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CORRIGENDUM TO *NEW BOUNDS FOR* $\psi(X)$

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Abstract. In this note we correct a mistake in *New bounds for* $\psi(x)$, Math. Comp. (84) **296** (2015), 1339–1357.

We correct a mistake in the definition of B_5 in [2, Equation (2.30)]: it should be $\frac{R_0}{\log x}$ instead of $\frac{R_0}{2\log x}$. As a consequence we correct the calculations displayed in [2, Tables 6.4 and 6.5]. We also correct a typo in [2, Equations (3.7) (3.8) and Line 9 Page 1351]: it should be $M(a,b,0)B_1(T_1)$ instead of $\frac{\delta}{2}B_1(T_1)$. We take this opportunity to restate [2, Theorem 1.1] and [2, Theorem 3.1] as our main result:

Theorem 0.1.

Let $H>0$ such that if $\zeta(\beta+i\gamma)=0$ and $0<\gamma<H$, then $\beta=1/2$.

Let $T_0>0$ such that $s_0=\displaystyle\sum_{0<\gamma<T_0}\frac{1}{\gamma}$ can be directly computed.

Let T_1 satisfy $T_0<T_1<H$.

Let $R_0\geq 1$ such that $\zeta(\sigma+it)$ does not vanish in the region $\sigma\geq 1-\dfrac{1}{R_0\log|t|}$, and $|t|\geq 2$.

Let $a_1,a_2,a_3>0$ such that $\left|N(T)-\dfrac{T}{2\pi}\log\dfrac{T}{2\pi}-\dfrac{T}{2\pi}+\dfrac{7}{8}\right|\leq R(T)$, with $R(T)=a_1\log T+a_2\log\log T+a_3$.

Let $3/5\leq\sigma_0<1$ and let c_1,c_2,c_3 depending on σ_0 such that, for all $T\geq H$, $N(\sigma_0,T)\leq c_1T+c_2\log T+c_3$.

Let $\delta>0$, m an integer, $m\geq 2$, and $b_0>0$ such that $b_0<mR_0(\log H)^2$. We denote $E(x)=\left|\frac{\psi(x)-x}{x}\right|$.

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Then for every $x \geq e^{b_0}$, there exists $\epsilon_0 > 0$ such that $E(x) < \epsilon_0$, with

$$\begin{aligned}
\epsilon_0 =& \max\left(\epsilon(b_0, 1, 1+\delta, m, \sigma_0, T_1), \epsilon(b_0, 1-\delta, 1, m, \sigma_0, T_1)\right),\\
\epsilon(b_0, a, b, m, \sigma_0, T_1) =& \frac{\delta}{2} + \frac{2M(a,b,m)B_5(e^{b_0}, m, \sigma_0)}{\delta^m} + \frac{2M(a,b,m)B_3(m)}{\delta^m} e^{-(1-\sigma_0)b_0}\\
&+ \frac{2M(a,b,m)B_3(m)}{\delta^m} e^{-\sigma_0 b_0}\\
&+ \frac{2M(a,b,m)B_4(m,H,\sigma_0)}{\delta^m} e^{-(1-\frac{1}{R_0 \log H})b_0}\\
&+ \Big(M(a,b,0)B_1(T_1) + \frac{M(a,b,m)B_2(m,T_1)}{\delta^m}\Big) e^{-b_0/2}\\
&+ \log(2\pi) e^{-b_0} + \frac{M(a,b,0)}{2} e^{-3b_0},
\end{aligned}$$

$$\begin{aligned}
q(T) =& \frac{a_1 \log T + a_2}{T(\log T)(\log \frac{T}{2\pi})},\\
B_1(T_1) =& s_0 + \Big(\frac{1}{2\pi} + q(T_0)\Big)\Big(\log(T_1/T_0)\ \log(\sqrt{T_1 T_0}/(2\pi))\Big) + \frac{2R(T_0)}{T_0},\\
B_2(m, T_1) =& \Big(\frac{1}{2\pi} + q(T_1)\Big)\Big(\frac{1 + m\log(T_1/2\pi)}{m^2 T_1^m} - \frac{1 + m\log(H/2\pi)}{m^2 H^m}\Big) + \frac{2R(T_1)}{T_1^{m+1}},\\
B_3(m) =& \Big(\frac{1}{2\pi} + q(H)\Big)\frac{1 + m\log(H/2\pi)}{m^2 H^m} + \frac{2R(H)}{H^{m+1}},\\
B_4(m, \sigma_0) =& \left(c_1\big(1 + \frac{1}{m}\big) + c_2\frac{\log H}{H} + \big(c_3 + \frac{c_2}{m+1}\big)\frac{1}{H}\right)\frac{1}{H^m},\\
B_5(x, m, \sigma_0) =& \Big(c_1 + c_2\frac{\log H}{H} + \frac{c_3}{H} + \big(c_1 + \frac{c_2}{H}\big)\frac{R_0}{\log x}\frac{(\log H)^2}{\big(\frac{mR_0}{\log x}\big)(\log H)^2 - 1}\Big)\frac{x^{-\frac{1}{R_0 \log H}}}{H^m},\\
M(a, b, 0) =& \frac{a+b}{2},\\
M(a, b, m) =& \frac{(2m+1)!}{m!}\int_0^1 |P_m(1-2u)|\,((b-a)u+a)^{m+1}\,du,
\end{aligned}$$

where P_m is the m^{th} Legendre polynomial.

Here we take the values $T_0 = 1\,132\,491$, $s_0 = 11.637732$, $R_0 = 5.69693$ from [4], and $a_1 = 0.137, a_2 = 0.443, a_3 = 1.588$ from [6].

In Table 1 we display calculations of ϵ_0 using Platt's $H = 30\,610\,046\,000$ from [5], and in 2, we use Gourdon's $H = 2\,445\,999\,556\,030$ from [3]. While the first is the most reliable result, we also do computations using the second to allow comparison with [1]. For the zero density estimate for $N(\sigma_0, T)$, we use values for c_i's from [2, Table 2]. In particular, we use the c_i's that make $N(\sigma_0, H)$ small when $45 \leq b_0 \leq 1000$ for Table 1 and when $60 \leq b_0 \leq 2000$ for Table 2. Otherwise we use the bound for $N(\sigma_0, T)$ that makes c_1 small.

Table 1: For all $x \geq e^{b_0}$, $E(x) \leq \epsilon_0$ (with Platt's H)

b_0	σ_0	m	δ	T_1	ϵ_0
20	0.89	4	$1.363 \cdot 10^{-5}$	1132491	$5.3688 \cdot 10^{-4}$
25	0.89	3	$7.256 \cdot 10^{-6}$	1132491	$4.8208 \cdot 10^{-5}$
30	0.90	2	$2.811 \cdot 10^{-6}$	1132491	$5.6685 \cdot 10^{-6}$
35	0.91	3	$1.760 \cdot 10^{-7}$	16655735	$7.4474 \cdot 10^{-7}$
40	0.93	5	$2.160 \cdot 10^{-8}$	243156644	$8.6386 \cdot 10^{-8}$
45	0.94	14	$3.890 \cdot 10^{-9}$	4465903456	$1.0412 \cdot 10^{-8}$
50	0.94	23	$3.200 \cdot 10^{-9}$	9414182663	$2.4075 \cdot 10^{-9}$
60	0.94	23	$3.180 \cdot 10^{-9}$	9473269034	$1.6665 \cdot 10^{-9}$
70	0.95	23	$3.170 \cdot 10^{-9}$	9503091533	$1.6553 \cdot 10^{-9}$
80	0.95	23	$3.160 \cdot 10^{-9}$	9533102596	$1.6495 \cdot 10^{-9}$
90	0.96	25	$3.154 \cdot 10^{-9}$	10459294072	$1.6403 \cdot 10^{-9}$
100	0.96	25	$3.144 \cdot 10^{-9}$	10492498721	$1.6351 \cdot 10^{-9}$
200	0.97	24	$3.045 \cdot 10^{-9}$	10362393882	$1.5859 \cdot 10^{-9}$
300	0.98	23	$2.950 \cdot 10^{-9}$	10210287058	$1.5386 \cdot 10^{-9}$
400	0.98	23	$2.860 \cdot 10^{-9}$	10530920743	$1.4920 \cdot 10^{-9}$
500	0.98	22	$2.764 \cdot 10^{-9}$	10378825816	$1.4451 \cdot 10^{-9}$
600	0.98	21	$2.670 \cdot 10^{-9}$	10208914692	$1.3986 \cdot 10^{-9}$
700	0.98	20	$2.576 \cdot 10^{-9}$	10027477419	$1.3525 \cdot 10^{-9}$
800	0.98	20	$2.489 \cdot 10^{-9}$	10377134449	$1.3065 \cdot 10^{-9}$
900	0.98	19	$2.394 \cdot 10^{-9}$	10193844799	$1.2601 \cdot 10^{-9}$
1000	0.98	18	$2.300 \cdot 10^{-9}$	9992276391	$1.2140 \cdot 10^{-9}$
1500	0.99	14	$1.834 \cdot 10^{-9}$	9447885887	$9.8234 \cdot 10^{-10}$
2000	0.99	10	$1.338 \cdot 10^{-9}$	8786967376	$7.3582 \cdot 10^{-10}$
2500	0.99	7	$8.590 \cdot 10^{-10}$	8938346983	$4.9090 \cdot 10^{-10}$
3000	0.99	3	$3.700 \cdot 10^{-10}$	7020936277	$2.4341 \cdot 10^{-10}$
3500	0.99	2	$1.040 \cdot 10^{-10}$	12895131073	$7.8012 \cdot 10^{-11}$
4000	0.99	2	$3.300 \cdot 10^{-11}$	25109069167	$2.3978 \cdot 10^{-11}$
4500	0.99	2	$9.900 \cdot 10^{-12}$	29954226185	$7.4269 \cdot 10^{-12}$
5000	0.99	2	$3.200 \cdot 10^{-12}$	30539413052	$2.3413 \cdot 10^{-12}$
5500	0.99	2	$1.012 \cdot 10^{-12}$	30602958659	$7.5880 \cdot 10^{-13}$
6000	0.99	2	$3.600 \cdot 10^{-13}$	30609148853	$2.6324 \cdot 10^{-13}$
6500	0.99	2	$1.650 \cdot 10^{-13}$	30609857532	$1.2377 \cdot 10^{-13}$
7000	0.99	3	$2.800 \cdot 10^{-13}$	H	$1.8540 \cdot 10^{-13}$
7500	0.99	3	$1.150 \cdot 10^{-13}$	H	$7.6641 \cdot 10^{-14}$
8000	0.99	3	$4.790 \cdot 10^{-14}$	H	$3.1960 \cdot 10^{-14}$
8500	0.99	3	$2.028 \cdot 10^{-14}$	H	$1.3523 \cdot 10^{-14}$
9000	0.99	3	$9.100 \cdot 10^{-15}$	H	$5.8937 \cdot 10^{-15}$
9500	0.99	3	$4.200 \cdot 10^{-15}$	H	$2.7623 \cdot 10^{-15}$

Table 2: For all $x \geq e^{b_0}$, $E(x) \leq \epsilon_0$ (with Gourdon's H)

b_0	σ_0	m	δ	T_1	ϵ_0
20	0.89	4	$1.363 \cdot 10^{-5}$	1132491	$5.3688 \cdot 10^{-4}$
25	0.89	3	$7.256 \cdot 10^{-6}$	1132491	$4.8208 \cdot 10^{-5}$
30	0.89	2	$2.806 \cdot 10^{-6}$	1132491	$5.6646 \cdot 10^{-6}$
35	0.90	2	$1.605 \cdot 10^{-7}$	11353619	$7.0193 \cdot 10^{-7}$
40	0.91	3	$1.600 \cdot 10^{-8}$	174242715	$8.0215 \cdot 10^{-8}$
45	0.92	4	$1.621 \cdot 10^{-9}$	2381966616	$8.7009 \cdot 10^{-9}$
50	0.94	7	$2.070 \cdot 10^{-10}$	36748017581	$9.4627 \cdot 10^{-10}$
55	0.96	21	$5.084 \cdot 10^{-11}$	531790503047	$1.1245 \cdot 10^{-10}$
60	0.95	28	$4.720 \cdot 10^{-11}$	785125401939	$3.1732 \cdot 10^{-11}$
65	0.95	30	$4.720 \cdot 10^{-11}$	846100223428	$2.4990 \cdot 10^{-11}$
70	0.95	30	$4.710 \cdot 10^{-11}$	847894273982	$2.4407 \cdot 10^{-11}$
75	0.95	30	$4.710 \cdot 10^{-11}$	847894273982	$2.4333 \cdot 10^{-11}$
80	0.95	29	$4.697 \cdot 10^{-11}$	819588350463	$2.4294 \cdot 10^{-11}$
85	0.95	29	$4.691 \cdot 10^{-11}$	820635226690	$2.4265 \cdot 10^{-11}$
90	0.96	29	$4.685 \cdot 10^{-11}$	821684782611	$2.4235 \cdot 10^{-11}$
95	0.96	29	$4.680 \cdot 10^{-11}$	822561466966	$2.4206 \cdot 10^{-11}$
100	0.96	29	$4.674 \cdot 10^{-11}$	823615962460	$2.4178 \cdot 10^{-11}$
200	0.97	29	$4.573 \cdot 10^{-11}$	841781697615	$2.3656 \cdot 10^{-11}$
300	0.98	28	$4.469 \cdot 10^{-11}$	829158420353	$2.3145 \cdot 10^{-11}$
400	0.98	27	$4.367 \cdot 10^{-11}$	815589845520	$2.2645 \cdot 10^{-11}$
500	0.98	27	$4.272 \cdot 10^{-11}$	833700245994	$2.2152 \cdot 10^{-11}$
600	0.98	26	$4.171 \cdot 10^{-11}$	819438195834	$2.1656 \cdot 10^{-11}$
700	0.98	26	$4.077 \cdot 10^{-11}$	838302579182	$2.1169 \cdot 10^{-11}$
800	0.98	25	$3.980 \cdot 10^{-11}$	822668712312	$2.0672 \cdot 10^{-11}$
900	0.98	24	$3.874 \cdot 10^{-11}$	808165873349	$2.0180 \cdot 10^{-11}$
1000	0.99	24	$3.780 \cdot 10^{-11}$	828229975474	$1.9686 \cdot 10^{-11}$
1500	0.99	21	$3.288 \cdot 10^{-11}$	821597164118	$1.7222 \cdot 10^{-11}$
2000	0.99	18	$2.800 \cdot 10^{-11}$	812211162080	$1.4779 \cdot 10^{-11}$
2500	0.99	14	$2.278 \cdot 10^{-11}$	750447831104	$1.2204 \cdot 10^{-11}$
3000	0.99	10	$1.745 \cdot 10^{-11}$	661230711518	$9.5984 \cdot 10^{-12}$
3500	0.99	8	$1.246 \cdot 10^{-11}$	707190917562	$7.0072 \cdot 10^{-12}$
4000	0.99	4	$6.950 \cdot 10^{-12}$	522374711336	$4.3466 \cdot 10^{-12}$
4500	0.99	2	$2.615 \cdot 10^{-12}$	511445186230	$1.9612 \cdot 10^{-12}$
5000	0.99	2	$9.600 \cdot 10^{-13}$	1212568911828	$7.1651 \cdot 10^{-13}$
5500	0.99	2	$3.600 \cdot 10^{-13}$	2035957199105	$2.6290 \cdot 10^{-13}$
6000	0.99	2	$1.300 \cdot 10^{-13}$	2377658700355	$9.6781 \cdot 10^{-14}$
6500	0.99	2	$4.780 \cdot 10^{-14}$	2436401450394	$3.5885 \cdot 10^{-14}$
7000	0.99	2	$1.790 \cdot 10^{-14}$	2444646487144	$1.3434 \cdot 10^{-14}$
7500	0.99	2	$6.800 \cdot 10^{-15}$	2445804143837	$5.1039 \cdot 10^{-15}$
8000	0.99	2	$2.700 \cdot 10^{-15}$	2445968744958	$1.9902 \cdot 10^{-15}$
8500	0.99	2	$1.091 \cdot 10^{-15}$	2445994525246	$8.1824 \cdot 10^{-16}$
9000	0.99	2	$5.500 \cdot 10^{-16}$	2445998277497	$4.0269 \cdot 10^{-16}$
9500	0.99	3	$1.556 \cdot 10^{-15}$	H	$1.0372 \cdot 10^{-15}$
10000	0.99	3	$7.500 \cdot 10^{-16}$	H	$4.8869 \cdot 10^{-16}$

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Table 2 – continued from previous page

b_0	σ_0	m	δ	T_1	ϵ_0
10500	0.99	3	$3.500 \cdot 10^{-16}$	H	$2.3069 \cdot 10^{-16}$
11000	0.99	3	$1.641 \cdot 10^{-16}$	H	$1.0944 \cdot 10^{-16}$
11500	0.99	3	$8.000 \cdot 10^{-17}$	H	$5.2288 \cdot 10^{-17}$
12000	0.99	3	$3.800 \cdot 10^{-17}$	H	$2.5199 \cdot 10^{-17}$
12500	0.99	3	$1.853 \cdot 10^{-17}$	H	$1.2351 \cdot 10^{-17}$
13000	0.99	3	$9.600 \cdot 10^{-18}$	H	$6.2560 \cdot 10^{-18}$

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