THE CHROMATIC NUMBER
OF THE SQUARE OF THE 8-CUBE

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Abstract. A cube-like graph is a Cayley graph for the elementary abelian
group of order 2^n. In studies of the chromatic number of cube-like graphs,
the kth power of the n-dimensional hypercube, Q_n^k, is frequently considered.
This coloring problem can be considered in the framework of coding theory,
as the graph Q_n^k can be constructed with one vertex for each binary word
of length n and edges between vertices exactly when the Hamming distance
between the corresponding words is at most k. Consequently, a proper coloring
of Q_n^k corresponds to a partition of the n-dimensional binary Hamming space
into codes with minimum distance at least $k + 1$. The smallest open case,
the chromatic number of Q_8^2, is here settled by finding a 13-coloring. Such
13-colorings with specific symmetries are further classified.

1. Introduction

A *cube-like graph* is a Cayley graph for the elementary abelian group of order
2^n. One of the original motivations for studying cube-like graphs was the fact that
they have only integer eigenvalues [5]. Cube-like graphs also form a generalization
of the hypercube.

There has been a lot of interest in the chromatic number of cube-like graphs
[6, Sect. 9.7]. In the early studies, people realized that many types of such graphs
have a chromatic number that is a power of two [4]. This observation inspired work
into one of the two main research directions that have emerged: Determine the
spectrum of chromatic numbers of cube-like graphs. Payan [19] showed that there
are gaps in the spectrum by proving that 3 is not a possible chromatic number; he
also found a cube-like graph with chromatic number 7, disproving earlier conjectures
that the chromatic number might always be a power of two.

The other main research direction is that of determining the chromatic number
for specific families of cube-like graphs. The n-dimensional hypercube, also called
the n-cube and denoted by Q_n, is the graph with one vertex for each binary word of
length n and with an edge between two vertices exactly when the Hamming distance
between the corresponding words is 1. The kth power of a graph $\Gamma = (V, E)$ is the
graph $\Gamma^k = (V', E')$, where $V' = V$ and in which two vertices are adjacent exactly
when their distance in Γ is at most k. In the current work, we focus on the chromatic
number of (the cube-like graph) Q_n^k, denoted by $\chi_{\bar{k}}(n)$. The chromatic number

$\chi_{\bar{k}}(n)$ has been studied, for example, in [9, 14, 16, 23] and is further motivated by the problem of scalability of certain optical networks [22].

The value of $\chi_{\bar{2}}(n)$ has been determined for $n \leq 7$, and for $n = 8$ it has been known that $13 \leq \chi_{\bar{2}}(8) \leq 14$, where the upper bound follows from 14-colorings found independently by Hougardy [23] and Royle [6, Section 9.7]. By finding a 13-coloring of Q_8^2, we shall prove here that $\chi_{\bar{2}}(8) = 13$. The result is obtained by computer search, where the search space is reduced by prescribing symmetries. An exhaustive classification is further carried out in the reduced search space. We also show that none of the colorings of Q_8^2 that were found can occur as a subgraph in a 13-coloring of Q_9^2.

The remainder of the paper is structured as follows. In Section 2, we review the relation between binary codes and the graph Q_n^k, give definitions, and survey some old results. Properties of a putative 13-coloring of Q_8^2 that are utilized in the computer search are discussed in Section 3. The method for computational classification is explained in Section 4.1, the results are reported in Section 4.2, and a consistency check of the computational results is discussed in Section 4.3. Finally, a method for searching for 13-colorings of Q_9^2 starting from the available 13-colorings of Q_8^2 is described in Section 5.

2. BINARY CODES AND POWERS OF THE n-CUBE

We have seen that the graphs Q_n and Q_n^k are conveniently defined via the corresponding Hamming space. Similarly, the problem of studying the chromatic number of these graphs benefits from a coding-theoretic framework.

A *binary code* of *length* n and *size* M is a subset of $\mathbb{Z}_2^n$ of size M. Since all codes in this work are binary, we frequently omit that term and just talk about codes. The elements of a code are called *codewords*, and the *minimum distance* of a code is the smallest Hamming distance between any two distinct codewords. A binary code with length n, size M, and minimum distance at least d is called an (n, M, d) code. A binary code is called *even* if the Hamming weight of all codewords is even. We denote the set of all even-weight binary words of length n by $\mathbb{E}^n$ and the set of all odd-weight binary words of length n by $\mathbb{O}^n$. Determining $A(n, d)$, the largest possible size of a code with given n and d, is one of the main problems in combinatorial coding theory.

A proper coloring of Q_n^k corresponds to a partition of $\mathbb{Z}_2^n$ into binary codes with minimum distance at least $k + 1$. The maximum size of a color class is $A(n, k+1)$, which implies the lower bound

$$(1) \qquad \chi_{\bar{k}}(n) \geq \lceil 2^n / A(n, k+1) \rceil.$$

For colorings of Q_n^2, general constructions [12, 22] give

$$(2) \qquad \chi_{\bar{2}}(n) \leq 2^{\lceil \log_2(n+1) \rceil}.$$

When $n = 2^t - j$ where $j = 1, 2, 3, 4$, we have $A(n, 3) = 2^{2^t - t - j}$ because the $j - 1$ times shortened Hamming code is optimal [2], so (1) and (2) coincide. With $n \leq 15$, $\chi_{\bar{2}}(n)$ is unknown for $n = 8, 9, 10, 11$. For these values of n, the values of $A(n, 3)$ are 20, 40, 72, and 144, respectively [1, 3, 18], and (1) yields the lower bounds 13, 13, 15, and 15, respectively. The upper bound $\chi_{\bar{2}}(8) \leq 14$ follows from 14-colorings of Q_8^2 found independently by Hougardy in 1991 [23] and Royle in 1993 [6, Section 9.7]. In this work, we shall show that $\chi_{\bar{2}}(8) = 13$. Recently, Lauri [11] reported a 14-coloring of Q_9^2, which implies that $\chi_{\bar{2}}(9) \leq 14$.

The concept of symmetry is essential for the results of the current study. Two binary codes are called *equivalent* if one can be obtained from the other by a permutation of coordinates and addition of a word in $\mathbb{Z}_2^n$ to each codeword. The operations maintaining equivalence of binary codes are the isometries of the Hamming space $\mathbb{Z}_2^n$. For even-weight binary codes of length n, we require that the addition be carried out with even-weight words and denote the group of operations maintaining equivalence by G_n.

The halved n-cube, $\frac{1}{2}Q_n$, is the graph over the words of $\mathbb{E}^n$ that has edges between any two vertices whose Hamming distance is 2. It is well known that Q_n^2 is isomorphic to $\frac{1}{2}Q_{n+1}$: adding a parity bit to each word in $\mathbb{Z}_2^n$ gives an isomorphism. For $n \geq 4$, the automorphism group of $\frac{1}{2}Q_{n+1}$ has order $(n+1)!2^n$. The automorphisms are precisely the operations maintaining equivalence of even binary codes. Further, an independent set in $\frac{1}{2}Q_{n+1}$ corresponds to an even binary code of length $n+1$ with minimum distance at least 4. Therefore, it is convenient to use even binary codes of length $n+1$ when discussing colorings of the square of the n-cube. A proper coloring of Q_n^2 thus corresponds to a partition of $\mathbb{E}^{n+1}$ into even binary codes of minimum distance at least 4. We call partitions of $\mathbb{E}^{n+1}$ and partitions of a subset of $\mathbb{E}^{n+1}$ that contain only codes with minimum distance at least 4 *admissible*.

For an element $g \in G_n$ and a codeword $c \in \mathbb{E}^n$, we use the notation gc for g acting on c. Further, for a code $C \subseteq \mathbb{E}^n$, we denote $gC = \{gc : c \in C\}$, and for a set of codes $\mathcal{C} \subseteq \mathcal{P}(\mathbb{E}^n)$ we denote $g\mathcal{C} = \{gC : C \in \mathcal{C}\}$. Two codes, C and D, are equivalent if $C = gD$ for some $g \in G_n$. The automorphism group of a code C is the group $\mathrm{Aut}(C) = \{g : gC = C\}$. The *orbit* of a code $C \subseteq \mathbb{E}^n$ under a group $H \leq G_n$ is the set $\{hC : h \in H\}$. We call two partitions $\mathcal{C}$ and $\mathcal{C}'$ of $\mathbb{E}^n$ *isomorphic* if $g\mathcal{C} = \mathcal{C}'$ for some $g \in G_n$. The automorphism group of a partition $\mathcal{C}$ is the group $\mathrm{Aut}(\mathcal{C}) = \{g : g\mathcal{C} = \mathcal{C}\}$.

The *Hamming code* of length 7 and size 16 is the unique $(7, 16, 3)$ code (up to equivalence) that is a subspace of the vector space $\mathbb{F}_2^n$. The extended Hamming code is the even $(8, 16, 4)$ code obtained by adding a parity bit to each codeword of the Hamming code. Adding another parity bit to each codeword (0 for all codewords) gives the doubly extended Hamming code, which is an even $(9, 16, 4)$ code.

3. Partitions of $\mathbb{E}^9$

As discussed above, a proper coloring of the square of the 8-cube, Q_8^2, corresponds to a partition of $\mathbb{E}^9$ into even codes with minimum distance at least 4. Let us now consider the distribution of code sizes in such a partition containing 13 codes. As the maximum size of a code is $A(8, 3) = A(9, 4) = 20$, there are five different distributions:

- one code of size 16, twelve of size 20,
- one code of size 17, one of size 19, eleven of size 20,
- two codes of size 18, eleven of size 20,
- one code of size 18, two of size 19, ten of size 20,
- four codes of size 19, nine of size 20.

All attempts by the authors to exhaustively search for partitions of these types failed, like in (unpublished) earlier studies. See also [20]. The authors then decided to restrict the search to partitions with prescribed automorphism groups, which turned out to be successful as we shall see.

Specifically, we search for, and classify up to isomorphism, all admissible partitions $\mathcal{C}$ of $\mathbb{E}^9$ for which $|\mathrm{Aut}(\mathcal{C})| \geq 3$. The case when one code is the doubly extended Hamming code of size 16 leads to many admissible partitions, and this case is also considered for $|\mathrm{Aut}(\mathcal{C})| = 2$.

We shall next discuss the theoretical framework for our search. Prescribing automorphism groups in the construction of combinatorial objects is a standard technique [7, Section 9], but there are a few more details to take into account when considering sets of objects rather than single objects. For example, an automorphism of such a set may map an object (here, code) onto itself or onto another object.

A common way to construct all objects with nontrivial automorphism group is to consider groups H of prime order from each conjugacy class, and for each such H, construct all objects that have H as a group of automorphisms. However, we use a slightly different approach that allows making use of classification results for codes to limit the search space.

As shown in Theorem 3.2 below, all admissible partitions $\mathcal{C}$ that we wish to find contain a code C such that there is a nontrivial group H that is subgroup of both $\mathrm{Aut}(C)$ and $\mathrm{Aut}(\mathcal{C})$. The search can then be carried out by first fixing the size distribution of $\mathcal{C}$ and a code C and then considering subgroups H of $\mathrm{Aut}(C)$. After fixing the size distribution of $\mathcal{C}$, we only need to consider codes C and groups H of certain sizes, as listed in Theorem 3.2.

The proof of Theorem 3.2 uses the following corollary of the Sylow theorems.

Theorem 3.1 ([21, Corollary 4.15]). *Let G be a finite group and let q be a prime power. If q divides $|G|$, then G contains a subgroup of order q.*

Theorem 3.2. *Let $\mathcal{C}$ be a partition of $\mathbb{E}^9$ into 13 codes of minimum distance at least 4.*

(i) *If $\mathcal{C}$ contains one code C of size 16 and 12 codes of size 20 and $|\mathrm{Aut}(\mathcal{C})| \geq 2$, then there is a subgroup $H \leq \mathrm{Aut}(\mathcal{C})$ of prime order which is also a subgroup of $\mathrm{Aut}(C)$.*

(ii) *If $|\mathrm{Aut}(\mathcal{C})| \geq 3$, then there is a subgroup $H \leq \mathrm{Aut}(\mathcal{C})$ so that $H \leq \mathrm{Aut}(C)$ for some code $C \in \mathcal{C}$ and $|H|$ and $|C|$ are one of the possibilities listed in Table 1 based on the size distribution of $\mathcal{C}$.*

TABLE 1. Possible sizes of H and C in Theorem 3.2 3.2

Size distribution	$|H|$	$|C|$
$16 + 12 \times 20$	$3, 4, 5, 7$	16
$17 + 19 + 11 \times 20$	$3, 4, 5, 7$	17
$2 \times 18 + 11 \times 20$	$3, 4, 5, 7$	20
$18 + 2 \times 19 + 10 \times 20$	$3, 4, 5, 7$	18
$4 \times 19 + 9 \times 20$	4	20
$4 \times 19 + 9 \times 20$	$3, 5, 7$	19

Proof.

(i) Because $\mathrm{Aut}(\mathcal{C})$ is nontrivial, Theorem 3.1 implies it contains a subgroup H of prime order. Because C is the only code of size 16 in $\mathcal{C}$, each element in H must map C onto itself, so $H \leq \mathrm{Aut}(C)$.

(ii) Because $|\mathrm{Aut}(\mathcal{C})| \geq 3$, Theorem 3.1 implies it contains a subgroup H of order 4 or odd prime. Further, $|H|$ must divide $|G_9| = 9!2^8$. Finally, H divides the codes in $\mathcal{C}$ into orbits such that the codes in an orbit have the same size and the size of each orbit is divisible by $|H|$. Therefore, for every size distribution and $|H|$, there must be an orbit of size 1 containing one code C of the size $|C|$ given in Table 1. For that code C, we have $H \leq \mathrm{Aut}(C)$. □

Because we are eventually interested only in constructing nonisomorphic partitions $\mathcal{C}$, we can reduce the search space by the following observations. Starting from two equivalent codes would generate isomorphic partitions, so it is enough to consider one candidate C from each equivalence class. Further, after fixing a code C, considering two subgroups $H \leq \mathrm{Aut}(C)$ that are conjugate in $\mathrm{Aut}(C)$ would generate partitions that are isomorphic by an element in $\mathrm{Aut}(C)$, so it is enough to consider only one subgroup H from each conjugacy class of subgroups of $\mathrm{Aut}(C)$.

4. COMPUTATIONAL CLASSIFICATION

4.1. **Algorithm.** Before the main search, the authors classified the even $(9, M, 4)$ codes for $16 \leq M \leq 20$; the number of equivalence classes is 343566, 41499, 2041, 33, and 2, respectively. This classification was carried out and validated with software developed for [17]; some of these codes were classified already in [18]. The automorphism groups of the codes can be obtained as a by-product of this classification or by separately using a standard transformation to a colored graph [18] (see also [7, pp. 86–87]) which is fed to the graph isomorphism software *nauty* [13]. We use the notation $\mathcal{C}_M$ for a set of representatives of the equivalence classes of even $(9, M, 4)$ codes.

The main idea of the search algorithm is to start by fixing a code C in the partition and a group H that is a subgroup of the automorphism groups of C and the partition. The other codes in the partition are divided into orbits by H, so the search proceeds by finding possible orbits and combining them into partitions of $\mathbb{E}^n$. To save memory and enable more efficient parallelization, finding the other codes is carried out in two phases.

The search algorithm is given as Algorithm 1 in pseudocode. The search is carried out by calling $\mathrm{SEARCH}(M, N_1, M_1, N_2, M_2)$ for each of the six possible cases regarding size distributions of codes and choice of the code size $|C|$ particularized in Theorem 3.2. The parameters of the call are as follows. The value of M is the size of the particularized code C in Theorem 3.2. Disregarding C, there are one or two sizes for the remaining codes. Let N_1 be the number of codes of size M_1 and N_2 the number of codes of size M_2, where $0 \leq N_1 \leq N_2$ (so $N_1 = 0$ if there is only one size of remaining codes; then M_1 is undefined) and $N_1 + N_2 = 12$.

The following subroutines are called from the search algorithm. We use the notation $\mathcal{P}(X)$, where X is a set, for the set of all subsets of X.

- $\mathrm{PACK}(X, \mathcal{S}, N)$, where X is a set, $\mathcal{S} \subseteq \mathcal{P}(\mathcal{P}(X))$, and N is an integer: Finds all subsets S of $\mathcal{S}$ where each element of X appears *at most* once and $\sum_{\mathcal{O} \in S} |\mathcal{O}| = N$, and returns the set of all such sets S.
- $\mathrm{EXACT}(X, \mathcal{S})$, where X is a set and $\mathcal{S} \subseteq \mathcal{P}(\mathcal{P}(Y))$ for some $Y \supseteq X$: Finds all subsets S of $\mathcal{S} \cap \mathcal{P}(\mathcal{P}(X))$ so that each element of X appears *exactly* once in S, and returns the set of all such sets S.

Algorithm 1. Main search procedure

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procedure SEARCH(M, N_1, M_1, N_2, M_2: integers)
    for all $C \in \mathcal{C}_M$ **do**
        for all $H \in$ NONCONJUGATESUBGROUPS(Aut(C)) **do**
            $\mathcal{S}_1 \leftarrow$ FINDORBITS(C, N_1, M_1, H)
            $\mathcal{S}_2 \leftarrow$ FINDORBITS(C, N_2, M_2, H)
            for all $S_1 \in$ PACK($\mathbb{E}^9 \setminus C, \mathcal{S}_1, N_1$) **do**
                for all $S_2 \in$ EXACT($\mathbb{E}^9 \setminus (C \cup \bigcup_{\mathcal{O} \in S_1} \bigcup_{C' \in \mathcal{O}} C'), \mathcal{S}_2$) **do**
                    Report $\{C\} \cup \left(\bigcup_{\mathcal{O} \in S_1} \mathcal{O}\right) \cup \left(\bigcup_{\mathcal{O} \in S_2} \mathcal{O}\right)$

- NONCONJUGATESUBGROUPS(G) returns a set containing one representative H from each conjugacy class of subgroups of G such that $|H|$ is one of the orders listed in Theorem 3.2 for the particular case.
- FINDORBITS(C, N, M, H) returns the set of all orbits of even $(9, M, 4)$ codes under the group H such that the size of the orbit is at most N and the codes in the orbit are pairwise disjoint and disjoint from C.

The first routine essentially finds cliques in a graph with vertices for sets of words and edges whenever the corresponding sets are nonintersecting. In this work a tailored backtrack algorithm was used due to the large number of vertices in the corresponding graph. For EXACT, one may use the *libexact* software [8]. For NONCONJUGATESUBGROUPS, any computer algebra software can be used (actually, the groups are so small that even brute force search performs well). For the last routine, a naive method is sufficient, looping over all even $(9, M, 4)$ codes (which are obtained by constructing gC' for every $g \in G_n$ and $C' \in \mathcal{C}_M$) and all $h \in H$.

Let $H \leq$ Aut(C) for a prescribed code C. To find all admissible partitions of $\mathbb{E}^n \setminus C$ into codes with the given size distribution that are divided into orbits by H, we search for sets $\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k\}$ for which each $\mathcal{O}_i$ is an orbit of a code under H and $\bigcup_i \mathcal{O}_i$ is an admissible partition of $\mathbb{E}^n \setminus C$. The algorithm does this by first finding the orbits of codes of size M_1 and then finding the orbits of codes of size M_2. In the search for an admissible partition, one needs to make sure that the codes are nonintersecting. When searching for the orbits of codes of size M_2, the additional requirement that all words should be included into some code is beneficial for the search; compare the difference between the routines PACK and EXACT.

Once the entire search is ready, isomorphic partitions are rejected and the automorphism group orders are determined for all solutions. One may consider the partitions as colorings of the graph Q_{n-1}^2 and use *nauty* for those graphs. Handling colorings with indistinguishable colors is described in the *nauty* manual.

4.2. **Results.** The search for admissible partitions with automorphism group order at least 2 containing the doubly extended Hamming code yielded 2266 nonisomorphic partitions. Out of these, 266 have an automorphism group of order 2 and the other 2000 have an automorphism group of order 4. The search required 5650 days of CPU time on a single core of Intel Core i7 870 processor. CPU times reported later are for a single core of that processor. The computations were carried out in a compute cluster.

For other cases, the numbers of partitions found are shown in Table 2 along with the required CPU time, grouped by the size distribution and the size of the

TABLE 2. Number of partitions

Size distribution	$|C|$	#	CPU time
$16 + 12 \times 20$	16	125	128 days
$17 + 19 + 11 \times 20$	17	0	162 days
$2 \times 18 + 11 \times 20$	20	5	291 hours
$18 + 2 \times 19 + 10 \times 20$	18	0	66 hours
$4 \times 19 + 9 \times 20$	19	1	42 days
$4 \times 19 + 9 \times 20$	20	5	32 hours

TABLE 3. Automorphism group orders

$|\mathrm{Aut}(\mathcal{C})|$	#
3	1
4	2099
6	5
8	25
9	1
12	2
24	1
48	2

initial code C in the search. Note that the line corresponding to the distribution $16+12\times 20$ does not include the search starting from the doubly extended Hamming code. The two separate cases with size distribution $4\times 19+9\times 20$ yielded no common partitions.

The number of partitions with each automorphism group order at least 3 are listed in Table 3. In addition, there are 266 partitions that have automorphism group of order 2 where one code is the doubly extended Hamming code.

Two of the partitions found, one with distribution $2 \times 18 + 11 \times 20$ and one with distribution $4 \times 19 + 9 \times 20$, contain codes that are not maximal. Augmenting these codes yield five new nonisomorphic partitions in total, two with trivial automorphism group, which have distributions $18 + 2 \times 19 + 10 \times 20$ and $4 \times 19 + 10 \times 20$, and three with automorphism group order 2, one of which have distribution $18+2\times 19+10\times 20$ and two of which have distribution $4\times 19+10\times 20$.

We present here a partition with distribution $16 + 12 \times 20$ that has an automorphism group of order 48. Because all codes of size 20 in this partition lie on the same orbit under the automorphism group, it suffices to list an even $(9, 16, 4)$ code C_0, an even $(9, 20, 4)$ code C_1, and two isomorphisms g_1, g_2 that generate the automorphism group. An isomorphism g is given as a pair (π, c), where π is a permutation of $\{1, 2, \ldots, 9\}$ and c is a word in $\mathbb{E}^9$ such that g maps each word $c' \in \mathbb{E}^9$ to a word that has $c'_{\pi^{-1}(i)} \oplus c_i$ at the ith coordinate for each i.

$$C_0 = \{000000000, 000011011, 100100101, 100111110, 101001010, 101010111,$$
$$001101001, 001110100, 110010001, 010011100, 110100010, 010101111,$$
$$011000110, 111001101, 011110011, 111111000\},$$
$$C_1 = \{000000011, 100001101, 100011010, 100110100, 000111001, 101000110,$$
$$001010101, 101101000, 001101111, 101110011, 010010000, 110010111,$$
$$010100101, 110101011, 010111110, 111000001, 011001100, 011011011,$$
$$011100010, 111111101\},$$
$$g_1 = ((23)(47)(68), 100100101),$$
$$g_2 = ((1857)(29)(46), 000011011).$$

This result gives an infinite family of colorings of Q_n^2.

Theorem 4.1. $\chi_{\bar{2}}(9 \cdot 2^i - 1) \leq 13 \cdot 2^i$ for $i \geq 0$.

Proof. The result follows from $\chi_{\bar{2}}(8) = 13$ and the bound $\chi_{\bar{2}}(2n + 1) \leq \chi_{\bar{2}}(n)$ [16, Theorem 1]. □

For example, Theorem 4.1 gives that $\chi_{\bar{2}}(17) \leq 26$, but we are not able to determine the exact chromatic number in that case. By $5632 \leq A(17, 3) \leq 6552$ [3, 10] and (1), we know that $\chi_{\bar{2}}(17) \geq 21$, and finding better bounds for $A(17, 3)$ would not be able to improve the bound given by (1) beyond 24.

4.3. **Double counting.** To increase confidence in the computational results, we perform a consistency check by double counting. The counting is done separately for every size distribution of $\mathcal{C}$ and size of the code C listed in Theorem 3.2. We find the number of triples $(\mathcal{C}, C, H)$ where $\mathcal{C}$ is an admissible partition of $\mathbb{E}^n$ with the given size distribution, C is a code in $\mathcal{C}$ of the given size, H is a subgroup of $\mathrm{Aut}(\mathcal{C})$ and $\mathrm{Aut}(C)$, and $|H|$ is one of the sizes listed in Theorem 3.2. We obtain this number in two ways.

The first way is as follows. For each partition $\mathcal{C}$, let $N(\mathcal{C})$ be the number of pairs (C, H) such that $(\mathcal{C}, C, H)$ is a triple to be counted. This can be found computationally by looping over all subgroups H of $\mathrm{Aut}(\mathcal{C})$ of admissible order and every code C of the fixed size and checking whether $H \leq \mathrm{Aut}(C)$. Because $N(\mathcal{C}) = N(\mathcal{D})$ when $\mathcal{C}$ and $\mathcal{D}$ are isomorphic, and the number of partitions isomorphic to $\mathcal{C}$ is $|G_n|/|\mathrm{Aut}(\mathcal{C})|$, the count can be obtained by

$$\sum_{\mathcal{C}} N(\mathcal{C}) \frac{|G_n|}{|\mathrm{Aut}(\mathcal{C})|},$$

where the sum is taken over equivalence class representatives of colorings that are found in the search.

On the other hand, for each pair (C, H) where C is a code of the fixed size and $H \leq \mathrm{Aut}(\mathcal{C})$ is of admissible size, let $N(C, H)$ be the number of colorings for which $(\mathcal{C}, C, H)$ is a triple to be counted. This is the number of colorings found in the search starting from C and H. Because $N(C, H) = N(gC, gHg^{-1})$ for every $g \in G_n$, the count can be obtained by looking at only one C from each equivalence class of codes and only one H from each conjugacy class of subgroups of $\mathrm{Aut}(C)$. As this

is exactly what is done in the search, the count can be obtained computationally by

$$\sum_{C,H} N(C,H) X(C,H) \frac{|G_n|}{|\mathrm{Aut}(C)|},$$

where $X(C,H)$ is the number of subgroups of $\mathrm{Aut}(C)$ conjugate to H and the sum is taken over all pairs (C,H) for which the search was performed. To this end, the numbers $X(C,H)$ and $N(C,H)$ are stored during the search.

5. Extending colorings

In an attempt to find a 13-coloring of Q_9^2, one may check whether the classified 13-colorings of Q_8^2 can occur as a subgraph of such a coloring. Consider an admissible partition $\mathcal{C} = \{C_1, C_2, \ldots, C_{13}\}$ of $\mathbb{E}^{10}$. Each code C_i can be written as $C_i = 0D_i \cup 1E_i$, where D_i is an even-weight code of length 9 with minimum distance 4 and E_i is an odd-weight code of length 9 with minimum distance 4. Now $\mathcal{D} = \{D_1, D_2, \ldots, D_{13}\}$ is an admissible partition of $\mathbb{E}^9$ and $\mathcal{E} = \{E_1, E_2, \ldots, E_{13}\}$ also corresponds to an admissible partition of $\mathbb{E}^9$ if for example the last bit is complemented in each codeword.

Given an admissible partition $\mathcal{D} = \{D_1, D_2, \ldots, D_{13}\}$ of $\mathbb{E}^9$, we are to determine whether it can be extended to a coloring of $\mathbb{E}^{10}$ in the way described above. The number of ways to express 256, the size of $\mathbb{E}^9$, as a sum of 13 integers smaller than or equal to 20 equals the number of ways to express $13 \times 20 - 256 = 4$ as a sum of 13 nonnegative integers, which is 1820. Therefore, there are 1820 possible choices for the sizes of the codes in $\mathcal{E}$ when the order matters. The algorithm now runs as follows. In Steps 1 and 2 of a complete search, representatives of all classified colorings and all possible choices of sizes M_i are considered, respectively.

(1) Consider an admissible partition $\mathcal{D} = \{D_1, D_2, \ldots, D_{13}\}$ of $\mathbb{E}^9$.
(2) Fix the sizes of the codes in $\mathcal{E}$, denoted by $(M_1, \ldots, M_{13})$, where $M_i \leq 20$ and $\sum_i M_i = 256$.
(3) For all $i = 1, \ldots, 13$, find all possible codes E_i such that $|E_i| = M_i$ and $C_i = 0D_i \cup 1E_i$ has minimum distance 4.
(4) Find a partition of $\mathbb{E}^{10}$ from all possible sets $0D_i \cup 1E_i$.

The task in Step 3 can be formulated in the framework of clique search: starting from $\mathbb{O}^9$, we remove the words that have distance less than 3 to a word in D_i (because having such a codeword in E_i would result to a pair of codewords in C_i that would have distance less than 4) and consider the graph over the remaining codewords that has an edge between each pair of codewords with Hamming distance at least 4. Now a possible code E_i corresponds to a clique of the size M_i in that graph. To search for cliques in the graph, we use the software Cliquer [15].

The task in Step 4 is an instance of the *exact cover problem*: Given a set X and a family $\mathcal{S}$ of subsets of X, enumerate all subsets of $\mathcal{S}$ that contain each element of X exactly once. We use *libexact* to solve the instances. Actually, it suffices to let $X = \{1, 2, \ldots, 13\} \cup \mathbb{O}^9$ with all possible sets $\{i\} \cup E_i$ in $\mathcal{S}$.

None of the known partitions of $\mathbb{E}^9$ could be extended to an admissible partition of $\mathbb{E}^{10}$ containing 13 codes. The search required 146 hours of CPU time.

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