E}\left[\|X_t - Y_t^N\|^r\right]\right) = 0$, *it holds that*

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\frac{U(Y_t^N)}{e^{\rho t}} + \int_0^{t\wedge\tau_N} \frac{\bar{U}(Y_s^N)}{e^{\rho s}}\, ds\right)\right] \leq \mathbb{E}\left[e^{U(X_0)}\right] < \infty, \tag{11}$$

and it holds that $\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(e^{-\rho t}\, U(Y_t^N) + \int_0^{t\wedge\tau_N} e^{-\rho s}\, \bar{U}(Y_s^N)\, ds\right)\right] < \infty$.

Theorem 1.1 is a special case of our main result, Corollary 3.8 below, in which the state space of the exact solution of the SDE under consideration is an open subset of $\mathbb{R}^d$. Corollary 3.8, in turn, follows from our general result on exponential integrability properties of stopped increment-tamed Euler-Maruyama schemes, Theorem 2.9 below, and from convergence in probability of stopped increment-tamed Euler-Maruyama schemes, Corollary 3.7 below (see also Subsection 1.1 below for a brief outline of the proof of Corollary 3.8). To the best of our knowledge, Theorem 1.1 and its generalization in Corollary 3.8, respectively, are the first results in the literature which imply exponential integrability properties for numerical approximations of the stochastic Ginzburg-Landau equation in Subsection 4.2, for numerical approximations of the stochastic Lorenz equation with additive noise in Subsection 4.3, for numerical approximations of the stochastic van der Pol oscillator in Subsection 4.4, for numerical approximations of the stochastic Duffing-van der Pol oscillator in Subsection 4.5, for numerical approximations of the model from experimental psychology in Subsection 4.6, for numerical approximations of the stochastic SIR model in Subsection 4.7, or—under additional assumptions on the model—for numerical approximations of the Langevin dynamics in Subsection 4.8.

1.1. **A brief outline of the proof of the main result of this article.** In this section we give a brief and rough outline of the proof of Theorem 1.1. Theorem 1.1 is a special case of Corollary 3.8, which is the main result of this article. For our outline of the proof of Theorem 1.1, suppose the assumptions of Theorem 1.1, let $D_t \subseteq \mathbb{R}^d$, $t \in (0,\infty)$, be the sets with the property that for all $t \in (0,\infty)$ it holds that $D_t = \big\{x \in \mathbb{R}^d \colon \|x\| \leq \exp\big(|\ln(t)|^{1/2}\big)\big\}$, let $\mathcal{G}^U_{\mu,\sigma} \colon \mathbb{R}^d \to \mathbb{R}$ be the function with the property that for all $x \in \mathbb{R}^d$ it holds that $\mathcal{G}^U_{\mu,\sigma}(x) = U'(x)\mu(x) + \frac{1}{2}\mathrm{tr}\big(\sigma(x)\sigma(x)^*(\mathrm{Hess}\, U)(x)\big)$ (cf. (2) above), let $\Phi_h \colon \mathbb{R}^d \times [0,h] \times \mathbb{R}^m \to \mathbb{R}^d$, $h \in [0,T]$, be the functions with the property that for all $h \in [0,T]$, $(x,t,y) \in \mathbb{R}^d \times [0,h] \times \mathbb{R}^m$ it holds that

$$\Phi_h(x,t,y) = x + \tfrac{\mu(x)t+\sigma(x)y}{1+\|\mu(x)t+\sigma(x)y\|^2}, \tag{12}$$

and let $Z^{s,x,h} \colon [0,h] \times \Omega \to \mathbb{R}^d$, $h \in [0,T-s]$, $s \in [0,T]$, $x \in \mathbb{R}^d$, be the stochastic processes with the property that for all $s \in [0,T]$, $h \in [0,T-s]$, $x \in \mathbb{R}^d$, $t \in [0,h]$ it holds that $Z^{s,x,h}_t = \Phi_h(x,t,W_{s+t}-W_s)$. Observe that for all $N \in \mathbb{N}$, $t \in [0,\frac{T}{N}]$, $\omega \in \Omega$ with $X_0(\omega) \in D_{T/N}$ it holds that $Y^N_t(\omega) = Z^{0,X_0(\omega),T/N}_t(\omega)$. A key step in the proof of Theorem 1.1 is to show that there exists a real number $c \in (0,\infty)$ such that for all $s \in [0,T]$, $h \in [0,\min\{T-s,1\}]$, $x \in D_h$, $t \in [0,h]$ it holds that

$$\mathbb{E}\bigg[\exp\bigg(\tfrac{U(Z^{s,x,h}_t)}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(Z^{s,x,h}_r)}{e^{\rho r}}\,dr\bigg)\bigg] \leq \exp\Big(c\,t^{1+1/c} + U(x)\Big) \tag{13}$$

(cf. (105) in the proof of Theorem 2.9 in Subsection 2.3 below and cf. also Lemma 2.8 in Subsection 2.2 below). We prove (13) by exploiting the assumption that $\forall\, x \in \mathbb{R}^d \colon \mathcal{G}^U_{\mu,\sigma}(x) + \frac{1}{2}\left\|\sigma(x)^*(\nabla U)(x)\right\|^2 + \bar{U}(x) \leq \rho\, U(x)$ (see (2) above) and by applying the Itô formula and the fundamental theorem of calculus, respectively. More formally, the fundamental theorem of calculus and the assumption that $\forall\, x \in \mathbb{R}^d \colon \mathcal{G}^U_{\mu,\sigma}(x) + \frac{1}{2}\left\|\sigma(x)^*(\nabla U)(x)\right\|^2 + \bar{U}(x) \leq \rho\, U(x)$ prove that for all $s \in [0,T]$, $h \in [0,T-s]$, $x \in D_h$, $t \in [0,h]$ it holds that

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(\tfrac{U(Z^{s,x,h}_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_h}(x)\,\bar{U}(Z^{s,x,h}_r)}{e^{\rho r}}\,dr\bigg)\bigg] - e^{U(x)} \\
&= \mathbb{E}\bigg[\exp\bigg(\tfrac{U(Z^{s,x,h}_t)}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(Z^{s,x,h}_r)}{e^{\rho r}}\,dr\bigg)\bigg] - e^{U(x)} \\
&\leq \int_0^t \bigg|\tfrac{\partial}{\partial u}\mathbb{E}\bigg[\exp\bigg(\tfrac{U(Z^{s,x,h}_u)}{e^{\rho u}} + \int_0^u \tfrac{\bar{U}(Z^{s,x,h}_r)}{e^{\rho r}}\,dr\bigg)\bigg] \\
&\qquad - \Big(\mathcal{G}^U_{\mu,\sigma}(x) + \tfrac{1}{2}\left\|\sigma(x)^*(\nabla U)(x)\right\|^2 + \bar{U}(x) - \rho\, U(x)\Big)\bigg|\,du.
\end{aligned} \tag{14}$$

In the next step we apply Itô's formula, we exploit the fact that $\forall\, h \in (0,1], u \in (0,h] \colon D_h = \big\{x \in \mathbb{R}^d \colon \|x\| \leq \exp\big(|\ln(h)|^{1/2}\big)\big\} \subseteq \big\{x \in \mathbb{R}^d \colon \|x\| \leq \exp\big(|\ln(u)|^{1/2}\big)\big\}$, we exploit (12), and we use a number of elementary estimates (see the proof of Lemma 2.8 in Subsection 2.2 for details) to obtain that there exists a real number

$c \in (0,\infty)$ such that for all $s \in [0,T]$, $h \in [0, \min\{T-s, 1\}]$, $x \in D_h$, $u \in (0,h]$ it holds that

$$\begin{aligned} &\Big| \tfrac{\partial}{\partial u} \mathbb{E}\bigg[\exp\bigg(\tfrac{U(Z_u^{s,x,h})}{e^{\rho u}} + \int_0^u \tfrac{\bar{U}(Z_r^{s,x,h})}{e^{\rho r}}\, dr \bigg) \bigg] \\ &\qquad - \Big(\mathcal{G}_{\mu,\sigma}^U(x) + \tfrac{1}{2} \left\| \sigma(x)^* (\nabla U)(x) \right\|^2 + \bar{U}(x) - \rho\, U(x) \Big) \Big| \\ &\le c\, u^{1/c}\, e^{U(x)}. \end{aligned} \tag{15}$$

Putting (15) into (14) then results in (13). Using (13) iteratively, in turn, will allow us to prove that there exists a real number $c \in (0,\infty)$ such that for all $N \in \mathbb{N} \cap [T,\infty)$ it holds that

$$\begin{aligned} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\tfrac{U(Y_t^N)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_{T/N}}(Y^N_{\lfloor r \rfloor_\theta})\, \bar{U}(Y_r^N)}{e^{\rho r}}\, dr \bigg) \bigg] &\le \exp\Big(cN \left[\tfrac{T}{N}\right]^{1+1/c} \Big)\, \mathbb{E}\Big[e^{U(Y_0^N)} \Big] \\ &= \exp\bigg(\frac{cT^{1+1/c}}{N^{1/c}} \bigg)\, \mathbb{E}\Big[e^{U(X_0)} \Big] \end{aligned} \tag{16}$$

(see Corollary 2.3 and (107) and (108) in the proof of Theorem 2.9 for details). Clearly, (16) implies

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\tfrac{U(Y_t^N)}{e^{\rho t}} + \int_0^{t\wedge\tau_N} \tfrac{\bar{U}(Y_s^N)}{e^{\rho s}}\, ds \bigg) \bigg] \le \mathbb{E}\Big[e^{U(X_0)} \Big] < \infty \tag{17}$$

and

$$\sup_{N\in\mathbb{N}\cap[T,\infty)} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\Big(e^{-\rho t}\, U(Y_t^N) + \int_0^{t\wedge\tau_N} e^{-\rho s}\, \bar{U}(Y_s^N)\, ds \Big) \bigg] < \infty. \tag{18}$$

Display (17) shows display (11) in Theorem 1.1 and the inequality

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\big[\exp\big(e^{-\rho t}\, U(Y_t^N) + \int_0^{t\wedge\tau_N} e^{-\rho s}\, \bar{U}(Y_s^N)\, ds \big) \big] < \infty$$

in Theorem 1.1 follows immediately from estimate (18) (cf. (163) in the proof of Corollary 3.8 below). Moreover, extensions of the notions and the results in Sections 3.2–3.4 in [18] will allow us to prove that for all $r \in (0,\infty)$ it holds that $\lim_{N\to\infty} \big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y_t^N\|^r \big] \big) = 0$ (see Section 3 below for details). This completes this sketch of the proof of Theorem 1.1.

1.2. **Notation.** Throughout this article the following notation is used. By $\mathbb{N} = \{1,2,3,\dots\}$ we denote the set of natural numbers and by $\mathbb{N}_0 = \{0,1,2,\dots\} = \mathbb{N} \cup \{0\}$ we denote the union of the set of natural numbers and zero. Additionally, for a natural number $d \in \mathbb{N}$ and a set $D \subseteq \mathbb{R}^d$ we denote by $\mathring{D}$ the interior of D, that is, the set given by

$$\mathring{D} = \left\{ x \in D \colon \left(\exists\, \varepsilon \in (0,\infty) \colon \left\{ y \in \mathbb{R}^d \colon \|x-y\| < \varepsilon \right\} \subseteq D \right) \right\}. \tag{19}$$

Furthermore, let $\|\cdot\| \colon \big(\bigcup_{n\in\mathbb{N}} \mathbb{R}^n\big) \to [0,\infty)$ and $\langle\cdot,\cdot\rangle \colon \big(\bigcup_{n\in\mathbb{N}} (\mathbb{R}^n \times \mathbb{R}^n)\big) \to [0,\infty)$ be the functions with the property that for all $n \in \mathbb{N}$, $v = (v_1,\dots,v_n)$, $w = (w_1,\dots,w_n) \in \mathbb{R}^n$ it holds that $\|v\| = \big[|v_1|^2 + \ldots + |v_n|^2 \big]^{1/2}$ and $\langle v,w\rangle = v_1 w_1 + \ldots + v_n w_n$. Moreover, for natural numbers $d,m \in \mathbb{N}$ and a $d\times m$-matrix $A \in \mathbb{R}^{d\times m}$ we denote by $A^* \in \mathbb{R}^{m\times d}$ the transpose of the matrix A and by $\|A\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}$ the Hilbert-Schmidt norm of the matrix A. Furthermore, for natural numbers

$k, d, m \in \mathbb{N}$ we denote by $L^{(k)}(\mathbb{R}^d, \mathbb{R}^m)$ the set of all k-linear mappings from $(\mathbb{R}^d)^k = \mathbb{R}^d \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ to $\mathbb{R}^m$ and we denote by $\|\cdot\|_{L^{(k)}(\mathbb{R}^d,\mathbb{R}^m)} : L^{(k)}(\mathbb{R}^d, \mathbb{R}^m) \to [0, \infty)$ the mapping with the property that for all $A\colon \mathbb{R}^d \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}^m \in L^{(k)}(\mathbb{R}^d, \mathbb{R}^m)$ it holds that

$$\|A\|_{L^{(k)}(\mathbb{R}^d,\mathbb{R}^m)} = \sup_{v_1,v_2,\ldots,v_k \in \mathbb{R}^d \setminus \{0\}} \left(\frac{\|A(v_1, v_2, \ldots, v_k)\|}{\|v_1\| \cdot \|v_2\| \cdot \ldots \cdot \|v_k\|} \right) \in [0, \infty). \tag{20}$$

Additionally, for natural numbers $k, d, m \in \mathbb{N}$, an open set $U \subseteq \mathbb{R}^d$, and a k-times continuously differentiable function $f \in C^k(U, \mathbb{R}^m)$ we denote by $f^{(k)}\colon U \to L^{(k)}(\mathbb{R}^d, \mathbb{R}^m)$ the k-th derivative of f. Observe that for all $k, d, m \in \mathbb{N}$, $v^{(1)} = (v^{(1)}_1, \ldots, v^{(1)}_d)$, $\ldots$, $v^{(k)} = (v^{(k)}_1, \ldots, v^{(k)}_d) \in \mathbb{R}^d$, all open sets $U \subseteq \mathbb{R}^d$, all k-times continuously differentiable functions $f \in C^k(U, \mathbb{R}^m)$, and all $x = (x_1, \ldots, x_d) \in U$ it holds that

$$f^{(k)}(x)(v_1, \ldots, v_k) = \sum_{l_1,\ldots,l_k=1}^{d} \big(\tfrac{\partial^k}{\partial x_{l_1} \ldots \partial x_{l_k}} f\big)(x) \cdot v^{(1)}_{l_1} \cdot v^{(2)}_{l_2} \cdot \ldots \cdot v^{(k)}_{l_k}. \tag{21}$$

Moreover, for sets A and B we denote by $\mathbb{M}(A, B)$ the set of all mappings from A to B. In addition, for natural numbers $d, m \in \mathbb{N}$ and arbitrary functions $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ we denote by $\mathcal{G}_{\mu,\sigma}\colon C^2(\mathbb{R}^d, \mathbb{R}) \to \mathbb{M}(\mathbb{R}^d, \mathbb{R})$ the formal generator associated to μ and σ, that is, we denote by $\mathcal{G}_{\mu,\sigma}\colon C^2(\mathbb{R}^d, \mathbb{R}) \to \mathbb{M}(\mathbb{R}^d, \mathbb{R})$ the mapping with the property that for all $\varphi \in C^2(\mathbb{R}^d, \mathbb{R})$, $x \in \mathbb{R}^d$ it holds that

$$(\mathcal{G}_{\mu,\sigma}\varphi)(x) = \langle \mu(x), (\nabla\varphi)(x)\rangle + \tfrac{1}{2} \operatorname{trace}\big(\sigma(x)\sigma(x)^*(\operatorname{Hess} \varphi)(x)\big). \tag{22}$$

Furthermore, for a natural number $d \in \mathbb{N}$ and a Borel measurable set $A \in \mathcal{B}(\mathbb{R}^d)$ we denote by $\lambda_A\colon \mathcal{B}(A) \to [0, \infty]$ the Lebesgue-Borel measure on $A \subseteq \mathbb{R}^d$. Moreover, for measurable spaces $(A, \mathcal{A})$ and $(B, \mathcal{B})$ we denote by $\mathcal{M}(\mathcal{A}, \mathcal{B})$ the set of all $\mathcal{A}/\mathcal{B}$-measurable mappings. In addition, for numbers $n, d \in \mathbb{N}$, $p, c \in (0, \infty)$, a set $B \subseteq \mathbb{R}$, and an open and convex set $A \subseteq \mathbb{R}^d$ we denote by $C^n_{p,c}(A, B)$ (cf. (1.12) in [18]) the set given by

$$C^n_{p,c}(A, B) = \left\{ f \in C^{n-1}(A, B)\colon \begin{array}{l} \forall\, x, y \in A, i \in \mathbb{N}_0 \cap [0, n)\colon \\ \|f^{(i)}(x) - f^{(i)}(y)\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})} \leq c\, \|x - y\| \\ \cdot \big[1 + \sup_{r\in[0,1]} |f(rx + (1 - r)y)|\big]^{\max\{1-(i+1)/p, 0\}} \end{array} \right\}. \tag{23}$$

Next let $(\cdot) \vee (\cdot)\colon \mathbb{R}^2 \to \mathbb{R}$ and $(\cdot) \wedge (\cdot)\colon \mathbb{R}^2 \to \mathbb{R}$ be the mappings with the property that for all $x, y \in \mathbb{R}$ it holds that $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. In addition, for a set Ω we denote by 2^Ω the power set of Ω (the set of all subsets of Ω) and for a set Ω we denote by $\#_\Omega\colon 2^\Omega \to [0, \infty]$ the counting measure on Ω. Furthermore, for a real number $T \in [0, \infty)$ we denote by $\mathcal{P}_T$ the set given by $\mathcal{P}_T = \{A \subseteq [0, T]\colon \#_{\mathbb{R}}(A) < \infty \text{ and } \{0, T\} \subseteq A\}$ (the set of all partitions of the interval $[0, T]$). Moreover, let $\ell\colon 2^{\mathbb{R}} \to (-\infty, \infty]$ be the mapping with the property that for all $\theta \subseteq \mathbb{R}$ it holds that $\ell(\theta) = \#_{\mathbb{R}}(\theta) - 1$. In addition, for a real number $T \in [0, \infty)$ we denote by $|\cdot|_T : \mathcal{P}_T \to [0, T]$ the mapping with the property that for all $\theta \in \mathcal{P}_T$ it holds that

$$|\theta|_T = \max\Big(\{0\} \cup \Big\{x \in (0, \infty)\colon \big(\exists\, a, b \in \theta\colon \big[x = b - a \text{ and } \theta \cap (a, b) = \emptyset\big]\big)\Big\}\Big) \in [0, T]. \tag{24}$$

Note for every $T \in [0,\infty)$ and every $\theta \in \mathcal{P}_T$ that $|\theta|_T \in [0,T]$ is the maximum step size of the partition θ. Finally, let $\lfloor \cdot \rfloor_\theta \colon [0,\infty) \to [0,\infty)$, $\theta \in [(0,\infty) \cup (\bigcup_{T\in[0,\infty)} \mathcal{P}_T)]$, and $\llcorner \cdot \lrcorner_\theta \colon [0,\infty) \to [0,\infty)$, $\theta \in (\bigcup_{T\in[0,\infty)} \mathcal{P}_T)$, be the mappings which satisfy $\lfloor t \rfloor_h = \max(\{0,h,2h,3h,\dots\} \cap [0,t])$, $\lfloor t \rfloor_\theta = \max(\theta \cap [0,t])$, $\llcorner t \lrcorner_\theta = \max(\theta \cap [0,t))$, and $\lfloor 0 \rfloor_h = \lfloor 0 \rfloor_\theta = \llcorner 0 \lrcorner_\theta = 0$ for all $\theta \in (\bigcup_{T\in[0,\infty)} \mathcal{P}_T)$, $h,t \in (0,\infty)$.

2. Exponential moments for numerical approximation processes

In this section we establish exponential integrability properties for certain numerical approximation processes of stochastic differential equations. In Subsection 2.1 we show how exponential moment bounds for the numerical approximation processes can be derived from suitable one-step estimates. We first prove a general Lyapunov-type estimate in Proposition 2.1 in Subsection 2.1. Thereafter, we establish appropriate Lyapunov-type estimates for exponentially growing Lyapunov-type functions in Lemma 2.2 and Corollary 2.3 in Subsection 2.1. The results in Subsection 2.1 are elementary extensions of known results in the literature (cf., e.g., Section 2.1.1 in [18] and Section 3.1 in Schurz [33]). Proposition 2.1, Lemma 2.2, and Corollary 2.3 all assume suitable one-step estimates for the considered numerical approximation processes; see (25) in the case of Proposition 2.1, see (2.2) in the case of Lemma 2.2, and see (37) in the case of Corollary 2.3. The purpose of Subsection 2.2 is to prove that an appropriate class of stopped numerical approximation schemes fulfills the one-step estimate (37) (see Lemma 2.8 in Subsection 2.2, which is the key result of this article) so that Corollary 2.3 in Subsection 2.1 can be applied. In Subsection 2.3 we then combine Corollary 2.3 in Subsection 2.1 and Lemma 2.8 in Subsection 2.2 to finally obtain exponential integrability properties for a class of stopped increment-tamed Euler-Maruyama schemes; see Theorem 2.9 and Corollary 2.10 in Subsection 2.3.

2.1. **From one-step estimates to exponential moments.**

The following proposition is an extended and generalized version of Corollary 2.2 in [18]. The proof of Proposition 2.1 is similar to the proof of Proposition 2.1 in [18].

Proposition 2.1. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $(E,\mathcal{E})$ be a measurable space, let $V \in \mathcal{M}(\mathcal{E},\mathcal{B}([0,\infty]))$, let $Z\colon \mathbb{N}_0 \times \Omega \to E$ be a stochastic process, let $\gamma_n \in [0,\infty)$, $n\in\mathbb{N}_0$, $\delta_n \in (0,\infty]$, $n \in \mathbb{N}_0$, and $\Omega_n \in \mathcal{F}$, $n \in \mathbb{N}_0$, be sequences which satisfy for all $n \in \mathbb{N}_0$ that $\Omega_0 = \Omega$, that $\Omega_n \backslash \Omega_{n+1} \subseteq \{V(Z_n) > \delta_n\}$, and that*

$$\mathbb{E}\Big[\mathbb{1}_{\Omega_{n+1}} V(Z_{n+1})\Big] \leq \gamma_n \cdot \mathbb{E}\Big[\mathbb{1}_{\Omega_n} V(Z_n)\Big]. \tag{25}$$

Then it holds for all $n \in \mathbb{N}_0$, $p \in [1,\infty]$, $\bar{V} \in \mathcal{M}(\mathcal{E},\mathcal{B}([0,\infty]))$ with $\bar{V} \leq V$ that

$$\mathbb{E}\big[\mathbb{1}_{\Omega_n} V(Z_n)\big] \leq \left(\prod_{k=0}^{n-1} \gamma_k\right) \cdot \mathbb{E}\big[V(Z_0)\big], \qquad \mathbb{P}[(\Omega_n)^c] \leq \left(\sum_{k=0}^{n-1} \frac{\prod_{l=0}^{k-1} \gamma_l}{\delta_k}\right) \mathbb{E}\big[V(Z_0)\big], \tag{26}$$

$$\begin{aligned} \mathbb{E}\big[\bar{V}(Z_n)\big] &\leq \left(\prod_{k=0}^{n-1} \gamma_k\right) \cdot \mathbb{E}\big[V(Z_0)\big] \\ &\quad + \|\bar{V}(Z_n)\|_{L^p(\Omega;\mathbb{R})} \left[\left(\sum_{k=0}^{n-1} \frac{\prod_{l=0}^{k-1} \gamma_l}{\delta_k}\right) \mathbb{E}\big[V(Z_0)\big]\right]^{(1-\frac{1}{p})}. \end{aligned} \tag{27}$$

Proof of Proposition 2.1. The first inequality in (26) is an immediate consequence of (25). To arrive at the second estimate in (26), note first for all $n \in \mathbb{N}$ that

$$(\Omega_n)^c = \big(\Omega_{n-1}\backslash\Omega_n\big) \uplus \big(\,(\Omega_{n-1})^c\,\backslash\Omega_n\big) \subseteq \big(\Omega_{n-1}\backslash\Omega_n\big) \cup \big(\,(\Omega_{n-1})^c\,\big). \tag{28}$$

Iterating inclusion (28) and using $\Omega_0 = \Omega$ shows for all $n \in \mathbb{N}_0$ that

$$\begin{aligned}(\Omega_n)^c &\subseteq \left(\bigcup_{k=0}^{n-1}(\Omega_k\backslash\Omega_{k+1})\right) \cup \big(\,(\Omega_0)^c\,\big)\\ &= \bigcup_{k=0}^{n-1}(\Omega_k\backslash\Omega_{k+1}) = \bigcup_{k=0}^{n-1}\Big(\Omega_k \cap \big(\Omega_k\backslash\Omega_{k+1}\big)\Big)\\ &\subseteq \bigcup_{k=0}^{n-1}(\Omega_k \cap \{V(Z_k) > \delta_k\}) = \bigcup_{k=0}^{n-1}\{\mathbb{1}_{\Omega_k}V(Z_k) > \delta_k\}.\end{aligned} \tag{29}$$

Additivity of the probability measure $\mathbb{P}$, Markov's inequality and the first inequality in (26) therefore imply for all $n \in \mathbb{N}_0$ that

$$\begin{aligned}\mathbb{P}\big[(\Omega_n)^c\big] &\leq \sum_{k=0}^{n-1}\mathbb{P}\Big[\,\mathbb{1}_{\Omega_k}V(Z_k) > \delta_k\Big] \leq \sum_{k=0}^{n-1}\left[\frac{\mathbb{E}\big[\mathbb{1}_{\Omega_k}V(Z_k)\big]}{\delta_k}\right]\\ &\leq \sum_{k=0}^{n-1}\left[\frac{\Big(\prod_{l=0}^{k-1}\gamma_l\Big)\cdot\mathbb{E}[V(Z_0)]}{\delta_k}\right].\end{aligned} \tag{30}$$

This is the second inequality in (26) and the proof of (26) is thus completed. Next observe that Hölder's inequality implies for all $\tilde{\Omega} \in \mathcal{F}$, $p \in [1,\infty]$ and all $\mathcal{F}/\mathcal{B}([0,\infty])$-measurable mappings $X\colon \Omega \to [0,\infty]$ that

$$\mathbb{E}[X] \leq \mathbb{E}[\mathbb{1}_{\tilde{\Omega}}X] + \big(\mathbb{P}\big[(\tilde{\Omega})^c\big]\big)^{(1-1/p)}\|X\|_{L^p(\Omega;\mathbb{R})}. \tag{31}$$

Combining (26) and (31) finally shows that for all $n \in \mathbb{N}_0$, $p \in [1,\infty]$ and all $\mathcal{E}/\mathcal{B}([0,\infty])$-measurable functions $\bar{V}\colon E \to [0,\infty]$ with $\forall\, x \in E\colon \bar{V}(x) \leq V(x)$ it holds that

$$\begin{aligned}\mathbb{E}\big[\bar{V}(Z_n)\big] &\leq \mathbb{E}\big[\mathbb{1}_{\Omega_n}V(Z_n)\big] + \|\bar{V}(Z_n)\|_{L^p(\Omega;\mathbb{R})}\big(\mathbb{P}\big[(\Omega_n)^c\big]\big)^{(1-1/p)}\\ &\leq \left(\prod_{k=0}^{n-1}\gamma_k\right)\cdot\mathbb{E}\big[V(Z_0)\big]\\ &\qquad + \|\bar{V}(Z_n)\|_{L^p(\Omega;\mathbb{R})}\left[\left(\sum_{k=0}^{n-1}\frac{\prod_{l=0}^{k-1}\gamma_l}{\delta_k}\right)\mathbb{E}\big[V(Z_0)\big]\right]^{(1-\frac{1}{p})}.\end{aligned} \tag{32}$$

The proof of Proposition 2.1 is thus completed. □

The next elementary lemma (Lemma 2.2) establishes an a priori bound based on a specific class of path dependent Lyapunov-type functions (see (33) below for details and cf., e.g., also Section 3.1 in Schurz [33]). For completeness the proof of Lemma 2.2 is given below.

Lemma 2.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(E, \mathcal{E})$ be a measurable space, let $T, \rho \in [0, \infty)$, $\theta \in \mathcal{P}_T$, $c \in \mathbb{R}$, $U, \bar{U} \in \mathcal{M}(\mathcal{E}, \mathcal{B}(\mathbb{R}))$, $A \in \mathcal{E}$, and let $Y \colon [0,T] \times \Omega \to E$ be a product measurable stochastic process which satisfies that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^t \mathbb{1}_A(Y_{\lfloor r \rfloor_\theta}) \, |\bar{U}(Y_r)| \, dr < \infty$ and*

$$\begin{aligned} &\mathbb{E}\bigg[\exp\!\bigg(-c\,t + \tfrac{U(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\bigg) \,\bigg|\, (Y_r)_{r \in [0, \llcorner t \lrcorner_\theta]}\bigg] \\ &\quad \leq \exp\!\bigg(-c\,\llcorner t \lrcorner_\theta + \tfrac{U(Y_{\llcorner t \lrcorner_\theta})}{e^{\rho \llcorner t \lrcorner_\theta}} + \int_0^{\llcorner t \lrcorner_\theta} \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\bigg). \end{aligned} \tag{33}$$

Then it holds for all $t \in [0,T]$ that

$$\mathbb{E}\bigg[\exp\!\bigg(\tfrac{U(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\bigg)\bigg] \leq e^{ct}\,\mathbb{E}\Big[e^{U(Y_0)}\Big]. \tag{34}$$

Proof of Lemma 2.2. Assumption (33) implies for all $t \in [0,T]$ that

$$\begin{aligned} &\mathbb{E}\bigg[\exp\!\bigg(-ct + \tfrac{U(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\bigg)\bigg] \\ &= \mathbb{E}\bigg[\mathbb{E}\bigg[\exp\!\bigg(-ct + \tfrac{U(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\bigg) \,\bigg|\, (Y_s)_{s \in [0, \llcorner t \lrcorner_\theta]}\bigg]\bigg] \\ &\leq \mathbb{E}\bigg[\exp\!\bigg(-c\,\llcorner t \lrcorner_\theta + \tfrac{U(Y_{\llcorner t \lrcorner_\theta})}{e^{\rho\,\llcorner t \lrcorner_\theta}} + \int_0^{\llcorner t \lrcorner_\theta} \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\bigg)\bigg] \\ &\leq \ldots \leq \mathbb{E}\Big[\exp\!\Big(\tfrac{U(Y_0)}{e^{\rho \cdot 0}}\Big)\Big] = \mathbb{E}\Big[e^{U(Y_0)}\Big]. \end{aligned} \tag{35}$$

This completes the proof of Lemma 2.2. □

The next corollary, Corollary 2.3, specialises Lemma 2.2 to the case where the product measurable stochastic process appearing in (33) and (34) is an appropriate one-step approximation process for an SDE driven by a standard Brownian motion; see (36) below for details.

Corollary 2.3. *Let $T, \rho, c \in [0, \infty)$, $\theta \in \mathcal{P}_T$, $d, m \in \mathbb{N}$, $\Phi \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d \times [0,T] \times \mathbb{R}^m), \mathcal{B}(\mathbb{R}^d))$, $A \in \mathcal{B}(\mathbb{R}^d)$, $U \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}([0,\infty)))$, $\bar{U} \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]})$ be a filtered probability space, let $W \colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t \in [0,T]}$-Brownian motion with continuous sample paths, let $Y \colon [0,T] \times \Omega \to \mathbb{R}^d$ be an $(\mathcal{F}_t)_{t \in [0,T]}$-adapted stochastic process which satisfies for all $t \in [0,T]$ that*

$$Y_t = \mathbb{1}_{\mathbb{R}^d \setminus A}(Y_{\llcorner t \lrcorner_\theta}) \cdot Y_{\llcorner t \lrcorner_\theta} + \mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta}) \cdot \Phi(Y_{\llcorner t \lrcorner_\theta}, t - \llcorner t \lrcorner_\theta, W_t - W_{\llcorner t \lrcorner_\theta}), \tag{36}$$

assume that for all $x \in A$ it holds $\mathbb{P}$-a.s. that

$$\int_0^T \mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\,|\bar{U}(Y_r)|\,dr + \int_0^{|\theta|_T} |\bar{U}(\Phi(x, r, W_r))|\,dr < \infty,$$

and assume for all $(t, x) \in \big(0, |\theta|_T\big] \times A$ that

$$\mathbb{E}\bigg[\exp\!\bigg(\tfrac{U(\Phi(x,t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(\Phi(x,s,W_s))}{e^{\rho s}}\,ds\bigg)\bigg] \leq e^{ct + U(x)}. \tag{37}$$

Then it holds for all $t \in [0,T]$ that

$$\mathbb{E}\bigg[\exp\!\bigg(\tfrac{U(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\bigg)\bigg] \leq e^{ct}\,\mathbb{E}\Big[e^{U(Y_0)}\Big]. \tag{38}$$

Proof of Corollary 2.3. We prove Corollary 2.3 through an application of Lemma 2.2. For this we observe that assumption (37) implies for all $(t,x) \in \big(0, |\theta|_T\big] \times \mathbb{R}^d$ that

$$\mathbb{E}\left[\exp\left(\tfrac{\mathbb{1}_A(x)\,U(\Phi(x,t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(x)\,\bar{U}(\Phi(x,s,W_s))}{e^{\rho s}}\,ds\right)\right] \leq e^{ct+\mathbb{1}_A(x)\,U(x)}. \tag{39}$$

Next note that equation (36), Jensen's inequality, and inequality (39) imply that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,U(Y_t)}{e^{\rho t}} + \int_{\llcorner t \lrcorner_\theta}^t \tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,\bar{U}(Y_s)}{e^{\rho s}}\,ds\right) \,\Big|\, (Y_s)_{s\in[0,\llcorner t \lrcorner_\theta]}\right] \\
&\quad = \mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,U(\Phi(Y_{\llcorner t \lrcorner_\theta},t-\llcorner t \lrcorner_\theta,W_t-W_{\llcorner t \lrcorner_\theta}))}{e^{\rho t}} \\
&\qquad + \int_{\llcorner t \lrcorner_\theta}^t \tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,\bar{U}(\Phi(Y_{\llcorner t \lrcorner_\theta},s-\llcorner t \lrcorner_\theta,W_s-W_{\llcorner t \lrcorner_\theta}))}{e^{\rho s}}\,ds\Bigg) \,\Big|\, (Y_s)_{s\in[0,\llcorner t \lrcorner_\theta]}\Bigg] \\
&\quad = \mathbb{E}\Bigg[\Bigg|\exp\Bigg(\tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,U(\Phi(Y_{\llcorner t \lrcorner_\theta},t-\llcorner t \lrcorner_\theta,W_t-W_{\llcorner t \lrcorner_\theta}))}{e^{\rho(t-\llcorner t \lrcorner_\theta)}} \\
&\qquad + \int_0^{t-\llcorner t \lrcorner_\theta} \tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,\bar{U}(\Phi(Y_{\llcorner t \lrcorner_\theta},s,W_{\llcorner t \lrcorner_\theta+s}-W_{\llcorner t \lrcorner_\theta}))}{e^{\rho s}}\,ds\Bigg)\Bigg|^{\exp(-\rho\,\llcorner t \lrcorner_\theta)} \,\Big|\, (Y_s)_{s\in[0,\llcorner t \lrcorner_\theta]}\Bigg] \\
&\quad \leq \Bigg|\mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,U(\Phi(Y_{\llcorner t \lrcorner_\theta},t-\llcorner t \lrcorner_\theta,W_t-W_{\llcorner t \lrcorner_\theta}))}{e^{\rho(t-\llcorner t \lrcorner_\theta)}} \\
&\qquad + \int_0^{t-\llcorner t \lrcorner_\theta} \tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,\bar{U}(\Phi(Y_{\llcorner t \lrcorner_\theta},s,W_{\llcorner t \lrcorner_\theta+s}-W_{\llcorner t \lrcorner_\theta}))}{e^{\rho s}}\,ds\Bigg) \,\Big|\, (Y_s)_{s\in[0,\llcorner t \lrcorner_\theta]}\Bigg]\Bigg|^{\exp(-\rho\,\llcorner t \lrcorner_\theta)} \\
&\quad \leq \left|e^{c(t-\llcorner t \lrcorner_\theta)+\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,U(Y_{\llcorner t \lrcorner_\theta})}\right|^{\exp(-\rho\,\llcorner t \lrcorner_\theta)} \\
&\quad \leq \exp\left(c\,(t-\llcorner t \lrcorner_\theta) + \tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,U(Y_{\llcorner t \lrcorner_\theta})}{e^{\rho\,\llcorner t \lrcorner_\theta}}\right).
\end{aligned} \tag{40}$$

Combining this with (36) shows that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-ct + \tfrac{U(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\right) \,\Big|\, (Y_r)_{r\in[0,\llcorner t \lrcorner_\theta]}\right] \\
&\quad = \mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,U(Y_t)}{e^{\rho t}} + \tfrac{\mathbb{1}_{\mathbb{R}^d\setminus A}(Y_{\llcorner t \lrcorner_\theta})\,U(Y_t)}{e^{\rho t}} \\
&\qquad + \int_{\llcorner t \lrcorner_\theta}^t \tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\Bigg) \,\Big|\, (Y_r)_{r\in[0,\llcorner t \lrcorner_\theta]}\Bigg] \\
&\qquad \cdot \exp\left(-ct + \int_0^{\llcorner t \lrcorner_\theta} \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\right) \\
&\quad = \mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,U(Y_t)}{e^{\rho t}} + \tfrac{\mathbb{1}_{\mathbb{R}^d\setminus A}(Y_{\llcorner t \lrcorner_\theta})\,U(Y_{\llcorner t \lrcorner_\theta})}{e^{\rho t}} \\
&\qquad + \int_{\llcorner t \lrcorner_\theta}^t \tfrac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\Bigg) \,\Big|\, (Y_r)_{r\in[0,\llcorner t \lrcorner_\theta]}\Bigg]
\end{aligned} \tag{41}$$

$$
\begin{aligned}
&\cdot \exp\!\left(-ct + \int_0^{\llcorner t \lrcorner_\theta} \frac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\, \bar{U}(Y_r)}{e^{\rho r}}\, dr \right) \\
&\leq \exp\!\Bigg(c\,(t - \llcorner t \lrcorner_\theta) + \frac{\mathbb{1}_A(Y_{\llcorner t \lrcorner_\theta})\, U(Y_{\llcorner t \lrcorner_\theta})}{e^{\rho \llcorner t \lrcorner_\theta}} - ct \\
&\qquad + \frac{\mathbb{1}_{\mathbb{R}^d \setminus A}(Y_{\llcorner t \lrcorner_\theta})\, U(Y_{\llcorner t \lrcorner_\theta})}{e^{\rho t}} + \int_0^{\llcorner t \lrcorner_\theta} \frac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\, \bar{U}(Y_r)}{e^{\rho r}}\, dr \Bigg) \\
&\leq \exp\!\left(-c \llcorner t \lrcorner_\theta + \frac{U(Y_{\llcorner t \lrcorner_\theta})}{e^{\rho \llcorner t \lrcorner_\theta}} + \int_0^{\llcorner t \lrcorner_\theta} \frac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\, \bar{U}(Y_r)}{e^{\rho r}}\, dr \right).
\end{aligned}
$$

Combining this with Lemma 2.2 yields for all $t \in [0, T]$ that

$$
\mathbb{E}\!\left[\exp\!\left(\frac{U(Y_t)}{e^{\rho t}} + \int_0^t \frac{\mathbb{1}_A(Y_{\lfloor r \rfloor_\theta})\, \bar{U}(Y_r)}{e^{\rho r}}\, dr \right) \right] \leq e^{ct}\, \mathbb{E}\!\left[e^{U(Y_0)} \right].
$$

This finishes the proof of Corollary 2.3. □

2.2. **A one-step estimate for exponential moments.** In Lemma 2.8 below a one-step estimate for exponential moments (see (37) in Corollary 2.3 above) is proved for a general class of stopped one-step numerical approximation schemes. The proof of Lemma 2.8 uses the elementary estimate in Lemma 2.5 below. Moreover, the proof of Lemma 2.5 exploits the following well-known result, Lemma 2.4. For completeness the proof of Lemma 2.4 is given below.

Lemma 2.4. *It holds for all $x \in \mathbb{R}$ that*

$$
e^x = 2 \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right) - \frac{1}{e^x} \leq 2 \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right). \tag{42}
$$

Proof of Lemma 2.4. Note for all $x \in \mathbb{R}$ that

$$
e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right) - \left(\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \right). \tag{43}
$$

This implies for all $x \in \mathbb{R}$ that

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + \left(\left[\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right] - e^{-x} \right) \\
&= 2 \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right) - e^{-x} \leq 2 \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right).
\end{aligned} \tag{44}
$$

The proof of Lemma 2.4 is thus completed. □

Lemma 2.5. *Let $T \in [0, \infty)$, $d, m \in \mathbb{N}$, $A \in \mathbb{R}^{d \times m}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W \colon [0, T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion. Then it holds for all $t \in [0, T]$ that $\mathbb{E}\big[e^{\|AW_t\|}\big] \leq 2 \exp\!\big(\frac{t}{2} \|A\|^2_{\mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d)}\big)$.*

Proof of Lemma 2.5. Throughout this proof let $f_n \colon \mathbb{R}^m \to [0, \infty)$, $n \in \mathbb{N}_0$, be the functions with the property that for all $x \in \mathbb{R}^m$ and all $n \in \mathbb{N}_0$ it holds that

$f_n(x) = \|Ax\|^{2n}$. Then note for all $x \in \mathbb{R}^m$ and all $n \in \mathbb{N}$ that

$$
\begin{aligned}
&\operatorname{trace}((\operatorname{Hess} f_n)(x)) \\
&\quad = \operatorname{trace}\Big(2n\,\|Ax\|^{(2n-2)} A^*A \\
&\qquad\qquad + \mathbb{1}_{\{x\neq 0\}}\, 2n\,(2n-2)\,\|Ax\|^{(2n-4)}\,(A^*Ax)\,(A^*Ax)^*\Big) \\
&\quad = 2n\,\|Ax\|^{(2n-2)}\,\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \mathbb{1}_{\{x\neq 0\}}\, 2n\,(2n-2)\,\|Ax\|^{(2n-4)}\,\|A^*Ax\|^2 \qquad (45)\\
&\quad \le 2n\,\|Ax\|^{(2n-2)}\,\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + 2n\,(2n-2)\,\|Ax\|^{(2n-2)}\,\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\quad = 2n\,(2n-1)\,\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\; f_{n-1}(x)\,.
\end{aligned}
$$

Itô's formula hence shows for all $s_0 \in [0,T]$ and all $n \in \mathbb{N}$ that

$$
\begin{aligned}
\mathbb{E}\Big[\|AW_{s_0}\|^{2n}\Big] &= \mathbb{E}\big[f_n(W_{s_0})\big] = \frac{1}{2}\int_0^{s_0} \mathbb{E}\big[\operatorname{trace}((\operatorname{Hess} f_n)(W_{s_1}))\big]\,ds_1 \\
&\le \frac{1}{2}\int_0^{s_0} 2n\,(2n-1)\,\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\,\mathbb{E}\big[f_{n-1}(W_{s_1})\big]\,ds_1 \qquad (46)\\
&\le \ldots \le \frac{(2n)!}{2^n}\,\|A\|^{2n}_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \int_0^{s_0}\int_0^{s_1}\cdots\int_0^{s_{n-1}} \mathbb{E}\big[f_0(W_{s_n})\big]\,ds_n\cdots ds_2\,ds_1 \\
&= \frac{(2n)!}{2^n n!}\,\|A\|^{2n}_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\,(s_0)^n\,.
\end{aligned}
$$

Combining this with Lemma 2.4 implies for all $t \in [0,T]$ that

$$
\begin{aligned}
\mathbb{E}\Big[e^{\|AW_t\|}\Big] &\le 2\left(\sum_{n=0}^{\infty} \frac{\mathbb{E}\big[\|AW_t\|^{2n}\big]}{(2n)!}\right) \\
&\le 2\left(\sum_{n=0}^{\infty} \frac{t^n\|A\|^{2n}_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{2^n n!}\right) = 2e^{\frac{t}{2}\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}. \qquad (47)
\end{aligned}
$$

This finishes the proof of Lemma 2.5. □

Beside Lemma 2.5, the proof of Lemma 2.8 also uses the following two lemmas (Lemma 2.6 and Lemma 2.7). The proof of Lemma 2.7 uses inequality (2.56) in Lemma 2.11 in [18].

Lemma 2.6. *Let $d \in \mathbb{N}$, $n \in \mathbb{N}_0$, $c, p \in (0,\infty)$, $x, y \in \mathbb{R}^d$, $V \in C^{n+1}_{p,c}(\mathbb{R}^d, [0,\infty))$. Then*

(i) *it holds for all $t \in \{s \in [0,1] \colon \mathbb{R} \ni u \mapsto V(x+uy) \in \mathbb{R}$ is differentiable at $s\}$ that $|\frac{\partial}{\partial t} V(x+ty)| \le c\,\|y\|\,[1+V(x+ty)]^{\max\{1-1/p,0\}}$ and*

(ii) *it holds for all $i \in \mathbb{N} \cap [0,n]$, $z_1,\ldots,z_i \in \mathbb{R}^d$, $t \in \{s \in [0,1] \colon \mathbb{R} \ni u \mapsto V^{(i)}(x+uy)(z_1,\ldots,z_i) \in \mathbb{R}$ is differentiable at $s\}$ that*

$$
\begin{aligned}
&\big|\tfrac{\partial}{\partial t}\big(V^{(i)}(x+ty)(z_1,\ldots,z_i)\big)\big| \\
&\qquad \le c\,\|z_1\|\cdots\|z_i\|\,\|y\|\,[1+V(x+ty)]^{\max\{1-(i+1)/p,0\}}. \qquad (48)
\end{aligned}
$$

Proof of Lemma 2.6. First, note that the assumption $V \in C^{n+1}_{p,c}(\mathbb{R}^d, [0,\infty))$ ensures that for all $t \in [0,1]$, $h \in \mathbb{R}$ it holds that

$$\begin{aligned} &|V(x+ty) - V(x+(t+h)y)| \\ &\leq c\,|h|\,\|y\| \left[1 + \sup\nolimits_{r\in[0,1]} V\big(x+(t+(1-r)h)y\big)\right]^{\max\{1-1/p,0\}} \\ &= c\,|h|\,\|y\| \left[1 + \sup\nolimits_{r\in[0,1]} V\big(x+(t+rh)y\big)\right]^{\max\{1-1/p,0\}}. \end{aligned} \tag{49}$$

Next observe that again the assumption $V \in C^{n+1}_{p,c}(\mathbb{R}^d, [0,\infty))$ ensures that V is locally Lipschitz continuous. Hence, we obtain for all $t \in [0,1]$ that

$$\limsup\nolimits_{(\mathbb{R}\backslash\{0\})\ni h\to 0} \left| \sup\nolimits_{r\in[0,1]} V(x+(t+rh)y) - V(x+ty)\right| = 0. \tag{50}$$

Combining this with (49) proves (i). In the next step we note again that the assumption $V \in C^{n+1}_{p,c}(\mathbb{R}^d, [0,\infty))$ shows that for all $i \in \mathbb{N} \cap [0,n]$, $z_1, \ldots, z_i \in \mathbb{R}^d\backslash\{0\}$, $t \in [0,1]$, $h \in \mathbb{R}$ it holds that

$$\begin{aligned} &\tfrac{|V^{(i)}(x+ty)(z_1,\ldots,z_i) - V^{(i)}(x+(t+h)y)(z_1,\ldots,z_i)|}{\|z_1\|\cdots\|z_i\|} \\ &\leq \|V^{(i)}(x+ty) - V^{(i)}(x+(t+h)y)\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})} \\ &\leq c|h|\|y\|[1 + \sup\nolimits_{r\in[0,1]} V(x+(t+(1-r)h)y)]^{\max\{1-(i+1)/p,0\}} \\ &= c|h|\|y\|[1 + \sup\nolimits_{r\in[0,1]} V(x+(t+rh)y)]^{\max\{1-(i+1)/p,0\}}. \end{aligned} \tag{51}$$

This and (50) establish (ii). The proof of Lemma 2.6 is thus completed. ☐

Lemma 2.7. *Let $c, p \in [1,\infty)$, $d \in \mathbb{N}$, $x, y \in \mathbb{R}^d$, $V \in C^1_{p,c}(\mathbb{R}^d, [0,\infty))$. Then it holds that $1 + V(x+y) \leq c^p 2^{p-1}(1 + V(x) + \|y\|^p)$.*

Proof of Lemma 2.7. Throughout this proof let $f\colon \mathbb{R} \to \mathbb{R}$ be the function with the property that for all $t \in \mathbb{R}$ it holds that $f(t) = V(x+ty)$. Next observe that the fact that V is locally Lipschitz continuous ensures that f is globally Lipschitz continuous. Moreover, note that Item (i) in Lemma 2.6 implies for all $t \in \{s \in [0,1]\colon \mathbb{R} \ni u \mapsto f(u) \in \mathbb{R}$ is differentiable at $s\}$ that

$$\begin{aligned} &\tfrac{\partial}{\partial t}(1 + f(t)) \leq \left|\tfrac{\partial}{\partial t}(1+f(t))\right| \\ &\qquad\leq c\,\|y\|\,[1 + f(t)]^{(1-1/p)} = c\,\|y\|\,|1+f(t)|^{(1-1/p)}. \end{aligned} \tag{52}$$

The fact that f is globally Lipschitz continuous and inequality (2.56) in Lemma 2.11 in Hutzenthaler and Jentzen [18] (with $T = 1$, $c = c\,\|y\|$, $p = p$, $y = ([0,1] \ni t \mapsto 1 + f(t) \in \mathbb{R})$ in the notation of Lemma 2.11 in Hutzenthaler and Jentzen [18]) hence prove for all $t \in [0,1]$ that

$$\begin{aligned} 1 + f(t) &\leq 2^{p-1}\left[1 + f(0) + \left|\tfrac{c\|y\|t}{p}\right|^p\right] = 2^{p-1}\left[1 + V(x) + \tfrac{c^p\|y\|^p t^p}{p^p}\right] \\ &\leq 2^{p-1}\big[1 + V(x) + c^p\|y\|^p\big]. \end{aligned} \tag{53}$$

This implies that

$$1 + V(x+y) \leq 2^{p-1}\left[1 + V(x) + c^p\|y\|^p\right] \leq c^p\, 2^{p-1}\left[1 + V(x) + \|y\|^p\right].$$

The proof of Lemma 2.7 is thus completed. ☐

Lemma 2.8. *Let $\alpha, h \in (0,\infty)$, $d, m \in \mathbb{N}$, $c, p \in [1,\infty)$, $\gamma_0, \gamma_1, \ldots, \gamma_6, \gamma_7, \rho \in [0,\infty)$, $\mu \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, $\sigma \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^{d\times m}))$, $\bar{U} \in C(\mathbb{R}^d, \mathbb{R})$, $U \in C^3_{p,c}(\mathbb{R}^d, [0,\infty))$, $D \in 2^{\{x\in\mathbb{R}^d\colon U(x)\leq ch^{-\alpha}\}}$, let $\Phi \in C^{0,1,2}(D \times [0,h] \times \mathbb{R}^m, \mathbb{R}^d)$, let*

$(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W\colon [0,h] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths, assume for all $x \in \mathbb{R}^d$ that

$$\|\mu(x)\| \leq c\,(1 + |U(x)|^{\gamma_0}), \qquad \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,(1 + |U(x)|^{\gamma_1}), \tag{54}$$

$$(\mathcal{G}_{\mu,\sigma} U)(x) + \tfrac{1}{2}\,\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \leq \rho \cdot U(x), \tag{55}$$

assume for all $r \in [1,\infty)$, $x \in D$, $s \in (0,h]$ that $\Phi(x,0,0) = x$ and

$$\big\|\big(\tfrac{\partial}{\partial s}\Phi\big)(x,s,W_s) - \mu(x)\big\|_{L^4(\Omega;\mathbb{R}^d)} \leq c s^{\gamma_2}, \tag{56}$$

$$\big\|\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s) - \sigma(x)\big\|_{L^8(\Omega;\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))} \leq c s^{\gamma_3}, \tag{57}$$

$$\big\|(\triangle_y \Phi)(x,s,W_s)\big\|_{L^4(\Omega;\mathbb{R}^d)} \leq c s^{\gamma_4}, \tag{58}$$

$$\begin{aligned} &\|\Phi(x,s,W_s) - x\|_{L^r(\Omega;\mathbb{R}^d)} \\ &\quad \leq c \min\!\big(r, 1 + |U(x)|^{\gamma_5}, (1 + |U(x)|^{\gamma_5})\, \|\mu(x)s + \sigma(x)W_s\|_{L^r(\Omega;\mathbb{R}^d)}\big), \end{aligned} \tag{59}$$

and assume for all $x, y \in \mathbb{R}^d$ that $|\bar{U}(x)| \leq c\,(1 + |U(x)|^{\gamma_6})$ and $|\bar{U}(x) - \bar{U}(y)| \leq c\,(1 + |U(x)|^{\gamma_7} + |U(y)|^{\gamma_7})\, \|x - y\|$. Then it holds for all $(t,x) \in (0,h] \times D$ that

$$\begin{aligned} &\mathbb{E}\bigg[\exp\bigg(\tfrac{U(\Phi(x,t,W_t))}{e^{\rho t}} + \textstyle\int_0^t \tfrac{\bar{U}(\Phi(x,s,W_s))}{e^{\rho s}}\, ds\bigg)\bigg] \\ &\quad \leq e^{U(x)} \bigg[1 + \textstyle\int_0^t \exp\bigg(\tfrac{s\,[2c]^{4p(\gamma_6+2)\max(\gamma_0,\gamma_1,\gamma_5,2)}}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\bigg) \\ &\qquad\qquad \cdot \tfrac{\max(\rho,1)\,[2pc\max(s,1)]^{6p(\gamma_7+3)\max(1,\gamma_0,\gamma_1,\ldots,\gamma_5)}}{[\min(s,1)]^{[\alpha(2\gamma_0+4\gamma_1+2\gamma_5+(p\gamma_5+1)\gamma_7+2)-\min(1/2,\gamma_2,\gamma_3,\gamma_4)]}}\, ds\bigg]. \end{aligned} \tag{60}$$

Proof of Lemma 2.8. Throughout this proof let $Y^x\colon [0,h] \times \Omega \to \mathbb{R}^d$, $x \in D$, be the stochastic processes with the property that for all $s \in [0,h]$, $x \in D$ it holds that $Y^x_s = \Phi(x,s,W_s)$ and let $\tau_n\colon \Omega \to [0,h]$, $n \in \mathbb{N}$, be the functions with the property that for all $n \in \mathbb{N}$ it holds that $\tau_n = \inf\!\big(\{s \in [0,h]\colon \|W_s\| > n\} \cup \{h\}\big)$. Next observe that Itô's formula implies that for all $x \in D$, $t \in [0,h]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} &\exp\bigg(e^{-\rho t} U(Y^x_t) + \int_0^t e^{-\rho r} \bar{U}(Y^x_r)\, dr\bigg) - e^{U(x)} \\ &= \int_0^t \exp\bigg(e^{-\rho s} U(Y^x_s) + \int_0^s e^{-\rho r} \bar{U}(Y^x_r)\, dr\bigg) \\ &\qquad \cdot e^{-\rho s}\, U'(Y^x_s)\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\, dW_s \\ &\quad + \int_0^t \exp\bigg(e^{-\rho s} U(Y^x_s) + \int_0^s e^{-\rho r} \bar{U}(Y^x_r)\, dr\bigg) \\ &\qquad \cdot e^{-\rho s}\bigg(\bar{U}(Y^x_s) - \rho U(Y^x_s) + U'(Y^x_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(x,s,W_s) \\ &\qquad + \tfrac{1}{2}\,\mathrm{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\mathrm{Hess}\, U)(Y^x_s)\Big) \\ &\qquad + \tfrac{1}{2}\, e^{-\rho s}\, \big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y^x_s)\big\|^2 \\ &\qquad + \tfrac{1}{2}\textstyle\sum_{i=1}^m U'(Y^x_s)\big(\tfrac{\partial^2}{\partial y_i^2}\Phi\big)(x,s,W_s)\bigg)\, ds. \end{aligned} \tag{61}$$

This shows for all $t \in [0,h]$, $x \in D$, $n \in \mathbb{N}$ that

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(e^{-\rho(t\wedge\tau_n)}U(Y^x_{t\wedge\tau_n}) + \int_0^{t\wedge\tau_n} e^{-\rho r}\bar{U}(Y^x_r)\,dr\bigg)\bigg] - e^{U(x)} \\
&= \mathbb{E}\bigg[\int_0^{t\wedge\tau_n} \exp\bigg(e^{-\rho s}U(Y^x_s) + \int_0^s e^{-\rho r}\bar{U}(Y^x_r)\,dr\bigg)\, e^{-\rho s} \\
&\qquad \cdot \bigg(\bar{U}(Y^x_s) - \rho U(Y^x_s) + U'(Y^x_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(x,s,W_s) \\
&\qquad + \tfrac{1}{2}\operatorname{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\operatorname{Hess}U)(Y^x_s)\Big) \\
&\qquad + \tfrac{1}{2}e^{-\rho s}\Big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y^x_s)\Big\|^2 + \tfrac{1}{2}U'(Y^x_s)\,(\triangle_y\Phi)(x,s,W_s)\bigg)\,ds\bigg].
\end{aligned} \tag{62}$$

Assumption (55) therefore yields for all $t \in [0,h]$, $x \in D$, $n \in \mathbb{N}$ that

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(e^{-\rho(t\wedge\tau_n)}U(Y^x_{t\wedge\tau_n}) + \int_0^{t\wedge\tau_n} e^{-\rho r}\bar{U}(Y^x_r)\,dr\bigg)\bigg] - e^{U(x)} \\
&= \mathbb{E}\bigg[\int_0^{t\wedge\tau_n} \exp\bigg(e^{-\rho s}U(Y^x_s) + \int_0^s e^{-\rho r}\bar{U}(Y^x_r)\,dr\bigg)\, e^{-\rho s} \\
&\quad \cdot \bigg(-\rho U(x) + (\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}e^{-\rho s}\,\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \\
&\quad + \bar{U}(Y^x_s) - \bar{U}(x) - \rho(U(Y^x_s) - U(x)) + U'(Y^x_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(x,s,W_s) - U'(x)\mu(x) \\
&\quad + \tfrac{1}{2}\operatorname{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\operatorname{Hess}U)(Y^x_s)\Big) \\
&\quad - \tfrac{1}{2}\operatorname{trace}\big(\sigma(x)\sigma(x)^*(\operatorname{Hess}U)(x)\big) \\
&\quad + \tfrac{1}{2}e^{-\rho s}\Big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y^x_s)\Big\|^2 \\
&\quad - \tfrac{1}{2}e^{-\rho s}\,\|\sigma(x)^*(\nabla U)(x)\|^2 + \tfrac{U'(Y^x_s)\,(\triangle_y\Phi)(x,s,W_s)}{2}\bigg)\,ds\bigg] \\
&\leq \mathbb{E}\bigg[\int_0^{t\wedge\tau_n} \exp\bigg(e^{-\rho s}U(Y^x_s) + \int_0^s e^{-\rho r}\bar{U}(Y^x_r)\,dr\bigg) \\
&\quad \cdot \bigg(\big|\bar{U}(Y^x_s) - \bar{U}(x)\big| + \rho\,|U(Y^x_s) - U(x)| + \big|U'(Y^x_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(x,s,W_s) - U'(x)\mu(x)\big| \\
&\quad + \tfrac{1}{2}\Big|\operatorname{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\operatorname{Hess}U)(Y^x_s)\Big) \\
&\quad - \operatorname{trace}\big(\sigma(x)\sigma(x)^*(\operatorname{Hess}U)(x)\big)\Big| \\
&\quad + \tfrac{1}{2}e^{-\rho s}\bigg|\Big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y^x_s)\Big\|^2 - \|\sigma(x)^*(\nabla U)(x)\|^2\bigg| \\
&\quad + \tfrac{\big|U'(Y^x_s)\,(\triangle_y\Phi)(x,s,W_s)\big|}{2}\bigg)\,ds\bigg].
\end{aligned} \tag{63}$$

Hence, Fatou's lemma, Fubini's theorem, and Hölder's inequality imply for all $t \in [0, h]$, $x \in D$ that

(64)
$$
\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(e^{-\rho t} U(Y_t^x) + \int_0^t e^{-\rho r} \bar{U}(Y_r^x)\, dr\bigg)\bigg] - e^{U(x)} \\
&\leq \liminf_{n\to\infty} \mathbb{E}\bigg[\exp\bigg(e^{-\rho(t\wedge\tau_n)} U(Y_{t\wedge\tau_n}^x) + \int_0^{t\wedge\tau_n} e^{-\rho r} \bar{U}(Y_r^x)\, dr\bigg)\bigg] - e^{U(x)} \\
&\leq \mathbb{E}\bigg[\int_0^t \exp\bigg(U(Y_s^x) + \int_0^s e^{-\rho r} \bar{U}(Y_r^x)\, dr\bigg) \\
&\qquad \cdot \bigg(\big|\bar{U}(Y_s^x) - \bar{U}(x)\big| + \rho |U(Y_s^x) - U(x)| + \big|U'(Y_s^x)\left(\tfrac{\partial}{\partial s}\Phi\right)(x, s, W_s) - U'(x)\mu(x)\big| \\
&\qquad + \tfrac{1}{2}\bigg| \operatorname{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x, s, W_s)\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x, s, W_s)\big]^* (\operatorname{Hess} U)(Y_s^x)\Big) \\
&\qquad - \operatorname{trace}\left(\sigma(x)\sigma(x)^* (\operatorname{Hess} U)(x)\right) \bigg| \\
&\qquad + \tfrac{1}{2} e^{-\rho s} \bigg| \Big\| \big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x, s, W_s)\big]^* (\nabla U)(Y_s^x) \Big\|^2 - \|\sigma(x)^* (\nabla U)(x)\|^2 \bigg| \\
&\qquad + \tfrac{\left|U'(Y_s^x)\,(\triangle_y \Phi)(x,s,W_s)\right|}{2} \bigg)\, ds \bigg] \\
&\leq \int_0^t \bigg\| \exp\bigg(U(Y_s^x) + \int_0^s e^{-\rho r} \bar{U}(Y_r^x)\, dr\bigg) \bigg\|_{L^2(\Omega;\mathbb{R})} \\
&\qquad \cdot \bigg[\rho \|U(Y_s^x) - U(x)\|_{L^2(\Omega;\mathbb{R})} + \big\|U'(Y_s^x)\big(\tfrac{\partial}{\partial s}\Phi\big)(x, s, W_s) - U'(x)\mu(x)\big\|_{L^2(\Omega;\mathbb{R})} \\
&\qquad + \tfrac{1}{2}\bigg\| \operatorname{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x, s, W_s)\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x, s, W_s)\big]^* (\operatorname{Hess} U)(Y_s^x) \\
&\qquad - \sigma(x)\sigma(x)^* (\operatorname{Hess} U)(x)\Big) \bigg\|_{L^2(\Omega;\mathbb{R})} \\
&\qquad + \tfrac{1}{2} e^{-\rho s} \bigg\| \big\| \big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x, s, W_s)\big]^* (\nabla U)(Y_s^x) \big\|^2 - \|\sigma(x)^* (\nabla U)(x)\|^2 \bigg\|_{L^2(\Omega;\mathbb{R})} \\
&\qquad + \tfrac{\left\|U'(Y_s^x)\,(\triangle_y \Phi)(x,s,W_s)\right\|_{L^2(\Omega;\mathbb{R})}}{2} + \big\|\bar{U}(Y_s^x) - \bar{U}(x)\big\|_{L^2(\Omega;\mathbb{R})} \bigg]\, ds\,.
\end{aligned}
$$

Next we estimate the L^2-norms on the right-hand side separately. Combining the assumption that $U \in C^3_{p,c}(\mathbb{R}^d, [0,\infty)) \subseteq C^1_{p,c}(\mathbb{R}^d, [0,\infty))$ with Lemma 2.7 (with $c = c$, $p = p$, $V = U$, $x = x$, $y = r(y - x)$ for $r \in [0, 1]$, $x, y \in \mathbb{R}^d$ in the notation of Lemma 2.7) implies for all $x, y \in \mathbb{R}^d$, $i \in \{0, 1, 2\}$ that

(65)
$$
\begin{aligned}
&\big\|U^{(i)}(y) - U^{(i)}(x)\big\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})} \\
&\qquad \leq c\,\|y - x\| \Big[1 + \sup\nolimits_{r\in[0,1]} U\big(ry + (1 - r)x\big)\Big]^{\max\{1-(i+1)/p,0\}} \\
&\qquad = c \sup\nolimits_{r\in[0,1]} \big[1 + U\big(x + r(y - x)\big)\big]^{\frac{\max(p-i-1,0)}{p}} \|y - x\| \\
&\qquad \leq c \sup\nolimits_{r\in[0,1]} \Big[c^p\, 2^{p-1}\big(1 + U(x) + \|r(y - x)\|^p\big)\Big]^{\frac{\max(p-i-1,0)}{p}} \|y - x\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{c\,(2c)^{\max\{p-i-1,0\}}}{2^{\max\{p-i-1,0\}/p}} \left([1+U(x)]^{\frac{\max(p-i-1,0)}{p}} + \|y-x\|^{\max(p-i-1,0)} \right) \|y-x\| \\
&\leq \tfrac{(2c)^p}{2} \left(1+|U(x)|^{\frac{\max(p-i-1,0)}{p}} + \|y-x\|^{\max(p-i-1,0)} \right) \|y-x\| .
\end{aligned}
$$

This, in particular, shows for all $x, y \in \mathbb{R}^d$ that

$$
|U(y) - U(x)| \leq \tfrac{(2c)^p}{2} \left(1 + |U(x)|^{\frac{p-1}{p}} + \|y-x\|^{(p-1)} \right) \|y-x\| . \tag{66}
$$

Combining this with Hölder's inequality and the fact that $\forall\, x \in \mathbb{R}^d \colon |\bar{U}(x)| \leq c\,(1+|U(x)|^{\gamma_6})$ yields for all $s \in (0,h]$, $x \in \mathbb{R}^d$ that

$$
\begin{aligned}
&\mathbb{E}\Big[\exp\Big(2U(Y_s^x) + 2\int_0^s e^{-\rho r}\bar{U}(Y_r^x)\,dr - 2U(x)\Big)\Big] \\
&\quad\leq \mathbb{E}\Big[\exp\Big(2\,|U(Y_s^x) - U(x)| + 2\int_0^s \big|\bar{U}(Y_r^x)\big|\,dr\Big)\Big] \\
&\quad\leq \mathbb{E}\Big[\exp\Big((2c)^p\big[1+|U(x)|^{\frac{p-1}{p}} + \|Y_s^x - x\|^{(p-1)}\,\big]\,\|Y_s^x - x\| \\
&\qquad + 2c\int_0^s (1+|U(Y_r^x)|^{\gamma_6})\,dr\Big)\Big] \\
&\quad\leq \Big\|\exp\Big((2c)^p\big[1+|U(x)|^{\frac{p-1}{p}} + \|Y_s^x - x\|^{(p-1)}\,\big]\,\|Y_s^x - x\|\Big)\Big\|_{L^1(\Omega;\mathbb{R})} \\
&\qquad \cdot \Big\|\exp\Big(2c\int_0^s (1+|U(Y_r^x)|^{\gamma_6})\,dr\Big)\Big\|_{L^\infty(\Omega;\mathbb{R})} \\
&\quad\leq \mathbb{E}\Big[\exp\Big((2c)^p\big[1+|U(x)|^{\frac{p-1}{p}} + \|Y_s^x - x\|^{(p-1)}\big]\,\|Y_s^x - x\|\Big)\Big] \\
&\qquad \cdot \exp\Big(2c\int_0^s \big(1+\|U(Y_r^x)\|^{\gamma_6}_{L^\infty(\Omega;\mathbb{R})}\big)\,dr\Big) .
\end{aligned} \tag{67}
$$

Next we estimate the two factors on the right-hand side of (67) separately. Using Hölder's inequality and assumption (59) shows for all $s \in (0,h]$, $x \in D$ that

(68)

$$
\begin{aligned}
&\mathbb{E}\Big[\exp\Big((2c)^p\big[1+|U(x)|^{\frac{p-1}{p}} + \|Y_s^x - x\|^{(p-1)}\big]\,\|Y_s^x - x\|\Big)\Big] \\
&= \mathbb{E}\Bigg[\sum_{n=0}^{\infty} \tfrac{(2c)^{pn}}{n!}\big[1+|U(x)|^{\frac{p-1}{p}} + \|\Phi(x,s,W_s) - x\|^{(p-1)}\big]^n\,\|\Phi(x,s,W_s) - x\|^n\Bigg] \\
&= \sum_{n=0}^{\infty} \Big\|\tfrac{(2c)^{pn}}{n!}\big[1+|U(x)|^{\frac{p-1}{p}} + \|\Phi(x,s,W_s) - x\|^{(p-1)}\big]^n\,\|\Phi(x,s,W_s) - x\|^n\Big\|_{L^1(\Omega;\mathbb{R})} \\
&\leq \sum_{n=0}^{\infty} \tfrac{(2c)^{pn}}{n!}\Big[1+|U(x)|^{\frac{p-1}{p}} + \|\Phi(x,s,W_s) - x\|^{(p-1)}_{L^\infty(\Omega;\mathbb{R}^d)}\Big]^n \\
&\qquad \cdot \|\Phi(x,s,W_s) - x\|^n_{L^n(\Omega;\mathbb{R}^d)} \\
&\leq \sum_{n=0}^{\infty} \tfrac{(2c)^{pn}}{n!}\Big[1+|U(x)|^{\frac{p-1}{p}} + c^{(p-1)}(1+|U(x)|^{\gamma_5})^{(p-1)}\Big]^n \\
&\qquad \cdot \Big[c\,(1+|U(x)|^{\gamma_5})\,\|\mu(x)s + \sigma(x)W_s\|_{L^n(\Omega;\mathbb{R}^d)}\Big]^n
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \tfrac{(2c)^{pn}}{n!} \Big[c\big(1 + |U(x)|^{\frac{p-1}{p}}\big)\big(1 + |U(x)|^{\gamma_5}\big) + c^p (1 + |U(x)|^{\gamma_5})^p \Big]^n \\
&\qquad \cdot \mathbb{E}\big[\|\mu(x)s + \sigma(x)W_s\|^n \big] \\
&= \mathbb{E}\Big[\exp\Big([2c]^p \Big[c\big(1 + |U(x)|^{\frac{p-1}{p}}\big)\big(1 + |U(x)|^{\gamma_5}\big) + c^p (1 + |U(x)|^{\gamma_5})^p \Big] \\
&\qquad \cdot \|\mu(x)s + \sigma(x)W_s\| \Big) \Big].
\end{aligned}
$$

Hence, assumption (54) and Lemma 2.5 yield for all $s \in (0, h]$, $x \in D$ that

(69)

$$
\begin{aligned}
&\mathbb{E}\Big[\exp\Big([2c]^p \big[1 + |U(x)|^{\frac{p-1}{p}} + \|Y_s^x - x\|^{(p-1)}\big] \, \|Y_s^x - x\|\Big)\Big] \\
&\leq \mathbb{E}\Big[\exp\Big(c\,[2c]^p \Big[1 + |U(x)|^{\frac{p-1}{p}} + |U(x)|^{\gamma_5} + |U(x)|^{(\gamma_5 + \frac{p-1}{p})} \\
&\qquad + [2c]^{(p-1)} \, (1 + |U(x)|^{p\gamma_5}) \Big] \, \|\mu(x)s + \sigma(x)W_s\| \Big) \Big] \\
&\leq \mathbb{E}\Big[\exp\Big(2^p c^{(p+1)} \Big[1 + \frac{c}{s^{\alpha(p-1)/p}} + \frac{c^{\gamma_5}}{s^{\alpha\gamma_5}} + \frac{c^{(\gamma_5+1)}}{s^{\alpha(\gamma_5 + (p-1)/p)}} \\
&\qquad + [2c]^{(p-1)} \left(1 + \frac{c^{p\gamma_5}}{s^{\alpha p \gamma_5}} \right) \Big] \, \|\mu(x)s + \sigma(x)W_s\| \Big) \Big] \\
&\leq \mathbb{E}\Big[\exp\Big(2^p c^{(p+1)} \, [\min(s,1)]^{-\alpha(p\gamma_5+1)} \Big[4c^{(\gamma_5+1)} + 2^p c^{(p+p\gamma_5-1)} \Big] \\
&\qquad \cdot \|\mu(x)s + \sigma(x)W_s\| \Big) \Big] \\
&\leq \mathbb{E}\Big[\exp\Big(2^{(2p+2)} c^{p(\gamma_5+3)} \, [\min(s,1)]^{-\alpha(p\gamma_5+1)} \, (\|\mu(x)\|s + \|\sigma(x)W_s\|) \Big)\Big] \\
&= \exp\Big(2^{(2p+2)} c^{p(\gamma_5+3)} \, [\min(s,1)]^{-\alpha(p\gamma_5+1)} \, \|\mu(x)\|s \Big) \\
&\qquad \cdot \mathbb{E}\Big[\exp\Big(2^{(2p+2)} c^{p(\gamma_5+3)} \, [\min(s,1)]^{-\alpha(p\gamma_5+1)} \, \|\sigma(x)W_s\| \Big)\Big] \\
&\leq \exp\Big(2^{(2p+2)} c^{p(\gamma_5+3)} \, [\min(s,1)]^{-\alpha(p\gamma_5+1)} \, \|\mu(x)\|s \Big) \\
&\qquad \cdot 2 \exp\left(\tfrac{s 2^{(4p+4)} c^{2p(\gamma_5+3)} \, \|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d)}}{2\,[\min(s,1)]^{2\alpha(p\gamma_5+1)}} \right) \\
&\leq 2 \exp\Big(s\, 2^{(4p+3)} c^{2p(\gamma_5+3)} \, [\min(s,1)]^{-2\alpha(p\gamma_5+1)} \, \Big[\|\mu(x)\| + \|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d)} \Big] \Big) \\
&\leq 2 \exp\Big(s\, 2^{(4p+3)} \, c^{2p(\gamma_5+3)} \, [\min(s,1)]^{-2\alpha(p\gamma_5+1)} \\
&\qquad \cdot \Big[c\,(1 + |U(x)|^{\gamma_0}) + c^2 \, (1 + |U(x)|^{\gamma_1})^2 \Big] \Big) \\
&\leq 2 \exp\Big(s\, 2^{(4p+3)} \, c^{2p(\gamma_5+3)} \, [\min(s,1)]^{-2\alpha(p\gamma_5+1)} \\
&\qquad \cdot \Big[c + c^{(1+\gamma_0)} s^{-\alpha\gamma_0} + 2c^2 + 2c^{2(1+\gamma_1)} s^{-2\alpha\gamma_1} \Big] \Big) \\
&\leq 2 \exp\Big(s\, 2^{(4p+3)} \, c^{2p(\gamma_5+3)} \, [\min(s,1)]^{-\alpha(2p\gamma_5+2+\gamma_0+2\gamma_1)} \Big[2c^{(1+\gamma_0)} + 4c^{2(1+\gamma_1)} \Big] \Big) \\
&\leq 2 \exp\Big(s\, 2^{(4p+6)} \, c^{2p(\max(\gamma_0/2,\gamma_1)+\gamma_5+4)} \, [\min(s,1)]^{-\alpha(2p\gamma_5+2+\gamma_0+2\gamma_1)} \Big) .
\end{aligned}
$$

Next we combine (66) with assumption (59) to obtain for all $r \in (0,h]$, $x \in D$ that

$$
\begin{aligned}
&\|U(Y_r^x)\|_{L^\infty(\Omega;\mathbb{R})} \leq U(x) + \|U(Y_r^x) - U(x)\|_{L^\infty(\Omega;\mathbb{R})} \\
&\leq U(x) + \tfrac{(2c)^p}{2} \left\| \Big(1 + |U(x)|^{(p-1)/p} + \|Y_r^x - x\|^{(p-1)}\Big) \|Y_r^x - x\| \right\|_{L^\infty(\Omega;\mathbb{R})} \\
&\leq U(x) + \tfrac{(2c)^p}{2} \Big[c\big(1 + |U(x)|^{(p-1)/p}\big)\left(1 + |U(x)|^{\gamma_5}\right) + c^p \left(1 + |U(x)|^{\gamma_5}\right)^p \Big] \\
&\leq U(x) + \tfrac{(2c)^p}{2} \left[2c \max(1, U(x)) \cdot 2 \max(1, |U(x)|^{\gamma_5}) + [2c]^p \max(1, |U(x)|^{p\gamma_5}) \right] \\
&\leq U(x) + \tfrac{(2c)^p}{2} \left[4c \max(1, |U(x)|^{(\gamma_5+1)}) + [2c]^p \max(1, |U(x)|^{p\gamma_5}) \right] \\
&\leq U(x) + \tfrac{(2c)^p}{2} \max(1, |U(x)|^{(p\gamma_5+1)}) \left[4c + (2c)^p\right] \\
&\leq U(x) + \tfrac{3}{2} \left[2c\right]^{2p} \max(1, |U(x)|^{(p\gamma_5+1)}) \\
&\leq 2^{(2p+1)} c^{2p} \max(1, |U(x)|^{(p\gamma_5+1)}).
\end{aligned}
\tag{70}
$$

Therefore, it holds for all $s \in (0,h]$, $x \in D$ that

$$
\begin{aligned}
&2c \int_0^s \left(1 + \|U(Y_r^x)\|_{L^\infty(\Omega;\mathbb{R})}^{\gamma_6}\right) dr \leq 2sc + 2sc \left(2^{(2p+1)} c^{2p} \max\big(1, |U(x)|^{(p\gamma_5+1)}\big)\right)^{\gamma_6} \\
&\qquad \leq 2sc + 2sc\, 2^{(2p+1)\gamma_6}\, c^{2p\gamma_6} \max\big(1, |U(x)|^{(p\gamma_5+1)\gamma_6}\big) \\
&\qquad \leq 2sc + s\, 2^{[(2p+1)\gamma_6+1]}\, c^{(2p\gamma_6+1+(p\gamma_5+1)\gamma_6)} \max\big(1, s^{-\alpha(p\gamma_5+1)\gamma_6}\big) \\
&\qquad \leq s\, 2^{[(2p+1)\gamma_6+2]}\, c^{[(p(\gamma_5+2)+1)\gamma_6+1]} \left[\min(s,1)\right]^{-\alpha(p\gamma_5+1)\gamma_6}.
\end{aligned}
\tag{71}
$$

Inserting (69) and (71) into (67) then shows for all $s \in (0,h]$, $x \in D$ that

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(2U(Y_s^x) + 2\int_0^s e^{-\rho r}\bar{U}(Y_r^x)\,dr - 2U(x)\right)\right] \\
&\leq 2\exp\left(s\, 2^{(4p+6)}\, c^{2p(\max(\gamma_0/2,\gamma_1)+\gamma_5+4)} \left[\min(s,1)\right]^{-\alpha(2p\gamma_5+2+\gamma_0+2\gamma_1)}\right) \\
&\quad \cdot \exp\left(s\, 2^{[(2p+1)\gamma_6+2]}\, c^{[(p(\gamma_5+2)+1)\gamma_6+1]} \left[\min(s,1)\right]^{-\alpha(p\gamma_5+1)\gamma_6}\right) \\
&\leq 2\exp\left(\left[2^{(4p+6)}\, c^{2p(\max(\gamma_0/2,\gamma_1)+\gamma_5+4)} + 2^{[(2p+1)\gamma_6+2]}\, c^{[p\gamma_6(\gamma_5+3)+1]}\right] \right. \\
&\quad \left. \cdot \tfrac{s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}} \right) \\
&\leq 2\exp\Big(2^{[1+(2p+3)(\gamma_6+2)]}\, c^{p[\max(\gamma_0,2\gamma_1)+(\gamma_5+4)(\gamma_6+2)]} \\
&\quad \cdot \left[\min(s,1)\right]^{-\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}\, s\Big).
\end{aligned}
\tag{72}
$$

Therefore, we obtain for all $s \in (0,h]$, $x \in D$ that

$$
\begin{aligned}
&\left\|\exp\left(U(Y_s^x) + \int_0^s \tfrac{\bar{U}(Y_r^x)}{e^{\rho r}}\,dr\right)\right\|_{L^2(\Omega;\mathbb{R})} \\
&\qquad \leq \sqrt{2}\exp\left(\tfrac{2^{(2p+3)(\gamma_6+2)}\, c^{p[\max(\gamma_0,2\gamma_1)+(\gamma_5+4)(\gamma_6+2)]}\, s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\right) e^{U(x)}.
\end{aligned}
\tag{73}
$$

Moreover, the fact that $\forall\, r \in [2,\infty), s \in [0,h], x \in \mathbb{R}^d$: $\|\sigma(x)W_s\|_{L^r(\Omega;\mathbb{R}^m)} \leq \sqrt{sr(r-1)/2}\, \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}$ (see, e.g., Lemma 7.7 in Da Prato and Zabczyk [4]), assumption (54), and assumption (59) imply for all $r \in [2,\infty)$, $s \in (0,h]$, $x \in D$ that

$$
\begin{aligned}
&\|Y_s^x - x\|_{L^r(\Omega;\mathbb{R}^d)} = \|\Phi(x,s,W_s) - x\|_{L^r(\Omega;\mathbb{R}^d)} \\
&\quad\leq c\,(1 + |U(x)|^{\gamma_5})\, \|\mu(x)s + \sigma(x)W_s\|_{L^r(\Omega;\mathbb{R}^d)} \\
&\quad\leq c\left(1 + c^{\gamma_5} s^{-\alpha\gamma_5}\right)\left(\|\mu(x)\|\, s + \sqrt{sr(r-1)/2}\, \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\right) \\
&\quad\leq c\left(1 + c^{\gamma_5} s^{-\alpha\gamma_5}\right)\left(cs\left(1 + c^{\gamma_0} s^{-\alpha\gamma_0}\right) + c\sqrt{sr(r-1)/2}\left(1 + c^{\gamma_1} s^{-\alpha\gamma_1}\right)\right) \\
&\quad\leq 2c^{(2+\max(\gamma_0,\gamma_1)+\gamma_5)}\, [\min(s,1)]^{-\alpha\gamma_5}\left(1 + \sqrt{r(r-1)/2}\right)\sqrt{s} \\
&\qquad\quad \cdot \max\!\left(\sqrt{s}\left(1 + s^{-\alpha\gamma_0}\right), 1 + s^{-\alpha\gamma_1}\right) \\
&\quad\leq 2c^{(2+\max(\gamma_0,\gamma_1)+\gamma_5)}\, [\min(s,1)]^{-\alpha\gamma_5}\, r\sqrt{s}\, [\max(s,1)]^{1/2} \\
&\qquad\quad \cdot \max\!\left(1 + s^{-\alpha\gamma_0}, 1 + s^{-\alpha\gamma_1}\right) \\
&\quad\leq 4rc^{(2+\max(\gamma_0,\gamma_1)+\gamma_5)} s^{1/2}\, [\max(s,1)]^{1/2}\, [\min(s,1)]^{-\alpha(\gamma_0+\gamma_1+\gamma_5)} \\
&\quad= 4rc^{(2+\max(\gamma_0,\gamma_1)+\gamma_5)} \max(s,1)\, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5)]}\,.
\end{aligned}
\tag{74}
$$

Combining (59) and (74) with Hölder's inequality and inequality (65) yields for all $r \in [2,\infty)$, $i \in \{0,1,2\}$, $s \in (0,h]$, $x \in D$ that

$$
\begin{aligned}
&\left\|U^{(i)}(Y_s^x) - U^{(i)}(x)\right\|_{L^r(\Omega;L^{(i)}(\mathbb{R}^d,\mathbb{R}))} \\
&\leq \left\|\tfrac{(2c)^p}{2}\left(1 + |U(x)|^{\frac{\max(p-i-1,0)}{p}} + \|Y_s^x - x\|^{\max(p-i-1,0)}\right)\|Y_s^x - x\|\right\|_{L^r(\Omega;\mathbb{R})} \\
&\leq \tfrac{(2c)^p}{2}\Big(\|Y_s^x - x\|_{L^r(\Omega;\mathbb{R}^d)} + |U(x)|^{\frac{\max(p-i-1,0)}{p}}\, \|Y_s^x - x\|_{L^r(\Omega;\mathbb{R}^d)} \\
&\qquad\qquad + \|Y_s^x - x\|^{\max(p-i,1)}_{L^{r\cdot\max(p-i,1)}(\Omega;\mathbb{R}^d)}\Big) \\
&\leq \tfrac{(2c)^p}{2}\left(1 + \tfrac{c}{s^{\alpha\max(p-i-1,0)/p}} + \|Y_s^x - x\|^{\max(p-i-1,0)}_{L^{r\cdot\max(p-i,1)}(\Omega;\mathbb{R}^d)}\right) \\
&\qquad\qquad \cdot \|Y_s^x - x\|_{L^{r\cdot\max(p-i,1)}(\Omega;\mathbb{R}^d)} \\
&\leq \tfrac{(2c)^p}{2}\left[\tfrac{2c}{[\min(s,1)]^{\alpha}} + [crp]^{\max(p-i-1,0)}\right] 4rp\, c^{(2+\max(\gamma_0,\gamma_1)+\gamma_5)} \\
&\qquad\qquad \cdot \max(s,1)\, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5)]} \\
&\leq 2^{(p+1)}\, c^{(p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\, [2crp + [crp]^p] \\
&\qquad\qquad \cdot \max(s,1)\, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} \\
&\leq 6\, c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\, [2rp]^p \max(s,1)\, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]}\,.
\end{aligned}
\tag{75}
$$

This, the assumption that $U \in C^3_{p,c}(\mathbb{R}^d, [0,\infty))$, Lemma 2.6, Hölder's inequality, assumption (54), and assumption (56) show for all $s \in (0,h]$, $x \in D$ that

(76)
$$\begin{aligned}
&\big\|U'(Y_s^x)\left(\tfrac{\partial}{\partial s}\Phi\right)(x,s,W_s) - U'(x)\,\mu(x)\big\|_{L^2(\Omega;\mathbb{R})} \\
&\leq \Big\|\|U'(Y_s^x)\|_{L(\mathbb{R}^d,\mathbb{R})}\,\big\|\left(\tfrac{\partial}{\partial s}\Phi\right)(x,s,W_s) - \mu(x)\big\|\Big\|_{L^2(\Omega;\mathbb{R})} \\
&\quad + \|U'(Y_s^x) - U'(x)\|_{L^2(\Omega;L(\mathbb{R}^d,\mathbb{R}))}\,\|\mu(x)\| \\
&\leq \|U'(x)\|_{L(\mathbb{R}^d,\mathbb{R})}\,\big\|\left(\tfrac{\partial}{\partial s}\Phi\right)(x,s,W_s) - \mu(x)\big\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\quad + \|U'(Y_s^x) - U'(x)\|_{L^4(\Omega;L(\mathbb{R}^d,\mathbb{R}))}\left[\|\mu(x)\| + \big\|\left(\tfrac{\partial}{\partial s}\Phi\right)(x,s,W_s) - \mu(x)\big\|_{L^4(\Omega;\mathbb{R}^d)}\right] \\
&\leq c\,[1+U(x)]^{\frac{(p-1)}{p}}\,cs^{\gamma_2} + 6\,c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\,[8p]^p \\
&\quad \cdot \max(s,1)\,[\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]}\left[c\left(1+c^{\gamma_0}s^{-\alpha\gamma_0}\right)+cs^{\gamma_2}\right] \\
&\leq c^2 s^{\gamma_2}\left[1+cs^{-\alpha}\right]^{(p-1)/p} + 6\,c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_0+\gamma_5)}\,[8p]^p \\
&\quad \cdot \max(s,1)\,[\min(s,1)]^{[1/2-\alpha(2\gamma_0+\gamma_1+\gamma_5+1)]}\,[2+s^{\gamma_2}] \\
&\leq 2c^3 s^{\gamma_2}\,[\min(s,1)]^{-\alpha} + 18\,c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_0+\gamma_5)}\,[8p]^p \\
&\quad \cdot [\max(s,1)]^{(1+\gamma_2)}\,[\min(s,1)]^{[1/2-\alpha(2\gamma_0+\gamma_1+\gamma_5+1)]} \\
&= 2c^3\,[\max(s,1)]^{\gamma_2}\,[\min(s,1)]^{(\gamma_2-\alpha)} + 18\,c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_0+\gamma_5)}\,[8p]^p \\
&\quad \cdot [\max(s,1)]^{(1+\gamma_2)}\,[\min(s,1)]^{[1/2-\alpha(2\gamma_0+\gamma_1+\gamma_5+1)]} \\
&\leq 20\,[8p]^p\,c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_0+\gamma_5)} \\
&\quad \cdot [\max(s,1)]^{(1+\gamma_2)}\,[\min(s,1)]^{[\min(\gamma_2,1/2)-\alpha(2\gamma_0+\gamma_1+\gamma_5+1)]}\,.
\end{aligned}$$

Analogously, (75), the assumption that $U \in C^3_{p,c}(\mathbb{R}^d, [0,\infty))$, Lemma 2.6, Hölder's inequality, assumption (54), and assumption (57) show for all $s \in (0,h]$, $x \in D$ that

$$\begin{aligned}
&\big\|U'(Y_s^x)\left(\tfrac{\partial}{\partial y}\Phi\right)(x,s,W_s) - U'(x)\,\sigma(x)\big\|_{L^4(\Omega;L(\mathbb{R}^m,\mathbb{R}))} \\
&\leq \Big\|\|U'(Y_s^x)\|_{L(\mathbb{R}^d,\mathbb{R})}\,\big\|\left(\tfrac{\partial}{\partial y}\Phi\right)(x,s,W_s) - \sigma(x)\big\|_{L(\mathbb{R}^m,\mathbb{R}^d)}\Big\|_{L^4(\Omega;\mathbb{R})} \\
&\quad + \|U'(Y_s^x) - U'(x)\|_{L^4(\Omega;L(\mathbb{R}^d,\mathbb{R}))}\,\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \\
&\leq \|U'(x)\|_{L(\mathbb{R}^d,\mathbb{R})}\,\big\|\left(\tfrac{\partial}{\partial y}\Phi\right)(x,s,W_s) - \sigma(x)\big\|_{L^4(\Omega;L(\mathbb{R}^m,\mathbb{R}^d))} \\
&\quad + \|U'(Y_s^x) - U'(x)\|_{L^8(\Omega;L(\mathbb{R}^d,\mathbb{R}))} \\
&\quad \cdot \left[\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} + \big\|\left(\tfrac{\partial}{\partial y}\Phi\right)(x,s,W_s) - \sigma(x)\big\|_{L^8(\Omega;L(\mathbb{R}^m,\mathbb{R}^d))}\right] \\
&\leq c\,[1+U(x)]^{(p-1)/p}\,cs^{\gamma_3} + 6\,c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\,[16p]^p \qquad (77)\\
&\quad \cdot \max(s,1)\,[\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]}\left[c\left(1+c^{\gamma_1}s^{-\alpha\gamma_1}\right)+cs^{\gamma_3}\right] \\
&\leq c^2 s^{\gamma_3}\left[1+cs^{-\alpha}\right]^{(p-1)/p} + 6\,c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)}\,[16p]^p \\
&\quad \cdot \max(s,1)\,[\min(s,1)]^{[1/2-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]}\,[2+s^{\gamma_3}] \\
&\leq 2c^3 s^{\gamma_3}\,[\min(s,1)]^{-\alpha} + 18\,c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)}\,[16p]^p \\
&\quad \cdot [\max(s,1)]^{(1+\gamma_3)}\,[\min(s,1)]^{[1/2-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]}
\end{aligned}$$

$$\leq 20\,[16p]^p\, c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)} \cdot [\max(s,1)]^{(1+\gamma_3)}\, [\min(s,1)]^{[\min(\gamma_3,1/2)-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]}.$$

In the next step we note for all $A_1, A_2 \in \mathbb{R}^{d\times m}$, $B_1, B_2 \in \mathbb{R}^{d\times d}$ that

(78)

$$\begin{aligned}
&|\mathrm{trace}\,(A_1 A_1^* B_1 - A_2 A_2^* B_2)| \\
&\quad = |\mathrm{trace}\,(A_1^* B_1 A_1 - A_2^* B_2 A_2)| \\
&\quad = \left| \langle A_1, B_1 A_1 \rangle_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} - \langle A_2, B_2 A_2 \rangle_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \right| \\
&\quad = \Big| \langle A_2, (B_1 - B_2) A_2 \rangle_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \langle A_1 - A_2, B_1 A_1 \rangle_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\qquad + \langle A_2, B_1(A_1 - A_2) \rangle_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \Big| \\
&\quad \leq \|A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\, \|(B_1 - B_2) A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \|A_1 - A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\, \|B_1 A_1\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\qquad + \|A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\, \|B_1(A_1 - A_2)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\quad \leq \|B_1 - B_2\|_{L(\mathbb{R}^d)}\, \|A_2\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \|A_1 - A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\, \|B_1\|_{L(\mathbb{R}^d)} \\
&\qquad \cdot \Big[\|A_1\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \|A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \Big] \\
&\quad \leq \|B_1 - B_2\|_{L(\mathbb{R}^d)}\, \|A_2\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\qquad + \Big[\|A_1 - A_2\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + 2\, \|A_1 - A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\, \|A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \Big] \\
&\qquad \cdot \Big[\|B_1 - B_2\|_{L(\mathbb{R}^d)} + \|B_2\|_{L(\mathbb{R}^d)} \Big].
\end{aligned}$$

Next we apply this inequality with $A_1 = (\frac{\partial}{\partial y}\Phi)(x,s,W_s)$, $A_2 = \sigma(x)$, $B_1 = (\mathrm{Hess}\, U)(Y_s^x)$, and $B_2 = (\mathrm{Hess}\, U)(x)$ for $s \in [0,h]$, we take expectations, we apply Hölder's inequality, we use the assumption that $U \in C^3_{p,c}(\mathbb{R}^d, [0,\infty))$, we use Lemma 2.6 (ii) (with $d = d, n = 2, c = c, p = p, x = x, y = w, V = U, i = 1, z_1 = v, t = 0$ for $x \in D, v, w \in \{u \in \mathbb{R}^d \colon \|u\| \leq 1\}$ in the notation of Lemma 2.6 (ii)), and we apply inequalities (54), (57), and (75) to obtain for all $s \in (0,h]$, $x \in D$ that

$$\begin{aligned}
&\Big\| \mathrm{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s) \big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s) \big]^* \\
&\qquad \cdot (\mathrm{Hess}\, U)(Y_s^x) - \sigma(x)\, \sigma(x)^*\, (\mathrm{Hess}\, U)(x) \Big) \Big\|_{L^2(\Omega;\mathbb{R})} \\
&\quad \leq \|(\mathrm{Hess}\, U)(Y_s^x) - (\mathrm{Hess}\, U)(x)\|_{L^2(\Omega; L(\mathbb{R}^d))}\, \|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\qquad + \Big\| \big\| (\tfrac{\partial}{\partial y}\Phi)(x,s,W_s) - \sigma(x) \big\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\qquad + 2\, \big\| (\tfrac{\partial}{\partial y}\Phi)(x,s,W_s) - \sigma(x) \big\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\, \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \Big\|_{L^4(\Omega;\mathbb{R})} \\
&\qquad \cdot \Big[\|(\mathrm{Hess}\, U)(Y_s^x) - (\mathrm{Hess}\, U)(x)\|_{L^4(\Omega;L(\mathbb{R}^d))} + \|(\mathrm{Hess}\, U)(x)\|_{L(\mathbb{R}^d)} \Big] \\
&\quad \leq 6\, c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\, [4p]^p \max(s,1) \\
&\qquad \cdot [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]}\, 2c^2 \left(1 + c^{2\gamma_1} s^{-2\alpha\gamma_1}\right) \\
&\qquad + \left[c^2 s^{2\gamma_3} + 2cs^{\gamma_3} c \left(1 + c^{\gamma_1} s^{-\alpha\gamma_1}\right) \right]
\end{aligned}$$

$$
\begin{aligned}
&\cdot \Big[6\, c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\, [8p]^p \max(s,1) \\
&\cdot [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} + c\,[1+U(x)]^{\max(p-2,0)/p}\Big] \qquad (79)\\
&\leq 24\,[4p]^p\, c^{(2p+4+\max(\gamma_0,\gamma_1)+2\gamma_1+\gamma_5)} \max(s,1)\,[\min(s,1)]^{[1/2-\alpha(\gamma_0+3\gamma_1+\gamma_5+1)]} \\
&\quad + c^{(2+\gamma_1)}\, s^{\gamma_3}\, \big[s^{\gamma_3}+2+2s^{-\alpha\gamma_1}\big] \Big[6\, c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\, [8p]^p \max(s,1) \\
&\quad \cdot [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} + 2c^2\, [\min(s,1)]^{-\alpha}\Big] \\
&\leq 24\,[4p]^p\, c^{(2p+4+\max(\gamma_0,\gamma_1)+2\gamma_1+\gamma_5)} \max(s,1)\,[\min(s,1)]^{[1/2-\alpha(\gamma_0+3\gamma_1+\gamma_5+1)]} \\
&\quad + 5\, c^{(2p+4+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)}\, [\max(s,1)]^{(1+2\gamma_3)} \\
&\quad \cdot [\min(s,1)]^{[\gamma_3-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]}\, [6\,[8p]^p+2] \\
&\leq 24\,[4p]^p\, c^{(2p+4+\max(\gamma_0,\gamma_1)+2\gamma_1+\gamma_5)} \max(s,1)\,[\min(s,1)]^{[1/2-\alpha(\gamma_0+3\gamma_1+\gamma_5+1)]} \\
&\quad + 35\,[8p]^p\, c^{(2p+4+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)}\, [\max(s,1)]^{(1+2\gamma_3)} \\
&\quad \cdot [\min(s,1)]^{[\gamma_3-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]} \\
&\leq 47\,[8p]^p\, c^{(2p+4+\max(\gamma_0,\gamma_1)+2\gamma_1+\gamma_5)}\, [\max(s,1)]^{(2\gamma_3+1)} \\
&\quad \cdot [\min(s,1)]^{[\min(\gamma_3,1/2)-\alpha(\gamma_0+3\gamma_1+\gamma_5+1)]}\,.
\end{aligned}
$$

Furthermore, the fact that $\forall\, a,b\in\mathbb{R}^m\colon \big|\|a\|^2-\|b\|^2\big| \leq \|a-b\|\,(\|a-b\|+2\,\|b\|)$, Hölder's inequality, inequality (77), and inequality (54) prove for all $s\in(0,h]$, $x\in D$ that

$$
\begin{aligned}
&\Big\|\big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y_s^x)\big\|^2 - \|\sigma(x)^*(\nabla U)(x)\|^2\Big\|_{L^2(\Omega;\mathbb{R})} \\
&\leq \big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y_s^x) - \sigma(x)^*(\nabla U)(x)\big\|_{L^4(\Omega;\mathbb{R}^m)} \\
&\quad \cdot \Big\|\big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y_s^x) - \sigma(x)^*(\nabla U)(x)\big\| \\
&\quad + 2\,\|\sigma(x)^*\|_{L(\mathbb{R}^d,\mathbb{R}^m)}\, \|(\nabla U)(x)\|\,\Big\|_{L^4(\Omega;\mathbb{R})} \\
&\leq 20\,[16p]^p\, c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)}\, [\max(s,1)]^{(1+\gamma_3)} \\
&\quad \cdot [\min(s,1)]^{[\min(\gamma_3,1/2)-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]} \\
&\quad \cdot \Big[\tfrac{20[16p]^p c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)}[\max(s,1)]^{(1+\gamma_3)}}{[\min(s,1)]^{-[\min(\gamma_3,1/2)-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]}} \qquad (80)\\
&\quad + 2c\,\big(1+c^{\gamma_1}s^{-\alpha\gamma_1}\big)\, c\,\big[1+cs^{-\alpha}\big]^{(p-1)/p}\Big] \\
&\leq 20\,[16p]^p\, c^{2(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)}\, [\max(s,1)]^{(1+\gamma_3)} \\
&\quad \cdot [\min(s,1)]^{[\min(\gamma_3,1/2)-2\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]} \\
&\quad \cdot \Big[20\,[16p]^p\, [\max(s,1)]^{(1+\gamma_3)} + 8\Big] \\
&\leq \big[2^9 p\big]^{2p}\, c^{2(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)}\, [\max(s,1)]^{(2+2\gamma_3)} \\
&\quad \cdot [\min(s,1)]^{[\min(\gamma_3,1/2)-2\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]}\,.
\end{aligned}
$$

In addition, we note that Hölder's inequality, the assumption that $U \in C^3_{p,c}(\mathbb{R}^d, [0,\infty))$, Lemma 2.6, inequality (58), and inequality (75) imply for all $s \in (0,h]$, $x \in D$ that

$$
\begin{aligned}
&\|U'(Y^x_s)\,(\triangle_y \Phi)(x,s,W_s)\|_{L^2(\Omega;\mathbb{R})} \\
&\quad\le \|U'(Y^x_s)\|_{L^4(\Omega;L(\mathbb{R}^d,\mathbb{R}))}\,\|(\triangle_y\Phi)(x,s,W_s)\|_{L^4(\Omega;\mathbb{R}^d)} \\
&\quad\le \Big(\|U'(Y^x_s)-U'(x)\|_{L^4(\Omega;L(\mathbb{R}^d,\mathbb{R}))} + \|U'(x)\|_{L(\mathbb{R}^d,\mathbb{R})}\Big) \\
&\qquad\cdot \|(\triangle_y\Phi)(x,s,W_s)\|_{L^4(\Omega;\mathbb{R}^d)} \\
&\quad\le \Big(6\,[8p]^p\,c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\max(s,1) \\
&\qquad\cdot[\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} + c\,[1+U(x)]^{(p-1)/p}\Big)cs^{\gamma_4} \\
&\quad\le \Big(6\,[8p]^p\,c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\max(s,1) \\
&\qquad\cdot[\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} + 2c^2\,[\min(s,1)]^{-\alpha}\Big)cs^{\gamma_4} \\
&\quad\le (6\,[8p]^p+2)\,c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_5)}\,[\max(s,1)]^{(\gamma_4+1)} \\
&\qquad\cdot[\min(s,1)]^{[\gamma_4-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} \\
&\quad\le 7\,[8p]^p\,c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_5)}\,[\max(s,1)]^{(\gamma_4+1)}\,[\min(s,1)]^{[\gamma_4-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]}.
\end{aligned}
\tag{81}
$$

Moreover, the fact that $\forall\, x,y\in\mathbb{R}^d\colon \big|\bar U(x)-\bar U(y)\big| \le c\,(1+|U(x)|^{\gamma_7}+|U(y)|^{\gamma_7})\,\|x-y\|$, inequality (70), and inequality (74) show for all $s\in(0,h]$, $x\in D$ that

$$
\begin{aligned}
&\big\|\bar U(Y^x_s)-\bar U(x)\big\|_{L^2(\Omega;\mathbb{R})} \le \|c\,(1+|U(x)|^{\gamma_7}+|U(Y^x_s)|^{\gamma_7})\,\|Y^x_s-x\|\|_{L^2(\Omega;\mathbb{R})} \\
&\le c\Big[1+|U(x)|^{\gamma_7}+\|U(Y^x_s)\|^{\gamma_7}_{L^\infty(\Omega;\mathbb{R})}\Big]\,\|Y^x_s-x\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\le c\Big[1+|U(x)|^{\gamma_7}+\Big[2^{(2p+1)}c^{2p}\max(1,|U(x)|^{(p\gamma_5+1)})\Big]^{\gamma_7}\Big]\,\tfrac{8\,c^{(2+\max(\gamma_0,\gamma_1)+\gamma_5)}\max(s,1)}{[\min(s,1)]^{-[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5)]}} \\
&\le \Big[1+c^{\gamma_7}s^{-\alpha\gamma_7}+2^{(2p+1)\gamma_7}\,c^{(2p+p\gamma_5+1)\gamma_7}\,[\min(s,1)]^{-\alpha\gamma_7(p\gamma_5+1)}\Big] \\
&\quad\cdot\tfrac{8\,c^{(3+\max(\gamma_0,\gamma_1)+\gamma_5)}\max(s,1)}{[\min(s,1)]^{-[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5)]}} \\
&\le \Big[2+2^{(2p+1)\gamma_7}\Big]\,8\,c^{[3+\max(\gamma_0,\gamma_1)+\gamma_5+(p(\gamma_5+2)+1)\gamma_7]}\max(s,1) \\
&\quad\cdot[\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+\gamma_7(p\gamma_5+1))]} \\
&\le 24\cdot 2^{(2p+1)\gamma_7}\,c^{[3+\max(\gamma_0,\gamma_1)+\gamma_5+(p(\gamma_5+2)+1)\gamma_7]}\max(s,1) \\
&\quad\cdot[\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+(p\gamma_5+1)\gamma_7)]}.
\end{aligned}
\tag{82}
$$

In the next step we insert (73), (75), (76), (79), (80), (81), and (82) into (64) to obtain for all $t \in (0,h]$, $x \in D$ that

(83)
$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(e^{-\rho t}U(Y_t^x) + \int_0^t e^{-\rho r}\bar{U}(Y_r^x)\,dr\bigg)\bigg] - e^{U(x)} \\
&\quad\leq \int_0^t \sqrt{2}\exp\Big(\tfrac{2^{(2p+3)(\gamma_6+2)}\,c^{p[\max(\gamma_0,2\gamma_1)+(\gamma_5+4)(\gamma_6+2)]}\,s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\Big)\,e^{U(x)} \\
&\qquad\cdot\Big[6\rho\,c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\,[4p]^p\max(s,1)\,[\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} \\
&\qquad+20\,[8p]^p\,c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_0+\gamma_5)}\,[\max(s,1)]^{(1+\gamma_2)} \\
&\qquad\cdot[\min(s,1)]^{[\min(\gamma_2,1/2)-\alpha(2\gamma_0+\gamma_1+\gamma_5+1)]} \\
&\qquad+\tfrac{47}{2}\,[8p]^p\,c^{(2p+4+\max(\gamma_0,\gamma_1)+2\gamma_1+\gamma_5)}\,[\max(s,1)]^{(2\gamma_3+1)} \\
&\qquad\cdot[\min(s,1)]^{[\min(\gamma_3,1/2)-\alpha(\gamma_0+3\gamma_1+\gamma_5+1)]}+\tfrac{1}{2}\,\big[2^9p\big]^{2p}\,c^{2(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)} \\
&\qquad\cdot[\max(s,1)]^{(2+2\gamma_3)}\,[\min(s,1)]^{[\min(\gamma_3,1/2)-2\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]} \\
&\qquad+4\,[8p]^p\,c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_5)}\,[\max(s,1)]^{(\gamma_4+1)}\,[\min(s,1)]^{[\gamma_4-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} \\
&\qquad+24\cdot 2^{(2p+1)\gamma_7}\,c^{[3+\max(\gamma_0,\gamma_1)+\gamma_5+(p(\gamma_5+2)+1)\gamma_7]}\max(s,1) \\
&\qquad\cdot[\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+(p\gamma_5+1)\gamma_7)]}\Big]\,ds \\
&\quad\leq e^{U(x)}\int_0^t \sqrt{2}\exp\Big(\tfrac{2^{(2p+3)(\gamma_6+2)}\,c^{p[\max(\gamma_0,2\gamma_1)+(\gamma_5+4)(\gamma_6+2)]}\,s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\Big) \\
&\qquad\cdot c^{[6+4p+6\max(\gamma_0,\gamma_1,\gamma_5)+p\gamma_7(\gamma_5+3)]}\Big[6\rho\,[4p]^p+48\,[8p]^p+\tfrac{1}{2}\,\big[2^9p\big]^{2p}+2^{(3p\gamma_7+5)}\Big] \\
&\qquad\cdot[\max(s,1)]^{\max(1+\gamma_2,2+2\gamma_3,1+\gamma_4)} \\
&\qquad\cdot[\min(s,1)]^{[\min(1/2,\gamma_2,\gamma_3,\gamma_4)-\alpha(2\gamma_0+4\gamma_1+2\gamma_5+(p\gamma_5+1)\gamma_7+2)]}\,ds\,.
\end{aligned}$$

This implies for all $t \in (0,h]$, $x \in D$ that

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(\tfrac{U(Y_t^x)}{e^{\rho t}}+\textstyle\int\limits_0^t \tfrac{\bar{U}(Y_r^x)}{e^{\rho r}}\,dr\bigg)\bigg] \\
&\quad\leq e^{U(x)}+e^{U(x)}\int_0^t \max(\rho,1)\,\big[2^9p\big]^{2p}\,2^{3p\gamma_7}\exp\Big(\tfrac{2^{(2p+3)(\gamma_6+2)}\,c^{p[\max(\gamma_0,2\gamma_1)+(\gamma_5+4)(\gamma_6+2)]}\,s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\Big) \\
&\qquad\cdot\frac{c^{[6+4p+6\max(\gamma_0,\gamma_1,\gamma_5)+p\gamma_7(\gamma_5+3)]}\,[\max(s,1)]^{[\max(\gamma_2,1+2\gamma_3,\gamma_4)+1]}}{[\min(s,1)]^{[\alpha(2\gamma_0+4\gamma_1+2\gamma_5+(p\gamma_5+1)\gamma_7+2)-\min(1/2,\gamma_2,\gamma_3,\gamma_4)]}}\,ds \\
&\quad\leq e^{U(x)}\bigg[1+\textstyle\int\limits_0^t \exp\Big(\tfrac{2^{(2p+3)(\gamma_6+2)}\,c^{4p(\gamma_6+2)\max(\gamma_0,\gamma_1,\gamma_5,2)}\,s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\Big) \\
&\qquad\cdot\tfrac{\max(\rho,1)[2pc]^{6p(\gamma_7+3)\max(1,\gamma_0,\gamma_1,\gamma_5)}[\max(s,1)]^{[\max(\gamma_2,1+2\gamma_3,\gamma_4)+1]}}{[\min(s,1)]^{[\alpha(2\gamma_0+4\gamma_1+2\gamma_5+(p\gamma_5+1)\gamma_7+2)-\min(1/2,\gamma_2,\gamma_3,\gamma_4)]}}\,ds\bigg]
\end{aligned}$$

$$\leq e^{U(x)} \Bigg[1 + \int_0^t \exp\Big(\tfrac{[2c]^{4p(\gamma_6+2)\max(\gamma_0,\gamma_1,\gamma_5,2)}\, s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}} \Big) \\ \cdot \tfrac{\max(\rho,1)[2pc]^{6p(\gamma_7+3)\max(1,\gamma_0,\gamma_1,\gamma_5)}[\max(s,1)]^{[\max(\gamma_2,1+2\gamma_3,\gamma_4)+1]}}{[\min(s,1)]^{[\alpha(2\gamma_0+4\gamma_1+2\gamma_5+(p\gamma_5+1)\gamma_7+2)-\min(1/2,\gamma_2,\gamma_3,\gamma_4)]}}\, ds \Bigg].$$

This proves (60) and thereby finishes the proof of Lemma 2.8. □

2.3. **Exponential moments for stopped increment-tamed Euler-Maruyama schemes.** Using Corollary 2.3 and Lemma 2.8 above, we are now ready to establish exponential moment bounds for a class of stopped increment-tamed Euler-Maruyama schemes in the next theorem.

Theorem 2.9. *Let $\gamma, \rho \in [0,\infty)$, $T \in (0,\infty)$, $d, m \in \mathbb{N}$, $p, c \in [1,\infty)$, $q \in (1,\infty)$, $\mu \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, $\sigma \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^{d\times m}))$, $U \in C^3_{p,c}(\mathbb{R}^d, [0,\infty))$, $\bar U \in C(\mathbb{R}^d, \mathbb{R})$, $\alpha \in \big(0, \frac{1}{2}\min\{\frac{1}{7\gamma+2}, \frac{q-1}{(q+8)\gamma+2}\}\big)$, let $D_h \in \mathcal{B}(\{x \in \mathbb{R}^d \colon U(x) \leq ch^{-\alpha}\})$, $h \in (0,T]$, be a nonincreasing family of sets, assume for all $h \in (0,T]$ that $\mu|_{D_h} \in C(D_h, \mathbb{R}^d)$ and $\sigma|_{D_h} \in C(D_h, \mathbb{R}^{d\times m})$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $W \colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, let $Y^\theta \colon [0,T]\times\Omega \to \mathbb{R}^d$, $\theta \in \mathcal{P}_T$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths which satisfy for all $t \in [0,T]$, $\theta \in \mathcal{P}_T$ that*

(84)
$$Y^\theta_t = Y^\theta_{\lfloor t\rfloor_\theta} + \mathbb{1}_{D_{|\theta|_T}}(Y^\theta_{\lfloor t\rfloor_\theta}) \left[\frac{\mu(Y^\theta_{\lfloor t\rfloor_\theta})\,(t-\lfloor t\rfloor_\theta) + \sigma(Y^\theta_{\lfloor t\rfloor_\theta})\,\big(W_t - W_{\lfloor t\rfloor_\theta}\big)}{1 + \big\|\mu(Y^\theta_{\lfloor t\rfloor_\theta})\,(t-\lfloor t\rfloor_\theta) + \sigma(Y^\theta_{\lfloor t\rfloor_\theta})\,\big(W_t - W_{\lfloor t\rfloor_\theta}\big)\big\|^q} \right],$$

and assume for all $x, y \in \mathbb{R}^d$ with $x \neq y$ that

$$\|\mu(x)\| + \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + |\bar U(x)| \leq c\,(1 + |U(x)|^\gamma), \tag{85}$$
$$\tfrac{|\bar U(x)-\bar U(y)|}{\|x-y\|} \leq c\,(1 + |U(x)|^\gamma + |U(y)|^\gamma),$$
$$(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}\,\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar U(x) \leq \rho \cdot U(x). \tag{86}$$

Then it holds for all $t \in [0,T]$, $\theta \in \mathcal{P}_T$ that

(87)
$$\limsup_{|\vartheta|_T \searrow 0} \sup_{u\in[0,T]} \mathbb{E}\left[\exp\left(\tfrac{U(Y^\vartheta_u)}{e^{\rho u}} + \int_0^u \tfrac{\mathbb{1}_{D_{|\vartheta|_T}}(Y^\vartheta_{\lfloor s\rfloor_\vartheta})\,\bar U(Y^\vartheta_s)}{e^{\rho s}}\, ds \right)\right] \leq \limsup_{|\vartheta|_T \searrow 0} \mathbb{E}\Big[e^{U(Y^\vartheta_0)}\Big]$$

and

$$\mathbb{E}\left[\exp\left(\tfrac{U(Y^\theta_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_{|\theta|_T}}(Y^\theta_{\lfloor s\rfloor_\theta})\,\bar U(Y^\theta_s)}{e^{\rho s}}\, ds \right)\right] \\ \leq \exp\left(\tfrac{\max(\rho,1)\,[\min(|\theta|_T,1)]^{[\min(1/2,(q-1)/2-\alpha(q+1)\gamma)-\alpha(7\gamma+2)]}}{\exp\left(-[5cpq\max(T,1)]^{9p(q+1)\max(\gamma,1)\max(\gamma,q,2)(\gamma+2)}\right)} \right) \mathbb{E}\Big[e^{U(Y^\theta_0)}\Big].$$

Proof of Theorem 2.9. Throughout this proof let $\gamma_0, \gamma_1, \ldots, \gamma_7, \hat c \in \mathbb{R}$ be the real numbers given by $\gamma_0 = \gamma_1 = \gamma_6 = \gamma_7 = \gamma$, $\gamma_2 = \gamma_3 = q/2 - \alpha\gamma\,(q+1)$, $\gamma_4 = (q-1)/2 - \alpha\gamma\,(q+1)$, $\gamma_5 = 0$, and $\hat c = \big[16c^{(1+\gamma)}q\max(T,1)\big]^{(q+1)}$, let $\Psi_h \colon \mathbb{R}^d \times [0,h] \times \mathbb{R}^m \to \mathbb{R}^d$, $h \in (0,T]$, be the functions which satisfy for all $h \in (0,T]$, $s \in [0,h]$, $y \in \mathbb{R}^m$, $x \in \mathbb{R}^d$ that

$$\Psi_h(x,s,y) = x + \mathbb{1}_{D_h}(x)\Big[\tfrac{\mu(x)s+\sigma(x)y}{1+\|\mu(x)s+\sigma(x)y\|^q}\Big] \tag{88}$$

and let $\Phi_h \colon D_h \times [0,h] \times \mathbb{R}^m \to \mathbb{R}^d$, $h \in (0,T]$, be the functions which satisfy for all $h \in (0,T]$, $s \in [0,h]$, $y \in \mathbb{R}^m$, $x \in D_h$ that

$$\Phi_h(x,s,y) = \Psi_h(x,s,y). \tag{89}$$

We now verify step by step all assumptions of Lemma 2.8. First, note for all $h \in (0,T]$, $x \in D_h$ that $\Phi_h(x,0,0) = x$. Moreover, observe that (85) ensures that (54) in Lemma 2.8 is fulfilled. Furthermore, note for all $h \in (0,T]$, $(x,s,y) \in D_h \times (0,h] \times \mathbb{R}^m$ that

$$\begin{aligned}
&\big(\tfrac{\partial}{\partial s}\Phi_h\big)(x,s,y) \\
&\quad= \tfrac{\mu(x)(1+\|\mu(x)s+\sigma(x)y\|^q)-(\mu(x)s+\sigma(x)y)q\|\mu(x)s+\sigma(x)y\|^{(q-2)}\langle\mu(x)s+\sigma(x)y,\mu(x)\rangle}{(1+\|\mu(x)s+\sigma(x)y\|^q)^2} \\
&\quad= \mu(x) - \tfrac{\mu(x)\|\mu(x)s+\sigma(x)y\|^q}{1+\|\mu(x)s+\sigma(x)y\|^q} - \tfrac{q(\mu(x)s+\sigma(x)y)\|\mu(x)s+\sigma(x)y\|^{(q-2)}\langle\mu(x)s+\sigma(x)y,\mu(x)\rangle}{(1+\|\mu(x)s+\sigma(x)y\|^q)^2}.
\end{aligned} \tag{90}$$

This implies for all $h \in (0,T]$, $s \in (0,h]$, $x \in D_h$, $y \in \mathbb{R}^m$ that

$$\big\|\big(\tfrac{\partial}{\partial s}\Phi_h\big)(x,s,y) - \mu(x)\big\| \leq (q+1)\,\|\mu(x)s+\sigma(x)y\|^q\,\|\mu(x)\|. \tag{91}$$

Moreover, the inequality $\forall\, r \in [2,\infty), s \in [0,T], x \in \mathbb{R}^d \colon \|\sigma(x)W_s\|_{L^r(\Omega;\mathbb{R}^d)} \leq \sqrt{sr(r-1)/2}\,\|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}$ shows for all $r \in [2,\infty)$, $h \in (0,T], s \in (0,h]$, $x \in D_h \subseteq D_s$ that

$$\begin{aligned}
&\|\mu(x)s+\sigma(x)W_s\|_{L^r(\Omega;\mathbb{R}^d)} \\
&\quad\leq cs\,(1+|U(x)|^\gamma) + c\sqrt{sr(r-1)/2}\,(1+|U(x)|^\gamma) \\
&\quad\leq c\left(s + c^\gamma s^{(1-\alpha\gamma)}\right) + c\sqrt{r(r-1)/2}\left(s^{1/2} + c^\gamma s^{(1/2-\alpha\gamma)}\right) \\
&\quad\leq c^{(1+\gamma)}\max(T,1)\,s^{(1/2-\alpha\gamma)}\left(2+2\sqrt{r(r-1)/2}\right) \\
&\quad\leq 2c^{(1+\gamma)}r\max(T,1)\,s^{(1/2-\alpha\gamma)}.
\end{aligned} \tag{92}$$

This together with (91) and the fact $\alpha\gamma < 1$ implies for all $h \in (0,T]$, $s \in (0,h]$, $x \in D_h \subseteq D_s$ that

$$\begin{aligned}
&\big\|\big(\tfrac{\partial}{\partial s}\Phi_h\big)(x,s,W_s) - \mu(x)\big\|_{L^4(\Omega;\mathbb{R}^d)} \\
&\quad\leq (q+1)\,\|\mu(x)s+\sigma(x)W_s\|^q_{L^{4q}(\Omega;\mathbb{R}^d)}\,\|\mu(x)\| \\
&\quad\leq (q+1)\left[8c^{(1+\gamma)}q\max(T,1)\right]^q s^{q(1/2-\gamma\alpha)}\,c\,(1+|U(x)|^\gamma) \\
&\quad\leq c\,(q+1)\left[8c^{(1+\gamma)}q\max(T,1)\right]^q s^{q(1/2-\gamma\alpha)}\left(1+c^\gamma s^{-\alpha\gamma}\right) \\
&\quad\leq c^{(1+\gamma)}\,(q+1)\left[8c^{(1+\gamma)}q\max(T,1)\right]^q s^{(q/2-\alpha(q+1)\gamma)}\,(s^{\alpha\gamma}+1) \\
&\quad\leq 2qc^{(1+\gamma)}\left[8c^{(1+\gamma)}q\max(T,1)\right]^q s^{(q/2-\alpha(q+1)\gamma)}\,2\,[\max(T,1)] \\
&\quad\leq \left[8c^{(1+\gamma)}q\max(T,1)\right]^{(q+1)} s^{(q/2-\alpha(q+1)\gamma)} \leq \hat{c}s^{\gamma_2}.
\end{aligned} \tag{93}$$

This proves that (56) in Lemma 2.8 is fulfilled. Similarly, it holds for all $h \in (0,T]$, $(x,s,y) \in D_h \times (0,h] \times \mathbb{R}^m$ that

(94)
$$\begin{aligned}
&\big(\tfrac{\partial}{\partial y}\Phi_h\big)(x,s,y) \\
&\quad = \tfrac{\sigma(x)(1+\|\mu(x)s+\sigma(x)y\|^q)-(\mu(x)s+\sigma(x)y)q\|\mu(x)s+\sigma(x)y\|^{(q-2)}(\mu(x)s+\sigma(x)y)^*\sigma(x)}{(1+\|\mu(x)s+\sigma(x)y\|^q)^2} \\
&\quad = \sigma(x) - \tfrac{\sigma(x)\|\mu(x)s+\sigma(x)y\|^q}{1+\|\mu(x)s+\sigma(x)y\|^q} - \tfrac{q(\mu(x)s+\sigma(x)y)\|\mu(x)s+\sigma(x)y\|^{(q-2)}(\mu(x)s+\sigma(x)y)^*\sigma(x)}{(1+\|\mu(x)s+\sigma(x)y\|^q)^2}.
\end{aligned}$$

This implies for all $h \in (0,T]$, $s \in (0,h]$, $x \in D_h$, $y \in \mathbb{R}^m$ that

$$\begin{aligned}
&\big\|\big(\tfrac{\partial}{\partial y}\Phi_h\big)(x,s,y) - \sigma(x)\big\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\qquad \leq (q+1)\,\|\mu(x)s+\sigma(x)y\|^q\,\|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\,.
\end{aligned} \tag{95}$$

This together with (92) implies for all $h \in (0,T]$, $s \in (0,h]$, $x \in D_h \subseteq D_s$ that

$$\begin{aligned}
&\Big\|\big(\tfrac{\partial}{\partial y}\Phi_h\big)(x,s,W_s) - \sigma(x)\Big\|_{L^8(\Omega;\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))} \\
&\leq (q+1)\,\|\mu(x)s+\sigma(x)W_s\|^q_{L^{8q}(\Omega;\mathbb{R}^d)}\,\|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\leq (q+1)\Big[16c^{(1+\gamma)}q\max(T,1)\Big]^q s^{q(1/2-\gamma\alpha)}c\left(1+c^\gamma s^{-\alpha\gamma}\right) \\
&\leq c^{(1+\gamma)}\,(q+1)\Big[16c^{(1+\gamma)}q\max(T,1)\Big]^q s^{[q/2-\alpha(q+1)\gamma]}\,(s^{\alpha\gamma}+1) \\
&\leq 2c^{(1+\gamma)}q\Big[16c^{(1+\gamma)}q\max(T,1)\Big]^q s^{[q/2-\alpha(q+1)\gamma]}\,2\max(T,1) \\
&\leq \Big[16c^{(1+\gamma)}q\max(T,1)\Big]^{(q+1)} s^{[q/2-\alpha(q+1)\gamma]} = \hat{c}s^{\gamma_3}.
\end{aligned} \tag{96}$$

This shows that (57) in Lemma 2.8 is fulfilled. In the next step let $\psi\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma_i\colon \mathbb{R}^d \to \mathbb{R}^d$, $i \in \{1,\dots,m\}$, be the functions with the property that for all $z \in \mathbb{R}^d$ it holds that $\psi(z) = \frac{z}{1+\|z\|^q}$ and $\sigma(z) = (\sigma_1(z),\sigma_2(z),\dots,\sigma_m(z))$. Observe that $\psi \in C^2(\mathbb{R}^d,\mathbb{R}^d)$ and that for all $z = (z_1,z_2,\dots,z_d)$, $u = (u_1,u_2,\dots,u_d)$, $v = (v_1,v_2,\dots,v_d) \in \mathbb{R}^d$ it holds that

(97)
$$\psi'(z)u = \sum_{k=1}^d \big(\tfrac{\partial}{\partial z_k}\psi\big)(z)\cdot u_k = \begin{cases} u & : z = 0, \\ \frac{u}{1+\|z\|^q} - \frac{qz\|z\|^{(q-2)}\langle z,u\rangle}{(1+\|z\|^q)^2} & : z \neq 0, \end{cases}$$

and

$$\begin{aligned}
\psi''(z)(u,v) &= \sum_{k,l=1}^d \big(\tfrac{\partial^2}{\partial z_k \partial z_l}\psi\big)(z)\cdot u_k\cdot u_l \\
&= \begin{cases} 0 & : z = 0, \\ -\frac{q\|z\|^{(q-2)}[u\langle z,v\rangle+v\langle z,u\rangle+z\langle u,v\rangle]}{(1+\|z\|^q)^2} - \frac{q(q-2)\|z\|^{(q-4)}z\langle z,v\rangle\langle z,u\rangle}{(1+\|z\|^q)^2} & \\ \qquad\qquad + \frac{2q^2\|z\|^{2(q-2)}z\langle z,u\rangle\langle z,v\rangle}{(1+\|z\|^q)^3} & : z \neq 0. \end{cases}
\end{aligned}$$

This implies for all $z, u \in \mathbb{R}^d$ that

$$\psi''(z)(u,u) = \begin{cases} 0 & : z = 0, \\ \frac{2q^2\|z\|^{2(q-2)}z|\langle z,u\rangle|^2}{(1+\|z\|^q)^3} - \frac{q\|z\|^{(q-2)}\left[2u\langle z,u\rangle + z\|u\|^2\right]}{(1+\|z\|^q)^2} \\ \qquad - \frac{q(q-2)\|z\|^{(q-4)}z|\langle z,u\rangle|^2}{(1+\|z\|^q)^2} & : z \neq 0. \end{cases} \tag{98}$$

Hence, we obtain for all $i \in \{1, 2, \ldots, m\}$, $(x, s, y) \in \mathbb{R}^d \times (0, h] \times \mathbb{R}^m$ that

$$\begin{aligned} &\frac{\partial^2}{\partial y_i^2}\Big(\psi\big(\mu(x)s + \sigma(x)y\big)\Big) \\ &= \frac{\partial}{\partial y_i}\Big(\psi'\big(\mu(x)s + \sigma(x)y\big)\big(\sigma_i(x)\big)\Big) = \psi''\big(\mu(x)s + \sigma(x)y\big)\big(\sigma_i(x), \sigma_i(x)\big) \\ &= \frac{\mathbb{1}_{\mathbb{R}^d\setminus\{0\}}(\mu(x)s+\sigma(x)y)\,2q^2\|\mu(x)s+\sigma(x)y\|^{2(q-2)}(\mu(x)s+\sigma(x)y)|\langle\mu(x)s+\sigma(x)y,\sigma_i(x)\rangle|^2}{(1+\|\mu(x)s+\sigma(x)y\|^q)^3} \\ &- \frac{\mathbb{1}_{\mathbb{R}^d\setminus\{0\}}(\mu(x)s+\sigma(x)y)\,q\,\|\mu(x)s+\sigma(x)y\|^{(q-2)}\left[2\sigma_i(x)\langle\mu(x)s+\sigma(x)y,\sigma_i(x)\rangle+(\mu(x)s+\sigma(x)y)\|\sigma_i(x)\|^2\right]}{(1+\|\mu(x)s+\sigma(x)y\|^q)^2} \\ &- \frac{\mathbb{1}_{\mathbb{R}^d\setminus\{0\}}(\mu(x)s+\sigma(x)y)\,q(q-2)\|\mu(x)s+\sigma(x)y\|^{(q-4)}(\mu(x)s+\sigma(x)y)|\langle\mu(x)s+\sigma(x)y,\sigma_i(x)\rangle|^2}{(1+\|\mu(x)s+\sigma(x)\|^q)^2}. \end{aligned} \tag{99}$$

This and the Cauchy-Schwarz inequality show for all $(x, s, y) \in \mathbb{R}^d \times (0, h] \times \mathbb{R}^m$ that

$$\begin{aligned} &\sum_{i=1}^m \left\|\frac{\partial^2}{\partial y_i^2}\Big(\psi\big(\mu(x)s + \sigma(x)y\big)\Big)\right\| \\ &\quad \leq \frac{2q^2\,\|\mu(x)s + \sigma(x)y\|^{(2q-1)}\,\|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{(1 + \|\mu(x)s + \sigma(x)y\|^q)^3} \\ &\qquad + \frac{3q\,\|\mu(x)s + \sigma(x)y\|^{(q-1)}\,\|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{(1 + \|\mu(x)s + \sigma(x)y\|^q)^2} \\ &\qquad + \frac{q\,|q-2|\,\|\mu(x)s + \sigma(x)y\|^{(q-1)}\,\|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{(1 + \|\mu(x)s + \sigma(x)\|^q)^2} \\ &\quad \leq \left[2q^2 + 3q + q\,|q-2|\right]\|\mu(x)s + \sigma(x)y\|^{(q-1)}\,\|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}. \end{aligned} \tag{100}$$

Consequently, it follows for all $h \in (0, T]$, $s \in (0, h]$, $x \in D_h \subseteq D_s$, $y \in \mathbb{R}^m$ that

$$\begin{aligned} \big\|(\triangle_y \Phi_h)(x, s, y)\big\| &\leq \sum_{i=1}^m \big\|\big(\tfrac{\partial^2}{\partial y_i^2}\Phi_h\big)(x, s, y)\big\| \\ &\leq \sum_{i=1}^m \left[\big(2q^2 + 3q + q^2\big)\,\|\mu(x)s + \sigma(x)y\|^{(q-1)}\,\|\sigma_i(x)\|^2\right] \\ &= 3q\,(q+1)\,\|\mu(x)s + \sigma(x)y\|^{(q-1)}\,\|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}. \end{aligned} \tag{101}$$

This together with (92) and the fact $2\alpha\gamma < 1$ yields for all $h \in (0,T]$, $s \in (0,h]$, $x \in D_h \subseteq D_s$ that

$$\begin{aligned}
&\left\|(\triangle_y \Phi_h)(x,s,W_s)\right\|_{L^4(\Omega;\mathbb{R}^d)} \\
&\leq 3q\,(q+1)\,\|\mu(x)s+\sigma(x)W_s\|^{(q-1)}_{L^{4(q-1)}(\Omega;\mathbb{R}^d)}\,\|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\leq 3q\,(q+1)\,\|\mu(x)s+\sigma(x)W_s\|^{(q-1)}_{L^{4q}(\Omega;\mathbb{R}^d)}\,\|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\leq 3q\,(q+1)\left[8c^{(1+\gamma)}q\max(T,1)\right]^{(q-1)} s^{(q-1)(1/2-\gamma\alpha)}\,c^2\,(1+|U(x)|^{\gamma})^2 \\
&\leq 6q\,(q+1)\,c^2\left[8c^{(1+\gamma)}q\max(T,1)\right]^{(q-1)} s^{(q-1)(1/2-\gamma\alpha)}\left(1+c^{2\gamma}s^{-2\alpha\gamma}\right) \\
&\leq 12c^{(2+2\gamma)}q^2\left[8c^{(1+\gamma)}q\max(T,1)\right]^{(q-1)} s^{[(q-1)/2-\alpha(q+1)\gamma]}\,2\,[\max(T,1)] \\
&\leq \left[8c^{(1+\gamma)}q\max(T,1)\right]^{(q+1)} s^{[(q-1)/2-\alpha(q+1)\gamma]} \leq \hat{c}s^{\gamma_4}.
\end{aligned} \tag{102}$$

This proves that (58) in Lemma 2.8 is fulfilled. Next observe for all $h \in (0,T]$, $s \in (0,h]$, $x \in D_h$, $r \in [1,\infty)$ that

$$\begin{aligned}
&\|\Phi_h(x,s,W_s)-x\|_{L^r(\Omega;\mathbb{R}^d)} \\
&\leq \min\Big\{1, \|\mu(x)s+\sigma(x)W_s\|_{L^r(\Omega;\mathbb{R}^d)}\Big\} \\
&\leq \hat{c}\min\Big\{r, 1+|U(x)|^{\gamma_5}, (1+|U(x)|^{\gamma_5})\,\|\mu(x)s+\sigma(x)W_s\|_{L^r(\Omega;\mathbb{R}^d)}\Big\}.
\end{aligned} \tag{103}$$

This shows that (59) in Lemma 2.8 is fulfilled. Thus all assumptions of Lemma 2.8 are satisfied. Next let $\varrho_h \in (0,\infty)$, $h \in (0,T]$, be the real numbers with the property that for all $h \in (0,T]$ it holds that

$$\begin{aligned}
\varrho_h &= \exp\!\left(\tfrac{h\,[2\hat{c}]^{4p(\gamma_6+2)\max(\gamma_0,\gamma_1,\gamma_5,2)}}{[\min(h,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\right) \\
&\quad\cdot\tfrac{\max(\rho,1)\,[2p\hat{c}\max(h,1)]^{6p(\gamma_7+3)\max(1,\gamma_0,\gamma_1,\dots,\gamma_5)}}{[\min(h,1)]^{[\alpha(2\gamma_0+4\gamma_1+2\gamma_5+(p\gamma_5+1)\gamma_7+2)-\min(1/2,\gamma_2,\gamma_3,\gamma_4)]}} \\
&= \exp\!\left(\tfrac{h\,[2\hat{c}]^{4p(\gamma+2)\max(\gamma,2)}}{[\min(h,1)]^{\alpha[4\gamma+2]}}\right)\tfrac{\max(\rho,1)\,[2p\hat{c}\max(h,1)]^{3p(\gamma+3)\max\{2,2\gamma,q-2\alpha\gamma(q+1)\}}}{[\min(h,1)]^{[\alpha[7\gamma+2]-\min(1/2,(q-1)/2-\alpha(q+1)\gamma)]}}.
\end{aligned} \tag{104}$$

Note that the estimates $\alpha\,[7\gamma+2] - \min(1/2,(q-1)/2-\alpha(q+1)\gamma) < 0$ and $\alpha\,[4\gamma+2]-1 < 0$ ensure that the function $(0,T] \ni h \mapsto \varrho_h \in (0,\infty)$ is nondecreasing and that $\lim_{h\searrow 0}\varrho_h = 0$. Combining Lemma 2.8 with the fact that $(0,T] \ni h \mapsto \varrho_h \in (0,\infty)$ is nondecreasing implies for all $h \in (0,T]$, $(t,x) \in (0,h] \times D_h$ that

$$\begin{aligned}
&\mathbb{E}\!\left[\exp\!\left(\tfrac{U(\Phi_h(x,t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(\Phi_h(x,s,W_s))}{e^{\rho s}}\,ds\right)\right] \\
&\leq \left(1+\int_0^t \varrho_s\,ds\right)e^{U(x)} \leq (1+\varrho_h t)\,e^{U(x)}.
\end{aligned} \tag{105}$$

Clearly, this implies for all $\theta \in \mathcal{P}_T$, $(t,x) \in \big(0,|\theta|_T\big] \times D_{|\theta|_T}$ that

$$\mathbb{E}\!\left[\exp\!\left(\tfrac{U(\Psi_{|\theta|_T}(x,t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(\Psi_{|\theta|_T}(x,s,W_s))}{e^{\rho s}}\,ds\right)\right] \leq e^{\varrho_{|\theta|_T}t+U(x)}. \tag{106}$$

Corollary 2.3 hence yields for all $t \in [0,T]$, $\theta \in \mathcal{P}_T$ that

$$\mathbb{E}\!\left[\exp\!\left(\tfrac{U(Y_t^\theta)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_{|\theta|_T}}(Y^\theta_{\lfloor r\rfloor_\theta})\,\bar{U}(Y_r^\theta)}{e^{\rho r}}\,dr\right)\right] \leq e^{\varrho_{|\theta|_T}t}\,\mathbb{E}\!\left[e^{U(Y_0^\theta)}\right]. \tag{107}$$

This implies for all $\theta \in \mathcal{P}_T$ that

$$\sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\tfrac{U(Y_t^\theta)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_{|\theta|_T}}(Y^\theta_{\lfloor s\rfloor_\theta})\,\bar{U}(Y_s^\theta)}{e^{\rho s}}\,ds\bigg)\bigg] \leq e^{\varrho_{|\theta|_T} T}\,\mathbb{E}\Big[e^{U(Y_0^\theta)}\Big]. \tag{108}$$

This and the fact $\lim_{h\searrow 0}\varrho_h = 0$ then show (87). Next observe that the estimate $\forall\, x \in [5^{72},\infty)\colon x \leq \exp(x^{1/20})$ shows for all $\theta \in \mathcal{P}_T$ that

$$\begin{aligned}
&\varrho_{|\theta|_T} T\\
&= \exp\left(\frac{|\theta|_T\,\Big[2\big[16c^{(\gamma+1)}q\max(T,1)\big]^{(q+1)}\Big]^{4p(\gamma+2)\max(\gamma,2)}}{\big[\min(|\theta|_T,1)\big]^{\alpha[4\gamma+2]}}\right)\\
&\quad\cdot \frac{\max(\rho,1)\,T\,\Big[2p\max(|\theta|_T,1)\big[16c^{(\gamma+1)}q\max(T,1)\big]^{(q+1)}\Big]^{3p(\gamma+3)\max\{2,2\gamma,q-2\alpha\gamma(q+1)\}}}{\big[\min(|\theta|_T,1)\big]^{[\alpha[7\gamma+2]-\min(1/2,(q-1)/2-\alpha(q+1)\gamma)]}}\\
&\leq \frac{\max(\rho,1)\,\exp\!\big([5cq\max(T,1)]^{8p\max(\gamma,1)(q+1)(\gamma+2)\max(\gamma,2)}\big)}{\big[\min(|\theta|_T,1)\big]^{[\alpha[7\gamma+2]-\min(1/2,(q-1)/2-\alpha(q+1)\gamma)]}}\\
&\quad\cdot [5cpq\max(T,1)]^{6p(q+1)\max(\gamma,1)(\gamma+3)\max(2,2\gamma,q)}\\
&\leq \frac{\max(\rho,1)\,\exp\!\big([5cq\max(T,1)]^{8p(q+1)\max(\gamma,1)\max(\gamma,2)(\gamma+2)}\big)}{\big[\min(|\theta|_T,1)\big]^{[\alpha[7\gamma+2]-\min(1/2,(q-1)/2-\alpha(q+1)\gamma)]}}\\
&\quad\cdot \exp\Big([5cpq\max(T,1)]^{3p(q+1)\max(\gamma,1)\max(2,2\gamma,q)(\gamma+3)/10}\Big)\\
&\leq \frac{\max(\rho,1)\,\exp\!\big(2\,[5cpq\max(T,1)]^{8p(q+1)\max(\gamma,1)\max(\gamma,q,2)(\gamma+2)}\big)}{\big[\min(|\theta|_T,1)\big]^{[\alpha[7\gamma+2]-\min(1/2,(q-1)/2-\alpha(q+1)\gamma)]}}.
\end{aligned}\tag{109}$$

Combining (108) with (109) completes the proof of Theorem 2.9. □

The next corollary of Theorem 2.9 considers the case in which there exists a Borel measurable set $D \subseteq \mathbb{R}^d$ such that the sets $D_h \in \mathcal{B}(\mathbb{R}^d)$, $h \in (0,T]$, satisfy for all $h \in (0,T]$ that $D_h \subseteq \{x \in D\colon U(x) \leq c\exp(c\,|\ln(h)|^{1/2})\}$ (see Corollary 2.10 below for details).

Corollary 2.10. *Let $d,m \in \mathbb{N}$, $\rho \in [0,\infty)$, $T \in (0,\infty)$, $c,q \in (1,\infty)$, $\bar{U} \in C(\mathbb{R}^d,\mathbb{R})$, $D \in \mathcal{B}(\mathbb{R}^d)$, $U \in \bigcup_{p\in[1,\infty)} C^3_{p,c}(\mathbb{R}^d,[0,\infty))$, $\mu \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d),\mathcal{B}(\mathbb{R}^d))$, $\sigma \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d),\mathcal{B}(\mathbb{R}^{d\times m}))$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $D_h \in \mathcal{B}(\{x \in D\colon U(x) \leq c\exp(c\,|\ln(h)|^{1/2})\})$, $h \in (0,T]$, be a non-increasing family of sets such that for all $h \in (0,T]$ it holds that $\mu|_{D_h} \in C(D_h,\mathbb{R}^d)$ and $\sigma|_{D_h} \in C(D_h,\mathbb{R}^{d\times m})$, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, let $Y^\theta\colon [0,T]\times\Omega \to \mathbb{R}^d$, $\theta \in \mathcal{P}_T$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths which satisfy $\sup_{\theta\in\mathcal{P}_T}\mathbb{E}\big[e^{U(Y_0^\theta)}\big] < \infty$ and which satisfy for all $t \in [0,T]$, $\theta \in \mathcal{P}_T$ that*

$$Y_t^\theta = Y^\theta_{\lfloor t\rfloor_\theta} + \mathbb{1}_{D_{|\theta|_T}}(Y^\theta_{\lfloor t\rfloor_\theta})\left[\frac{\mu(Y^\theta_{\lfloor t\rfloor_\theta})\,(t-\lfloor t\rfloor_\theta)+\sigma(Y^\theta_{\lfloor t\rfloor_\theta})\big(W_t-W_{\lfloor t\rfloor_\theta}\big)}{1+\|\mu(Y^\theta_{\lfloor t\rfloor_\theta})\,(t-\lfloor t\rfloor_\theta)+\sigma(Y^\theta_{\lfloor t\rfloor_\theta})\big(W_t-W_{\lfloor t\rfloor_\theta}\big)\|^q}\right], \tag{110}$$

and assume for all $x,y \in \mathbb{R}^d$ that

$$\|\mu(x)\| + \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,(1+\|x\|^c)\,, \tag{111}$$
$$|\bar{U}(x)-\bar{U}(y)| \leq c\,(1+\|x\|^c+\|y\|^c)\,\|x-y\|,$$
$$(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}\,\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \leq \rho\cdot U(x), \tag{112}$$
$$\|x\|^{1/c} \leq c\,(1+U(x))\,.$$

Then it holds that

$$\sup_{\theta\in\mathcal{P}_T}\sup_{t\in[0,T]}\mathbb{E}\big[\exp(e^{-\rho t}\,U(Y_t^\theta)+\int_0^t e^{-\rho s}\,\mathbb{1}_{D_{|\theta|_T}}(Y^\theta_{\lfloor s\rfloor_\theta})\,\bar{U}(Y_s^\theta)\,ds)\big]<\infty$$

and

$$\begin{aligned}&\limsup_{|\theta|_T\searrow 0}\sup_{t\in[0,T]}\mathbb{E}\big[\exp(e^{-\rho t}\,U(Y_t^\theta)+\int_0^t e^{-\rho s}\,\mathbb{1}_{D_{|\theta|_T}}(Y^\theta_{\lfloor s\rfloor_\theta})\,\bar{U}(Y_s^\theta)\,ds)\big]\\&\quad\le\limsup_{|\theta|_T\searrow 0}\mathbb{E}\big[e^{U(Y_0^\theta)}\big].\end{aligned}$$

Proof of Corollary 2.10. We show Corollary 2.10 through an application of Theorem 2.9. For this let $\gamma,\alpha\in\mathbb{R}$ be the real numbers with the property that $\gamma=c\,(c+1)$ and $\alpha=\frac{1}{4}\min\{\frac{1}{7\gamma+2},\frac{q-1}{(q+8)\gamma+2}\}$ and observe that (111), (112), and the assumption that $U\in\bigcup_{p\in[1,\infty)}C^3_{p,c}(\mathbb{R}^d,[0,\infty))$ ensure that there exist real numbers $p\in[1,\infty)$ and $\tilde{c}\in[c,\infty)$ such that $U\in C^3_{p,\tilde{c}}(\mathbb{R}^d,[0,\infty))$, such that for all $h\in(0,T]$ it holds that $c\exp\big(c\,|\ln(h)|^{1/2}\big)\le\tilde{c}\,h^{-\alpha}$, and such that for all $x,y\in\mathbb{R}^d$ with $x\neq y$ it holds that

$$\begin{aligned}&\|\mu(x)\|+\|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}+|\bar{U}(x)|\le\tilde{c}\,(1+|U(x)|^\gamma)\,,\\&|\bar{U}(x)-\bar{U}(y)|\le\tilde{c}\,(1+|U(x)|^\gamma+|U(y)|^\gamma)\,\|x-y\|.\end{aligned}\tag{113}$$

An application of Theorem 2.9 thus completes the proof of Corollary 2.10. □

Theorem 2.9 and Corollary 2.10 above establish exponential integrability properties for a family of stopped increment-tamed Euler-Maruyama approximation schemes. Another interesting class of approximation schemes which might admit exponential integrability properties are certain *rejection- or reflection-type methods*. More formally, let $d,m\in\mathbb{N}$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $D_t\in\mathcal{B}(\mathbb{R}^d)$, $t\in(0,T]$, be an appropriate nonincreasing family of sets, let $W\colon[0,T]\times\Omega\to\mathbb{R}^m$ be a standard Brownian motion, and let $Y^N\colon\{0,1,\dots,N\}\times\Omega\to\mathbb{R}^d$, $N\in\mathbb{N}$, be stochastic processes which satisfy for all $N\in\mathbb{N}$, $n\in\{0,1,\dots,N-1\}$ that

$$\begin{aligned}Y^N_{n+1}=Y^N_n+&\mathbb{1}_{\{Y^N_n+\mu(Y^N_n)\frac{T}{N}+\sigma(Y^N_n)(W_{(n+1)T/N}-W_{nT/N})\in D_{T/N}\}}\\&\cdot\Big[\mu(Y^N_n)\tfrac{T}{N}+\sigma(Y^N_n)(W_{\frac{(n+1)T}{N}}-W_{\frac{nT}{N}})\Big].\end{aligned}\tag{114}$$

Under suitable additional assumptions, we suspect that the stochastic processes Y^N, $N\in\mathbb{N}$, also admit exponential integrability properties. In the setting of the Langevin equation, a similar class of approximation methods has been considered in Bou-Rabee and Hairer [1]. Further related approximation methods have been studied in Milstein and Tretjakov [27]. In [18] (see, e.g., Section 3.6.3 in [18]) several types of appropriately tamed schemes have been investigated. The taming often constitutes by dividing the increment of an Euler-Maruyama step through a possibly large number and thereby decreasing the increment of the tamed scheme (cf., e.g., (3.140), (3.141) and (3.145) in [18]). The larger the number by which we divide the original increment of the Euler-Maruyama step the stronger the a priori bound that we can expect for the tamed scheme. In particular, if the increment of the Euler-Maruyama step is tamed by an appropriate exponential term, then we might obtain a scheme that admits exponential integrability properties. For instance, consider stochastic processes $Z^N\colon \{0,1,\ldots,N\}\times\Omega\to\mathbb{R}^d$, $N\in\mathbb{N}$, which satisfy for all $N\in\mathbb{N}$, $n\in\{0,1,\ldots,N-1\}$ that

$$Z_{n+1}^N = Z_n^N + \frac{\mu(Z_n^N)\frac{T}{N} + \sigma(Z_n^N)(W_{(n+1)T/N} - W_{nT/N})}{\exp\Big(\big\|\mu(Z_n^N)\frac{T}{N} + \sigma(Z_n^N)(W_{(n+1)T/N} - W_{nT/N})\big\|^2\Big)} \tag{115}$$

or, more generally, consider stochastic processes $Z^N\colon \{0,1,\ldots,N\}\times\Omega\to\mathbb{R}^d$, $N\in\mathbb{N}$, which satisfy that there exist (appropriate) $\alpha,\beta,\gamma\in\mathbb{R}$ such that for all $N\in\mathbb{N}$, $n\in\{0,1,\ldots,N-1\}$ it holds that

$$Z_{n+1}^N = Z_n^N + \frac{\mu(Z_n^N)\frac{T}{N} + \sigma(Z_n^N)(W_{(n+1)T/N} - W_{nT/N})}{\max\Big\{1, \frac{T^\alpha}{N^\alpha}\exp\Big(\frac{T^\beta}{N^\beta}\big\|\mu(Z_n^N)\frac{T}{N} + \sigma(Z_n^N)(W_{(n+1)T/N} - W_{nT/N})\big\|^\gamma\Big)\Big\}}. \tag{116}$$

Under suitable assumptions, it might be the case that schemes of the form (115) and (116) admit exponential integrability properties.

3. Consistency and convergence of a class of stopped and tamed schemes

In Section 2 exponential integrability properties for certain numerical approximation processes of SDEs have been established. In this section we show under suitable assumptions that these approximation processes converge in probability and strongly to the exact solution process of the considered SDE; see Corollary 3.7 and Corollary 3.8 in Subsection 3.3.3. For this we extend the notions and the convergence results in Sections 3.2–3.4 in [18]. More specifically, in Theorem 3.3 in [18] convergence in probability has, under suitable assumptions, been established for numerical approximations that are (μ,σ)-consistent in the sense of Definition 3.1 in [18]. In this article we slightly generalize this notion (Definition 3.1 in [18]) and the corresponding convergence in probability result (Theorem 3.3 in [18]) in Definition 3.1, Proposition 3.4, and Proposition 3.5 below. In addition, we establish several auxiliary results that provide sufficient conditions to ensure that a considered approximation scheme is (μ,σ)-consistent in the sense of Definition 3.1 below; see Lemma 3.2 for consistency for a class of stopped schemes, see Lemma 3.3 for consistency for a class of increment-tamed Euler-Maruyama schemes, and see Corollary 3.6 (which is an immediate consequence from Lemma 3.2 and Lemma 3.3) for consistency for a class of stopped increment-tamed Euler-Maruyama schemes. As a consequence of Corollary 3.6 and Proposition 3.5 we then obtain convergence in probability (and, under additional assumptions, also strong convergence) of the stopped increment-tamed Euler-Maruyama schemes; see Corollary 3.7. Combining Corollary 3.7, in turn, with the exponential integrability result in Corollary 2.10 in Section 2 will then allow us to derive Corollary 3.8 (the main result of this article).

Definition 3.1 (Consistency)**.** We say that ϕ is (μ,σ)-consistent with respect to Brownian motion if and only if there exist real numbers $T\in(0,\infty)$, $d,m\in\mathbb{N}$, an open set $D\subseteq\mathbb{R}^d$, a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and a standard Brownian motion

$W\colon [0,T]\times\Omega\to\mathbb{R}^m$ such that

(i) $\phi \in \mathbb{M}(\mathbb{R}^d \times (0,T] \times \mathbb{R}^m, \mathbb{R}^d)$,
(ii) $\mu \in \mathbb{M}(D, \mathbb{R}^d)$,
(iii) $\sigma \in \mathbb{M}(D, \mathbb{R}^{d\times m})$,
(iv) $\forall\, t \in (0,T]\colon (\mathbb{R}^d \times \mathbb{R}^m \| !(x,y) \mapsto \phi(x,t,y) \in \mathbb{R}^d \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^m), \mathcal{B}(\mathbb{R}^d))$, and
(v) for all nonempty compact sets $K \subseteq D$ it holds that

$$\begin{aligned}\limsup_{t\searrow 0}\left(\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\big[\|\sigma(x)W_t-\phi(x,t,W_t)\|\big]\right)&=0\\ =\limsup_{t\searrow 0}\left(\sup_{x\in K}\big\|\mu(x)-\tfrac{1}{t}\cdot\mathbb{E}[\phi(x,t,W_t)]\big\|\right).\end{aligned}\tag{117}$$

In Definition 3.1 in [18], the increment function ϕ is assumed to be Borel measurable while in Definition 3.1 above the increment function ϕ does not need to be Borel measurable in all three arguments $(x,t,y)\in\mathbb{R}^d\times(0,T]\times\mathbb{R}^m$ (see Definition 3.1 for details). In Proposition 3.4 below it is shown under suitable assumptions that if a numerical one-step scheme is consistent in the sense of Definition 3.1, then it converges in probability to the exact solution of the considered SDE (cf. also Corollaries 3.11–3.13 in [18] for strong convergence results based on consistency).

3.1. **Consistency of stopped schemes.** The next lemma establishes consistency for appropriately stopped numerical approximation schemes.

Lemma 3.2. *Let $T\in(0,\infty)$, $d,m\in\mathbb{N}$, let $D\subseteq\mathbb{R}^d$ be an open set, let $\mu\colon D\to\mathbb{R}^d$ and $\sigma\colon D\to\mathbb{R}^{d\times m}$ be functions, let $\phi\colon\mathbb{R}^d\times(0,T]\times\mathbb{R}^m\to\mathbb{R}^d$ be (μ,σ)-consistent with respect to Brownian motion, and let $D_t\in\mathcal{B}(\mathbb{R}^d)$, $t\in(0,T]$, be a nonincreasing family of sets satisfying $D\subseteq\bigcup_{t\in(0,T]}\mathring{D}_t$. Then the function $\mathbb{R}^d\times(0,T]\times\mathbb{R}^m\ni(x,t,y)\mapsto\mathbb{1}_{D_t}(x)\cdot\phi(x,t,y)\in\mathbb{R}^d$ is (μ,σ)-consistent with respect to Brownian motion.*

Proof of Lemma 3.2. Throughout this proof assume w.l.o.g. that $D\neq\emptyset$, let $K\subseteq D$ be an arbitrary nonempty compact subset of D, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $W\colon[0,T]\times\Omega\to\mathbb{R}^m$ be a standard Brownian motion. The fact that K is a compact set and the assumption that $D\subseteq\bigcup_{t\in(0,T]}\mathring{D}_t$ together with the assumption that ϕ is (μ,σ)-consistent with respect to Brownian motion ensures that there exists a real number $t_K\in(0,T]$ such that $K\subseteq\mathring{D}_{t_K}$ and

$$\sup_{t\in(0,t_K]}\left(\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\big[\|\sigma(x)W_t-\phi(x,t,W_t)\|\big]\right)<\infty.\tag{118}$$

The fact that the family D_t, $t\in(0,T]$, is nonincreasing hence shows that for all $t\in(0,t_K]$ it holds that

$$\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\Big[\big\|\sigma(x)W_t-\mathbb{1}_{D_t}(x)\,\phi(x,t,W_t)\big\|\Big]=\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\Big[\big\|\sigma(x)W_t-\phi(x,t,W_t)\big\|\Big]\tag{119}$$

and

$$\sup_{x\in K}\big\|\mu(x)-\tfrac{1}{t}\cdot\mathbb{E}\big[\mathbb{1}_{D_t}(x)\,\phi(x,t,W_t)\big]\big\|=\sup_{x\in K}\big\|\mu(x)-\tfrac{1}{t}\cdot\mathbb{E}\big[\phi(x,t,W_t)\big]\big\|.\tag{120}$$

Combining this with the assumption that ϕ is (μ, σ)-consistent with respect to Brownian motion implies

$$\limsup_{t \searrow 0} \left(\tfrac{1}{\sqrt{t}} \cdot \sup_{x \in K} \mathbb{E}\Big[\big\| \sigma(x) W_t - \mathbb{1}_{D_t}(x) \phi(x, t, W_t) \big\| \Big] \right) = 0 \tag{121}$$

and

$$\limsup_{t \searrow 0} \left(\sup_{x \in K} \big\| \mu(x) - \tfrac{1}{t} \cdot \mathbb{E}\big[\mathbb{1}_{D_t}(x) \phi(x, t, W_t) \big] \big\| \right) = 0. \tag{122}$$

Combining (121) and (122) with Definition 3.1 completes the proof of Lemma 3.2. □

3.2. **Consistency of a class of incremented-tamed Euler-Maruyama schemes.** The following lemma proves consistency for a class of increment-tamed Euler-Maruyama approximation schemes for SDEs.

Lemma 3.3. *Let $T \in (0, \infty)$, $q \in (1, \infty)$, $d, m \in \mathbb{N}$, let $D \subseteq \mathbb{R}^d$ be an open set, and let $\mu \in \mathcal{M}(\mathcal{B}(D), \mathcal{B}(\mathbb{R}^d))$ and $\sigma \in \mathcal{M}(\mathcal{B}(D), \mathcal{B}(\mathbb{R}^{d \times m}))$ be locally bounded. Then it holds that the function*

$$\mathbb{R}^d \times (0, T] \times \mathbb{R}^m \ni (x, t, y) \mapsto \left\{ \begin{array}{ll} \frac{\mu(x)t + \sigma(x)y}{1 + \|\mu(x)t + \sigma(x)y\|^q} & : x \in D \\ 0 & : x \in \mathbb{R}^d \backslash D \end{array} \right\} \in \mathbb{R}^d \tag{123}$$

is (μ, σ)-consistent with respect to Brownian motion.

Proof of Lemma 3.3. Throughout this proof let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $W \colon [0, T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion. Observe that for all nonempty compact sets $K \subseteq D$ it holds that

$$\begin{aligned}
&\limsup_{t \searrow 0} \left(\tfrac{1}{\sqrt{t}} \cdot \sup_{x \in K} \mathbb{E}\left[\left\| \sigma(x) W_t - \frac{\mu(x)t + \sigma(x)W_t}{1 + \|\mu(x)t + \sigma(x)W_t\|^q} \right\| \right] \right) \\
&= \limsup_{t \searrow 0} \left(\tfrac{1}{\sqrt{t}} \cdot \sup_{x \in K} \mathbb{E}\left[\left\| \frac{\sigma(x)W_t \, \|\mu(x)t + \sigma(x)W_t\|^q - \mu(x)t}{1 + \|\mu(x)t + \sigma(x)W_t\|^q} \right\| \right] \right) \\
&\leq \limsup_{t \searrow 0} \left(\tfrac{1}{\sqrt{t}} \cdot \sup_{x \in K} \mathbb{E}\Big[\big\| \sigma(x)W_t \, \|\mu(x)t + \sigma(x)W_t\|^q - \mu(x)t \big\| \Big] \right) \\
&\leq \limsup_{t \searrow 0} \left(\tfrac{1}{\sqrt{t}} \cdot \sup_{x \in K} \mathbb{E}\big[\, \|\sigma(x)W_t\| \, \|\mu(x)t + \sigma(x)W_t\|^q \, \big] \right) \\
&\quad + \limsup_{t \searrow 0} \left(\sqrt{t} \cdot \sup_{x \in K} \|\mu(x)\| \right) \\
&\leq 2^{(q-1)} \cdot \limsup_{t \searrow 0} \left(t^{(q - \frac{1}{2})} \left[\sup_{x \in K} \|\sigma(x)\|_{L(\mathbb{R}^m, \mathbb{R}^d)} \|\mu(x)\|^q \right] \mathbb{E}\big[\|W_t\| \big] \right) \\
&\quad + 2^{(q-1)} \cdot \limsup_{t \searrow 0} \left(\tfrac{1}{\sqrt{t}} \left[\sup_{x \in K} \|\sigma(x)\|_{L(\mathbb{R}^m, \mathbb{R}^d)}^{(1+q)} \right] \mathbb{E}\Big[\|W_t\|^{(1+q)} \Big] \right) = 0.
\end{aligned} \tag{124}$$

In addition, note that for all nonempty compact sets $K \subseteq D$ it holds that

$$\begin{aligned}
&\limsup_{t \searrow 0} \left(\sup_{x \in K} \left\| \mu(x) - \tfrac{1}{t} \cdot \mathbb{E}\left[\frac{\mu(x)t + \sigma(x)W_t}{1 + \|\mu(x)t + \sigma(x)W_t\|^q} \right] \right\| \right) \\
&\leq \limsup_{t \searrow 0} \left(\sup_{x \in K} \left\| \mathbb{E}\left[\frac{\mu(x) \, \|\mu(x)t + \sigma(x)W_t\|^q}{1 + \|\mu(x)t + \sigma(x)W_t\|^q} \right] \right\| \right) \\
&\quad + \limsup_{t \searrow 0} \left(\tfrac{1}{t} \cdot \sup_{x \in K} \left\| \mathbb{E}\left[\frac{\sigma(x)W_t}{1 + \|\mu(x)t + \sigma(x)W_t\|^q} \right] \right\| \right) \\
&\leq \limsup_{t \searrow 0} \left(\sup_{x \in K} \mathbb{E}\big[\|\mu(x)\| \, \|\mu(x)t + \sigma(x)W_t\|^q \big] \right) \\
&\quad + \limsup_{t \searrow 0} \left(\tfrac{1}{t} \cdot \sup_{x \in K} \left\| \mathbb{E}\left[\frac{\sigma(x)W_t}{1 + \|\mu(x)t + \sigma(x)W_t\|^q} - \sigma(x)W_t \right] \right\| \right) \\
&= \limsup_{t \searrow 0} \left(\tfrac{1}{t} \cdot \sup_{x \in K} \left\| \mathbb{E}\left[\frac{\sigma(x)W_t \, \|\mu(x)t + \sigma(x)W_t\|^q}{1 + \|\mu(x)t + \sigma(x)W_t\|^q} \right] \right\| \right) \\
&\leq \limsup_{t \searrow 0} \left(\tfrac{1}{t} \cdot \sup_{x \in K} \mathbb{E}\big[\|\sigma(x)\|_{L(\mathbb{R}^m, \mathbb{R}^d)} \, \|W_t\| \, \|\mu(x)t + \sigma(x)W_t\|^q \big] \right) = 0.
\end{aligned} \tag{125}$$

Combining (124) and (125) with Definition 3.1 completes the proof of Lemma 3.3. □

3.3. **Convergence of stopped increment-tamed Euler-Maruyama schemes.** This subsection establishes consistency, convergence in probability, strong convergence, and numerically weak convergence of a class of stopped increment-tamed Euler-Maruyama schemes.

3.3.1. *Setting.* Throughout Subsection 3.3 the following setting is frequently used. Let $T \in (0, \infty)$, $d, m \in \mathbb{N}$, let $D \subseteq \mathbb{R}^d$ be an open set, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0,T]}$, let $W\colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t \in [0,T]}$-Brownian motion with continuous sample paths, let $\mu\colon D \to \mathbb{R}^d$ and $\sigma\colon D \to \mathbb{R}^{d \times m}$ be locally Lipschitz continuous functions, and let $X\colon [0,T] \times \Omega \to D$ be an $(\mathcal{F}_t)_{t \in [0,T]}$-adapted stochastic process with continuous sample paths which satisfies that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = X_0 + \int_0^t \mu(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s. \tag{126}$$

3.3.2. *Convergence in probability of appropriate time-continuous interpolations.* The next proposition is a slight generalization of Theorem 3.3 in [18]. The proof of Proposition 3.4 is entirely analogous to the proof of Theorem 3.3 in [18] and therefore omitted.

Proposition 3.4. *Assume the setting in Subsection* 3.3.1*, let $\phi\colon \mathbb{R}^d \times (0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ be (μ, σ)-consistent, and let $Y^N\colon [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be mappings satisfying for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$, $t \in (\frac{nT}{N}, \frac{(n+1)T}{N}]$ that $Y_0^N = X_0$ and $Y_t^N = Y_{\frac{nT}{N}}^N + \big(\frac{tN}{T} - n\big) \cdot \phi\big(Y_{\frac{nT}{N}}^N, \frac{T}{N}, W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\big)$. Then it holds for all $\varepsilon \in (0, \infty)$ that $\limsup_{N \to \infty} \mathbb{P}\big[\sup_{t \in [0,T]} \|X_t - Y_t^N\| \geq \varepsilon \big] = 0$.*

The next proposition is an extension of Proposition 3.4 and proves convergence in probability of suitable time-continuous interpolations of numerical approximation processes of consistent schemes.

Proposition 3.5. *Assume the setting in Subsection* 3.3.1*, let* $\Psi\colon \mathbb{R}^d \times (0,T]^2 \times \mathbb{R}^m \to \mathbb{R}^d$ *be a function satisfying that* $\mathbb{R}^d \times (0,T] \times \mathbb{R}^m \ni (x,t,y) \mapsto \Psi(x,t,t,y) \in \mathbb{R}^d$ *is* (μ,σ)*-consistent, let* $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^d$*,* $N \in \mathbb{N}$*, be stochastic processes with continuous sample paths satisfying for all* $N \in \mathbb{N}$*,* $n \in \{0,1,\dots,N-1\}$*,* $t \in (\frac{nT}{N}, \frac{(n+1)T}{N}]$ *that* $Y_0^N = X_0$ *and* $Y_t^N = Y^N_{\frac{nT}{N}} + \Psi\big(Y^N_{\frac{nT}{N}}, \frac{T}{N}, t-\frac{nT}{N}, W_t - W_{\frac{nT}{N}}\big)$*, assume for all nonempty compact sets* $K \subseteq D$ *that there exists an* $h_K \in (0,T]$ *such that for all* $h \in (0,h_K]$ *it holds that* $\sup_{x\in K}\sup_{t\in[0,T]} \|\Psi(x,h,t-\lfloor t\rfloor_h, W_t - W_{\lfloor t\rfloor_h})\|$ *is* $\mathcal{F}/\mathcal{B}(\mathbb{R})$*-measurable, and assume for all nonempty compact sets* $K \subseteq D$ *that*

$$\limsup_{h\searrow 0} \mathbb{E}\Bigg[\sup_{x\in K}\sup_{t\in[0,T]} \big\|\Psi(x,h,t-\lfloor t\rfloor_h, W_t - W_{\lfloor t\rfloor_h})\big\|\Bigg] = 0. \tag{127}$$

Then it holds for all $\varepsilon \in (0,\infty)$ *that* $\limsup_{N\to\infty} \mathbb{P}\big[\sup_{t\in[0,T]} \|X_t - Y_t^N\| \geq \varepsilon\big] = 0$.

Proof of Proposition 3.5. The triangle inequality implies for all $N \in \mathbb{N}$ that

$$\begin{aligned}\sup_{t\in[0,T]} \big\|X_t - Y_t^N\big\| \leq \sup_{t\in[0,T]} \Big\|X_t - X_{\lfloor t\rfloor_{T/N}}\Big\| + \sup_{t\in[0,T]} \Big\|X_{\lfloor t\rfloor_{T/N}} - Y^N_{\lfloor t\rfloor_{T/N}}\Big\| \\ + \sup_{t\in[0,T]} \Big\|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\Big\|.\end{aligned} \tag{128}$$

Combining this, Proposition 3.4, and sample paths continuity of X_t, $t \in [0,T]$, implies for all $\varepsilon \in (0,\infty)$ that

$$\begin{aligned}&\limsup_{N\to\infty} \mathbb{P}\Bigg[\sup_{t\in[0,T]} \big\|X_t - Y_t^N\big\| \geq \varepsilon\Bigg] \\ &\quad\leq \limsup_{N\to\infty} \mathbb{P}\Bigg[\sup_{t\in[0,T]} \Big\|X_t - X_{\lfloor t\rfloor_{T/N}}\Big\| \geq \tfrac{\varepsilon}{3}\Bigg] \\ &\qquad+ \limsup_{N\to\infty} \mathbb{P}\Bigg[\sup_{t\in[0,T]} \Big\|X_{\lfloor t\rfloor_{T/N}} - Y^N_{\lfloor t\rfloor_{T/N}}\Big\| \geq \tfrac{\varepsilon}{3}\Bigg] \\ &\qquad+ \limsup_{N\to\infty} \mathbb{P}\Bigg[\sup_{t\in[0,T]} \Big\|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\Big\| \geq \tfrac{\varepsilon}{3}\Bigg] \\ &\quad= \limsup_{N\to\infty} \mathbb{P}\Bigg[\sup_{t\in[0,T]} \Big\|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\Big\| \geq \tfrac{\varepsilon}{3}\Bigg].\end{aligned} \tag{129}$$

It thus remains to prove that $\sup_{t\in[0,T]} \|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\|$ converges to zero in probability as $N \to \infty$. To prove this let $D_v \subseteq D$, $v \in \mathbb{N}$, be open sets with the property that for all $v \in \mathbb{N}$ it holds that $D_v = \{x \in D\colon \|x\| < v \text{ and } \operatorname{dist}(x, D^c) > \frac{1}{v}\}$. Then

(127) and Markov's inequality show for all $\varepsilon \in (0,\infty)$, $v \in \mathbb{N}$ that

$$\begin{aligned}
&\limsup_{N\to\infty} \mathbb{P}\Big[\Big\{\sup\nolimits_{t\in[0,T]} \big\|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\big\| \geq \varepsilon\Big\} \cap \Big\{\forall\, t \in [0,T]\colon Y^N_{\lfloor t\rfloor_{T/N}} \in D_v\Big\}\Big] \\
&= \limsup_{N\to\infty} \mathbb{P}\Big[\big(\sup\nolimits_{t\in[0,T]} \big\|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\big\|\big)\, \mathbb{1}_{\{\forall\, t\in[0,T]\colon Y^N_{\lfloor t\rfloor_{T/N}} \in D_v\}} \geq \varepsilon\Big] \\
&\leq \frac{1}{\varepsilon} \limsup_{N\to\infty} \mathbb{E}\Big[\big(\sup\nolimits_{t\in[0,T]} \big\|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\big\|\big)\, \mathbb{1}_{\{\forall\, t\in[0,T]\colon Y^N_{\lfloor t\rfloor_{T/N}} \in D_v\}}\Big] \\
&= \frac{1}{\varepsilon} \limsup_{N\to\infty} \mathbb{E}\Big[\big(\sup\nolimits_{t\in[0,T]} \big\|\Psi\big(Y^N_{\lfloor t\rfloor_{T/N}}, \tfrac{T}{N}, t - \lfloor t\rfloor_{T/N}, W_t - W_{\lfloor t\rfloor_{T/N}}\big)\big\|\big) \\
&\qquad\qquad \cdot \mathbb{1}_{\{\forall\, t\in[0,T]\colon Y^N_{\lfloor t\rfloor_{T/N}} \in D_v\}}\Big] \\
&\leq \frac{1}{\varepsilon} \limsup_{N\to\infty} \mathbb{E}\Big[\sup_{x\in\overline{D_v}} \sup_{t\in[0,T]} \big\|\Psi\big(x, \tfrac{T}{N}, t - \lfloor t\rfloor_{T/N}, W_t - W_{\lfloor t\rfloor_{T/N}}\big)\big\|\Big] = 0.
\end{aligned} \tag{130}$$

In addition, Proposition 3.4 and the continuity of the sample paths of X imply for all $\varepsilon \in (0,\infty)$ that

$$\begin{aligned}
&\limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big[\Big\{\sup\nolimits_{t\in[0,T]} \big\|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\big\| \geq \varepsilon\Big\} \\
&\qquad\qquad \cap \Big\{\exists\, t \in [0,T]\colon Y^N_{\lfloor t\rfloor_{T/N}} \notin D_{2v}\Big\}\Big] \\
&\leq \limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big[\exists\, t \in [0,T]\colon Y^N_{\lfloor t\rfloor_{T/N}} \notin D_{2v}\Big] \\
&\leq \limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big[\Big\{\sup\nolimits_{t\in[0,T]} \big\|X_{\lfloor t\rfloor_{T/N}} - Y^N_{\lfloor t\rfloor_{T/N}}\big\| < \tfrac{1}{2v}\Big\} \\
&\qquad\qquad \cap \Big\{\exists\, t \in [0,T]\colon Y^N_{\lfloor t\rfloor_{T/N}} \notin D_{2v}\Big\}\Big] \\
&\quad + \limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big[\sup\nolimits_{t\in[0,T]} \big\|X_{\lfloor t\rfloor_{T/N}} - Y^N_{\lfloor t\rfloor_{T/N}}\big\| \geq \tfrac{1}{2v}\Big] \\
&= \limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big[\Big\{\sup\nolimits_{t\in[0,T]} \big\|X_{\lfloor t\rfloor_{T/N}} - Y^N_{\lfloor t\rfloor_{T/N}}\big\| < \tfrac{1}{2v}\Big\} \\
&\qquad\qquad \cap \Big\{\exists\, t \in [0,T]\colon Y^N_{\lfloor t\rfloor_{T/N}} \notin D_{2v}\Big\}\Big] \\
&\leq \limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big[\exists\, t \in [0,T]\colon X_{\lfloor t\rfloor_{T/N}} \notin D_v\Big] \\
&\leq \limsup_{v\to\infty} \mathbb{P}\big[\exists\, t \in [0,T]\colon X_t \notin D_v\big] = 0.
\end{aligned} \tag{131}$$

Combining (130) and (131) proves that $\sup_{t\in[0,T]} \|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\|$ converges in probability to zero as N tends to infinity. The proof of Proposition 3.5 is thus completed. ☐

3.3.3. *Convergence of stopped increment-tamed Euler-Maruyama schemes.* Combining Lemma 3.2 and Lemma 3.3 immediately proves the following consistency result.

Corollary 3.6. *Let $T \in (0,\infty)$, $q \in (1,\infty)$, $d, m \in \mathbb{N}$, let $D \subseteq \mathbb{R}^d$ be an open set, let $D_t \in \mathcal{B}(\mathbb{R}^d)$, $t \in (0,T]$, be a nonincreasing family of sets satisfying $D \subseteq \bigcup_{t\in(0,T]} \mathring{D}_t$, and let $\mu \in \mathcal{M}(\mathcal{B}(D), \mathcal{B}(\mathbb{R}^d))$ and $\sigma \in \mathcal{M}(\mathcal{B}(D), \mathcal{B}(\mathbb{R}^{d\times m}))$ be locally bounded. Then it holds that the function*

$$\mathbb{R}^d \times (0,T] \times \mathbb{R}^m \ni (x,t,y) \mapsto \left\{ \begin{array}{ll} \frac{\mathbb{1}_{D_t}(x)\,[\mu(x)t+\sigma(x)y]}{1+\|\mu(x)t+\sigma(x)y\|^q} & : x \in D \\ 0 & : x \in \mathbb{R}^d \backslash D \end{array} \right\} \in \mathbb{R}^d \tag{132}$$

is (μ,σ)-consistent with respect to Brownian motion.

Combining Corollary 3.6 with Proposition 3.5 shows that the stopped increment-tamed Euler-Maruyama schemes converge in probability. This is the subject of the next result.

Corollary 3.7. *Assume the setting in Subsection 3.3.1, let $q \in (1,\infty)$, let $D_t \in \mathcal{B}(\mathbb{R}^d)$, $t \in (0,T]$, be a nonincreasing family of sets satisfying $D \subseteq \bigcup_{t\in(0,T]} \mathring{D}_t$, and let $Y^N \colon [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be mappings satisfying for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N-1\}$, $t \in \left[\frac{nT}{N}, \frac{(n+1)T}{N}\right]$, $\omega \in \Omega$ that $Y_0^N(\omega) = X_0(\omega)$ and*

$$Y_t^N(\omega) = Y_{\frac{nT}{N}}^N(\omega) + \begin{cases} \dfrac{\mu\big(Y_{\frac{nT}{N}}^N(\omega)\big)\big(t-\frac{nT}{N}\big)+\sigma\big(Y_{\frac{nT}{N}}^N(\omega)\big)\big(W_t(\omega)-W_{\frac{nT}{N}}(\omega)\big)}{1+\Big\|\mu\big(Y_{\frac{nT}{N}}^N(\omega)\big)\big(t-\frac{nT}{N}\big)+\sigma\big(Y_{\frac{nT}{N}}^N(\omega)\big)\big(W_t(\omega)-W_{\frac{nT}{N}}(\omega)\big)\Big\|^q} \\ \qquad\qquad\qquad\qquad : Y_{\frac{nT}{N}}^N(\omega) \in D_{\frac{T}{N}}, \\ 0 \qquad\qquad\qquad\quad : Y_{\frac{nT}{N}}^N(\omega) \in \mathbb{R}^d \backslash D_{\frac{T}{N}}. \end{cases} \tag{133}$$

Then:

(i) *for all $\varepsilon \in (0,\infty)$ it holds that*

$$\limsup_{N\to\infty} \mathbb{P}\big[\sup_{t\in[0,T]} \|X_t - Y_t^N\| \geq \varepsilon\big] = 0,$$

(ii) *for all $p \in (0,\infty)$, $r \in (0,p)$ with $\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big[\|Y_t^N\|^p\big] < \infty$ it holds that*

$$\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t\|^p\big] < \infty \quad \textit{and} \quad \limsup_{N\to\infty} \big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\big) = 0,$$

and

(iii) *for all continuous $f \colon C([0,T],\mathbb{R}^d) \to \mathbb{R}$ with*

$$\limsup_{p\searrow 1} \limsup_{N\to\infty} \mathbb{E}\big[|f(Y^N)|^p\big] < \infty$$

it holds that

$$\mathbb{E}\big[|f(X)|\big] < \infty \quad \textit{and} \quad \limsup_{N\to\infty} \big|\mathbb{E}\big[f(Y^N)\big] - \mathbb{E}\big[f(X)\big]\big| = 0.$$

Proof of Corollary 3.7. Throughout this proof let $\Psi \colon \mathbb{R}^d \times (0,T]^2 \times \mathbb{R}^m \to \mathbb{R}^d$ be the function which satisfies for all $(x,t,s,y) \in \mathbb{R}^d \times (0,T]^2 \times \mathbb{R}^m$ that

$$\Psi(x,t,s,y) = \begin{cases} \mathbb{1}_{D_t}(x)\left[\frac{\mu(x)s+\sigma(x)y}{1+\|\mu(x)s+\sigma(x)y\|^q}\right] & : x \in D, \\ 0 & : x \in \mathbb{R}^d \backslash D. \end{cases} \tag{134}$$

Next observe that Corollary 3.6 implies that $\mathbb{R}^d \times (0,T] \times \mathbb{R}^m \ni (x,t,y) \mapsto \Psi(x,t,t,y) \in \mathbb{R}^d$ is (μ,σ)-consistent with respect to Brownian motion. In addition, observe that for all nonempty compact sets $K \subseteq D$ it holds that

(135)

$$\begin{aligned}
&\limsup_{h \searrow 0} \mathbb{E}\left[\sup_{x\in K}\sup_{t\in[0,T]} \left\|\Psi(x,h,t-\lfloor t\rfloor_h, W_t - W_{\lfloor t\rfloor_h})\right\|\right] \\
&= \limsup_{h \searrow 0} \mathbb{E}\left[\sup_{x\in K}\sup_{t\in[0,T]} \left\| \mathbb{1}_{D_h}(x)\left[\frac{\mu(x)(t-\lfloor t\rfloor_h) + \sigma(x)(W_t - W_{\lfloor t\rfloor_h})}{1+\left\|\mu(x)(t-\lfloor t\rfloor_h) + \sigma(x)(W_t - W_{\lfloor t\rfloor_h})\right\|^q}\right]\right\|\right] \\
&\leq \limsup_{h \searrow 0} \mathbb{E}\left[\sup_{x\in K}\sup_{t\in[0,T]} \left(\frac{\left\|\mu(x)(t-\lfloor t\rfloor_h) + \sigma(x)(W_t - W_{\lfloor t\rfloor_h})\right\|}{1+\left\|\mu(x)(t-\lfloor t\rfloor_h) + \sigma(x)(W_t - W_{\lfloor t\rfloor_h})\right\|^q}\right)\right] \\
&\leq \limsup_{h \searrow 0} \mathbb{E}\left[\sup_{x\in K}\sup_{t\in[0,T]} \left\|\mu(x)(t-\lfloor t\rfloor_h) + \sigma(x)(W_t - W_{\lfloor t\rfloor_h})\right\|\right] \\
&\leq \left(\limsup_{h \searrow 0} h\right)\left(\sup_{x\in K}\|\mu(x)\|\right) \\
&\quad + \left(\limsup_{h \searrow 0} \mathbb{E}\left[\sup_{t\in[0,T]}\left\|W_t - W_{\lfloor t\rfloor_h}\right\|\right]\right)\left(\sup_{x\in K}\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)}\right) = 0.
\end{aligned}$$

Proposition 3.5 hence shows that $\limsup_{N\to\infty} \mathbb{P}\big[\sup_{t\in[0,T]}\|X_t - Y_t^N\| \geq \varepsilon\big] = 0$ for all $\varepsilon \in (0,\infty)$. The proof of the strong convergence statement in Corollary 3.7 is entirely analogous to the proof of Corollary 3.12 in [18] and thus omitted. It thus remains to prove the weak convergence statement in Corollary 3.7. For this assume that $p \in (1,\infty)$ is a real number, that $N_0 \in \mathbb{N}$ is a natural number, and that $f\colon C([0,T],\mathbb{R}^d) \to \mathbb{R}$ is a continuous function with $\sup_{N\in\{N_0,N_0+1,\dots\}} \mathbb{E}\big[|f(Y^N)|^p\big] < \infty$. The fact that $\sup_{t\in[0,T]}\|X_t - Y_t^N\|$ converges in probability to zero as $N\to\infty$ together with, e.g., Lemma 3.10 in [18] then proves that

$$\mathbb{E}\big[|f(X)|^p\big] < \infty \quad \text{and } \forall\, \varepsilon \in (0,\infty)\colon \quad \limsup_{N\to\infty} \mathbb{P}\big[|f(X) - f(Y^N)| \geq \varepsilon\big] = 0. \tag{136}$$

This shows that the family $|f(X) - f(Y^N)|$, $N \in \{N_0, N_0+1,\dots\}$, of random variables is uniformly integrable. Combining this and (136) with, e.g., Theorem 6.25 in Klenke [22] proves that $\limsup_{N\to\infty} \mathbb{E}\big[|f(X) - f(Y^N)|\big] = 0$. The proof of Corollary 3.7 is thus completed. □

Combining Corollary 3.7 with Corollary 2.10 and Fatou's lemma results in Corollary 3.8. Corollary 3.8 establishes both exponential integrability properties and for any $r \in [0,\infty)$ strong L^r-convergence.

Corollary 3.8. *Let $d,m \in \mathbb{N}$, $\rho \in [0,\infty)$, $T \in (0,\infty)$, $c,q \in (1,\infty)$, $U \in \bigcup_{p\in[1,\infty)} C^3_{p,c}(\mathbb{R}^d,[0,\infty))$, $\bar{U} \in C(\mathbb{R}^d,[-c,\infty))$, $\mu \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d),\mathcal{B}(\mathbb{R}^d))$, $\sigma \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d),\mathcal{B}(\mathbb{R}^{d\times m}))$, let $D \subseteq \mathbb{R}^d$ be an open set, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, assume that $\mu|_D\colon D \to \mathbb{R}^d$ and $\sigma|_D\colon D \to \mathbb{R}^{d\times m}$ are locally Lipschitz continuous, let $X\colon [0,T]\times\Omega \to D$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths which satisfies $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$ and which satisfies that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that*

$X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$, *let* $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N\in\mathbb{N}$, *and* $\tau_N\colon \Omega\to[0,T]$, $N\in\mathbb{N}$, *be mappings satisfying for all* $N\in\mathbb{N}$, $n\in\{0,1,\dots,N-1\}$, $t\in\left[\frac{nT}{N},\frac{(n+1)T}{N}\right]$ *that* $\tau_N = \inf\big(\{s\in\{0,\frac{T}{N},\frac{2T}{N},\dots,T\}\colon Y_s^N\notin D$ *or* $\|Y_s^N\| > \exp(|\ln(T/N)|^{1/2})\}\cup\{T\}\big)$, $Y_0^N = X_0$, *and*

$$Y_t^N = Y^N_{\frac{nT}{N}} + \mathbb{1}_{\left\{Y^N_{nT/N}\in D \text{ and } \|Y^N_{nT/N}\|\le\exp(|\ln(T/N)|^{1/2})\right\}} \cdot \left[\frac{\mu(Y^N_{\frac{nT}{N}})\big(t-\frac{nT}{N}\big)+\sigma(Y^N_{\frac{nT}{N}})\big(W_t-W_{\frac{nT}{N}}\big)}{1+\big\|\mu(Y^N_{\frac{nT}{N}})\big(t-\frac{nT}{N}\big)+\sigma(Y^N_{\frac{nT}{N}})\big(W_t-W_{\frac{nT}{N}}\big)\big\|^q}\right], \tag{137}$$

and assume for all $x,y\in\mathbb{R}^d$ *that*

$$\|\mu(x)\| + \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \le c\,(1+\|x\|^c), \tag{138}$$
$$|\bar U(x)-\bar U(y)| \le c\,(1+\|x\|^c+\|y\|^c)\,\|x-y\|,$$

$$(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac12\,\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar U(x) \le \rho\cdot U(x), \tag{139}$$
$$\|x\|^{1/c} \le c\,(1+U(x)).$$

Then it holds for all $r\in(0,\infty)$ *that* $\limsup_{N\to\infty}\big(\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t-Y_t^N\|^r\big]\big)=0$, *that*

$$\sup_{t\in[0,T]}\mathbb{E}\left[\exp\left(\tfrac{U(X_t)}{e^{\rho t}}+\int_0^t \tfrac{\bar U(X_s)}{e^{\rho s}}\,ds\right)\right] \le \sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\left[\exp\left(\tfrac{U(Y_t^N)}{e^{\rho t}}+\int_0^{t\wedge\tau_N}\tfrac{\bar U(Y_s^N)}{e^{\rho s}}\,ds\right)\right]<\infty, \tag{140}$$

and that

$$\sup_{t\in[0,T]}\mathbb{E}\left[\exp\left(\tfrac{U(X_t)}{e^{\rho t}}+\int_0^t \tfrac{\bar U(X_s)}{e^{\rho s}}\,ds\right)\right] \le \limsup_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\exp\left(\tfrac{U(Y_t^N)}{e^{\rho t}}+\int_0^{t\wedge\tau_N}\tfrac{\bar U(Y_s^N)}{e^{\rho s}}\,ds\right)\right]\le\mathbb{E}\Big[e^{U(X_0)}\Big]. \tag{141}$$

Proof of Corollary 3.8. Throughout this proof let $\theta_N\in\mathcal{P}_T$, $N\in\mathbb{N}$, be the sets which satisfy for all $N\in\mathbb{N}$ that $\theta_N=\big\{0,\frac{T}{N},\frac{2T}{N},\dots,\frac{(N-1)T}{N},T\big\}$, let $D_h\subseteq\mathbb{R}^d$, $h\in(0,T]$, be the sets which satisfy for all $h\in(0,T]$ that $D_h=\big\{x\in D\colon \|x\|\le\exp\big(|\ln(\min\{1,h\})|^{1/2}\big)\big\}$, let $Z^\theta\colon[0,T]\times\Omega\to\mathbb{R}^d$, $\theta\in\mathcal{P}_T$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths which satisfy for all $\theta\in\mathcal{P}_T$, $t\in[0,T]$ that $Z_0^\theta=X_0$ and

$$Z_t^\theta = Z^\theta_{\lfloor t\rfloor_\theta} + \mathbb{1}_{D_{|\theta|_T}}(Z^\theta_{\lfloor t\rfloor_\theta})\left[\frac{\mu(Z^\theta_{\lfloor t\rfloor_\theta})\big(t-\lfloor t\rfloor_\theta\big)+\sigma(Z^\theta_{\lfloor t\rfloor_\theta})\big(W_t-W_{\lfloor t\rfloor_\theta}\big)}{1+\big\|\mu(Z^\theta_{\lfloor t\rfloor_\theta})\big(t-\lfloor t\rfloor_\theta\big)+\sigma(Z^\theta_{\lfloor t\rfloor_\theta})\big(W_t-W_{\lfloor t\rfloor_\theta}\big)\big\|^q}\right], \tag{142}$$

and let $\varrho_N\colon\Omega\to[0,T]$, $N\in\mathbb{N}$, be the functions which satisfy for all $N\in\mathbb{N}$ that $\varrho_N=\inf\big(\big\{s\in\theta_N\colon Z_s^{\theta_N}\notin D_{T/N}\big\}\cup\{T\}\big)$. Observe that the assumption that $U\in\bigcup_{p\in[1,\infty)}C^3_{p,c}(\mathbb{R}^d,[0,\infty))$ together with Lemma 2.7 shows that

$$\limsup_{p\to\infty}\sup_{x\in\mathbb{R}^d}\frac{U(x)}{[1+\|x\|]^p}<\infty.$$

This implies that there exists a real number $\hat{c} \in [c, \infty)$ such that for all $h \in (0, T]$ and $x \in \{y \in \mathbb{R}^d \colon \|y\| \leq \exp(|\ln(\min\{1, h\})|^{1/2})\}$ it holds that $U(x) \leq \hat{c} \exp(\hat{c} \, |\ln(\min\{1, h\})|^{1/2})$. This and the fact that $\hat{c} \exp(\hat{c} \, |\ln(\min\{1, h\})|^{1/2}) \leq \hat{c} \exp(\hat{c} \, |\ln(h)|^{1/2}) \; \forall \, h \in (0, T]$ assure that $D_h \subseteq \mathbb{R}^d$, $h \in (0, T]$, is a nonincreasing family of sets which satisfies for all $h \in (0, T]$ that $D_h \in \mathcal{B}(\{x \in D \colon U(x) \leq \hat{c} \exp(\hat{c} \, |\ln(h)|^{1/2})\})$, $\mu|_{D_h} \in C(D_h, \mathbb{R}^d)$, and $\sigma|_{D_h} \in C(D_h, \mathbb{R}^{d \times m})$. We can hence apply Corollary 2.10 to obtain that

$$\sup_{\theta \in \mathcal{P}_T} \sup_{t \in [0,T]} \mathbb{E}\left[\exp\left(e^{-\rho t} \, U(Z_t^\theta) + \int_0^t e^{-\rho s} \, \mathbb{1}_{D_{|\theta|_T}}(Z_{\lfloor s \rfloor_\theta}^\theta) \, \bar{U}(Z_s^\theta) \, ds\right)\right] < \infty \tag{143}$$

and

$$\begin{aligned} &\limsup_{|\theta|_T \searrow 0} \sup_{t \in [0,T]} \mathbb{E}\left[\exp\left(e^{-\rho t} \, U(Z_t^\theta) + \int_0^t e^{-\rho s} \, \mathbb{1}_{D_{|\theta|_T}}(Z_{\lfloor s \rfloor_\theta}^\theta) \, \bar{U}(Z_s^\theta) \, ds\right)\right] \\ &\quad \leq \limsup_{|\theta|_T \searrow 0} \mathbb{E}\big[e^{U(Z_0^\theta)}\big] = \mathbb{E}\big[e^{U(X_0)}\big]. \end{aligned} \tag{144}$$

Inequalities (143)–(144) and the assumption that $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$, in particular, ensure that

$$\sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \mathbb{E}\left[\exp\left(e^{-\rho t} \, U(Z_t^{\theta_N}) + \int_0^t e^{-\rho s} \, \mathbb{1}_{D_{T/N}}(Z_{\lfloor s \rfloor_{\theta_N}}^{\theta_N}) \, \bar{U}(Z_s^{\theta_N}) \, ds\right)\right] < \infty \tag{145}$$

and

$$\begin{aligned} &\limsup_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}\left[\exp\left(e^{-\rho t} \, U(Z_t^{\theta_N}) + \int_0^t e^{-\rho s} \, \mathbb{1}_{D_{T/N}}(Z_{\lfloor s \rfloor_{\theta_N}}^{\theta_N}) \, \bar{U}(Z_s^{\theta_N}) \, ds\right)\right] \\ &\quad \leq \mathbb{E}\big[e^{U(X_0)}\big] < \infty. \end{aligned} \tag{146}$$

The definition of ϱ_N, $N \in \mathbb{N}$, hence proves that

$$\sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \mathbb{E}\left[\exp\left(e^{-\rho t} \, U(Z_t^{\theta_N}) + \int_0^{t \wedge \varrho_N} e^{-\rho s} \, \bar{U}(Z_s^{\theta_N}) \, ds\right)\right] < \infty \tag{147}$$

and

$$\limsup_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}\left[\exp\left(e^{-\rho t} \, U(Z_t^{\theta_N}) + \int_0^{t \wedge \varrho_N} e^{-\rho s} \, \bar{U}(Z_s^{\theta_N}) \, ds\right)\right] \leq \mathbb{E}\big[e^{U(X_0)}\big] < \infty. \tag{148}$$

In the next step we observe that (147) and the assumption that $\inf_{x \in \mathbb{R}^d} \bar{U}(x) \geq -c$ prove that $\sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \mathbb{E}\big[\exp\big(e^{-\rho T}\{1 + U(Z_t^{\theta_N})\}\big)\big] < \infty$. This and the assumption that $\forall \, x \in \mathbb{R}^d \colon \|x\|^{1/c} \leq c \, (1 + U(x))$ ensure that

$$\sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \mathbb{E}\big[\exp\big(c^{-1} e^{-\rho T} \|Z_t^{\theta_N}\|^{1/c}\big)\big] < \infty.$$

Hence, we obtain for all $p \in (0, \infty)$ that $\sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \mathbb{E}\big[\|Z_t^{\theta_N}\|^p\big] < \infty$. Combining this with items (i)–(ii) in Corollary 3.7 assures for all $r \in (0, \infty)$ that

$$\begin{aligned} &\limsup\nolimits_{N \to \infty} \mathbb{P}\big[\sup\nolimits_{t \in [0,T]} \|X_t - Z_t^{\theta_N}\| \geq r\big] \\ &\quad + \limsup\nolimits_{N \to \infty} \sup\nolimits_{t \in [0,T]} \mathbb{E}\big[\|X_t - Z_t^{\theta_N}\|^r\big] = 0. \end{aligned} \tag{149}$$

In addition, observe for all $\varepsilon \in (0,\infty)$, $N \in \mathbb{N}$ that

(150)

$$\begin{aligned}
&\mathbb{P}\big[\mathbb{1}_{\{\varrho_N<T\}} \geq \varepsilon\big] \leq \mathbb{P}\big[\mathbb{1}_{\{\varrho_N<T\}} \geq 1\big] = \mathbb{P}\big[\varrho_N < T\big] \\
&\leq \mathbb{P}\big[\exists\, s \in \theta_N \colon Z^{\theta_N}_s \notin D_{T/N}\big] \leq \mathbb{P}\big[\exists\, s \in [0,T] \colon Z^{\theta_N}_s \notin D_{T/N}\big] \\
&\leq \mathbb{P}\big[\exists\, s \in [0,T] \colon Z^{\theta_N}_s \notin D\big] + \mathbb{P}\big[\exists\, s \in [0,T] \colon \|Z^{\theta_N}_s\| > \exp\big(|\ln(\min\{1, T/N\})|^{1/2}\big)\big] \\
&\leq \mathbb{P}\Bigg[\Big\{\exists\, t \in [0,T] \colon Z^{\theta_N}_t \notin D\Big\} \\
&\qquad \cap \left\{\sup_{t\in[0,T]} \|X_t - Z^{\theta_N}_t\| < \inf\big(\{\infty\} \cup \{\|X_t - v\| \in \mathbb{R} \colon t \in [0,T], v \in (\mathbb{R}^d \backslash D)\}\big)\right\}\Bigg] \\
&\qquad + \mathbb{P}\left[\sup_{t\in[0,T]} \|X_t - Z^{\theta_N}_t\| \geq \inf\big(\{\infty\} \cup \big\{\|X_t - v\| \in \mathbb{R} \colon t \in [0,T], v \in (\mathbb{R}^d \backslash D)\big\}\big)\right] \\
&\qquad + \mathbb{P}\left[\sup_{t\in[0,T]} \|Z^{\theta_N}_t\| > \exp\big(|\ln(\min\{1, T/N\})|^{1/2}\big)\right] \\
&\leq \mathbb{P}\Big[\sup\nolimits_{t\in[0,T]} \|X_t - Z^{\theta_N}_t\| \geq \inf\big(\{\infty\} \cup \big\{\|X_t - v\| \in \mathbb{R} \colon t \in [0,T], v \in (\mathbb{R}^d \backslash D)\big\}\big)\Big] \\
&\qquad + \mathbb{P}\Big[\sup\nolimits_{t\in[0,T]} \|X_t - Z^{\theta_N}_t\| + \sup\nolimits_{t\in[0,T]} \|X_t\| > \exp\big(|\ln(\min\{1, T/N\})|^{1/2}\big)\Big].
\end{aligned}$$

This, (149), and the assumption that $X \colon [0,T] \times \Omega \to D$ has continuous sample paths prove for all $\varepsilon \in (0,\infty)$ that

$$\begin{aligned}
&\limsup_{N\to\infty} \mathbb{P}\big[\mathbb{1}_{\{\varrho_N<T\}} \geq \varepsilon\big] \\
&\quad \leq \limsup_{N\to\infty} \mathbb{P}\Big[\sup\nolimits_{t\in[0,T]} \|X_t - Z^{\theta_N}_t\| \\
&\qquad\qquad \geq \inf\big(\{\infty\} \cup \big\{\|X_t - v\| \in \mathbb{R} \colon t \in [0,T], v \in (\mathbb{R}^d \backslash D)\big\}\big)\Big] \\
&\qquad + \limsup_{N\to\infty} \mathbb{P}\Big[\sup\nolimits_{t\in[0,T]} \|X_t - Z^{\theta_N}_t\| \geq 1\Big] \\
&\qquad + \limsup_{N\to\infty} \mathbb{P}\Big[\sup\nolimits_{t\in[0,T]} \|X_t\| > \exp\big(|\ln(\min\{1, T/N\})|^{1/2}\big) - 1\Big] = 0.
\end{aligned} \tag{151}$$

Furthermore, we note that the assumption that $\forall\, x, y \in \mathbb{R}^d \colon |\bar{U}(x) - \bar{U}(y)| \leq c\,\big(1 + \|x\|^c + \|y\|^c\big)\,\|x - y\|$ implies that

$$\forall\, x, y \in \mathbb{R}^d \colon \quad |\bar{U}(x) - \bar{U}(y)| \leq c\, 2^c\big(1 + \|x - y\|^c + \|y\|^c\big)\,\|x - y\|.$$

This together with the triangle inequality ensures for all $t \in [0,T]$ that

$$\begin{aligned}
&\left|\tfrac{U(Z^{\theta_N}_t)}{e^{\rho t}} + \int_0^{t\wedge\varrho_N} \tfrac{\bar{U}(Z^{\theta_N}_s)}{e^{\rho s}}\, ds - \tfrac{U(X_t)}{e^{\rho t}} - \int_0^t \tfrac{\bar{U}(X_s)}{e^{\rho s}}\, ds\right|] \\
&\quad \leq \big|U(Z^{\theta_N}_t) - U(X_t)\big| + \left|\int_0^{t\wedge\varrho_N} \tfrac{\bar{U}(Z^{\theta_N}_s)}{e^{\rho s}}\, ds - \int_0^{t\wedge\varrho_N} \tfrac{\bar{U}(X_s)}{e^{\rho s}}\, ds\right| \\
&\qquad + \left|\int_0^{t\wedge\varrho_N} \tfrac{\bar{U}(X_s)}{e^{\rho s}}\, ds - \int_0^t \tfrac{\bar{U}(X_s)}{e^{\rho s}}\, ds\right|
\end{aligned}$$

$$
\begin{aligned}
(152) \qquad &\le \left|U(Z_t^{\theta_N}) - U(X_t)\right| + \int_0^{t\wedge\varrho_N} \left|\tfrac{\bar{U}(Z_s^{\theta_N})}{e^{\rho s}} - \tfrac{\bar{U}(X_s)}{e^{\rho s}}\right| ds + \int_{t\wedge\varrho_N}^{t} \left|\tfrac{\bar{U}(X_s)}{e^{\rho s}}\right| ds \\
&\le \left|U(Z_t^{\theta_N}) - U(X_t)\right| + T\left[\sup_{s\in[0,T]} \left|\bar{U}(Z_s^{\theta_N}) - \bar{U}(X_s)\right|\right] \\
&\quad + (t - \min\{t, \varrho_N\}) \left[\sup_{s\in[0,T]} \left|\bar{U}(X_s)\right|\right] \\
&\le T c\, 2^c \left[1 + \sup_{s\in[0,T]} \left\|Z_s^{\theta_N} - X_s\right\|^c + \sup_{s\in[0,T]} \|X_s\|^c\right] \left[\sup_{s\in[0,T]} \left\|Z_s^{\theta_N} - X_s\right\|\right] \\
&\quad + \left|U(Z_t^{\theta_N}) - U(X_t)\right| + \mathbb{1}_{\{\varrho_N < T\}}\, T\left[\sup_{s\in[0,T]} \left|\bar{U}(X_s)\right|\right].
\end{aligned}
$$

Next note that (149) and the assumption that U is continuous establish that for all $\varepsilon \in (0,\infty)$, $t \in [0,T]$ it holds that $\limsup_{N\to\infty} \mathbb{P}\big[|U(Z_t^{\theta_N}) - U(X_t)| \ge \varepsilon\big] = 0$. This, (149), (152), and (151) prove for all $\varepsilon \in (0,\infty)$, $t \in [0,T]$ that

$$
\limsup_{N\to\infty} \mathbb{P}\left[\left|\tfrac{U(Z_t^{\theta_N})}{e^{\rho t}} + \int_0^{t\wedge\varrho_N} \tfrac{\bar{U}(Z_s^{\theta_N})}{e^{\rho s}}\, ds - \tfrac{U(X_t)}{e^{\rho t}} - \int_0^t \tfrac{\bar{U}(X_s)}{e^{\rho s}}\, ds\right| \ge \varepsilon\right] = 0. \tag{153}
$$

Combining this with a well-known modification of Fatou's lemma (see, e.g., Lemma 3.10 in [18]) proves for all $t \in [0,T]$ that

$$
\mathbb{E}\left[\exp\left(\tfrac{U(X_t)}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(X_s)}{e^{\rho s}}\, ds\right)\right] \le \liminf_{N\to\infty} \mathbb{E}\left[\exp\left(\tfrac{U(Z_t^{\theta_N})}{e^{\rho t}} + \int_0^{t\wedge\varrho_N} \tfrac{\bar{U}(Z_s^{\theta_N})}{e^{\rho s}}\, ds\right)\right]. \tag{154}
$$

Hence, we obtain that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\tfrac{U(X_t)}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(X_s)}{e^{\rho s}}\, ds\right)\right] \\
&\quad \le \liminf_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\tfrac{U(Z_t^{\theta_N})}{e^{\rho t}} + \int_0^{t\wedge\varrho_N} \tfrac{\bar{U}(Z_s^{\theta_N})}{e^{\rho s}}\, ds\right)\right].
\end{aligned} \tag{155}
$$

This together with (148) ensures that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\tfrac{U(X_t)}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(X_s)}{e^{\rho s}}\, ds\right)\right] \\
&\quad \le \limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\tfrac{U(Z_t^{\theta_N})}{e^{\rho t}} + \int_0^{t\wedge\varrho_N} \tfrac{\bar{U}(Z_s^{\theta_N})}{e^{\rho s}}\, ds\right)\right] \le \mathbb{E}\big[e^{U(X_0)}\big] < \infty.
\end{aligned} \tag{156}
$$

Combining this with the fact that for all $N \in \mathbb{N} \cap [T,\infty)$ it holds that $Y^N = Z^{\theta_N}$ and $\tau_N = \varrho_N$ proves (141). It thus remains to prove (140). For this observe that (145) together with the fact that for all $N \in \mathbb{N} \cap [T,\infty)$ it holds that $Y^N = Z^{\theta_N}$ and $\tau_N = \varrho_N$ assures that

$$
\sup_{N\in\mathbb{N}\cap[T,\infty)} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(e^{-\rho t}\, U(Y_t^N) + \int_0^{t\wedge\tau_N} e^{-\rho s}\, \bar{U}(Y_s^N)\, ds\right)\right] < \infty. \tag{157}
$$

In addition, we observe that the fact that $\forall\, N \in \mathbb{N}, t \in [0,T]\colon Y_t^N = Y_{t\wedge\tau_N}^N$ proves that for all $N \in \mathbb{N}$, $t \in [0,T]$ it holds that

$$
\begin{aligned}
&\exp\Bigg(e^{-\rho t}\, U(Y_t^N) + \int_0^{t\wedge\tau_N} e^{-\rho s}\, \bar{U}(Y_s^N)\, ds\Bigg) \\
&\quad= \exp\Bigg(e^{-\rho t}\, U(Y_{t\wedge\tau_N}^N) + \int_0^{t\wedge\tau_N} e^{-\rho s}\, \bar{U}(Y_{s\wedge\tau_N}^N)\, ds\Bigg) \\
&\quad= \mathbb{1}_{\{\tau_N=0\}} \exp\Bigg(e^{-\rho t}\, U(Y_{t\wedge\tau_N}^N) + \int_0^{t\wedge\tau_N} e^{-\rho s}\, \bar{U}(Y_{s\wedge\tau_N}^N)\, ds\Bigg) \\
&\qquad+ \mathbb{1}_{\{\tau_N>0\}} \exp\Bigg(e^{-\rho t}\, U(Y_{t\wedge\tau_N}^N) + \int_0^{t\wedge\tau_N} e^{-\rho s}\, \bar{U}(Y_{s\wedge\tau_N}^N)\, ds\Bigg) \\
&\quad= \mathbb{1}_{\{\tau_N=0\}} \exp\big(e^{-\rho t}\, U(X_0)\big) + \mathbb{1}_{\{\tau_N>0\}} \\
&\qquad\cdot \exp\Bigg(e^{-\rho t}\, U(Y_{t\wedge\tau_N}^N) + \int_0^{t\wedge\tau_N} e^{-\rho s}\, \bar{U}(Y_{s\wedge\tau_N}^N)\, ds\Bigg).
\end{aligned}
\tag{158}
$$

Moreover, we note the fact that $\forall\, x \in \mathbb{R}\colon |x| \leq 1 + |x|^q$ ensures for all $N \in \mathbb{N}$, $t \in [0,T]$ that

$$
\begin{aligned}
\big\|Y_t^N\big\| &\leq \big\|Y_{\llcorner t \lrcorner_{\theta_N}}^N\big\| + \frac{\big\|\mu(Y_{\llcorner t \lrcorner_{\theta_N}}^{\theta_N})\,(t - \llcorner t \lrcorner_{\theta_N}) + \sigma(Y_{\llcorner t \lrcorner_{\theta_N}}^{\theta_N})\,(W_t - W_{\llcorner t \lrcorner_{\theta_N}})\big\|}{1 + \big\|\mu(Y_{\llcorner t \lrcorner_{\theta_N}}^N)\,(t - \llcorner t \lrcorner_{\theta_N}) + \sigma(Y_{\llcorner t \lrcorner_{\theta_N}}^N)\,(W_t - W_{\llcorner t \lrcorner_{\theta_N}})\big\|^q} \\
&\leq \big\|Y_{\llcorner t \lrcorner_{\theta_N}}^N\big\| + 1.
\end{aligned}
\tag{159}
$$

This implies for all $N \in \mathbb{N}$, $t \in [0,T]$ that

$$
\big\|Y_{t\wedge\tau_N}^N \mathbb{1}_{\{\tau_N>0\}}\big\| \leq \big\|Y_{\llcorner t\wedge\tau_N \lrcorner_{\theta_N}}^N \mathbb{1}_{\{\tau_N>0\}}\big\| + 1 \leq \exp\Big(\big|\ln(T/N)\big|^{1/2}\Big) + 1.
\tag{160}
$$

Combining this with (158) establishes for all $N \in \mathbb{N}$, $t \in [0,T]$ that

$$
\begin{aligned}
&\exp\Bigg(e^{-\rho t}\, U(Y_t^N) + \int_0^{t\wedge\tau_N} e^{-\rho s}\, \bar{U}(Y_s^N)\, ds\Bigg) \\
&\quad\leq \mathbb{1}_{\{\tau_N=0\}} \exp(U(X_0)) + \mathbb{1}_{\{\tau_N>0\}} \exp\Bigg(U(Y_{t\wedge\tau_N}^N) + \int_0^{t\wedge\tau_N} |\bar{U}(Y_{s\wedge\tau_N}^N)|\, ds\Bigg) \\
&\quad\leq e^{U(X_0)} + \mathbb{1}_{\{\tau_N>0\}} \exp\Bigg(U\big(Y_{t\wedge\tau_N}^N \mathbb{1}_{\{\tau_N>0\}}\big) + \int_0^T \big|\bar{U}\big(Y_{s\wedge\tau_N}^N \mathbb{1}_{\{\tau_N>0\}}\big)\big|\, ds\Bigg) \\
&\quad\leq e^{U(X_0)} + \mathbb{1}_{\{\tau_N>0\}} \exp\Bigg(\Bigg[\sup_{v\in\mathbb{R}^d, \|v\|\leq \exp(|\ln(T/N)|^{1/2})+1} U(v)\Bigg] \\
&\qquad\qquad + T\Bigg[\sup_{v\in\mathbb{R}^d, \|v\|\leq \exp(|\ln(T/N)|^{1/2})+1} |\bar{U}(v)|\Bigg]\Bigg).
\end{aligned}
\tag{161}
$$

Hence, we obtain for all $N \in \mathbb{N}$ that

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\Bigg[\exp\Bigg(e^{-\rho t}\, U(Y_t^N) + \int\limits_0^{t\wedge\tau_N} e^{-\rho s}\, \bar{U}(Y_s^N)\, ds\Bigg)\Bigg] \\
&\leq \mathbb{E}\Big[e^{U(X_0)}\Big] + \exp\Bigg(\Bigg[\sup_{v\in\mathbb{R}^d, \|v\|\leq \exp(|\ln(T/N)|^{1/2})+1} U(v)\Bigg] \\
&\qquad + T\Bigg[\sup_{v\in\mathbb{R}^d, \|v\|\leq \exp(|\ln(T/N)|^{1/2})+1} |\bar{U}(v)|\Bigg]\Bigg).
\end{aligned} \tag{162}$$

Combining this with the assumption that $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$ and the assumption that U and $\bar{U}$ are continuous ensures that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}\cap[0,\max\{T,1\}]}\ \sup_{t\in[0,T]} \mathbb{E}\Bigg[\exp\Bigg(e^{-\rho t}\, U(Y_t^N) + \int\limits_0^{t\wedge\tau_N} e^{-\rho s}\, \bar{U}(Y_s^N)\, ds\Bigg)\Bigg] \\
&\leq \mathbb{E}\Big[e^{U(X_0)}\Big] + \exp\Bigg(\Bigg[\sup_{v\in\mathbb{R}^d, \|v\|\leq \exp(|\ln(T)|^{1/2})+1} U(v)\Bigg] \\
&\qquad + T\Bigg[\sup_{v\in\mathbb{R}^d, \|v\|\leq \exp(|\ln(T)|^{1/2})+1} |\bar{U}(v)|\Bigg]\Bigg) < \infty.
\end{aligned} \tag{163}$$

Inequality (163) together with inequality (157) establishes inequality (140). The proof of Corollary 3.8 is thus completed. $\square$

Observe, in the setting of Corollary 3.8, that the assumption that $X = (X_t)_{t\in[0,T]} \colon [0,T]\times\Omega \to D$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process, in particular, ensures that the initial random variable $X_0 \colon \Omega \to D$ is an $\mathcal{F}_0/\mathcal{B}(D)$-measurable mapping.

4. Examples of SDEs with exponential moments

In this section Corollary 3.8 is applied to a number of example SDEs from the literature. To keep this article at a reasonable length, we present the example SDEs here in a very brief way and refer to [3, 18] for references and further details for these example SDEs.

4.1. **Setting.** Throughout Section 4 the following setting is used. Let $T \in (0,\infty)$, $d, m \in \mathbb{N}$, $\mu \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, $\sigma \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^{d\times m}))$, let $D \subseteq \mathbb{R}^d$ be an open set, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, assume that $\mu|_D \colon D \to \mathbb{R}^d$ and $\sigma|_D \colon D \to \mathbb{R}^{d\times m}$ are locally Lipschitz continuous, let $W \colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, let $X = (X^1, \ldots, X^d) \colon [0,T]\times\Omega \to D$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths which satisfies that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = X_0 + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s, \tag{164}$$

and let $Y^N = (Y^{1,N}, \ldots, Y^{d,N}) \colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, and $\tau_N \colon \Omega \to [0,T]$, $N \in \mathbb{N}$, be functions satisfying for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$, $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$

that $Y_0^N = X_0$,

$$Y_t^N = Y_{\frac{nT}{N}}^N + \mathbb{1}_{\left\{Y_{nT/N}^N \in D\right\} \cap \left\{\|Y_{nT/N}^N\| \le \exp\left(|\ln(T/N)|^{1/2}\right)\right\}} \cdot \left[\frac{\mu(Y_{nT/N}^N)(t-\frac{nT}{N})+\sigma(Y_{nT/N}^N)(W_t - W_{nT/N})}{1+\|\mu(Y_{nT/N}^N)(t-\frac{nT}{N})+\sigma(Y_{nT/N}^N)(W_t-W_{nT/N})\|^2}\right] \tag{165}$$

and $\tau_N = \inf\left(\left\{s \in \{0, \frac{T}{N}, \frac{2T}{N}, \ldots, T\} \colon Y_s^N \notin D \text{ or } \|Y_s^N\| > \exp(|\ln(T/N)|^{1/2})\right\} \cup \{T\}\right)$. Then Corollary 3.7 ensures that $\limsup_{N\to\infty} \mathbb{P}\left[\sup_{t\in[0,T]} \|X_t - Y_t^N\| \ge \varepsilon\right] = 0$ for all $\varepsilon \in (0,\infty)$.

4.2. **Stochastic Ginzburg-Landau equation.** In this subsection assume the setting in Subsection 4.1, let $\alpha \in [0,\infty)$, $\beta, \delta \in (0,\infty)$, $\varepsilon \in (0, \frac{\delta}{\beta^2}]$, $U, \bar{U} \in C(\mathbb{R},\mathbb{R})$, and assume for all $x \in \mathbb{R}$ that $d = m = 1$, $D = \mathbb{R}$, $\mu(x) = \alpha x - \delta x^3$, $\sigma(x) = \beta x$, $U(x) = \varepsilon x^2$, $\bar{U}(x) = 2\varepsilon\,[\delta - \beta^2\varepsilon]\,x^4$, and $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$. Then it holds for all $x \in \mathbb{R}$ that

$$\begin{aligned}
&(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \\
&\quad = \varepsilon\left[2x\left[\alpha x - \delta x^3\right] + \beta^2 x^2\right] + 2\left(\beta\varepsilon\right)^2 x^4 + \bar{U}(x) \\
&\quad = \varepsilon\left[2\alpha + \beta^2\right] x^2 + 2\varepsilon\left[\beta^2\varepsilon - \delta\right] x^4 + \bar{U}(x) = \left[2\alpha + \beta^2\right] U(x),
\end{aligned} \tag{166}$$

and Corollary 3.8 hence shows for all $r \in (0,\infty)$ that

$$\limsup_{N\to\infty} \Big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\Big) = 0,$$

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\frac{\varepsilon\,(Y_t^N)^2}{e^{[2\alpha+\beta^2]t}} + \int_0^{t\wedge\tau_N} \frac{2\varepsilon\,[\delta-\beta^2\varepsilon]\,(Y_s^N)^4}{e^{[2\alpha+\beta^2]s}}\,ds\right)\right] < \infty, \tag{167}$$

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\frac{\varepsilon\,(Y_t^N)^2}{e^{[2\alpha+\beta^2]t}} + \int_0^{t\wedge\tau_N} \frac{2\varepsilon\,[\delta-\beta^2\varepsilon]\,(Y_s^N)^4}{e^{[2\alpha+\beta^2]s}}\,ds\right)\right] \le \mathbb{E}\left[e^{\varepsilon|X_0|^2}\right] < \infty. \tag{168}$$

4.3. **Stochastic Lorenz equation with additive noise.** In this subsection assume the setting in Subsection 4.1, let $\alpha_1, \alpha_2, \alpha_3, \beta \in [0,\infty)$, $\varepsilon \in (0,\infty)$, $U, \bar{U} \in C(\mathbb{R}^3,\mathbb{R})$, and $\vartheta = \min_{r\in(0,\infty)} \max\{(\alpha_1+\alpha_2)^2/r - 2\alpha_1, r-1, 0\} \in [0,\infty)$, and assume for all $x = (x_1,x_2,x_3), u = (u_1,u_2,u_3) \in \mathbb{R}^3$ that $d = m = 3$, $D = \mathbb{R}^3$, $\sigma(x)u = \sqrt{\beta}u$, $\mu(x) = \left(\alpha_1\,(x_2 - x_1), \alpha_2 x_1 - x_2 - x_1 x_3, x_1 x_2 - \alpha_3 x_3\right)$, $U(x) = \varepsilon\|x\|^2$, $\bar{U}(x) = -3\varepsilon\beta$, and $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$. Then it holds for all $x \in \mathbb{R}^3$ that $(\mathcal{G}_{\mu,\sigma}U)(x) + \frac{1}{2}\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \le [2\varepsilon\beta + \vartheta]\,U(x)$ (cf. Subsection 4.4 in Cox et al. [3]) and Corollary 3.8 hence shows for all $r \in (0,\infty)$ that

$$\limsup_{N\to\infty} \Big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\Big) = 0,$$

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\Big[\exp\Big(\varepsilon\,\|Y_t^N\|^2\,e^{-[2\varepsilon\beta+\vartheta]t}\Big)\Big] < \infty,$$

and

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\tfrac{\varepsilon\,\|Y_t^N\|^2}{e^{[2\varepsilon\beta+\vartheta]t}}\right)\right] \le \exp\left(\int_0^T \tfrac{3\varepsilon\beta}{e^{[2\varepsilon\beta+\vartheta]s}}\,ds\right) \mathbb{E}\left[e^{\varepsilon\|X_0\|^2}\right] < \infty. \tag{169}$$

4.4. **Stochastic van der Pol oscillator.** In this subsection assume the setting in Subsection 4.1, let $\alpha, \varepsilon \in (0, \infty)$, $\gamma, \delta, \eta_0, \eta_1 \in [0, \infty)$, $U, \bar{U} \in C(\mathbb{R}^2, \mathbb{R})$, $\vartheta = \min_{r \in (0,\infty)} \max\{|\delta - 1|/r + \eta_1, r\,|\delta - 1| + 2\gamma + 4\eta_0\varepsilon\} \in [0, \infty)$, let $g\colon \mathbb{R} \to \mathbb{R}^{1\times m}$ be a globally Lipschitz continuous function which satisfies for all $y \in \mathbb{R}$ that $\|g(y)\|^2 \leq \eta_0 + \eta_1|y|^2$, and assume for all $x = (x_1, x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}^m$ that $d = 2$, $D = \mathbb{R}^2$, $\mu(x) = \big(x_2, \big(\gamma - \alpha(x_1)^2\big)\, x_2 - \delta x_1\big)$, $\sigma(x)u = (0, g(x_1)u)$, $\varepsilon\eta_1 \leq \alpha$, $U(x) = \varepsilon\|x\|^2$, $\bar{U}(x) = 2\varepsilon\,[\alpha - \varepsilon\eta_1]\,(x_1x_2)^2 - \varepsilon\eta_0$, and $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$. Then it holds for all $x \in \mathbb{R}^2$ that $(\mathcal{G}_{\mu,\sigma}U)(x) + \frac{1}{2}\|\sigma(x)^*(\nabla U)(x)\|^2 + \overline{U}(x) \leq \vartheta\, U(x)$ (cf. Subsection 4.2 in Cox et al. [3]) and Corollary 3.8 hence shows for all $r \in (0, \infty)$ that $\limsup_{N\to\infty} \big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\big) = 0$ and

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\Bigg[\exp\Bigg(\frac{\varepsilon\,\|Y_t^N\|^2}{e^{\vartheta t}} + \int_0^{t\wedge\tau_N} \frac{2\varepsilon\,[\alpha - \varepsilon\eta_1]\,|Y_s^{1,N}Y_s^{2,N}|^2}{e^{\vartheta s}}\,ds\Bigg)\Bigg] < \infty, \tag{170}$$

$$\begin{aligned} \limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\Bigg[\exp\Bigg(\frac{\varepsilon\,\|Y_t^N\|^2}{e^{\vartheta t}} + \int_0^{t\wedge\tau_N} \frac{2\varepsilon\,[\alpha - \varepsilon\eta_1]\,|Y_s^{1,N}Y_s^{2,N}|^2}{e^{\vartheta s}}\,ds\Bigg)\Bigg] \\ \leq \exp\Bigg(\int_0^T \frac{\varepsilon\eta_0}{e^{\vartheta s}}\,ds\Bigg)\,\mathbb{E}\Big[e^{\varepsilon\|X_0\|^2}\Big] < \infty. \end{aligned} \tag{171}$$

4.5. **Stochastic Duffing-van der Pol oscillator.** In this subsection assume the setting in Subsection 4.1, let $\eta_0, \eta_1, \alpha_1 \in [0, \infty)$, $\alpha_2, \alpha_3, \varepsilon \in (0, \infty)$, $U, \bar{U} \in C(\mathbb{R}^2, \mathbb{R})$, let $g\colon \mathbb{R} \to \mathbb{R}^{1\times m}$ be a globally Lipschitz continuous function which satisfies for all $y \in \mathbb{R}$ that $\|g(y)\|^2 \leq \eta_0 + \eta_1|y|^2$, and assume for all $x = (x_1, x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}^m$ that $d = 2$, $D = \mathbb{R}^2$, $\sigma(x)u = (0, g(x_1)u)$, $\mu(x) = \big(x_2, \alpha_2 x_2 - \alpha_1 x_1 - \alpha_3 (x_1)^2 x_2 - (x_1)^3\big)$, $\varepsilon\eta_1 \leq \alpha_3$, $U(x_1, x_2) = \varepsilon\big[\frac{1}{2}\,(x_1)^4 + \alpha_1\,(x_1)^2 + (x_2)^2\,\big]$, $\bar{U}(x) = 2\varepsilon\,[\alpha_3 - \varepsilon\eta_1]\,(x_1x_2)^2 - \varepsilon\eta_0 - \frac{\varepsilon|0\vee(\eta_1 - 2\alpha_1(\varepsilon\eta_0 + \alpha_2))|^2}{4(\varepsilon\eta_0 + \alpha_2)}$, and $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$. Then it holds for all $x \in \mathbb{R}^2$ that $(\mathcal{G}_{\mu,\sigma}U)(x) + \frac{1}{2}\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \leq 2\,(\varepsilon\eta_0 + \alpha_2)\,U(x)$ (cf. Subsection 4.3 in Cox et al. [3]) and Corollary 3.8 hence shows for all $r \in (0, \infty)$ that $\limsup_{N\to\infty} \big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\big) = 0$ and

$$\begin{aligned} \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\Bigg[\exp\Bigg(&\frac{\frac{\varepsilon}{2}\big|Y_t^{1,N}\big|^4 + \varepsilon\alpha_1\big|Y_t^{1,N}\big|^2 + \varepsilon\big|Y_t^{2,N}\big|^2}{e^{2t[\varepsilon\eta_0+\alpha_2]}} \\ &+ \int_0^{t\wedge\tau_N} \frac{2\varepsilon\,[\alpha_3 - \varepsilon\eta_1]\,\big|Y_s^{1,N}Y_s^{2,N}\big|^2}{e^{2s[\varepsilon\eta_0+\alpha_2]}}\,ds\Bigg)\Bigg] < \infty, \end{aligned} \tag{172}$$

$$\begin{aligned} \limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\Bigg[\exp\Bigg(&\frac{\frac{\varepsilon}{2}\big|Y_t^{1,N}\big|^4 + \varepsilon\alpha_1\big|Y_t^{1,N}\big|^2 + \varepsilon\big|Y_t^{2,N}\big|^2}{e^{2t[\varepsilon\eta_0+\alpha_2]}} \\ &+ \int_0^{t\wedge\tau_N} \frac{2\varepsilon\,[\alpha_3 - \varepsilon\eta_1]\,\big|Y_s^{1,N}Y_s^{2,N}\big|^2}{e^{2s[\varepsilon\eta_0+\alpha_2]}}\,ds\Bigg)\Bigg] \\ \leq \exp\Bigg(\int_0^T &\frac{\varepsilon\eta_0}{e^{2s[\varepsilon\eta_0+\alpha_2]}} + \frac{\varepsilon\,|0\vee(\eta_1 - 2\alpha_1[\varepsilon\eta_0+\alpha_2])|^2}{4\,[\varepsilon\eta_0+\alpha_2]\,e^{2s[\varepsilon\eta_0+\alpha_2]}}\,ds\Bigg) \\ &\cdot \mathbb{E}\Big[e^{\frac{\varepsilon}{2}|X_0^1|^4 + \varepsilon\alpha_1|X_0^1|^2 + \varepsilon|X_0^2|^2}\Big] < \infty. \end{aligned} \tag{173}$$

4.6. **Experimental psychology model.** In this subsection assume the setting in Subsection 4.1, let $\alpha, \delta, \varepsilon \in (0, \infty)$, $\beta \in \mathbb{R}$, $q \in [3, \infty)$, $U \in C(\mathbb{R}^2, \mathbb{R})$, and assume for all $x = (x_1, x_2) \in \mathbb{R}^2$ that $d = 2$, $m = 1$, $D = \mathbb{R}^2$, $\mu(x_1, x_2) = \big((x_2)^2(\delta + 4\alpha x_1) - \frac{1}{2}\beta^2 x_1, -x_1x_2(\delta + 4\alpha x_1) - \frac{1}{2}\beta^2 x_2\big)$, $\sigma(x_1, x_2) = (-\beta x_2, \beta x_1)$,

$U(x) = \varepsilon\|x\|^q$, and $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$. Then it holds for all $x \in \mathbb{R}^2$ that $(\mathcal{G}_{\mu,\sigma}U)(x) + \frac{1}{2}\|\sigma(x)^* (\nabla U)(x)\|^2 = 0$ (cf. Subsection 4.8 in Cox et al. [3]) and Corollary 3.8 hence shows for all $r \in (0,\infty)$ that

$$\limsup_{N\to\infty}\Big(\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\Big) = 0,$$
$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\big[\exp\big(\varepsilon\|Y_t^N\|^q\big)\big] < \infty,$$

and

$$\limsup_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\big[\exp\big(\varepsilon\|Y_t^N\|^q\big)\big] \le \mathbb{E}[\exp(\varepsilon\|X_0\|^q)]\,.$$

4.7. **Stochastic SIR model.** In this subsection assume the setting in Subsection 4.1, let $\alpha,\beta,\gamma,\delta,\varepsilon \in (0,\infty)$, $U,\bar{U} \in C(\mathbb{R}^3,\mathbb{R})$, $\hat{\varepsilon} \in (0, \frac{4\varepsilon\delta}{\gamma}]$, assume that $d = 3$, $m = 1$, and $D = (0,\infty)^3$, assume for all $x = (x_1,x_2,x_3) \in D$ that $\mu(x_1,x_2,x_3) = (-\alpha x_1x_2 - \delta x_1 + \delta, \alpha x_1x_2 - (\gamma+\delta)x_2, \gamma x_2 - \delta x_3)$, $\sigma(x_1,x_2,x_3) = (-\beta x_1x_2, \beta x_1x_2, 0)$, assume for all $x \in \mathbb{R}^3\backslash D$ that $\mu(x) = \sigma(x) = 0$, let $\phi\colon \mathbb{R} \to [0,1]$ and $\psi\colon \mathbb{R}^2 \to [0,1]$ be infinitely often differentiable functions which satisfy for all $x \in (-\infty,0]$ that $\phi(x) = 0$, which satisfy for all $x \in [1,\infty)$ that $\phi(x) = 1$, and which satisfy for all $x = (x_1,x_2) \in \mathbb{R}^2$ that $\psi(x_1,x_2) = \phi(x_1)\cdot\phi(-x_2) + \phi(-x_1)\cdot\phi(x_2)$, and assume for all $x = (x_1,x_2,x_3) \in \mathbb{R}^3$ that $U(x) = \varepsilon\left[\frac{5}{2} + (x_1+x_2)^2 - 2\cdot x_1\cdot x_2\cdot\psi(x_1,x_2)\right] + \hat{\varepsilon}\,[x_3]^2$, $\bar{U}(x) = -2\varepsilon\delta$, and $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$. Then note for all $x = (x_1,x_2,x_3) \in D$ that

(174)
$$\begin{aligned}
&(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x)\\
&\quad= (\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}\,|\langle\sigma(x),(\nabla U)(x)\rangle|^2 + \bar{U}(x)\\
&\quad= (\mathcal{G}_{\mu,\sigma}U)(x) + \bar{U}(x) = 2\varepsilon\,[x_1+x_2]\,[-\delta x_1 + \delta - (\gamma+\delta)x_2]\\
&\qquad+ 2\hat{\varepsilon}x_3\,[\gamma x_2 - \delta x_3] + \bar{U}(x)\\
&\quad= -2\varepsilon\delta\,[x_1+x_2]\,[x_1+x_2-1] - 2\varepsilon\gamma\,[x_1+x_2]\,x_2 + 2\hat{\varepsilon}x_3\,[\gamma x_2 - \delta x_3] + \bar{U}(x)\\
&\quad= -2\varepsilon\delta\,[x_1+x_2-1]^2 - 2\varepsilon\delta\,[x_1+x_2] - 2\varepsilon\gamma\,[x_1+x_2]\,x_2 - 2\hat{\varepsilon}\delta\,[x_3]^2\\
&\qquad+ 2\varepsilon\delta + 2\hat{\varepsilon}\gamma x_2x_3 + \bar{U}(0)\\
&\quad\le \bar{U}(0) + 2\varepsilon\delta - 2\varepsilon\gamma\,[x_2]^2 - 2\hat{\varepsilon}\delta\,[x_3]^2 + [2\sqrt{\varepsilon\gamma}x_2]\left[\tfrac{\hat{\varepsilon}\sqrt{\gamma}x_3}{\sqrt{\varepsilon}}\right]\\
&\quad\le \bar{U}(0) + 2\varepsilon\delta + \hat{\varepsilon}\left[\tfrac{\hat{\varepsilon}\gamma}{2\varepsilon} - 2\delta\right][x_3]^2 \le 0.
\end{aligned}$$

Combining (4.34)–(4.35) in Section 4.6 in [18] with Corollary 3.8 therefore implies that for all $r \in (0,\infty)$ it holds that

$$\limsup_{N\to\infty}\Big(\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\Big) = 0,$$
$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\Big[e^{\frac{\varepsilon}{2}|Y_t^{1,N}|^2 + \frac{\varepsilon}{2}|Y_t^{2,N}|^2 + \hat{\varepsilon}|Y_t^{3,N}|^2}\Big] < \infty,$$

and

$$\limsup_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\Big[e^{U(Y_t^N) - 2\varepsilon\delta(t\wedge\tau_N)}\Big] \le \mathbb{E}\Big[e^{U(X_0)}\Big] < \infty.$$

4.8. **Langevin dynamics.** In this subsection assume the setting in Subsection 4.1, let $\beta, \gamma \in (0,\infty)$, $\varepsilon \in (0, \frac{2\gamma}{\beta}]$, $U, \bar{U} \in C(\mathbb{R}^{2m}, \mathbb{R})$, and $V \in C^3(\mathbb{R}^m, [0,\infty)) \cap (\bigcup_{p,c\in(0,\infty)} C^3_{p,c}(\mathbb{R}^m, [0,\infty)))$, and assume that $\limsup_{r \searrow 0} \sup_{z\in\mathbb{R}^m} \frac{\|z\|^r}{1+V(z)} < \infty$, $d = 2m$, $D = \mathbb{R}^d$, $\mu(x) = (x_2, -(\nabla V)(x_1) - \gamma x_2)$, $\sigma(x)u = (0, \sqrt{\beta}u)$, $U(x) = \varepsilon\, V(x_1) + \frac{\varepsilon}{2}\,\|x_2\|^2$, $\bar{U}(x) = \varepsilon\big[\gamma - \frac{\varepsilon\beta}{2}\big]\|x_2\|^2 - \frac{\varepsilon\beta m}{2}$, and $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$ for all $x = (x_1, x_2) \in \mathbb{R}^{2m}$, $u \in \mathbb{R}^m$. Then it holds for all $x \in \mathbb{R}^{2m}$ that $(\mathcal{G}_{\mu,\sigma}U)(x) + \frac{1}{2}\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) = 0$ (cf. Subsection 4.5 in Cox et al. [3]) and Corollary 3.8 hence shows for all $r \in (0,\infty)$ that $\limsup_{N\to\infty} \big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\big) = 0$ and

(175)
$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\varepsilon\, V(Y_t^{1,N}) + \tfrac{\varepsilon}{2}\,\|Y_t^{2,N}\|^2 + \int_0^{t\wedge\tau_N} \varepsilon\left[\gamma - \tfrac{\varepsilon\beta}{2}\right] \|Y_s^{2,N}\|^2\, ds\bigg)\bigg] < \infty,$$
$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\varepsilon\, V(Y_t^{1,N}) + \tfrac{\varepsilon}{2}\,\|Y_t^{2,N}\|^2 + \int_0^{t\wedge\tau_N} \varepsilon\left[\gamma - \tfrac{\varepsilon\beta}{2}\right] \|Y_s^{2,N}\|^2\, ds\bigg)\bigg] \leq \mathbb{E}\Big[e^{\frac{\varepsilon\beta m T}{2} + \varepsilon V(X_0^1) + \frac{\varepsilon}{2}\|X_0^2\|^2}\Big].$$

4.9. **Brownian dynamics (Overdamped Langevin dynamics).** In this subsection assume the setting in Subsection 4.1, let $\beta \in (0,\infty)$, $\eta_0, \eta_1 \in [0,\infty)$, $\eta_2 \in [0, \frac{2}{\beta})$, $V \in \bigcup_{p,c\in(0,\infty)} C^3_{p,c}(\mathbb{R}^d, [0,\infty))$, $\varepsilon \in (0, \frac{2}{\beta} - \eta_2]$, $U, \bar{U} \in C(\mathbb{R}^d, \mathbb{R})$, assume for all $x, u \in \mathbb{R}^d$ that $d = m$, $D = \mathbb{R}^d$, $\limsup_{r\searrow 0} \sup_{z\in\mathbb{R}^d} \frac{\|z\|^r}{1+V(z)} < \infty$, $\mu(x) = -(\nabla V)(x)$, $\sigma(x)u = \sqrt{\beta}u$, $(\triangle V)(x) \leq \eta_0 + 2\eta_1 V(x) + \eta_2\,\|(\nabla V)(x)\|^2$, $U(x) = \varepsilon V(x)$, $\bar{U}(x) = \varepsilon\,(1 - \frac{\beta}{2}(\eta_2 + \varepsilon))\,\|(\nabla V)(x)\|^2 - \frac{\varepsilon\beta\eta_0}{2}$, and $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$. Then it holds for all $x \in \mathbb{R}^d$ that $(\mathcal{G}_{\mu,\sigma}U)(x) + \frac{1}{2}\,\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \leq \beta\eta_1 U(x)$ (cf. Subsection 4.6 in Cox et al. [3]) and Corollary 3.8 hence shows for all $r \in (0,\infty)$ that $\limsup_{N\to\infty} \big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\big) = 0$ and

(176)
$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\tfrac{\varepsilon V(Y_t^N)}{e^{\beta\eta_1 t}} + \int_0^{t\wedge\tau_N} \tfrac{\varepsilon\left[1 - \frac{\beta}{2}(\eta_2+\varepsilon)\right]}{e^{\beta\eta_1 s}} \|(\nabla V)(Y_s^N)\|^2\, ds\bigg)\bigg] < \infty,$$
(177)
$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\tfrac{\varepsilon V(Y_t^N)}{e^{\beta\eta_1 t}} + \int_0^{t\wedge\tau_N} \tfrac{\varepsilon\left[1 - \frac{\beta}{2}(\eta_2+\varepsilon)\right]}{e^{\beta\eta_1 s}} \|(\nabla V)(Y_s^N)\|^2 - \tfrac{\varepsilon\beta\eta_0}{2e^{\beta\eta_1 s}}\, ds\bigg)\bigg] \leq \mathbb{E}\Big[e^{\varepsilon V(X_0)}\Big] < \infty.$$

5. Counterexamples to exponential integrability properties

Corollary 3.8 above establishes, under suitable assumptions, that stopped increment-tamed Euler-Maruyama approximations converge strongly to the exact solution process of the considered SDE and also inherit suitable exponential integrability properties of the exact solution process of the SDE. In this section we illustrate in the case of one simple example SDE that several other approximation schemes, which converge strongly to the exact solution process of this example SDE, fail to preserve appropriate exponential integrability properties of the exact solution process of the SDE.

5.1. **An example SDE with finite exponential moments.** Let $T \in (0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $W \colon [0,T] \times \Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, let $X \colon [0,T] \times \Omega \to \mathbb{R}$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths which satisfies that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = X_0 - \int_0^t (X_s)^3 \, ds + \int_0^t dW_s, \tag{178}$$

let $\mu, \sigma \colon \mathbb{R} \to \mathbb{R}$ be the functions with the property that for all $x \in \mathbb{R}$ it holds that $\mu(x) = -x^3$ and $\sigma(x) = 1$, let $\varepsilon \in (0, \frac{1}{2}]$ satisfy $\mathbb{E}\big[\exp(\varepsilon |X_0|^4)\big] < \infty$, and let $U_\delta \colon \mathbb{R} \to [0,\infty)$, $\delta \in [0,\infty)$, be the functions with the property that for all $x \in \mathbb{R}$, $\delta \in [0,\infty)$ it holds that $U_\delta(x) = \delta x^4$. Then observe for all $\delta \in [0,\infty)$, $x \in \mathbb{R}$ that

$$\begin{aligned}
&(\mathcal{G}_{\mu,\sigma} U_\delta)(x) + \tfrac{1}{2} \left|\sigma(x)^* (\nabla U_\delta)(x)\right|^2 \\
&= \langle (\nabla U_\delta)(x), \mu(x) \rangle + \tfrac{1}{2} \operatorname{trace}\!\big(\sigma(x)[\sigma(x)]^* (\operatorname{Hess} U_\delta)(x)\big) + \tfrac{1}{2} \left|\sigma(x)^* (\nabla U_\delta)(x)\right|^2 \\
&= \delta 4x^3 \cdot (-x^3) + \tfrac{1}{2} \delta 12 x^2 + \tfrac{1}{2} \left|\delta 4 x^3\right|^2 = 6\delta x^2 - (4\delta - 8\delta^2) x^6 \\
&= 6\delta x^2 - 4\delta\,(1 - 2\delta)\, x^6.
\end{aligned} \tag{179}$$

Corollary 2.4 in Cox et al. [3] (with $\bar{U} = [0,T] \times \mathbb{R} \ni (t,x) \mapsto 4\delta\,(1-2\delta)\,x^6 - 6\delta x^2 \in \mathbb{R}$ in the notation of Corollary 2.4 in Cox et al. [3]; see also Corollary 3.8 above) hence implies for all $\delta \in [0,\varepsilon]$ that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\exp\!\left(\delta \left|X_t\right|^4 + \int_0^t 4\delta\,(1-2\delta) \left|X_s\right|^6 - 6\delta \left|X_s\right|^2 ds\right)\right] \le \mathbb{E}\left[e^{\delta |X_0|^4}\right] < \infty. \tag{180}$$

This shows, in particular, that $\sup_{t\in[0,T]} \mathbb{E}\big[\exp\!\big(\delta\,|X_t|^4\big)\big] < \infty$ for all $\delta \in (-\infty, \varepsilon] \cap (-\infty, 1/2)$.

5.2. **Infinite exponential moments for (stopped) Euler approximation schemes.** The Euler scheme stopped after leaving certain sets is not suitable for approximating the exponential moments on the left-hand side of (180) as there is at least one Euler step and this results in tails of a normal distribution. Note that in the special case $D_t = \mathbb{R}$, $t \in (0,T]$, the numerical scheme (181) is the Euler scheme for the SDE (178). We also note that Liu and Mao consider in [24] a stopped Euler scheme with $D_t = [0,\infty)$, $t \in [0,T]$, for SDEs on the domain $[0,\infty)$.

Lemma 5.1. *Assume the setting in Subsection* 5.1, *let $D_t \in \mathcal{B}(\mathbb{R})$, $t \in (0,T]$, be a nonincreasing family of sets satisfying $\lambda_{\mathbb{R}}(D_T) \cdot \mathbb{P}\big[X_0 \in D_T\big] > 0$ and $\bigcup_{t\in(0,T]} \mathring{D}_t = \mathbb{R}$, and let $Y^N \colon [0,T] \times \Omega \to \mathbb{R}$, $N \in \mathbb{N}$, be the mappings which satisfy for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$, $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ that $Y_0^N = X_0$ and*

$$Y_t^N = Y_{\frac{nT}{N}}^N + \mathbb{1}_{D_{\frac{T}{N}}}\big(Y_{\frac{nT}{N}}^N\big) \left(W_t - W_{\frac{nT}{N}} - \big(Y_{\frac{nT}{N}}^N\big)^3 \big(t - \tfrac{nT}{N}\big)\right). \tag{181}$$

Then it holds that $\limsup_{M\to\infty} \mathbb{P}\big[\sup_{s\in[0,T]} |X_s - Y_s^M| > p\big] = 0$ *and* $\mathbb{E}\big[\exp\!\big(p\,|Y_t^N|^q\big)\big] = \infty$ *for all $t \in (0,T]$, $N \in \mathbb{N}$, $p \in (0,\infty)$, $q \in (2,\infty)$.*

Proof of Lemma 5.1. Throughout this proof let $N \in \mathbb{N}$, $n \in \{0, 1, \dots, N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $p \in (0, \infty)$, and $q \in (2, \infty)$ be numbers. Lemma 3.2 implies that the function $\mathbb{R} \times (0,T] \times \mathbb{R} \ni (x,h,y) \mapsto \mathbb{1}_{D_h}(x)\,\big(y - x^3 h\big) \in \mathbb{R}$ is (μ, σ)-consistent with respect to Brownian motion. Then Proposition 3.5 applied to the function $\mathbb{R} \times (0,T]^2 \times \mathbb{R} \ni (x,h,s,y) \mapsto \mathbb{1}_{D_h}(x)\big(y - x^3 s\big) \in \mathbb{R}$ shows that $\limsup_{M\to\infty} \mathbb{P}\big[\sup_{s\in[0,T]} \big|X_s - Y_s^M\big| > p\big] = 0$. Moreover, the fact that $\mathbb{P}\big[Y_0^N \in D_{T/N}\big] = \mathbb{P}\big[X_0 \in D_{T/N}\big] > 0$ and the fact that $\lambda_{\mathbb{R}}\big(D_{T/N}\big) \geq \lambda_{\mathbb{R}}(D_T) > 0$ prove that $\mathbb{P}\big[Y_{nT/N}^N \in D_{T/N}\big] > 0$. Combining this with the fact that $W_t - W_{\frac{nT}{N}}$ is independent from $Y_{\frac{nT}{N}}^N$ yields that

$$
\begin{aligned}
&\mathbb{E}\big[\exp\big(p\,|Y_t^N|^q\big)\big] \geq \mathbb{E}\Big[\mathbb{1}_{\{Y_{nT/N}^N \in D_{T/N}\}} \exp\big(p\,|Y_t^N|^q\big)\Big] \\
&= \int_{D_{\frac{T}{N}}} \mathbb{E}\Big[\exp\big(p\,|Y_t^N|^q\big) \,\big|\, Y_{\frac{nT}{N}}^N = x\Big]\, \mathbb{P}\Big[Y_{\frac{nT}{N}}^N \in dx\Big] \\
&= \int_{D_{\frac{T}{N}}} \mathbb{E}\Big[\exp\Big(p\,\big|W_t - W_{\frac{nT}{N}} + x - x^3\,(t - \tfrac{nT}{N})\big|^q\Big)\Big]\, \mathbb{P}\Big[Y_{\frac{nT}{N}}^N \in dx\Big].
\end{aligned} \tag{182}
$$

The fact that $\forall\, x, y \in \mathbb{R}\colon |y+x|^q \geq \frac{|y|^q}{2^q} - |x|^q$ and the fact that $\lim_{y\to\infty} \big(|y|^q / y^2\big) = \infty$ hence prove that

(183)

$$
\begin{aligned}
&\mathbb{E}\big[\exp\big(p\,|Y_t^N|^q\big)\big] \\
&\geq \int_{D_{T/N}} \mathbb{E}\Big[\exp\Big(\tfrac{p}{2^q}\,\big|W_t - W_{\frac{nT}{N}}\big|^q - p\,\big|x - x^3(t - \tfrac{nT}{N})\big|^q\Big)\Big]\, \mathbb{P}\Big[Y_{\frac{nT}{N}}^N \in dx\Big] \\
&= \mathbb{E}\Big[\exp\Big(\tfrac{p}{2^q}\,\big|W_{t-\frac{nT}{N}}\big|^q\Big)\Big] \left[\int_{D_{\frac{T}{N}}} \exp\Big(-p\,\big|x - x^3(t - \tfrac{nT}{N})\big|^q\Big)\, \mathbb{P}\Big[Y_{\frac{nT}{N}}^N \in dx\Big]\right] \\
&= \mathbb{E}\Big[\exp\Big(\tfrac{p}{2^q}\,\big|W_{t-\frac{nT}{N}}\big|^q\Big)\Big]\, \mathbb{E}\Big[\mathbb{1}_{\{Y_{nT/N}^N \in D_{T/N}\}} \exp\Big(-p\,\big|Y_{nT/N}^N - \big[Y_{nT/N}^N\big]^3 [t - \tfrac{nT}{N}]\big|^q\Big)\Big] \\
&= \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}} \exp\Big(\big(\tfrac{Nt - nT}{4N}\big)^{\frac{q}{2}} p\,|y|^q - \tfrac{y^2}{2}\Big)\, dy \\
&\qquad \cdot \mathbb{E}\Big[\mathbb{1}_{\{Y_{nT/N}^N \in D_{T/N}\}} \exp\Big(-p\,\big|Y_{nT/N}^N - \big[Y_{nT/N}^N\big]^3 [t - \tfrac{nT}{N}]\big|^q\Big)\Big] \\
&= \infty.
\end{aligned}
$$

This finishes the proof of Lemma 5.1. □

5.3. **Infinite exponential moments for a (stopped) linear-implicit Euler approximation scheme.** The following lemma shows that the stopped linear-implicit Euler scheme (184) is not suitable for approximating the exponential moments on the left-hand side of (180). Display (184) shows that the linear-implicit Euler scheme (184) with $\forall\, t \in (0,T]\colon D_t = \mathbb{R}$ belongs to the class of balanced implicit methods (choose $c^0 = [0,\infty) \times \mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$ and $c^1 = [0,\infty) \times \mathbb{R} \ni x \mapsto 0 \in \mathbb{R}$ in the notation of (3.3) in [26]) introduced in Milstein, Platen and Schurz [26].

Lemma 5.2. *Assume the setting in Subsection* 5.1, *let* $D_t \in \mathcal{B}(\mathbb{R})$, $t \in (0,T]$, *be a nonincreasing family of sets satisfying* $\lambda_{\mathbb{R}}(D_T) \cdot \mathbb{P}[X_0 \in D_T] > 0$ *and* $\bigcup_{t\in(0,T]} \mathring{D}_t = \mathbb{R}$, *and let* $Y^N \colon [0,T]\times\Omega \to \mathbb{R}$, $N \in \mathbb{N}$, *be the mappings which satisfy for all* $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$, $t \in \left[\frac{nT}{N}, \frac{(n+1)T}{N}\right]$ *that* $Y_0^N = X_0$ *and*

$$
\begin{aligned}
Y_t^N &= Y^N_{\frac{nT}{N}} + \mathbb{1}_{D_{\frac{T}{N}}}(Y^N_{\frac{nT}{N}}) \left(W_t - W_{\frac{nT}{N}} - Y_t^N \big(Y^N_{\frac{nT}{N}}\big)^2 \big(t - \tfrac{nT}{N}\big)\right) \\
&= Y^N_{\frac{nT}{N}} + \mathbb{1}_{D_{\frac{T}{N}}}(Y^N_{\frac{nT}{N}}) \left(W_t - W_{\frac{nT}{N}} - \big(Y^N_{\frac{nT}{N}}\big)^3 \big(t - \tfrac{nT}{N}\big)\right) \\
&\quad + \mathbb{1}_{D_{\frac{T}{N}}}(Y^N_{\frac{nT}{N}}) \big(Y^N_{\frac{nT}{N}}\big)^2 \big(Y^N_{\frac{nT}{N}} - Y_t^N\big) \big(t - \tfrac{nT}{N}\big).
\end{aligned}
\tag{184}
$$

Then it holds that $\limsup_{N\to\infty} \mathbb{E}\big[\sup_{n\in\{0,1,\ldots,N\}} |X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}|^p\big] = 0$ *and* $\mathbb{E}\big[\exp\big(p\,|Y_t^M|^q\big)\big] = \infty$ *for all* $t \in (0,T]$, $M \in \mathbb{N}$, $p \in (0,\infty)$, $q \in (2,\infty)$.

Proof of Lemma 5.2. Throughout this proof let $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, and $q \in (2,\infty)$ be numbers. Lemma 3.30 in [18] and Lemma 3.2 imply that the function $\mathbb{R} \times (0,T] \times \mathbb{R} \ni (x,h,y) \mapsto \mathbb{1}_{D_h}(x)\big(\frac{x+y}{1+x^2h} - x\big) \in \mathbb{R}$ is (μ,σ)-consistent with respect to Brownian motion. Then Proposition 3.5 applied to the function $\mathbb{R}\times(0,T]^2\times\mathbb{R} \ni (x,h,s,y) \mapsto \mathbb{1}_{D_h}(x)\big(\frac{x+y}{1+x^2s} - x\big) \in \mathbb{R}$ shows for all $p \in (0,\infty)$ that $\limsup_{M\to\infty} \mathbb{P}\big[\sup_{s\in[0,T]} |X_s - Y_s^M| > p\big] = 0$. In addition, Lemma 2.28 in [18] yields for all $p \in (0,\infty)$ that $\sup_{M\in\mathbb{N}} \big\| \sup_{m\in\{0,1,\ldots,M\}} |Y^M_{\frac{mT}{M}}| \big\|_{L^p(\Omega;\mathbb{R})} < \infty$ and this shows for all $p \in (0,\infty)$ that the family of random variables $\sup_{m\in\{0,1,\ldots,M\}} |X_{\frac{mT}{M}} - Y^M_{\frac{mT}{M}}|^p$, $M \in \mathbb{N}$, is uniformly integrable. Combining this with convergence in probability and, e.g., Theorem 6.25 in Klenke [22] proves for all $p \in (0,\infty)$ that

$$
\limsup_{M\to\infty} \mathbb{E}\Big[\sup_{m\in\{0,1,\ldots,M\}} |X_{\frac{mT}{M}} - Y^M_{\frac{mT}{M}}|^p\Big] = 0.
$$

Moreover, the fact that $\mathbb{P}\big[Y_0^N \in D_{T/N}\big] = \mathbb{P}\big[X_0 \in D_{T/N}\big] > 0$ and the fact that $\lambda_{\mathbb{R}}\big(D_{T/N}\big) \geq \lambda_{\mathbb{R}}\big(D_T\big) > 0$ prove that $\mathbb{P}\big[Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N}}\big] > 0$. This and the fact that $\lim_{y\to\infty} \big(|y|^q/y^2\big) = \infty$ imply that

$$
\begin{aligned}
\mathbb{E}\big[\exp\big(p\,|Y_t^N|^q\big)\big] &\geq \mathbb{E}\Big[\mathbb{1}_{\{Y^N_{nT/N} \in D_{T/N}\}} \exp\big(p\,|Y_t^N|^q\big)\Big] \\
&= \int_{D_{\frac{T}{N}}} \mathbb{E}\Big[\exp\big(p\,|Y_t^N|^q\big) \,\big|\, Y^N_{\frac{nT}{N}} = x\Big]\, \mathbb{P}\Big[Y^N_{\frac{nT}{N}} \in dx\Big] \\
&= \int_{D_{\frac{T}{N}}} \mathbb{E}\left[\exp\left(p \left|\frac{x + W_t - W_{\frac{nT}{N}}}{1+x^2\big(t-\frac{nT}{N}\big)}\right|^q\right)\right] \mathbb{P}\Big[Y^N_{\frac{nT}{N}} \in dx\Big] \\
&= \int_{D_{\frac{T}{N}}} \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}} \exp\left(p \left|\frac{x+\sqrt{t-\frac{nT}{N}}\,y}{1+x^2\big(t-\frac{nT}{N}\big)}\right|^q - \frac{y^2}{2}\right) dy\; \mathbb{P}\Big[Y^N_{\frac{nT}{N}} \in dx\Big] = \infty.
\end{aligned}
\tag{185}
$$

This finishes the proof of Lemma 5.2. ∎

5.4. **Unbounded exponential moments for a (stopped) increment-tamed Euler approximation scheme.**

Lemma 5.3. *Let $T, q, \delta, \alpha, \beta \in (0, \infty)$ satisfy $q\beta > 2\alpha + 1$, let $f\colon \mathbb{R} \to \mathbb{R}$ be a $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable and locally bounded function, let $D_{h,t} \in \mathcal{B}(\mathbb{R})$, $h, t \in (0, T]$, be sets satisfying $\forall\, n \in \mathbb{N}\colon \exists\, r \in (0, T]\colon [-n, n] \subseteq \bigcap_{h\in(0,r]} \bigcap_{t\in(0,h]} D_{h,t}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $W\colon [0, T] \times \Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, let $Y^N\colon [0, T] \times \Omega \to \mathbb{R}$, $N \in \mathbb{N}$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes which satisfy that for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ it holds $\mathbb{P}$-a.s. that*

$$|Y^N_t| \geq \Big[\tfrac{2\delta}{(t-\frac{nT}{N})^\beta} - f(Y^N_{\frac{nT}{N}})\Big]\, \mathbb{1}_{\left\{1 \leq (t-\frac{nT}{N})^\alpha (W_t - W_{\frac{nT}{N}}) \leq 2\right\} \cap \left\{Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N}, t-\frac{nT}{N}}\right\}}, \tag{186}$$

and let $X\colon [0, T] \times \Omega \to \mathbb{R}$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths which satisfies $\limsup_{N\to\infty} \sup_{n\in\{0,1,\ldots,N\}} \mathbb{P}\big[|X_{nT/N} - Y^N_{nT/N}| > 1\big] = 0$. Then $\liminf_{N\to\infty} \inf_{t\in(0,T]} \mathbb{E}\big[\exp\big(\delta\, |Y^N_t|^q\big)\big] = \infty$.

Proof of Lemma 5.3. Assumption (186) implies that for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}
&\exp\Big(\delta\, \big|Y^N_t\big|^q\Big)\\
&\geq \exp\Big(\delta\, \Big|\max\Big\{0, \tfrac{2\delta}{(t-\frac{nT}{N})^\beta} - f(Y^N_{\frac{nT}{N}})\Big\}\Big|^q\Big)\\
&\quad\cdot \mathbb{1}_{\left\{1 \leq (t-\frac{nT}{N})^\alpha (W_t - W_{\frac{nT}{N}}) \leq 2\right\} \cap \left\{Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N}, t-\frac{nT}{N}}\right\}}\\
&\geq \exp\Big(\delta\, \Big|\max\Big\{0, \tfrac{2\delta}{(t-\frac{nT}{N})^\beta} - f(Y^N_{\frac{nT}{N}})\Big\}\Big|^q\Big)\\
&\quad\cdot \mathbb{1}_{\left\{1 \leq (t-\frac{nT}{N})^\alpha (W_t - W_{\frac{nT}{N}}) \leq 2\right\} \cap \left\{Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N}, t-\frac{nT}{N}}\right\} \cap \left\{f\left(Y^N_{\frac{nT}{N}}\right) \leq \frac{\delta}{(t-\frac{nT}{N})^\beta}\right\}}\\
&\geq \exp\Big(\delta\, \Big|\tfrac{\delta}{(t-\frac{nT}{N})^\beta}\Big|^q\Big)\\
&\quad\cdot \mathbb{1}_{\left\{1 \leq (t-\frac{nT}{N})^\alpha (W_t - W_{\frac{nT}{N}}) \leq 2\right\} \cap \left\{Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N}, t-\frac{nT}{N}}\right\} \cap \left\{f\left(Y^N_{\frac{nT}{N}}\right) \leq \frac{\delta}{(t-\frac{nT}{N})^\beta}\right\}}\\
&= \exp\Big(\tfrac{\delta^{(q+1)}}{(t-\frac{nT}{N})^{q\beta}}\Big) \cdot \mathbb{1}_{\left\{1 \leq (t-\frac{nT}{N})^\alpha (W_t - W_{\frac{nT}{N}}) \leq 2\right\}}\\
&\quad\cdot \mathbb{1}_{\left\{Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N}, t-\frac{nT}{N}}\right\}} \cdot \mathbb{1}_{\left\{f\left(Y^N_{\frac{nT}{N}}\right) \leq \frac{\delta}{(t-\frac{nT}{N})^\beta}\right\}}.
\end{aligned} \tag{187}$$

The fact that for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ it holds that $W_t - W_{\frac{nT}{N}}$ and $Y^N_{\frac{nT}{N}}$ are independent hence implies for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ that

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\delta\, \big|Y^N_t\big|^q\Big)\Big]\\
&\geq \exp\Big(\tfrac{\delta^{(q+1)}}{(t-\frac{nT}{N})^{q\beta}}\Big) \cdot \mathbb{P}\Big[1 \leq \big(t - \tfrac{nT}{N}\big)^\alpha \big(W_t - W_{\frac{nT}{N}}\big) \leq 2\Big]\\
&\quad\cdot \mathbb{P}\Big[|f(Y^N_{\frac{nT}{N}})|\, \big(t - \tfrac{nT}{N}\big)^\beta \leq \delta,\ Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N}, t-\frac{nT}{N}}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \exp\Big(\tfrac{\delta^{(q+1)}}{(t-\frac{nT}{N})^{q\beta}}\Big) \cdot \int_1^2 \tfrac{1}{\sqrt{2\pi (t-\frac{nT}{N})^{(2\alpha+1)}}} \exp\Big(-\tfrac{y^2}{2\,(t-\frac{nT}{N})^{(2\alpha+1)}}\Big)\,dy \\
&\quad \cdot \mathbb{P}\Big[|f(Y^N_{\frac{nT}{N}})|\,\big(t-\tfrac{nT}{N}\big)^{\beta} \leq \delta,\ Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N},t-\frac{nT}{N}}\Big] \\
&\geq \tfrac{1}{\sqrt{2\pi T^{(2\alpha+1)}}} \cdot \exp\Big(\tfrac{\delta^{(q+1)}}{(t-\frac{nT}{N})^{q\beta}} - \tfrac{2}{(t-\frac{nT}{N})^{(2\alpha+1)}}\Big) \\
&\quad \cdot \mathbb{P}\Big[|f(Y^N_{\frac{nT}{N}})|\,\big(t-\tfrac{nT}{N}\big)^{\beta} \leq \delta,\ Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N},t-\frac{nT}{N}}\Big] \\
&\geq \tfrac{1}{\sqrt{2\pi T^{(2\alpha+1)}}} \cdot \exp\Bigg(\inf_{z\in(0,\frac{T}{N}]} \Big[\tfrac{\delta^{(q+1)}}{z^{q\beta}} - \tfrac{2}{z^{(2\alpha+1)}}\Big]\Bigg) \\
&\quad \cdot \mathbb{P}\Big[|f(Y^N_{\frac{nT}{N}})|\,\big(t-\tfrac{nT}{N}\big)^{\beta} \leq \delta,\ Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N},t-\frac{nT}{N}}\Big] \\
&= \tfrac{1}{\sqrt{2\pi T^{(2\alpha+1)}}} \cdot \exp\Bigg(\inf_{z\in[\frac{N}{T},\infty)} \Big[\delta^{(q+1)} z^{q\beta} - 2z^{(2\alpha+1)}\Big]\Bigg) \\
&\quad \cdot \mathbb{P}\Big[|f(Y^N_{\frac{nT}{N}})|\,\big(t-\tfrac{nT}{N}\big)^{\beta} \leq \delta,\ Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N},t-\frac{nT}{N}}\Big].
\end{aligned} \tag{188}
$$

In the next step we note that the fact that the sample paths of X are continuous ensures that there exists a natural number $k \in \mathbb{N}$ such that

$$
\mathbb{P}\big[\sup\nolimits_{s\in[0,t]} |X_s| \leq k-1\big] \geq \tfrac{1}{2}. \tag{189}
$$

The assumption that $\forall\, n \in \mathbb{N}\colon \exists\, r \in (0,T]\colon [-n,n] \subseteq \bigcap_{h\in(0,r]} \bigcap_{t\in(0,h]} D_{h,t}$ and the fact that $\sup_{x\in[-k,k]} |f(x)| < \infty$ yield that there exists a natural number $N_0 \in \mathbb{N}$ such that $N_0 \geq T\,\big[\frac{1}{\delta} \cdot \sup_{x\in[-k,k]} |f(x)|\big]^{1/\beta}$ and $[-k,k] \subseteq \big(\bigcap_{h\in(0,\frac{T}{N_0}]} \bigcap_{t\in(0,h]} D_{h,t}\big) = \big(\bigcap_{N=N_0}^{\infty} \bigcap_{h\in(0,\frac{T}{N}]} \bigcap_{t\in(0,h]} D_{h,t}\big)$. This shows for all $N \in \mathbb{N} \cap [N_0,\infty)$ that

$$
\begin{aligned}
[-k,k] &\subseteq \bigcap_{h\in(0,\frac{T}{N}]} \bigcap_{t\in(0,h]} \Big\{x \in D_{h,t} \cap [-k,k]\colon N \geq T\big[\tfrac{1}{\delta} \cdot \sup\nolimits_{y\in[-k,k]} |f(y)|\big]^{1/\beta}\Big\} \\
&\subseteq \bigcap_{h\in(0,\frac{T}{N}]} \bigcap_{t\in(0,h]} \Big\{x \in D_{h,t} \cap [-k,k]\colon \ \big[\tfrac{N}{T}\big]^{\beta} \geq \tfrac{1}{\delta} \cdot |f(x)|\Big\} \\
&\subseteq \bigcap_{h\in(0,\frac{T}{N}]} \bigcap_{t\in(0,h]} \Big\{x \in D_{h,t}\colon |f(x)|\, t^{\beta} \leq \delta\Big\} \\
&\subseteq \bigcap_{t\in(0,\frac{T}{N}]} \Big\{x \in D_{\frac{T}{N},t}\colon |f(x)|\, t^{\beta} \leq \delta\Big\} \\
&= \bigcap_{n=0}^{N-1} \bigcap_{t\in(\frac{nT}{N},\frac{(n+1)T}{N}]} \Big\{x \in D_{\frac{T}{N},t-\frac{nT}{N}}\colon |f(x)|\,\big[t-\tfrac{nT}{N}\big]^{\beta} \leq \delta\Big\}.
\end{aligned} \tag{190}
$$

Hence, we obtain that for all $N \in \mathbb{N} \cap [N_0,\infty)$, $n \in \{0,1,\ldots,N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ it holds that

$$
\Big\{x \in D_{\frac{T}{N},t-\frac{nT}{N}}\colon |f(x)|\,\big[t-\tfrac{nT}{N}\big]^{\beta} \leq \delta\Big\} \supseteq \Big\{x \in \mathbb{R}\colon |x| \leq k\Big\}. \tag{191}
$$

Combining this with (189) and the monotonicity of $\mathbb{P}$ yields that for all $N \in \mathbb{N} \cap [N_0, \infty)$, $n \in \{0, 1, \ldots, N-1\}$, $t \in \left(\frac{nT}{N}, \frac{(n+1)T}{N}\right]$ it holds that

(192)
$$\mathbb{P}\Big[|f(Y^N_{\frac{nT}{N}})| \left(t - \tfrac{nT}{N}\right)^{\beta} \leq \delta,\ Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N}, t - \frac{nT}{N}}\Big] \geq \mathbb{P}\Big[|Y^N_{\frac{nT}{N}}| \leq k\Big]$$
$$\geq \mathbb{P}\Big[|X_{\frac{nT}{N}}| \leq k-1,\ |X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}| \leq 1\Big] \geq \frac{1}{2} - \sup_{m \in \{0,1,\ldots,N\}} \mathbb{P}\Big[|X_{\frac{mT}{N}} - Y^N_{\frac{mT}{N}}| > 1\Big].$$

In the next step we combine inequalities (188) and (192) with the assumption that $q\beta > 2\alpha + 1$ and with the assumption that

$$\limsup_{N \to \infty} \sup_{n \in \{0,1,\ldots,N\}} \mathbb{P}\big[|X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}| > 1\big] = 0$$

to obtain

$$\begin{aligned} &\liminf_{N \to \infty} \inf_{t \in (0,T]} \mathbb{E}\Big[\exp\Big(\delta \left|Y^N_t\right|^q\Big)\Big] \\ &\geq \liminf_{N \to \infty} \Bigg[\tfrac{1}{\sqrt{2\pi T^{2\alpha+1}}} \cdot \exp\Bigg(\inf_{z \in [\frac{N}{T}, \infty)} \Big[\delta^{(q+1)} z^{q\beta} - 2z^{(2\alpha+1)}\Big]\Bigg) \\ &\quad \cdot \Bigg(\frac{1}{2} - \sup_{m \in \{0,1,\ldots,N\}} \mathbb{P}\Big[|X_{\frac{mT}{N}} - Y^N_{\frac{mT}{N}}| > 1\Big]\Bigg)\Bigg] \\ &= \infty. \end{aligned} \tag{193}$$

The proof of Lemma 5.3 is thus completed. □

Corollary 5.4. *Assume the setting in Subsection* 5.1*, let $D_t \in \mathcal{B}(\mathbb{R})$, $t \in (0,T]$, be a nonincreasing family of sets satisfying $\forall\, n \in \mathbb{N} \colon \exists\, t \in (0,T] \colon [-n, n] \subseteq D_t$, and let $Y^N \colon [0,T] \times \Omega \to \mathbb{R}$, $N \in \mathbb{N}$, be the mappings which satisfy for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$, $t \in \left(\frac{nT}{N}, \frac{(n+1)T}{N}\right]$ that $Y^N_0 = X_0$ and*

(194)
$$Y^N_t = Y^N_{\frac{nT}{N}} + \mathbb{1}_{D_{\frac{T}{N}}}(Y^N_{\frac{nT}{N}}) \left[\frac{W_t - W_{\frac{nT}{N}} - \big(Y^N_{\frac{nT}{N}}\big)^3 (t - \frac{nT}{N})}{\max\big\{1, \big(t - \frac{nT}{N}\big) \big|W_t - W_{\frac{nT}{N}} - \big(Y^N_{\frac{nT}{N}}\big)^3 (t - \frac{nT}{N})\big|\big\}}\right].$$

Then it holds that $\limsup_{N \to \infty} \mathbb{P}\big[\sup_{n \in \{0,1,\ldots,N\}} |X_{nT/N} - Y^N_{nT/N}| > p\big] = 0$ and $\liminf_{N \to \infty} \inf_{t \in (0,T]} \mathbb{E}\big[\exp\big(p \left|Y^N_t\right|^q\big)\big] = \infty$ for all $p \in (0, \infty)$, $q \in (3, \infty)$.

Proof of Corollary 5.4. Throughout this proof let $p \in (0,\infty)$ and $q \in (3, \infty)$ be real numbers and let $\psi \colon \mathbb{R} \times (0,T]^2 \times \mathbb{R} \to \mathbb{R}$ be the mapping with the property that for all $(x, h, t, y) \in \mathbb{R} \times (0,T]^2 \times \mathbb{R}$ it holds that $\psi(x,h,t,y) = \mathbb{1}_{D_h}(x) \frac{y - x^3 t}{\max\{1, t|y - x^3 t|\}}$. In the next step we apply Lemma 3.28 in [18] and Lemma 3.2 to obtain that the function $\mathbb{R} \times (0,T] \times \mathbb{R} \ni (x,t,y) \mapsto \psi(x,t,t,y) \in \mathbb{R}$ is (μ, σ)-consistent with respect to Brownian motion. Proposition 3.4 hence implies for all $r \in (0,\infty)$ that $\limsup_{N \to \infty} \mathbb{P}\big[\sup_{n \in \{0,1,\ldots,N\}} |X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}| > r\big] = 0$. For proving the divergence statement in Corollary 5.4, we intend to apply Lemma 5.3 above. To this end we first prove inequality (186). For this let $D_{h,t} \in \mathcal{B}(\mathbb{R})$, $h, t \in (0,T]$, be the sets which satisfy for all $h, t \in (0,T]$ that $D_{h,t} = D_h \cap [-t^{-2/3}, t^{-2/3}]$. Next note that for all

$r \in (0,T]$ it holds that

$$\begin{aligned}\bigcap_{h\in(0,r]}\bigcap_{t\in(0,h]} D_{h,t} &= \bigcap_{h\in(0,r]}\bigcap_{t\in(0,h]} \big(D_h \cap \big[-t^{-2/3}, t^{-2/3}\big]\big)\\ &= \bigcap_{h\in(0,r]} \big(D_h \cap \big[-h^{-2/3}, h^{-2/3}\big]\big)\\ &= D_r \cap \big[-r^{-2/3}, r^{-2/3}\big].\end{aligned} \tag{195}$$

This and the assumption that $\forall\, n \in \mathbb{N}\colon \exists\, t \in (0,T]\colon [-n,n] \subseteq D_t$ assure that for all $n \in \mathbb{N}$ there exists a real number $r \in (0,T]$ such that

$$[-n,n] \subseteq \bigcap_{h\in(0,r]}\bigcap_{t\in(0,h]} D_{h,t}. \tag{196}$$

Moreover, observe that for all $(x,h,t,y) \in \mathbb{R}\times(0,T]^2\times\mathbb{R}$ it holds that

$$\begin{aligned}|x+\psi(x,h,t,y)| &\geq \left(\tfrac{|y-x^3t|}{\max(1,t|y-x^3t|)} - |x|\right)\cdot \mathbb{1}_{D_{h,t}}(x)\cdot\mathbb{1}_{[1,2]}(ty)\\ &\geq \left(\tfrac{|y|}{\max(1,t|y-x^3t|)} - |x| - |x|^3t\right)\cdot \mathbb{1}_{D_{h,t}}(x)\cdot\mathbb{1}_{[1,2]}(ty)\\ &\geq \left(\tfrac{|y|}{1+t^2|x|^3+t|y|} - |x| - |x|^3t\right)\cdot \mathbb{1}_{D_{h,t}}(x)\cdot\mathbb{1}_{[1,2]}(ty)\\ &\geq \left(\tfrac{|y|}{2+t|y|} - |x| - |x|^3T\right)\cdot \mathbb{1}_{D_{h,t}}(x)\cdot\mathbb{1}_{[1,2]}(ty)\\ &\geq \left(\tfrac{1}{3t} - |x| - |x|^3T\right)\cdot \mathbb{1}_{D_{h,t}}(x)\cdot\mathbb{1}_{[1,2]}(ty).\end{aligned} \tag{197}$$

This together with the fact that for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ it holds that $Y_t^N = Y^N_{\frac{nT}{N}} + \psi\big(Y^N_{\frac{nT}{N}}, \frac{T}{N}, t-\frac{nT}{N}, W_t - W_{\frac{nT}{N}}\big)$ shows that for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ it holds that

$$\begin{aligned}|Y_t^N| \geq{}& \left[\tfrac{2\min\{\frac{1}{6},p\}}{(t-\frac{nT}{N})} - \left[\big|Y^N_{\frac{nT}{N}}\big| + \big|Y^N_{\frac{nT}{N}}\big|^3 T\right]\right]\\ &\cdot \mathbb{1}_{\left\{1\leq(t-\frac{nT}{N})(W_t - W_{\frac{nT}{N}})\leq 2\right\}\cap\left\{Y^N_{\frac{nT}{N}}\in D_{\frac{T}{N},t-\frac{nT}{N}}\right\}}.\end{aligned} \tag{198}$$

This and (196) allows us to apply Lemma 5.3 (with $\alpha = \beta = 1$ in the notation of Lemma 5.3) to obtain the divergence statement in Corollary 5.4. The proof of Corollary 5.4 is thus completed. $\square$

The proof of the following corollary is analogous to the proof of Corollary 5.4 and therefore omitted.

Corollary 5.5. *Assume the setting in Subsection* 5.1*, let $D_t \in \mathcal{B}(\mathbb{R})$, $t \in (0,T]$, be a nonincreasing family of sets satisfying $\forall\, n \in \mathbb{N}\colon \exists\, t \in (0,T]\colon [-n,n] \subseteq D_t$, and let $Y^N\colon [0,T]\times\Omega\to\mathbb{R}$, $N \in \mathbb{N}$, be mappings which satisfy for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ that $Y_0^N = X_0$ and*

$$Y_t^N = Y^N_{\frac{nT}{N}} + \mathbb{1}_{D_{\frac{T}{N}}}\big(Y^N_{\frac{nT}{N}}\big)\left[\frac{W_t - W_{\frac{nT}{N}} - (Y^N_{\frac{nT}{N}})^3\,\big(t-\frac{nT}{N}\big)}{1+\big(t-\frac{nT}{N}\big)\,\big|W_t - W_{\frac{nT}{N}} - (Y^N_{\frac{nT}{N}})^3\,\big(t-\frac{nT}{N}\big)\,\big|}\right]. \tag{199}$$

Then it holds that $\limsup_{N\to\infty}\mathbb{P}\big[\sup_{n\in\{0,1,\dots,N\}}|X_{nT/N} - Y^N_{nT/N}| > p\big] = 0$ and $\liminf_{N\to\infty}\inf_{t\in(0,T]}\mathbb{E}\big[\exp\big(p\,|Y_t^N|^q\big)\big] = \infty$ for all $p \in (0,\infty)$, $q \in (3,\infty)$.

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Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Strasse 9, 45127 Essen, Germany

E-mail address: `martin.hutzenthaler@uni-due.de`

Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland

E-mail address: `arnulf.jentzen@sam.math.ethz.ch`

School of Mathematics and Statistics, Central South University, Changsha 410083, Hunan, People's Republic of China

E-mail address: `x.j.wang7@csu.edu.cn`