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ON THE ERROR ESTIMATES OF THE VECTOR PENALTY-PROJECTION METHODS: SECOND-ORDER SCHEME

PHILIPPE ANGOT AND RIMA CHEAYTOU

Abstract. In this paper, we study the vector penalty-projection method for incompressible unsteady Stokes equations with Dirichlet boundary conditions. The time derivative is approximated by the backward difference formula of second-order scheme (BDF2), namely Gear's scheme, whereas the approximation in space is performed by the finite volume scheme on a Marker And Cell (MAC) grid. After proving the stability of the method, we show that it yields second-error estimates in the time step for both velocity and pressure in the norm of $l^\infty(\mathbf{L}^2(\Omega))$ and $l^2(L^2(\Omega))$, respectively. Also, we show that the splitting error for both velocity and pressure is of order $\mathcal{O}(\sqrt{\varepsilon\,\delta t})$ where ε is a penalty parameter chosen as small as desired and δt is the time step. Numerical results in agreement with the theoretical study are also provided.

1. Introduction

For $T > 0$, we consider the time-dependent incompressible Navier-Stokes equations in the primitive variables on a finite time interval [0,T]:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right) - \mu\,\Delta\mathbf{v} + \nabla p = \mathbf{f} \qquad \text{in } \Omega\times]0,T[, \tag{1.1}$$

$$\nabla\cdot\mathbf{v} = 0 \qquad \text{in } \Omega\times]0,T[, \tag{1.2}$$

$$\mathbf{v} = 0 \qquad \text{on } \Gamma\times]0,T[, \tag{1.3}$$

where $\Omega \subset R^d$ ($d = 2$ or 3 in practice) is an open bounded and connected domain with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. The generic point in Ω is denoted by $\mathbf{x}$. We denote by $\mathbf{v}=(u,v)^T$ the fluid velocity with initial value $\mathbf{v}(t=0) = \mathbf{v}_0$, p the pressure field, ρ the fluid density (the density is taken equal to 1), μ the dynamic viscosity (here, $\mu = 1/Re$ with Re a Reynolds number), and $\mathbf{f}$ the external body forces. We impose homogeneous Dirichlet condition (1.3) on the whole boundary Γ for the sake of simplicity. Finally, the reader will keep in mind that bold letters such as $\mathbf{v}$, $\mathbf{f}$, etc., indicate vector valued quantities.

One of the main numerical difficulties in solving (1.1)–(1.3) arises from the coupling between the velocity and the pressure by the incompressibility constraint at each time step. Undoubtedly, the most popular way to overcome this difficulty consists of using the projection methods introduced initially by Chorin [15] and Temam [43] in the late sixties. Projection methods are fractional-step schemes which consist in splitting the time evolution into two sub-steps. In the first step, an intermediate velocity, that does not satisfy the incompressibility constraint, is computed by solving an advection diffusion problem. In the second step, according to the Helmholtz-Hodge decomposition [17], the intermediate velocity is projected

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to the space of the divergence-free vector fields to get the pressure and the corrected velocity that satisfies the incompressibility condition. Projection methods gained popularity due to the fact that the computations of the velocity and the pressure are decoupled by the two-step predictor-corrector procedure which significantly reduces the computational cost. However, Chorin-Temam's projection method suffers from an inconsistent Neumann boundary condition satisfied by the pressure approximation. This artificial condition induces a loss of the temporal accuracy in the solution; hence the numerical scheme is not satisfactory since its splitting error is irreducibly of $\mathcal{O}(\delta t)$ [39] where δt is the time step. Over the years, several variants of projection methods have been developed to improve the temporal accuracy among which are pressure-correction methods [20, 21, 26, 32, 45] (incremental or rotational form), velocity-correction methods [25, 36, 37] (incremental or rotational form), consistent splitting scheme [24, 31, 42], scalar penalty-projection methods [18, 30, 40] and more recently vector penalty-projection methods [1, 3]. Hereafter we present a short review on some theoretical results obtained from some of these variants in the presence of Dirichlet boundary conditions.

Incremental pressure-correction methods [20, 32] were widely used in practice and have been rigorously analyzed by E and Li [16] and Shen [41] in the semi-discrete case and by Guermond [21] in the fully discrete case. Second-order accuracy in time on the velocity in L^2-norm has been proved but only $\mathcal{O}(\delta t)$ estimates on the pressure approximation are available due to the presence of a numerical boundary layer.

Timmermans et al. [45] proposed a modified version of the incremental pressure-correction methods, referred to by Guermond and Shen in [23, 26] as *incremental pressure-correction methods in rotational form.* Brown, Cortez, and Minion [12] showed, *using normal mode analysis in a periodic channel*, that the pressure approximation in this particular case is second-order accurate. In this regard, a rigorous normal mode error analysis was carried out by Pyo and Shen [38] for two second-order projection type methods. Finally, Guermond and Shen showed in [26] that the best possible convergence rate for pressure approximation in the L^2-norm is of order $3/2$ in general domains.

Another class of projection methods namely *velocity correction methods* has been introduced and rigorously analyzed (in its incremental and rotational form) by Guermond and Shen [25]. Error estimates lead to a second-order accuracy for the velocity in the L^2-norm for both versions. In addition, they proved better error estimates for the rotational form, i.e., $\mathcal{O}(\delta t^{3/2})$ in the H^1-norm of the velocity and the L^2-norm of the pressure (see also [28] for the fully discrete case). It was also shown that this family of projection methods can be related to a set of methods in [33, 36].

For more details regarding both numerical and theoretical results of different projection methods, the reader can refer to the complete review of Guermond et al. [27].

Moreover, the *scalar penalty projection method* is another variant of projection methods, proposed and numerically investigated by Jobelin et al. [30]. It was also theoretically analyzed in [10] and verified later by Février̀e et al. [18] using a spatial discretization by finite volumes on staggered grids. The basic idea behind the development of this scheme originated from a paper of Shen in 1992 [40] and consists in adding to the velocity prediction step a penalty term similar to the augmentation term used in the so-called Augmented Lagrangian method (e.g. [19]), which constrains the divergence of the intermediate velocity. The same idea has been exploited independently later, in 1999, by Caltagirone and Breil [13] with a different projection step called by the authors "vector projection step". From the point of view of convergence properties, the authors show in [10] that for low value of the penalty parameter r, splitting error estimates of the so-called rotational projection scheme are recovered, i.e., $\mathcal{O}(\delta t^2)$ and $\mathcal{O}(\delta t^{3/2})$ convergence for the velocity and the pressure, respectively. Indeed, for high values of the penalty parameter, they obtain the $\delta t/r$ behavior for the velocity splitting error known for the penalty scheme.

In 2008, Angot et al. [3] introduced a new fractional-step scheme called *Vector Penalty-Projection* (*VPP*) *methods* to solve incompressible fluid flows and to overcome most of the drawbacks of the usual projection methods. This family of methods represents a compromise between the best properties of both classes: the Augmented Lagrangian (without iterations) and splitting methods under a vector form. In fact, an original penalty-correction step for the velocity replaces the standard scalar pressure-correction step to calculate flows with divergence-free velocity. This allows us to impose the desired boundary condition to the end-of-step velocity pressure variables. The VPP methods were improved in [1, 4, 5] where they showed that such methods are also very efficient to compute incompressible multiphase viscous flows or Darcy flows whatever the density, viscosity or permeability jumps are and also in the presence of outflow boundary conditions [8, 9]. Indeed, they showed to favorably compete with the best incremental projection methods or Augmented Lagrangian methods in terms of accuracy, cheapness and robustness.

In [1, 3, 4], the VPP methods were implemented using the first-order Euler implicit scheme in time with Dirichlet conditions on the boundary. The authors found that the scheme is $\mathcal{O}(h^2)$ in space for velocity and pressure, where h is the spatial mesh step of the MAC scheme and $\mathcal{O}(\delta t)$ in time for velocity and pressure (δt is the time step). However, in the literature, the VPP methods concern only the case of the first-order time discretization with Dirichlet boundary conditions. The present paper is devoted to the extension of such methods to a second-order time discretization. Remember that in view of all the previous results of different types of projection methods, one can notice that, while a temporally second-order convergence for the velocity can be readily obtained analytically, the computed pressure can not reach the full second-order accuracy in time. We believe that this paper provides interesting results in this regard as well as for the splitting error of the scheme.

The main task of the present paper is to provide stability and rigorous error analysis of the second-order vector penalty-projection method with Dirichlet boundary conditions. Our results indicate that the VPP method guarantees $\mathcal{O}(\delta t^2)$ for both velocity and pressure in the norm of $l^\infty(\mathbf{L}^2(\Omega))$ and $l^2(L^2(\Omega))$, respectively. Also, we show that the splitting error of the method varies as $\mathcal{O}(\varepsilon)$ where the penalty parameter ε can be chosen as small as desired. This feature is very interesting since it offers the possibility to reduce the splitting error, up to make it negligible with respect to the consistency error of higher-order schemes.

The rest of the paper is organized as follows. In Section 2, we describe the vector penalty-projection method using a second-order scheme to discretize in time and we underline the role of each step of the method. In Section 3, we show the well-posedness and the stability of the method. Section 4 is devoted to giving the error estimates for the VPP method. In Section 5, we present some numerical experiments which are consistent with the theoretical results. Concluding remarks are reported in Section 6.

2. Vector penalty-projection method

In this section, we describe the vector penalty-projection method for the incompressible Navier-Stokes problem using a second-order backward difference formula (BDF2) to march in time. In addition, we highlight briefly the role of each step of the method.

2.1. **Description of the scheme.** Before presenting the scheme, let us first introduce the following functional spaces:

$$\begin{aligned}
&\mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^d,\\
&\mathbf{H}^1(\Omega) = \left\{\mathbf{u} \in \mathbf{L}^2(\Omega);\ \nabla u \in (L^2(\Omega))^{d\times d}\right\},\\
&L^2_0(\Omega) = \left\{q \in L^2(\Omega);\ \int_\Omega q\,dx = 0\right\}.
\end{aligned}$$

We denote $L^2(\Omega)$-norm by $\| \,.\, \|_0$, the $H^1(\Omega)$-norm by $\| \,.\, \|_1$, the $H^{-1}(\Omega)$-norm by $\| \,.\, \|_{-1}$ and $L^2(\Omega)$-inner product by $(.\,,\,.)_0$.

Now, let $0 = t^0 < t^1 < \cdots < t^N = T$ be a partition of the time interval of computation [0,T] which we suppose uniform for the sake of simplicity. We denote by $\delta t = t^{n+1} - t^n > 0$ the time step. Let $\phi^0, \phi^1, \ldots, \phi^N$ be a sequence of functions in a Hilbert space H. We denote this sequence by $\phi_{\delta t}$ and we define the following discrete norm: $\| \phi_{\delta t} \|_{l^2(H)} := (\delta t \sum_{n=0}^N \| \phi^n \|_H^2)^{1/2}$. The notation $\mathbf{v}^n$ is used to represent an approximation of $\mathbf{v}(t^n)$, where $t^n = n\,\delta t$.

We use a semi-implicit time-integration scheme. We approximate the time derivative the BDF2 scheme. The convective term is handled explicitly. Finally, the viscous term is treated implicitly. Hence, the VPP method reads as follows.

Let $n \geq 1$ such that $(n+1)\delta t \leq T$, $\widetilde{\mathbf{v}}^0, \widetilde{\mathbf{v}}^1, \mathbf{v}^0, \mathbf{v}^1 \in \mathbf{L}^2(\Omega)$ and $p^0, p^1 \in L^2_0(\Omega)$ given. Find $(\mathbf{v}^{n+1}, p^{n+1})$ such that:

- **Vector penalty-prediction step with an augmentation parameter** $r \geq 0$:

$$\begin{aligned}
\frac{3\widetilde{\mathbf{v}}^{n+1} - 4\widetilde{\mathbf{v}}^n + \widetilde{\mathbf{v}}^{n-1}}{2\delta t} + NLT_1 - \mu\,\Delta\widetilde{\mathbf{v}}^{n+1} \quad &- \quad r\,\nabla(\nabla\cdot\widetilde{\mathbf{v}}^{n+1}) \\
&+ \quad \nabla p^{\star,n+1} = \mathbf{f}^{n+1} \text{ in } \Omega,
\end{aligned} \tag{2.1}$$

$$\widetilde{\mathbf{v}}^{n+1} = 0 \ \text{ on } \Gamma, \tag{2.2}$$

where $p^{\star,n+1}$ is the second-order Richardson extrapolation for p^{n+1}:

$$p^{\star,n+1} = 2p^n - p^{n-1},$$

and NLT_1 is the second-order extrapolated nonlinear term:

$$NLT_1 = 2(\mathbf{v}^n\cdot\nabla)\widetilde{\mathbf{v}}^n - (\mathbf{v}^{n-1}\cdot\nabla)\widetilde{\mathbf{v}}^{n-1}.$$

- **Vector penalty-projection step with a penalty parameter** $0 < \varepsilon \leq 1$:

$$\begin{aligned}(2.3)\quad \frac{3\widehat{\mathbf{v}}^{n+1} - 4\widehat{\mathbf{v}}^{n} + \widehat{\mathbf{v}}^{n-1}}{2\delta t} + NLT_2 - \mu\,\Delta\widehat{\mathbf{v}}^{n+1} &- \frac{1}{\varepsilon}\nabla\left(\nabla\cdot\widehat{\mathbf{v}}^{n+1}\right) \\ &= \frac{1}{\varepsilon}\nabla\left(\nabla\cdot\widetilde{\mathbf{v}}^{n+1}\right) \quad \text{in } \Omega,\end{aligned}$$

$$(2.4)\qquad \widehat{\mathbf{v}}^{n+1} = 0 \quad \text{on } \Gamma,$$

where NLT_2 is the second-order extrapolated nonlinear term:

$$NLT_2 = 2(\mathbf{v}^n\cdot\nabla)\widehat{\mathbf{v}}^n - (\mathbf{v}^{n-1}\cdot\nabla)\widehat{\mathbf{v}}^{n-1}.$$

- **Correction step for velocity and pressure**:

$$(2.5)\qquad \mathbf{v}^{n+1} = \widetilde{\mathbf{v}}^{n+1} + \widehat{\mathbf{v}}^{n+1},$$

$$(2.6)\qquad p^{n+1} = 2p^n - p^{n-1} - \frac{1}{\varepsilon}\left(\nabla\cdot\mathbf{v}^{n+1}\right) - r\,\nabla\cdot\widetilde{\mathbf{v}}^{n+1}.$$

Remark 2.1 (**Nonlinear term in the projection step**). It is useful to mention that the nonlinear term in the velocity correction step can be omitted since the purpose of this step is to perform an approximate divergence-free projection; see [5, 7]. Hence, we can take $NLT_2 = 0$ and consequently replace the nonlinear term NLT_1 in the prediction step by

$$NLT_1 = 2(\mathbf{v}^n\cdot\nabla)\mathbf{v}^n - (\mathbf{v}^{n-1}\cdot\nabla)\mathbf{v}^{n-1},$$

which is better for the consistency of the scheme.

2.2. **Vector penalty-prediction step.** Contrary to the first penalty-projection method introduced by Shen in [40], the augmentation parameter r in the prediction step of the VPP method is totally independent from the time step δt as it is also the case of the scalar penalty-projection method presented by Jobelin et al. in [30]. It is useful to note that the augmentation parameter r plays the role of a preconditioner for the prediction step. Indeed, the parameter r is kept constant (r can be strictly positive or equal 0) and within small values ($r \leq 1$) to avoid to degrade too severely the conditioning of the linear system associated with the prediction step.

Remark 2.2. From a numerical point of view, we observe that for $r = 0$, there is a poor convergence in time for velocity and pressure with very small values of ε and this is due to the accumulation of the round-off errors when ε is relatively small [9]. Hence, in order to improve the convergence rate, it was proposed in [1, 4, 6] to reconstruct the pressure field from its gradient to avoid the effect of round-off errors when ε is very small. Thus, in the numerical experiments (see Section 5) with $r = 0$, the following estimation of the gradient of the pressure will be used directly for the pressure gradient correction:

$$(2.7)\qquad \nabla p^{n+1} = 2\nabla p^n - \nabla p^{n-1} - \frac{3\widehat{\mathbf{v}}^{n+1} - 4\widehat{\mathbf{v}}^{n} + \widehat{\mathbf{v}}^{n-1}}{2\delta t} + \mu\,\Delta\widehat{\mathbf{v}}^{n+1}.$$

2.3. **Vector penalty-projection step.** The vector penalty-projection step is based on the Helmholtz-Hodge decompositions of $\mathbf{L}^2(\Omega)$ vector fields for bounded domains (see e.g. [34, 35, 44]). Besides, we notice that the vector penalty-projection step can be written as follows:

$$
\left\{
\begin{array}{ll}
\varepsilon \left(\dfrac{\widehat{\mathbf{v}}^{n+1} - \widehat{\mathbf{v}}^{n}}{\delta t} + NLT_2 - \mu \, \Delta \widehat{\mathbf{v}}^{n+1} \right) & \\
\qquad\qquad - \nabla(\nabla \cdot \widehat{\mathbf{v}}^{n+1}) = \nabla(\nabla \cdot \widetilde{\mathbf{v}}^{n+1}) & \text{in } \Omega, \\
\widehat{\mathbf{v}}^{n+1} = 0 & \text{on } \Gamma,
\end{array}
\right. \tag{2.8}
$$

where we use the implicit Euler scheme to discretize in time for the sake of simplicity. Formally speaking, as ε is taken small enough, the right-hand side in the projection step lies in the range of the left-hand side. Hence, the vector penalty-projection step appears to be very fast and cheap in terms of the number of iterations whatever the spatial mesh size is. This crucial result was already shown theoretically in [6, Theorem 1.1 and Corollary 1.3] and in [5, Theorem 3.1] and also numerically confirmed in [5, 6, 9, 14]. Finally, the vector correction step (2.3)-(2.5)) carries out an approximate divergence-free projection of the velocity with the penalty parameter $\varepsilon > 0$ chosen as small as desired.

Remark 2.3 (**Vector penalty-projection methods with variable density**). The vector penalty-projection methods can be generalized in a natural way for variable density as done recently in [2] where it is shown that the velocity correction step can be made completely independent on the mass density. Thus, this step is fully kinematic and only concerned with the Helmholtz-Hodge decomposition of the predicted velocity.

Remark 2.4 (**Vector penalty-projection method with open boundary conditions**). The vector penalty-projection methods can naturally be extended also to the case of incompressible viscous flows with open boundary conditions. In fact, in [8, 9], the authors described in detail the VPP methods in this case and showed that for a second-order scheme used for time discretization, the VPP methods yield approximately $\mathcal{O}(\delta t^2)$ for both the velocity and the pressure for the homogeneous as well as and nonhomogeneous open boundary conditions.

3. Well-posedness and stability

Before starting the analysis, let us note that since the treatment of the nonlinear term does not affect in an essential way the analysis of the vector penalty-projection method, we shall carry out the well-posedness, the stability and later, the error estimates for the linearized Navier-Stokes equations only as in [25, 26], thus avoiding technicalities associated with the nonlinearities which obscure the essential difficulties. Moreover, we suppose that the temporal derivative of the velocity is approximated by a second-order scheme in time, the pressure field is approximated by a first-order scheme in time, i.e, $p^{\star,n+1} = p^n$ and the augmentation parameter r is set to 0 for the sake of simplicity.

Thus, to fix the ideas, the vector penalty-projection method is written now as follows. For given $\widetilde{\mathbf{v}}^0, \widetilde{\mathbf{v}}^1, \mathbf{v}^0, \mathbf{v}^1$ and p^1, we are looking for $(\mathbf{v}^{n+1},\, p^{n+1})$ such that for all $n \geq 1$ with $(n+1)\delta t \leq T$:

- **Vector penalty-prediction step**:

$$
(3.1)\qquad \begin{cases} \dfrac{3\widetilde{\mathbf{v}}^{n+1} - 4\widetilde{\mathbf{v}}^{n} + \widetilde{\mathbf{v}}^{n-1}}{2\delta t} - \mu\,\Delta\widetilde{\mathbf{v}}^{n+1} + \nabla p^{n} = \mathbf{f}^{n+1} & \text{in } \Omega,\\ \widetilde{\mathbf{v}}^{n+1} = 0 & \text{on } \Gamma. \end{cases}
$$

- **Vector penalty-projection step with a penalty parameter** $0 < \varepsilon \leq 1$:

$$
(3.2)\qquad \begin{cases} \dfrac{3\widehat{\mathbf{v}}^{n+1} - 4\widehat{\mathbf{v}}^{n} + \widehat{\mathbf{v}}^{n-1}}{2\delta t} - \mu\,\Delta\widehat{\mathbf{v}}^{n+1} - \dfrac{1}{\varepsilon}\nabla\left(\nabla\cdot\widehat{\mathbf{v}}^{n+1}\right)\\ \qquad\qquad = \dfrac{1}{\varepsilon}\nabla\left(\nabla\cdot\widetilde{\mathbf{v}}^{n+1}\right) \quad \text{in } \Omega,\\ \qquad\qquad \widehat{\mathbf{v}}^{n+1} = 0 \quad \text{on } \Gamma. \end{cases}
$$

- **Correction step for the velocity and the pressure**:

$$
(3.3)\qquad \begin{cases} \mathbf{v}^{n+1} = \widetilde{\mathbf{v}}^{n+1} + \widehat{\mathbf{v}}^{n+1},\\ p^{n+1} = p^{n} - \dfrac{1}{\varepsilon}\,(\nabla\cdot\mathbf{v}^{n+1}). \end{cases}
$$

Finally, the discrete problem resulting from the sum of the two steps, taking into account (3.3), becomes

$$
(3.4)\qquad \frac{3\mathbf{v}^{n+1} - 4\mathbf{v}^{n} + \mathbf{v}^{n-1}}{2\delta t} - \mu\,\Delta\mathbf{v}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1} \qquad \text{in } \Omega\times]0,T[,
$$

$$
(3.5)\qquad (\varepsilon\delta t)\frac{p^{n+1} - p^{n}}{\delta t} + \nabla\cdot\mathbf{v}^{n+1} = 0 \qquad \text{in } \Omega\times]0,T[,
$$

$$
(3.6)\qquad \mathbf{v}^{n+1} = 0 \qquad \text{on } \Gamma\times]0,T[.
$$

Remark 3.1. The initial condition on the velocity is $\mathbf{v}^0 = \mathbf{v}_0$ with $\widetilde{\mathbf{v}}^0 = \mathbf{v}^0 = \mathbf{v}_0$ and $\widehat{\mathbf{v}}^0 = 0$. To start the second-order VPP scheme, we need $\mathbf{v}^1$ and p^1. For this reason, we first solve the VPP method using Euler scheme of first-order for a given $\mathbf{v}^0$ instead of the BDF2 scheme. This permits us to calculate $\widetilde{\mathbf{v}}^1$ and $\widehat{\mathbf{v}}^1$ and consequently to find $\mathbf{v}^1$ and p^1.

3.1. **Well-posedness of the scheme.**

Lemma 1 (**Well-posedness of the prediction step**). *For given $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\widetilde{\mathbf{v}}^0$, $\widetilde{\mathbf{v}}^1 \in \mathbf{L}^2(\Omega)$, $p^1 \in L^2_0(\Omega)$ given, and for all $\delta t > 0$, there exists at each time step a unique solution $\widetilde{\mathbf{v}}^{n+1} \in \mathbf{H}^1_0(\Omega)$ to the penalty-prediction step* (3.1).

Sketch of the proof. We take first the inner product of (3.1) with a test function $\boldsymbol{\varphi}$ in $\mathbf{H}^1_0(\Omega)$.

It is an easy matter to prove with the Lax-Milgram theorem that there exists a unique solution $\widetilde{\mathbf{v}}^{n+1}$ to the prediction step (3.1) in the Hilbert space $\mathbf{H}^1_0(\Omega)$. □

Lemma 2 (**Well-posedness of the projection step**). *For given $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\widetilde{\mathbf{v}}^{n+1} \in \mathbf{H}^1_0(\Omega)$, with $0 < \varepsilon \leq 1$ and $\delta t > 0$, there exists at each time step a unique solution $\widehat{\mathbf{v}}^{n+1} \in \mathbf{H}^1_0(\Omega)$ to the penalty-projection step* (3.2).

Sketch of the proof. We take the inner product of (3.2) with a test function $\boldsymbol{\varphi} \in \mathbf{H}^1_0(\Omega)$.

Thanks to the Lax-Milgram theorem, it is an easy matter to show that the projection step (3.2) has a unique solution $\widehat{\mathbf{v}}^{n+1}$ in the Hilbert space $\mathbf{H}_0^1(\Omega)$. $\square$

Lemma 3 (**Global solvability of the** VPP **method**). *For given $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\mathbf{v}^0$, $\mathbf{v}^1 \in \mathbf{L}^2(\Omega)$ and $p^1 \in L_0^2(\Omega)$, for all $0 < \delta t \leq T$, $0 < \varepsilon \leq 1$ and for all $n \in \mathbb{N}^*$ such that $(n+1)\delta t \leq T$, there exists a unique solution $(\widetilde{\mathbf{v}}^{n+1}, \mathbf{v}^{n+1}, p^{n+1})$ in $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ to the VPP scheme such that:*

$$\frac{3\,\mathbf{v}^{n+1} - 4\,\mathbf{v}^n + \mathbf{v}^{n-1}}{2\,\delta t} - \mu\,\Delta \mathbf{v}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1} \quad \text{in } \Omega, \tag{3.7}$$

$$\varepsilon(p^{n+1} - p^n) + \nabla \cdot \mathbf{v}^{n+1} = 0 \quad \text{in } \Omega. \tag{3.8}$$

Proof. The proof is made by induction for all n in $\mathbb{N}^*$ such that $(n+1)\delta t \leq T$ starting with the given initial conditions $\mathbf{v}^0$, $\mathbf{v}^1$ in $L^2(\Omega)^d$ and p^1 in $L_0^2(\Omega)$. Thanks to Lemma 1, there exists a unique solution $\widetilde{\mathbf{v}}^{n+1}$ in $\mathbf{H}_0^1(\Omega)$ to the prediction step (3.1). Moveover, according to Lemma 2, there exists a unique solution $\widehat{\mathbf{v}}^{n+1}$ in $\mathbf{H}_0^1(\Omega)$ to the projection step (3.2). Hence, we deduce that $\mathbf{v}^{n+1} = \widetilde{\mathbf{v}}^{n+1} + \widehat{\mathbf{v}}^{n+1} \in \mathbf{H}_0^1(\Omega)$ with $\nabla \cdot \mathbf{v}^{n+1} \in L_0^2(\Omega)$.

Finally, since p^1, $\nabla \cdot \mathbf{v}^{n+1}$ and $\nabla \cdot \widetilde{\mathbf{v}}^{n+1} \in L_0^2(\Omega)$, it is easy to verify using the expression of the pressure (3.3), that $p^{n+1} \in L_0^2(\Omega)$, which concludes the proof. $\square$

We are now in position to establish the stability of the scheme.

3.2. Stability.

Theorem 3.2 (**Stability of the scheme**). *For given $\boldsymbol{f} \in L^2(0,T; L^2(\Omega)^d)$, $\mathbf{v}^0, \mathbf{v}^1 \in L^2(\Omega)^d$ and $p^1 \in L_0^2(\Omega)$, there exists a positive constant*

$$C_0 = C_0(\Omega, T, \mu, ||\mathbf{f}||_{L^2((0,T)\times\Omega)}, ||\mathbf{v}^0||_0, ||\mathbf{v}^1||_0, ||p^1||_0)$$

such that, for all $0 < \delta t \leq T$, for $0 < \varepsilon \leq 1$, the solution $(\mathbf{v}^{n+1}, p^{n+1})$ of the VPP method satisfies the following estimation for all $n \in \mathbb{N}^*$ with $(n+1)\,\delta t \leq T$:

$$\begin{aligned} ||\mathbf{v}^{n+1}||_0^2 \quad &+ \quad ||2\,\mathbf{v}^{n+1} - \mathbf{v}^n||_0^2 + \sum_{k=1}^{n} ||\delta^2 \mathbf{v}^{k+1}||_0^2 + 2\,\mu \sum_{k=1}^{n} \delta t ||\nabla \mathbf{v}^{k+1}||_0^2 \\ &+ \quad 2\,\varepsilon\,\delta t ||p^{n+1}||_0^2 + 2\,\varepsilon \sum_{k=1}^{n} \delta t ||p^{k+1} - p^k||_0^2 \\ &\leq C_0. \end{aligned}$$

Proof.

Step 1: Taking the inner product of (3.4) with $4\delta t\,\mathbf{v}^{n+1}$, applying Green's formula and using the algebraic relation

$$\begin{aligned} 2(a^{k+1}, 3\,a^{k+1} - 4\,a^k + a^{k-1}) = ||a^{k+1}||^2 \quad &+ \quad ||2\,a^{k+1} - a^k||^2 - ||a^k||^2 \\ &- \quad ||2\,a^k - a^{k-1}||^2 + ||\delta^2 a^{k+1}||^2, \end{aligned} \tag{3.9}$$

with $\delta^2 a^{k+1} = \delta(\delta a^{k+1})$ and $\delta a^{k+1} = a^{k+1} - a^k$, we obtain, taking into account the fact that $\mathbf{v}^{n+1} = 0$ on the whole boundary Γ,

$$\begin{aligned} ||\mathbf{v}^{n+1}||_0^2 - ||\mathbf{v}^n||_0^2 + ||2\,\mathbf{v}^{n+1} - \mathbf{v}^n||_0^2 - ||2\,\mathbf{v}^n - \mathbf{v}^{n-1}||_0^2 + ||\delta^2\mathbf{v}^{n+1}||_0^2 \\ + 4\,\mu\,\delta t\,||\nabla\mathbf{v}^{n+1}||_0^2 - 4\,\delta t\,(p^{n+1}, \nabla\cdot\mathbf{v}^{n+1})_0 \\ = 4\,\delta t\,(\mathbf{f}^{n+1}, \mathbf{v}^{n+1})_0. \end{aligned} \tag{3.10}$$

Step 2: Taking the inner product of (3.5) with $4\delta t p^{n+1}$ and using the algebraic identity

$$2(a^{k+1}, a^{k+1} - b^{k+1}) = ||a^{k+1}||^2 - ||b^{k+1}||^2 + ||a^{k+1} - b^{k+1}||^2, \tag{3.11}$$

one gets:

$$2\,\varepsilon\,\delta t\,\left(||p^{n+1}||_0^2 - ||p^n||_0^2 + ||p^{n+1} - p^n||_0^2\right) + 4\,\delta t\,(p^{n+1}, \nabla\cdot\mathbf{v}^{n+1})_0 = 0. \tag{3.12}$$

Step 3: Summing (3.10) and (3.12), we obtain the following relation:

$$\begin{aligned} ||\mathbf{v}^{n+1}||_0^2 - ||\mathbf{v}^n||_0^2 + ||2\,\mathbf{v}^{n+1} - \mathbf{v}^n||_0^2 - ||2\,\mathbf{v}^n - \mathbf{v}^{n-1}||_0^2 + ||\delta^2\mathbf{v}^{n+1}||_0^2 \\ + 4\,\mu\,\delta t\,||\nabla\mathbf{v}^{n+1}||_0^2 + 2\,\varepsilon\,\delta t\,\left(||p^{n+1}||_0^2 - ||p^n||_0^2 + ||p^{n+1} - p^n||_0^2\right) \\ = 4\,\delta t(\mathbf{f}^{n+1}, \mathbf{v}^{n+1})_0. \end{aligned} \tag{3.13}$$

The term in the right-hand side of the inequality above, which we denote by $T1$, may be bounded using the Cauchy-Schwarz inequality, Young inequality and Poincaré inequality. We shall repeatedly use this standard trick hereafter without mentioning it again:

$$\begin{aligned} |T1| \quad &\leq \quad 4\,\delta t\,||\mathbf{v}^{n+1}||_0 ||\mathbf{f}^{n+1}||_0 \\ &\leq \quad 2\,\mu\,\delta t\,||\nabla\mathbf{v}^{n+1}||_0^2 + \frac{2\,c_p(\Omega)^2}{\mu}\,\delta t||\mathbf{f}^{n+1}||_0^2, \end{aligned}$$

where $c_p(\Omega)$ is the Poincaré constant.

Using this bound in (3.13), we get

$$\begin{aligned} ||\mathbf{v}^{n+1}||_0^2 - ||\mathbf{v}^n||_0^2 + ||2\,\mathbf{v}^{n+1} - \mathbf{v}^n||_0^2 - ||2\,\mathbf{v}^n - \mathbf{v}^{n-1}||_0^2 + ||\delta^2\mathbf{v}^{n+1}||_0^2 \\ + 2\,\mu\,\delta t\,||\nabla\mathbf{v}^{n+1}||_0^2 + 2\,\varepsilon\,\delta t\,\left(||p^{n+1}||_0^2 - ||p^n||_0^2 + ||p^{n+1} - p^n||_0^2\right) \\ \leq \frac{2\,c_p(\Omega)^2}{\mu}\,\delta t||\mathbf{f}^{n+1}||_0^2. \end{aligned}$$

We now write the previous inequality with the index k instead of n, and then sum it up for $k = 1, \ldots, n$. This yields, for all $0 < \varepsilon \leq 1$ and $0 < \delta t \leq T$, the following energy estimate for all $n \in \mathbb{N}^*$ with $(n+1)\,\delta t \le T$:

$$\begin{aligned}
&||\mathbf{v}^{n+1}||_0^2 + ||2\,\mathbf{v}^{n+1} - \mathbf{v}^n||_0^2 + \sum_{k=1}^{n} ||\delta^2 \mathbf{v}^{k+1}||_0^2 + 2\,\mu \sum_{k=1}^{n} \delta t ||\nabla \mathbf{v}^{k+1}||_0^2 \\
&\qquad + 2\,\varepsilon\,\delta t ||p^{n+1}||_0^2 + 2\,\varepsilon \sum_{k=1}^{n} \delta t\, ||p^{k+1} - p^k||_0^2 \\
&\le ||\mathbf{v}^1||_0^2 + ||2\,\mathbf{v}^1 - \mathbf{v}^0||_0^2 + 2\,\varepsilon\,\delta t\,||p^1||_0^2 + \frac{2\,c_p(\Omega)^2}{\mu} \sum_{k=1}^{n} \delta t\, ||\mathbf{f}^{k+1}||_0^2 \\
&\le C_0(\Omega, T, \mu, ||\mathbf{f}||_{L^2((0,T)\times\Omega)}, ||\mathbf{v}^0||_0, ||\mathbf{v}^1||_0, ||p^1||_0),
\end{aligned}$$

which concludes the proof. □

Remark 3.3. The vector penalty-projection method is also stable for $r > 0$. We note that the VPP method with $r \ge 0$ was also studied in [3, 6] using a first-order scheme in time.

4. Error estimates

4.1. **Notations and assumptions.** Let $\mathbf{v}(t^{n+1}) = \overline{\mathbf{v}}^{n+1}$ and $p(t^{n+1}) = \overline{p}^{n+1}$ the exact solution of the Stokes problem at time t^{n+1} and let $\mathbf{v}^{n+1}$ and p^{n+1} the solution obtained by the vector penalty-projection method (3.1)–(3.3). Then, we define the velocity and the pressure error respectively:

$$\begin{aligned}
\mathbf{e}^{n+1} &= \overline{\mathbf{v}}^{n+1} - \mathbf{v}^{n+1} = \mathbf{v}(t^{n+1}) - \mathbf{v}^{n+1}, \\
\pi^{n+1} &= \overline{p}^{n+1} - p^{n+1} = p(t^{n+1}) - p^{n+1}.
\end{aligned}$$

In addition, we assume that the solution of the continuous Stokes problem satisfies the following regularity conditions:

$$\int_0^T \left\| \frac{\partial^3 \mathbf{v}}{\partial t^3} \right\|_0^2 dt \le M, \tag{4.1}$$

and

$$\int_0^T \left\| \frac{\partial p}{\partial t} \right\|_0^2 dt \le M. \tag{4.2}$$

We will use M as a generic positive constant which depends eventually on Ω, T, $\mathbf{f}$, μ and $\mathbf{v}_0$.

Moreover, we need to assume that the initial errors are sufficiently controlled, i.e., there exists a constant $c' > 0$ such that

$$||\mathbf{e}^1||_0^2 + ||2\,\mathbf{e}^1 - \mathbf{e}^0||_0^2 + 2\,\varepsilon\,\delta t\,||\pi^1||_0^2 \le c'^4 \tag{4.3}$$

and

$$\mu\,\delta t ||\nabla \mathbf{e}^1||_0^2 \le c'^4. \tag{4.4}$$

Finally, we define $\mathbf{R}^{n+1}$ as

$$\mathbf{R}^{n+1} = \frac{3\mathbf{v}(t^{n+1}) - 4\mathbf{v}(t^n) + \mathbf{v}(t^{n-1})}{2\delta t} - \frac{\partial \mathbf{v}(t^{n+1})}{\partial t}.$$

Finally, throughout this paper, we will make use of the following theorems whose proofs can be found in [11].

Theorem 4.1. *Let Ω be an open, bounded, and Lipshitz domain of $\mathbb{R}^d$. There exists a constant $c_1 > 0$ such that for all $u \in L^2(\Omega)$,*

$$||u||_0 \leq c_1 \left(||u||_{-1} + ||\nabla u||_{-1}\right). \tag{4.5}$$

Theorem 4.2. *Let Ω be an open, bounded, connected, lipshitz domain of $\mathbb{R}^d$. There exists a constant $c_2 > 0$ such that, for all $u \in L^2(\Omega)$,*

$$||u||_{-1} \leq c_2 \left(||\nabla u||_{-1} + \frac{1}{|\Omega|}\left|\int_\Omega u\,dx\right|\right). \tag{4.6}$$

4.2. **Basic error estimates.**

Lemma 4. *For given $\boldsymbol{f}$ and $\mathbf{v}_0$ which are smooth enough and assuming that the solution $(\mathbf{v}, p)$ of the Stokes problem is smooth enough in space and time such that $\boldsymbol{v} \in \mathbf{W}^{3,2}(0,T;\mathbf{L}^2(\Omega))$ and $p \in W^{1,2}(0,T;L^2(\Omega))$, then, there exists a constant $M(\Omega,T,\mu,\mathbf{f},\mathbf{v}_0) > 0$ such that the following estimations are satisfied for all $n \in \mathbb{N}^*$ with $(n+1)\,\delta t \leq T$:*

$$\text{(a)} \quad \sum_{k=1}^{n} \delta t||\mathbf{R}^{k+1}||_0^2 \leq M(\Omega,T,\mu,\mathbf{f},\mathbf{v}_0)\,\delta t^4,$$

$$\text{(b)} \quad \sum_{k=1}^{n} \delta t||\delta\overline{p}^{k+1}||_0^2 \leq M(\Omega,T,\mu,\mathbf{f},\mathbf{v}_0)\,\delta t^2,$$

$$\text{(c)} \quad \sum_{k=1}^{n} \delta t\left\|\frac{\delta\overline{p}^{k+1}}{\delta t}\right\|_0^2 \leq M(\Omega,T,\mu,\mathbf{f},\mathbf{v}_0).$$

Sketch of the Proof. Note that we can reformulate $\mathbf{R}^{k+1}$ as a residual integral of Taylor series

$$\begin{aligned}\mathbf{R}^{k+1} &= \frac{3\mathbf{v}(t^{k+1}) - 4\mathbf{v}(t^k) + \mathbf{v}(t^{k-1})}{2\delta t} - \frac{\partial \mathbf{v}(t^{k+1})}{\partial t}\\ &= \frac{4}{2\,\delta t}\int_{t^k}^{t^{k+1}} \frac{(t^k-t)^2}{2}\frac{\partial^3\mathbf{v}(t)}{\partial t^3}dt - \frac{1}{2\,\delta t}\int_{t^{k-1}}^{t^{k+1}} \frac{(t^{k-1}-t)^2}{2}\frac{\partial^3\mathbf{v}(t)}{\partial t^3}dt.\end{aligned}$$

Then, the proof of (a) can be concluded as done in [18].

For the proof of (b), we proceed as follows:

$$\delta\overline{p}^{k+1} = p(t^{k+1}) - p(t^k) = \int_{t^k}^{t^{k+1}} \frac{\partial p(t)}{\partial t}dt,$$

then

$$||\delta\overline{p}^{k+1}||_0^2 \leq \delta t\int_{t^k}^{t^{k+1}} \left\|\frac{\partial p(t)}{\partial t}\right\|_0^2 dt.$$

We obtain, after summing it for $k = 1,\ldots,n$ and using (4.2),

$$\sum_{k=1}^{n} \delta t||\delta\overline{p}^{k+1}||_0^2 \leq \delta t^2\int_0^T \left\|\frac{\partial p(t)}{\partial t}\right\|_0^2 dt \leq M\,\delta t^2,$$

which concludes the proof of (b).

The inequality (c) is a direct consequence of (b). □

Theorem 4.3. *Under Lemma* 4 *and the assumption* (4.3) *and for all* $0 < \varepsilon \leq 1$, $0 < \delta t \leq \max(1, T)$, *there exists a positive constant*

$$C_0 = C_0(\Omega, T, \mu, ||\mathbf{f}||_{L^2((0,T)\times\Omega)}, ||\mathbf{e}^0||_0, ||\mathbf{e}^1||_0, ||\pi^1||_0)$$

such that the solution of the VPP method (3.1)–(3.3) *verifies for all* $n \geq 1$ *such that* $(n+1)\,\delta t \leq T$, *the following:*

$$\begin{aligned}
&\text{(i)} \quad ||\mathbf{e}^{n+1}||_0^2 + ||2\,\mathbf{e}^{n+1} - \mathbf{e}^n||_0^2 + \sum_{k=1}^{n} ||\delta^2 \mathbf{e}^{k+1}||_0^2 + 2\,\mu \sum_{k=1}^{n} \delta t ||\nabla \mathbf{e}^{k+1}||_0^2 \\
&\qquad + 2\,\varepsilon\,\delta t ||\pi^{n+1}||_0^2 + \varepsilon \sum_{k=1}^{n} \delta t ||\pi^{k+1} - \pi^k||_0^2 \leq C_0\,(\delta t^4 + \varepsilon\,\delta t),
\end{aligned}$$

$$\text{(ii)} \quad \sum_{k=1}^{n} \delta t ||\nabla \cdot \mathbf{e}^{k+1}||_0^2 \leq C_0\,(\delta t^3 + \varepsilon)\,\varepsilon\,\delta t.$$

Proof. (i) *Error estimate for the velocity.*

Step 1: We have for the Stokes equations at time t^{n+1}:

$$\frac{3\mathbf{v}(t^{n+1}) - 4\mathbf{v}(t^n) + \mathbf{v}(t^{n-1})}{2\delta t} - \mu\,\Delta \mathbf{v}(t^{n+1}) + \nabla p(t^{n+1}) = \mathbf{f}(t^{n+1}) + \mathbf{R}^{n+1}, \tag{4.7}$$

with

$$\mathbf{R}^{n+1} = \frac{3\mathbf{v}(t^{n+1}) - 4\mathbf{v}(t^n) + \mathbf{v}(t^{n-1})}{2\delta t} - \frac{\partial \mathbf{v}(t^{n+1})}{\partial t}.$$

By subtracting (3.4) from (4.7), we get the following error equation:

$$\frac{3\,\mathbf{e}^{n+1} - 4\,\mathbf{e}^n + \mathbf{e}^{n-1}}{2\delta t} - \mu\,\Delta \mathbf{e}^{n+1} + \nabla \pi^{n+1} = \mathbf{R}^{n+1}. \tag{4.8}$$

Taking the inner product of (4.8) with $4\delta t\,\mathbf{e}^{n+1}$ and taking into account that $\mathbf{e}^{n+1} = 0$ on Γ, we obtain:

$$\begin{aligned}
&||\mathbf{e}^{n+1}||_0^2 - ||\mathbf{e}^n||_0^2 + ||2\,\mathbf{e}^{n+1} - \mathbf{e}^n||_0^2 \\
&\quad - ||2\,\mathbf{e}^n - \mathbf{e}^{n-1}||_0^2 + ||\delta^2 \mathbf{e}^{n+1}||_0^2 + 4\,\mu\,\delta t\,||\nabla \mathbf{e}^{n+1}||_0^2 \\
&\quad - 4\,\delta t(\pi^{n+1}, \nabla \cdot \mathbf{e}^{n+1})_0 = 4\,\delta t(\mathbf{R}^{n+1}, \mathbf{e}^{n+1})_0.
\end{aligned} \tag{4.9}$$

Step 2: By adding $+\varepsilon\,p(t^{n+1})$ and $-\varepsilon\,p(t^{n+1})$ to the pressure equation (3.5) and by adding $+\varepsilon\,p(t^n)$ and $-\varepsilon\,p(t^n)$ to (3.5), we get

$$\varepsilon(\pi^{n+1} - \pi^n) + \nabla \cdot \mathbf{e}^{n+1} = \varepsilon\,\delta \overline{p}^{n+1}, \tag{4.10}$$

where $\nabla \cdot \mathbf{v}^{n+1} = -\nabla \cdot \mathbf{e}^{n+1}$ and $\delta \overline{p}^{n+1} = \overline{p}^{n+1} - \overline{p}^n$.

Taking the inner product of (4.10) with $4\delta t\,\pi^{n+1}$ and using the identity (3.11), we obtain:

$$\begin{aligned}
&2\,\varepsilon\,\delta t\,\left(||\pi^{n+1}||_0^2 - ||\pi^n||_0^2 + ||\pi^{n+1} - \pi^n||_0^2\right) + 4\,\delta t(\pi^{n+1}, \nabla \cdot \mathbf{e}^{n+1})_0 \\
&\qquad = 4\,\varepsilon\,\delta t\,(\pi^{n+1}, \delta \overline{p}^{n+1})_0.
\end{aligned} \tag{4.11}$$

Step 3: Summing (4.9) and (4.11) and writing π^{n+1} in the right-hand side of (4.11) as $\pi^{n+1} = \pi^n + (\pi^{n+1} - \pi^n)$, we obtain

$$\begin{aligned}
&||\mathbf{e}^{n+1}||_0^2 - ||\mathbf{e}^n||_0^2 + ||2\,\mathbf{e}^{n+1} - \mathbf{e}^n||_0^2 - ||2\,\mathbf{e}^n - \mathbf{e}^{n-1}||_0^2 + ||\delta^2 \mathbf{e}^{n+1}||_0^2 \\
&\quad + 4\,\mu\,\delta t ||\nabla \mathbf{e}^{n+1}||_0^2 + 2\,\varepsilon\,\delta t\,\left(||\pi^{n+1}||_0^2 - ||\pi^n||_0^2 + ||\pi^{n+1} - \pi^n||_0^2\right) \\
&= 4\,\varepsilon\,\delta t(\pi^n, \delta \overline{p}^{n+1})_0 + 4\,\varepsilon\,\delta t(\pi^{n+1} - \pi^n, \delta \overline{p}^{n+1})_0 + 4\,\delta t\,(\mathbf{R}^{n+1}, \mathbf{e}^{n+1})_0,
\end{aligned} \tag{4.12}$$

where the terms in the right-hand side of (4.12) are denoted, respectively, by $T1$, $T2$, $T3$, so that

$$|T1| \leq 4\,\varepsilon\,\delta t||\pi^n||_0||\delta\overline{p}^{n+1}||_0 \leq 2\,\varepsilon\,\delta t^2||\pi^n||_0^2 + 2\,\varepsilon\,\delta t^2 \left\| \frac{\delta\overline{p}^{n+1}}{\delta t} \right\|_0^2,$$

$$|T2| \leq 4\,\varepsilon\,\delta t\,||\pi^{n+1} - \pi^n||_0||\delta\overline{p}^{n+1}||_0 \leq \varepsilon\,\delta t\,||\pi^{n+1} - \pi^n||_0^2 + 4\,\varepsilon\,\delta t^3 \left\| \frac{\delta\overline{p}^{n+1}}{\delta t} \right\|_0^2,$$

$$|T3| \leq 4\,\delta t||\mathbf{R}^{n+1}||_0||\mathbf{e}^{n+1}||_0 \leq \frac{2\,c_p(\Omega)^2}{\mu}\,\delta t||\mathbf{R}^{n+1}||_0^2 + 2\,\mu\,\delta t||\nabla\mathbf{e}^{n+1}||_0^2$$

with $c_p(\Omega)$ the constant of Poincaré.

Step 4: Combining the bounds obtained above with (4.12) and replacing the index n by k and then summing for $k = 0, \ldots, n$, yield the following energy estimate for all $n \in \mathbb{N}^*$ such that $(n+1)\delta t \leq T$:

$$\begin{aligned}
&||\mathbf{e}^{n+1}||_0^2 + ||2\,\mathbf{e}^{n+1} - \mathbf{e}^n||_0^2 + \sum_{k=1}^{n} ||\delta^2\mathbf{e}^{k+1}||_0^2 + 2\,\mu \sum_{k=1}^{n} \delta t||\nabla\mathbf{e}^{k+1}||_0^2 \\
&\qquad + 2\,\varepsilon\,\delta t||\pi^{n+1}||_0^2 + \varepsilon \sum_{k=1}^{n} \delta t||\pi^{k+1} - \pi^k||_0^2 \\
&\quad \leq 2\,\varepsilon\,\delta t \sum_{k=1}^{n} \delta t||\pi^k||_0^2 + ||\mathbf{e}^1||_0^2 + ||2\,\mathbf{e}^1 - \mathbf{e}^0||_0^2 + 2\,\varepsilon\,\delta t||\pi^1||_0^2 \\
&\qquad + \frac{2\,c_p(\Omega)^2}{\mu} \sum_{k=1}^{n} \delta t||\mathbf{R}^{k+1}||_0^2 + (2\,\varepsilon\,\delta t + 4\,\varepsilon\,\delta t^2) \sum_{k=1}^{n} \delta t \left\| \frac{\delta\overline{p}^{k+1}}{\delta t} \right\|_0^2.
\end{aligned}$$

Finally, using assumption (4.3), Lemma 4, and applying the discrete Gronwall Lemma, we conclude the proof. □

(ii) *Error estimate for the velocity divergence.*

Based on (4.10), the velocity divergence error can be written as follows:

$$\nabla \cdot \mathbf{e}^{n+1} = \varepsilon\,\delta\overline{p}^{n+1} - \varepsilon\,(\pi^{n+1} - \pi^n).$$

Thanks to Lemma 4 and to part (i) of Theorem 4.3, we get the desired estimation for $\delta t \leq \max(1, T)$:

$$\begin{aligned}
\sum_{k=1}^{n} \delta t||\nabla \cdot \mathbf{e}^{k+1}||_0^2 &\leq 2\,M\,\varepsilon^2\,\delta t^2 + 2\,\varepsilon\,C_0\,(\delta t^4 + \varepsilon\,\delta t) \\
&\leq C_0(\Omega, T, \mu, ||\mathbf{f}||_{L^2((0,T)\times\Omega)}, ||\mathbf{e}^0||_0, ||\mathbf{e}^1||_0, ||\pi^1||_0)\,(\delta t^3 + \varepsilon)\,\varepsilon\,\delta t,
\end{aligned}$$

which concludes the proof. □

Remark 4.4 (**Convergence rate and splitting error of the velocity approximation**). Theorem 4.3 (part (i)) shows that the second-order vector penalty-projection method yields optimal error estimates in time for the velocity; particularly, we obtain a convergence rate of $\mathcal{O}(\delta t^2)$ for the velocity in $l^\infty(\mathbf{L}^2(\Omega))$. The same result for the velocity (in l^∞-norm) was already obtained in [18] with the second-order scalar penalty-projection method and also in [26, 27] with the incremental and rotational pressure-correction methods (in l^2-norm).

Indeed, the interesting result is that the velocity splitting error is of order $\mathcal{O}(\sqrt{\varepsilon\,\delta t})$ in $l^\infty(\mathbf{L}^2(\Omega)) \cap l^2(\mathbf{H}^1(\Omega))$. Practically speaking, if we choose the penalty parameter ε equal to δt^5, for example, the splitting error of the velocity will be of order $\mathcal{O}(\delta t^3)$ and hence, the second-order accuracy in time for the velocity is provided.

Now, in order to derive an error estimate for the pressure approximation, we need the following lemma.

Lemma 5.

$$\sum_{k=1}^{n} \left\|3\,\mathbf{e}^{k+1} - 4\,\mathbf{e}^k + \mathbf{e}^{k-1}\right\|_0^2 \\ \leq C_0(\Omega, T, \mu, ||\mathbf{f}||_{L^2((0,T)\times\Omega)}, \mathbf{e}^0, \mathbf{e}^1, ||\nabla \mathbf{e}^1||_0, ||\pi^1||_0)\,(\delta t^4 + \varepsilon\,\delta t).$$

Sketch of the Proof.

Step 1: After rewriting $3\,\mathbf{e}^{n+1} - 4\,\mathbf{e}^n + \mathbf{e}^{n-1}$ in (4.8) as:

$$\begin{aligned} 3\,\mathbf{e}^{n+1} - 4\,\mathbf{e}^n + \mathbf{e}^{n-1} &= (\mathbf{e}^{n+1} - 2\,\mathbf{e}^n + \mathbf{e}^{n-1}) + 2\,(\mathbf{e}^{n+1} - \mathbf{e}^n) \\ &= \delta^2\mathbf{e}^{n+1} + 2\,(\mathbf{e}^{n+1} - \mathbf{e}^n), \end{aligned} \tag{4.13}$$

we take the inner product of (4.8) with $2\,\delta t(\mathbf{e}^{n+1} - \mathbf{e}^n)$. We derive

$$\begin{aligned} &(\delta^2\mathbf{e}^{n+1}, \mathbf{e}^{n+1} - \mathbf{e}^n)_0 + 2\,||\mathbf{e}^{n+1} - \mathbf{e}^n||_0^2 \\ &\qquad + \mu\,\delta t(||\nabla\mathbf{e}^{n+1}||_0^2 - ||\nabla\mathbf{e}^n||_0^2 + ||\nabla(\mathbf{e}^{n+1} - \mathbf{e}^n)||_0^2) \\ &\qquad - 2\,\delta t(\pi^{n+1}, \nabla\cdot(\mathbf{e}^{n+1} - \mathbf{e}^n))_0 = 2\,\delta t(\mathbf{R}^{n+1}, \mathbf{e}^{n+1} - \mathbf{e}^n)_0. \end{aligned} \tag{4.14}$$

Step 2: Now, we write (4.10) at time t^{n+1} and t^n, respectively, and we subtract after that the resulted equations. Hence, it yields

$$\varepsilon\,\delta^2\pi^{n+1} + \nabla\cdot(\mathbf{e}^{n+1} - \mathbf{e}^n) = \varepsilon\,\delta^2\overline{p}^{n+1}, \quad \text{where } \delta^2\pi^{n+1} = \pi^{n+1} - 2\pi^n + \pi^{n-1}.$$

Taking the inner product of the previous equation with $2\,\delta t\,\pi^{n+1}$ and rewriting $\delta^2\pi^{n+1}$ as $\delta^2\pi^{n+1} = (3\,\pi^{n+1} - 4\,\pi^n + \pi^{n-1}) - 2(\pi^{n+1} - \pi^n)$, yields

$$\begin{aligned} &\varepsilon\,\delta t\left(||\pi^{n+1}||_0^2 - ||\pi^n||_0^2 + ||2\,\pi^{n+1} - \pi^n||_0^2 - ||2\,\pi^n - \pi^{n-1}||_0^2 + ||\delta^2\pi^{n+1}||_0^2\right) \\ &\qquad -2\,\varepsilon\,\delta t\,\left(||\pi^{n+1}||_0^2 - ||\pi^n||_0^2 + ||\pi^{n+1} - \pi^n||_0^2\right) \\ &\qquad +2\,\delta t(\pi^{n+1}, \nabla\cdot(\mathbf{e}^{n+1} - \mathbf{e}^n))_0 = 2\,\varepsilon\,\delta t\,\left(\pi^{n+1}, \delta^2\overline{p}^{n+1}\right)_0. \end{aligned} \tag{4.15}$$

Step 3: Summing (4.14) and (4.15), we find after writing π^{n+1} in the right-hand side of (4.15) as $\pi^{n+1} = \pi^n + (\pi^{n+1} - \pi^n)$:

$$\begin{aligned} &2\,||\mathbf{e}^{n+1} - \mathbf{e}^n||_0^2 + \mu\,\delta t\left(||\nabla\mathbf{e}^{n+1}||_0^2 - ||\nabla\mathbf{e}^n||_0^2 + ||\nabla(\mathbf{e}^{n+1} - \mathbf{e}^n)||_0^2\right) \\ &\qquad + \varepsilon\,\delta t(||\pi^{n+1}||_0^2 - ||\pi^n||_0^2) \\ &\qquad + \varepsilon\,\delta t\,\left(||2\,\pi^{n+1} - \pi^n||_0^2 - ||2\,\pi^n - \pi^{n-1}||_0^2 + ||\delta^2\pi^{n+1}||_0^2\right) \\ &\quad = 2\,\varepsilon\,\delta t\,(||\pi^{n+1}||_0^2 - ||\pi^n||_0^2) + 2\,\varepsilon\,\delta t\,||\pi^{n+1} - \pi^n||_0^2 \\ &\qquad + 2\,\delta t(\mathbf{R}^{n+1}, \mathbf{e}^{n+1} - \mathbf{e}^n) - (\delta^2\mathbf{e}^{n+1}, \mathbf{e}^{n+1} - \mathbf{e}^n)_0 \\ &\qquad + 2\,\varepsilon\,\delta t^2\left(\pi^n, \frac{\delta^2\overline{p}^{n+1}}{\delta t}\right)_0 + 2\,\varepsilon\,\delta t^2\left(\pi^{n+1} - \pi^n, \frac{\delta^2\overline{p}^{n+1}}{\delta t}\right)_0, \end{aligned} \tag{4.16}$$

where we denote by $T1$, $T2$, $T3$, $T4$ the last four terms in the right-hand side of (4.16), so that

$$\begin{aligned}
|T1| &\leq 2\,\delta t||\mathbf{R}^{n+1}||_0||\mathbf{e}^{n+1}-\mathbf{e}^n||_0 \leq 2\,\delta t^2||\mathbf{R}^{n+1}||_0^2+\frac{1}{2}||\mathbf{e}^{n+1}-\mathbf{e}^n||_0^2,\\
|T2| &\leq ||\delta^2\mathbf{e}^{n+1}||_0||\mathbf{e}^{n+1}-\mathbf{e}^n||_0 \leq \frac{1}{2}||\delta^2\mathbf{e}^{n+1}||_0^2+\frac{1}{2}||\mathbf{e}^{n+1}-\mathbf{e}^n||_0^2,\\
|T3| &\leq 2\,\varepsilon\,\delta t^2||\pi^n||_0\left\|\frac{\delta^2\overline{p}^{n+1}}{\delta t}\right\|_0 \leq \varepsilon\,\delta t^2||\pi^n||_0^2+\varepsilon\,\delta t^2\left\|\frac{\delta^2\overline{p}^{n+1}}{\delta t}\right\|_0^2,\\
|T4| &\leq 2\,\varepsilon\,\delta t^2||\pi^{n+1}-\pi^n||_0\left\|\frac{\delta^2\overline{p}^{n+1}}{\delta t}\right\|_0 \leq \varepsilon\,\delta t^2||\pi^{n+1}-\pi^n||_0^2+\varepsilon\,\delta t^2\left\|\frac{\delta^2\overline{p}^{n+1}}{\delta t}\right\|_0^2.
\end{aligned}$$

Now combine the above inequalities with (4.16). Then, replace the index n by k in the resulting inequality and sum it for $k=1$ to n. After that, apply the discrete Gronwall lemma by taking into account assumptions (4.3) and (4.4), Lemma 4, Theorem 4.3, and the fact that $\sum_{k=1}^n \delta t\left\|\frac{\delta^2\overline{p}^{k+1}}{\delta t}\right\|_0^2 \leq M\,\delta t^2$ (see Lemma 6), it yields

$$\begin{aligned}
&\sum_{k=1}^n||\mathbf{e}^{k+1}-\mathbf{e}^k||_0^2+\mu\,\delta t||\nabla\mathbf{e}^{n+1}||_0^2+\mu\sum_{k=1}^n\delta t||\nabla(\mathbf{e}^{k+1}-\mathbf{e}^k)||_0^2\\
&\qquad+\varepsilon\,\delta t\,||\pi^{n+1}||_0^2+\varepsilon\,\delta t\,||2\,\pi^{n+1}-\pi^n||_0^2+\varepsilon\,\delta t\sum_{k=1}^n||\delta^2\pi^{k+1}||_0^2\\
&\qquad\leq C(\Omega,T,\mu,||\mathbf{f}||_{L^2((0,T)\times\Omega)},\mathbf{e}^0,\mathbf{e}^1,||\nabla\mathbf{e}^1||_0,||\pi^1||_0)\,(\delta t^4+\varepsilon\,\delta t).
\end{aligned}\tag{4.17}$$

Since we have $\left\|3\,\mathbf{e}^{n+1}-4\,\mathbf{e}^n+\mathbf{e}^{n-1}\right\|_0^2 \leq 2(||\delta^2\mathbf{e}^{n+1}||_0^2+4||\mathbf{e}^{n+1}-\mathbf{e}^n||_0^2)$, it is an easy matter to show (thanks to (4.17) and Theorem 4.3) that

$$\begin{aligned}
\sum_{k=1}^n\left\|3\,\mathbf{e}^{k+1}-4\,\mathbf{e}^k+\mathbf{e}^{k-1}\right\|_0^2 &\leq \sum_{k=1}^n 2(||\delta^2\mathbf{e}^{k+1}||_0^2+4||\mathbf{e}^{k+1}-\mathbf{e}^n||_0^2)\\
&\leq C_0(\Omega,T,\mu,||\mathbf{f}||_{L^2((0,T)\times\Omega)},\mathbf{e}^0,\mathbf{e}^1,||\nabla\mathbf{e}^1||_0,||\pi^1||_0)\,(\delta t^4+\varepsilon\,\delta t). \qquad\square
\end{aligned}$$

Now, we can get the approximation for the pressure.

Theorem 4.5. *Under the assumptions of Theorem* 4.3 *and using Lemma* 5 *and for all* $0<\varepsilon\leq 1$, $0<\delta t\leq\max(1,T)$, *there exists a positive constant,*

$$C_0 = C_0(\Omega,T,\mu,||\mathbf{f}||_{L^2((0,T)\times\Omega)},||\mathbf{e}^0||_0,||\mathbf{e}^1||_0,||\pi^1||_0),$$

such that the solution of the VPP *method* (3.1)-(3.3) *verifies for all* $n\geq 1$ *with* $(n+1)\,\delta t\leq T$ *the following inequality:*

$$\sum_{k=1}^n\delta t||\pi^{k+1}||_0^2 \leq C_0\,(\delta t^3+\varepsilon).$$

Sketch of the Proof.

Step 1: We rearrange (4.8) as

$$\nabla\pi^{n+1}=\mathbf{R}^{n+1}+\mu\,\Delta\mathbf{e}^{n+1}-\left(\frac{3\,\mathbf{e}^{n+1}-4\,\mathbf{e}^n+\mathbf{e}^{n-1}}{2\,\delta t}\right).$$

Using, respectively, both inequalities $(a-b)^2 \leq 2(a^2+b^2)$ and $(a+b)^2 \leq 2(a^2+b^2)$, owing to Sobolev injection $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ and to the injection of the Laplacian Δ: $H^1(\Omega) \longrightarrow H^{-1}(\Omega)$, one gets:

$$||\nabla\pi^{n+1}||_{-1}^2 \leq 4||\mathbf{R}^{n+1}||_0^2 + 4\,\mu^2||\mathbf{e}^{n+1}||_1^2 + 2\left\|\frac{3\,\mathbf{e}^{n+1} - 4\,\mathbf{e}^n + \mathbf{e}^{n-1}}{2\,\delta t}\right\|_0^2.$$

Taking into account that $\mathbf{e}^{n+1} \in H_0^1(\Omega)$, we have $||\mathbf{e}^{n+1}||_1^2 = ||\nabla\mathbf{e}^{n+1}||_0^2$. Multiplying then the above inequality by δt and summing it up for k from 1 to n, we get

$$\begin{aligned}\sum_{k=1}^{n}\delta t||\nabla\pi^{k+1}||_{-1}^2 \leq 4\sum_{k=1}^{n}\delta t||\mathbf{R}^{k+1}||_0^2 + 4\,\mu^2\sum_{k=1}^{n}\delta t\,||\nabla\mathbf{e}^{k+1}||_0^2 \\ + 2\sum_{k=1}^{n}\delta t\left\|\frac{3\,\mathbf{e}^{k+1} - 4\,\mathbf{e}^k + \mathbf{e}^{k-1}}{2\,\delta t}\right\|_0^2.\end{aligned} \tag{4.18}$$

Step 2: Using Nečás inequality, there exists a constant $C > 0$ such that

$$||\pi^{n+1}||_0 \leq C\,||\nabla\pi^{n+1}||_{-1}. \tag{4.19}$$

Finally, using (4.18) and thanks to Lemma 4, Lemma 5, and part (i) of Theorem 4.3, we derive the desired estimation.

4.3. **Improvement of the basic error estimates.** In order to improve the basic error estimates for the velocity divergence and the pressure, the critical step here consists in establishing estimates for the time increment.

First, we impose the following regularities on the continuous Stokes problem

$$\frac{\partial^3\mathbf{v}(t)}{\partial t^3} \in L^2(0,T;\mathbf{L}^2(\Omega)) \tag{4.20}$$

and

$$\frac{\partial^2 p(t)}{\partial t^2} \in L^2(0,T;L^2(\Omega)). \tag{4.21}$$

Then, we define the increment error in time

$$\delta\mathbf{R}^{n+1} = \mathbf{R}^{n+1} - \mathbf{R}^n,$$

$$\delta^2\overline{p}^{n+1} = \delta\overline{p}^{n+1} - \delta\overline{p}^n,$$

$$\delta\mathbf{e}^{n+1} = \mathbf{e}^{n+1} - \mathbf{e}^n,$$

$$\delta\pi^{n+1} = \pi^{n+1} - \pi^n.$$

Finally, we suppose that the initial errors are well-controlled as follows, i.e, there exists a constant $c' > 0$ such that

$$||\mathbf{e}^1||_0^2 + ||2\,\mathbf{e}^1 - \mathbf{e}^0||_0^2 + 2\,\varepsilon\,\delta t\,||\pi^1||_0^2 \leq c'^6, \tag{4.22}$$

$$||\delta\mathbf{e}^1||_0^2 \leq c'^6, \tag{4.23}$$

Lemma 6. *For $\boldsymbol{f}$ and $\mathbf{v}_0$ given and smooth enough, we suppose that the solution $(\mathbf{v}, p)$ of the Stokes problem is smooth enough in space and time such that*

$\boldsymbol{v} \in \mathbf{W}^{3,2}(0,T;\mathbf{L}^2(\Omega))$ *and* $p \in W^{2,2}(0,T;L^2(\Omega))$. *Then, there exists a constant* $M(\Omega,T,\mathbf{f},\mathbf{v}_0) > 0$ *such that*

$$\text{(a)}\quad \sum_{k=2}^{n} \delta t||\delta \mathbf{R}^{k+1}||_0^2 \leq M(\Omega,T,\mathbf{f},\mathbf{v}_0)\,\delta t^6,$$

$$\text{(b)}\quad \sum_{k=1}^{n} \delta t||\delta^2 \overline{p}^{k+1}||_0^2 \leq M(\Omega,T,\mathbf{f},\mathbf{v}_0)\,\delta t^4,$$

$$\text{(c)}\quad \sum_{k=1}^{n} \delta t \left\|\frac{\delta^2 \overline{p}^{k+1}}{\delta t}\right\|_0^2 \leq M(\Omega,T,\mathbf{f},\mathbf{v}_0)\,\delta t^2.$$

Sketch of the proof. We can reformulate $\mathbf{R}^{k+1}$ and $\mathbf{R}^k$ as the residual integral of the Taylor series. Then, thanks to the regularity hypothesis imposed on the velocity, we obtain as in [22, 26]:

$$\sum_{k=2}^{n} \delta t||\delta \mathbf{R}^{k+1}||_0^2 = \sum_{k=2}^{n} \delta t||\mathbf{R}^{k+1} - \mathbf{R}^k||_0^2 \leq M\,\delta t^6,$$

which concludes the proof of (a).

For the proof of (b), we note that $\delta^2\overline{p}^{k+1}$ is defined as follows:

$$\delta^2\overline{p}^{k+1} = \delta(\delta\overline{p}^{k+1}) = \delta(p(t^{k+1}) - p(t^k)) = p(t^{k+1}) - 2\,p(t^k) + p(t^{k-1}).$$

By reformulating $\delta^2\overline{p}^{k+1}$ as the residual integral of the Taylor series, it yields

$$\begin{aligned}
&||p(t^{k+1}) - 2p(t^k) + p(t^{k-1})||_0^2 \\
&\quad = \left\| \int_{t^{k-1}}^{t^k} (t - t^{k-1}) \frac{\partial^2 p(t)}{\partial t^2} dt + \int_{t^k}^{t^{k+1}} (t^{k+1} - t) \frac{\partial^2 p(t)}{\partial t^2} dt \right\|_0^2 \\
&\quad \leq 2 \int_{t^{k-1}}^{t^k} (t - t^{k-1})^2\, dt \int_{t^{k-1}}^{t^k} \left\| \frac{\partial^2 p(t)}{\partial t^2} \right\|_0^2 dt \\
&\qquad + 2 \int_{t^k}^{t^{k+1}} (t^{k+1} - t)^2\, dt \int_{t^k}^{t^{k+1}} \left\| \frac{\partial^2 p(t)}{\partial t^2} \right\|_0^2 dt.
\end{aligned}$$

Therefore, using the assumption (4.21), we deduce

$$\sum_{k=1}^{n} \delta t||\delta^2 \overline{p}^{k+1}||_0^2 \leq M\,\delta t^4,$$

which concludes (b).

Finally, the proof of (c) is deduced easily from (b). □

Lemma 7. *Provided that* $\frac{\partial^3 \mathbf{v}(t)}{\partial t^3} \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ *and* $\frac{\partial^2 p(t)}{\partial t^2} \in L^\infty(0,T;L^2(\Omega))$ *and using both assumptions* (4.22) *and* (4.23) *and Lemma* 6 *with* $0 < \varepsilon \leq 1$, $0 < \delta t \leq \max(1,T)$, *then there exists a positive constant*

$$C_0 = C_0\left(\Omega, T, \mu, ||\mathbf{f}||_{L^2((0,T)\times\Omega)}, ||\mathbf{e}^0||_0, ||\mathbf{e}^1||_0, ||\delta\mathbf{e}^1||_0, ||\pi^1||_0\right)$$

such that for all $n \geq 2$,

$$||\delta \mathbf{e}^{n+1}||_0^2 + ||2\,\delta \mathbf{e}^{n+1} - \delta \mathbf{e}^{n}||_0^2 + \sum_{k=2}^{n} ||\delta^2(\delta \mathbf{e}^{k+1})||_0^2 + 2\,\mu \sum_{k=2}^{n} \delta t ||\nabla(\delta \mathbf{e}^{k+1})||_0^2$$
$$+ 2\,\varepsilon\,\delta t||\delta\pi^{n+1}||_0^2 + \varepsilon \sum_{k=2}^{n} \delta t ||\delta\pi^{k+1} - \delta\pi^{k}||_0^2 \leq C_0\,(\delta t^6 + \varepsilon\,\delta t^3).$$

Sketch of the Proof. The proof of this lemma follows the same principle adopted in the proof of Theorem 4.3 (part (i)). First, we form the equation which governs the error increments $\delta\mathbf{e}^{n+1}$ by subtracting the error equation for the velocity at two consecutive discrete times. We do the same thing with the error equation for the pressure in order to form the equation governing the error increments $\delta\pi^{n+1}$.

Second, we take the scalar product of of the equation which has been obtained by $\delta\mathbf{e}^{n+1}$ and $\delta\pi^{n+1}$, respectively. Then, we proceed as for part (i) of the proof of Theorem 4.3 with the necessary modifications, i.e, by replacing $\mathbf{e}^{n+1}$ by $\delta\mathbf{e}^{n+1}$, π^{n+1} by $\delta\pi^{n+1}$, $\mathbf{R}^{n+1}$ by $\delta\mathbf{R}^{n+1}$ and $\delta\overline{p}^{n+1}$ by $\delta^2\overline{p}^{n+1}$.

Owing to assumptions (4.22) and (4.23), we can show that

$$||\delta \mathbf{e}^2\,||_0^2 + ||2\,\delta \mathbf{e}^2 - \delta \mathbf{e}^1||_0^2 + 2\,\varepsilon\,\delta t||\delta\pi^2||_0^2 \leq c'^6 + \varepsilon\,\delta t^3). \tag{4.24}$$

Finally, by using Lemma 6, the majoration (4.24) and by applying the discrete Gronwall lemma, we get the desired estimate.

Corollary 1. *Based on Lemma* 4 *and Lemma* 7 *and for all* $0 < \varepsilon \leq 1$, $0 < \delta t \leq \max(1, T)$, *there exists a positive constant*

$$C_0 = C_0\left(\Omega, T, \mu, ||\mathbf{f}||_{L^2((0,T)\times\Omega)}, ||\mathbf{e}^0||_0, ||\mathbf{e}^1||_0, ||\delta\mathbf{e}^1||_0, ||\pi^1||_0\right)$$

such that

$$(i) \quad \sum_{k=1}^{n} \delta t ||\nabla\cdot\mathbf{e}^{k+1}||_0^2 \leq C_0\,(\delta t^3 + \varepsilon)\varepsilon\,\delta t^2.$$
$$(ii) \quad \sum_{k=1}^{n} \delta t ||\pi^{k+1}||_0^2 \leq C_0\,(\delta t^4 + \varepsilon\,\delta t).$$

Sketch of the Proof. (i) *Error estimate for the velocity divergence.*

Using (4.10), we have

$$\nabla\cdot\mathbf{e}^{n+1} = \varepsilon\,\delta\overline{p}^{n+1} - \varepsilon\,(\pi^{n+1} - \pi^{n}).$$

Therefore, Lemma 4 and Lemma 7 allow us to conclude the proof as follows:

$$\begin{aligned} \sum_{k=1}^{n} \delta t ||\nabla\cdot\mathbf{e}^{k+1}||_0^2 &\leq 2\,\varepsilon^2\delta t^2 \sum_{k=1}^{n} \delta t \left\|\frac{\delta\overline{p}^{k+1}}{\delta t}\right\|_0^2 + \varepsilon \sum_{k=1}^{n} 2\,\varepsilon\,\delta t||\delta\pi^{k+1}||_0^2 \\ &\leq C_0\,(\delta t^3 + \varepsilon)\,\varepsilon\,\delta t^2. \end{aligned}$$

Remark 4.6. The error analysis carried out here shows that the splitting error of the velocity divergence is of order $\mathcal{O}(\varepsilon\,\delta t)$ in the norm $l^2(L^2(\Omega))$.

(ii) *Error estimate for the pressure.*

The key of the improvement of the approximation of the pressure lies in writing $3\,\mathbf{e}^{n+1} - 4\,\mathbf{e}^n + \mathbf{e}^{n-1}$ as the velocity error increment in time, i.e.:

$$3\,\mathbf{e}^{n+1} - 4\,\mathbf{e}^n + \mathbf{e}^{n-1} = 3(\mathbf{e}^{n+1} - \mathbf{e}^n) - (\mathbf{e}^n - \mathbf{e}^{n-1}) = 3\delta\mathbf{e}^{n+1} - \delta\mathbf{e}^n.$$

Thus,

$$\begin{aligned} ||3\,\mathbf{e}^{n+1} - 4\,\mathbf{e}^n + \mathbf{e}^{n-1}||_0^2 &\leq 2\left(9||\mathbf{e}^{n+1} - \mathbf{e}^n||_0^2 + ||\mathbf{e}^n - \mathbf{e}^{n-1}||_0^2\right) \\ &= 2\left(9||\delta\mathbf{e}^{n+1}||_0^2 + ||\delta\mathbf{e}^n||_0^2\right). \end{aligned}$$

Hence, thanks to Lemma 7, we infer

$$\sum_{k=1}^{n} \delta t||3\,\mathbf{e}^{k+1} - 4\,\mathbf{e}^k + \mathbf{e}^{k-1}||_0^2 \leq 20 \sum_{k=1}^{n} \delta t||\delta\mathbf{e}^{k+1}||_0^2 \leq C_0(\delta t^5 + \varepsilon\,\delta t^2). \tag{4.25}$$

Finally, thanks to Nečás' inequality and using inequality (4.25), Lemma 4, and Theorem 4.3, the desired estimate of the pressure is concluded for $\delta t \leq \max(1, T)$.

Remark 4.7 (**Another improvement of the splitting errors**). It is worth mentioning that the splitting error of the velocity can be also improved to reach the order of $\mathcal{O}(\sqrt{\varepsilon\,\delta t^3 + \varepsilon^2\,\delta t^{\frac{3}{2}}})$ in $l^\infty(\mathbf{L}^2(\Omega)) \cap l^2(\mathbf{H}^1(\Omega))$. The key improvement is to treat directly the term $4\,\varepsilon\,\delta t\,(\pi^{n+1}, \delta\overline{p}^{n+1})_0$ in (4.11) by using the Nečás lemma and the equation (4.8). Note that this improvement will consequently affect the splitting errors of Corollary 1 and improve them. Hence, it is no more useful to use the discrete Gronwall inequality and the resulting splitting error of the velocity in Theorem 4.3.

Remark 4.8 (**Convergence rate and splitting error of the pressure approximation**). There exists in the literature a large number of works dedicated to theoretical investigations on the convergence rate of the pressure. In fact, the standard form of the second-order pressure-correction scheme guarantees a convergence rate only of order 1 for the pressure in $l^\infty(L^2(\Omega))$. The rotational form of this method improves the convergence rate to 3/2 in $l^2(L^2(\Omega))$. Note also that the second-order velocity-correction method in its rotational form [25] as well as the scalar penalty-projection method [18] provide also a convergence rate of order 3/2 in $l^2(L^2(\Omega))$. To the best of our knowledge, this is the best possible convergence rate established for the pressure approximation.

However, the result in part (ii) of Corollary 1 deserves attention since it shows that the second-order vector penalty-projection method yields optimal error estimates in time for the pressure. In fact, the temporal convergence rate of the pressure obtained here is of order 2 in $l^2(L^2(\Omega))$ and this is because, contrary to the usual projection methods, there is no artificial Neumann boundary condition for the pressure, which, if it exists, will thus limit the accuracy of the scheme.

Finally, we notice that the pressure splitting error is of order $\mathcal{O}(\sqrt{\varepsilon\,\delta t})$ in $l^2(L^2(\Omega))$ which is a remarkable result because the splitting error can be made as small as desired (with ε small enough) until machine precision and thus completely negligible with respect to the time error of the scheme, i.e, $\mathcal{O}(\delta t^2)$ in the present case.

5. Numerical experiments

In this section, we give some numerical results in order to verify the theoretical results obtained in Section 4. First, we examine the accuracy of the method on a standard Navier-Stokes benchmark, namely the computation of Taylor-Green vortices. Second, we test the time accuracy of the velocity and the pressure in the case of the Stokes flow with Dirichlet boundary conditions. In addition, we check the L^2-norm of the velocity divergence. Finally, we conduct a comparative and qualitative study between the VPP method presented in this paper and some pressure-correction schemes often used in the literature for the solution of non-stationary incompressible flow problems (see, e.g., [18, 27]).

Before presenting the numerical experiments, we note that the simulations presented are performed with a formally second-order scheme in time, i.e., the second-order backward difference formula (BDF2) to march in time and the second-order Richardson's extrapolation to extrapolate the pressure. Concerning the spatial discretization, the VPP method is implemented with a finite volume solver on the classical Marker and Cells grid (MAC mesh) of Harlow and Welch [29]. In our implementations, pressure unknowns are calculated at the cell-center and velocity components at mid-faces. Additionally, the method is initialized with a first time step performed with a standard backward Euler scheme. Finally, in order to solve the symmetric linear systems obtained in the prediction and projection steps, we are running the Conjugate Gradient (CG) method with the zero-order Incomplete Cholesky (IC(0)) as a preconditioner. The stopping criterion for the iterative (CG) method is chosen such that $||res||_2 \leq 10^{-6}$, where res denotes the residuals at the current CG iteration.

5.1. **Taylor-Green vortex.** As a first benchmark for the proposed method, the nondimensional unsteady incompressible nonlinear Navier-Stokes equations are solved on a two-dimensional square domain for the Taylor-Green vortex decaying problem. In fluid dynamics, the Taylor-Green vortex is a two-dimensional, unsteady flow of a decaying vortex which has exactly the same closed form solution of incompressible Navier-Stokes equations in Cartesian coordinates. We adjust the source term $\mathbf{f}$ in such a way that the exact solutions of the nonlinear Navier-Stokes problem for velocity and pressure become

$$\begin{aligned} u(x,y,t) &= -\sin(\frac{\pi x}{2})\cos(\frac{\pi y}{2})\exp(-2\mu t), \\ v(x,y,t) &= \cos(\frac{\pi x}{2})\sin(\frac{\pi y}{2})\exp(-2\mu t), \\ p(x,y,t) &= \frac{1}{4\pi}(\cos(\pi x)+\cos(\pi y))\exp(-4\mu t). \end{aligned}$$

The chosen computational domain is the square $]0,1[\times]0,1[$ and the velocity is imposed on the whole boundary. The viscosity is set to $\mu = 0.01$ where $\mu = \frac{1}{Re}$. We vary the time step δt to investigate the temporal accuracy. We choose δt sufficiently small to satisfy the usual CFL condition.

Figure 1 shows the difference between the numerical and the analytical solution at $T = 2$ measured in the L^2-norm for the velocity and for the pressure. These curves are drawn for the 128×128 mesh with $r = 10^{-2}$ and ε=10^{-10}. In both cases, the error decreases with the time step. We observe that the convergence rate is of order 1.85 for the velocity and the pressure. Note that the saturations observed for very small time steps are due to the approximation error in space which becomes dominant for very small time steps.

Moreover, we compute the L^2-norm of the velocity divergence as a function of ε. We repeat this test for two different values of Reynolds number: $Re = 1$ and $Re = 100$. The time step δt is set to 5×10^{-1}. The results are illustrated in Figure 2

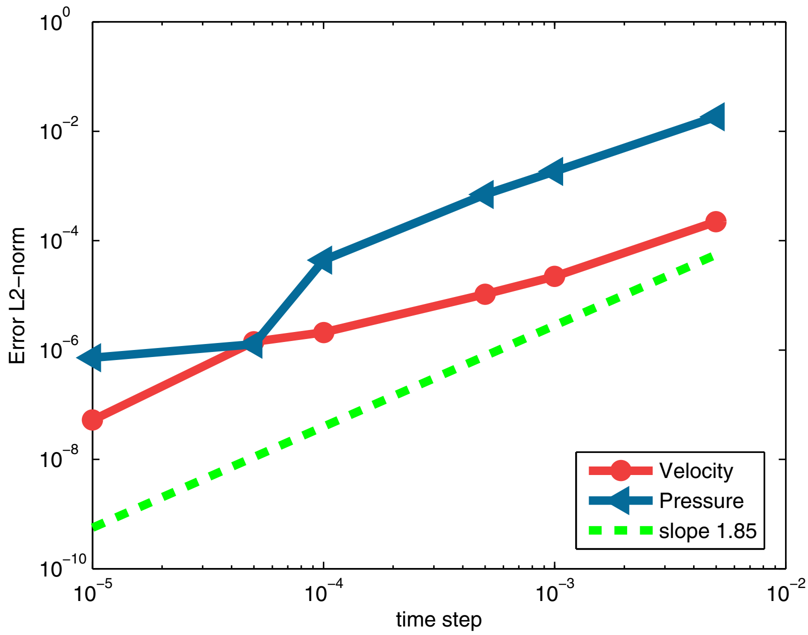

FIGURE 1. Taylor-Green vortex (Nonlinear case) - Error on the velocity and the pressure in L^2-norm vs δt for $Re = 100$, $r = 10^{-2}$ and $\varepsilon{=}10^{-10}$.

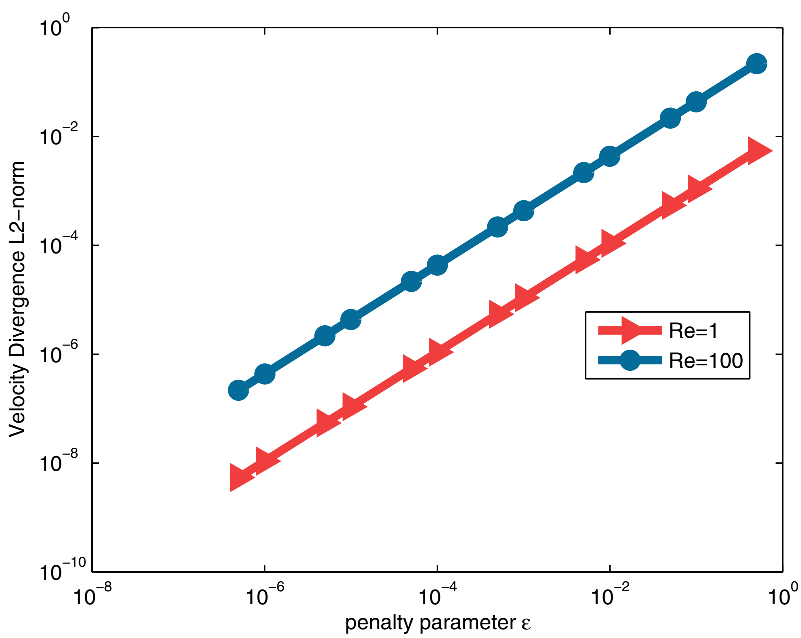

FIGURE 2. Taylor-Green vortex (Nonlinear case) - Velocity divergence L^2-norm vs ε at $T = 2$ with $1/h = 128$, for $Re = 1$ and $Re = 100$, respectively.

at the final time $T = 2$. Both curves show that when ε tends to 0, the L^2-norm of the velocity divergence tends also to 0. For example, taking $\varepsilon = 10^{-4}$ with $Re = 1$, the value of the L^2-norm of velocity divergence is approximately equal to 10^{-6}. It is equal to 10^{-5} for $Re = 100$. Moreover, we observe that the velocity divergence is vanishing approximately with an order of $\mathcal{O}(\varepsilon\,\delta t)$. Finally, we notice that the values of the L^2-norm of velocity divergence for $Re = 1$ seem smaller than those computed for $Re = 100$.

5.2. **A Stokes flow with Dirichlet boundary conditions.** We consider a square domain $\Omega =]0,1[^2$ and we enforce nonhomogeneous Dirichlet boundary conditions on $\partial\Omega$. The tests are performed using the following analytical solution which defines the right-hand side of the balance momentum equation of the linearized Navier-Stokes equations (known as Stokes equations).

$$\begin{aligned} \mathbf{v}(x,y,t) &= (\sin(x+t)\,\sin(y+t), \cos(x+t)\,\cos(y+t)),\\ p(x,y,t) &= \cos(x-y+t), \end{aligned}$$

This test case is the same studied in [18, 30]. In order to check the accuracy in time, we plot the errors of the velocity and the pressure (or the pressure gradient) in the L^2-norm for different values of the augmentation parameter r ranging between 0 and 1 at time $T = 2$. In the computations reported herein, the mesh size h is equal to $1/128$ so that the spatial discretization errors are negligible compared with the time discretization errors. The time steps tested are in the range $10^{-3} \le \delta t \le 10^0$. We choose a penalty parameter small enough: $\varepsilon = 10^{-10}$.

First, we present in Figures 3 and 4 the L^2-norm of the error of the velocity and the pressure gradient respectively as a function of the time step while choosing the augmentation parameter r equal 0. We observe in Figure 3 that the convergence rate in time for the velocity is clearly of order 2, as predicted by Theorem 4.3. In addition, a convergence order of 2 is observed for the pressure gradient in Figure 4. This result is in agreement with the error estimates established in Corollary 1.

Indeed, the vector penalty-projection method with three different nonzero values of r: 10^{-4}, 10^{-2} and 1 gives for the velocity and the pressure the same temporal convergence rate as the case of $r = 0$, i.e, we obtain a convergence order of 2 in L^2-norm for both velocity and pressure (see Figures 5 and 6, respectively).

As a conclusion on the convergence rate in time in presence of Dirichlet conditions on the boundaries, the VPP method improves the order of pressure from $\mathcal{O}(\delta t)$ to $\mathcal{O}(\delta t^2)$ compared to the standard incremental pressure-correction scheme [27]. The VPP method provides also a higher-order than the rotational incremental pressure-correction (order of $3/2$ in L^∞-norm) and the scalar penalty-projection scheme [18]. However, the convergence rate of order 2 for the velocity remains the same as in the standard and rotational pressure-correction methods [27] and also in the scalar penalty-projection scheme [18].

Moreover, we plot in Figure 7 the L^2-norm of the velocity divergence as a function of the penalty parameter ε. We fix δt at 10^{-1} and the augmentation parameter r at 0. The curve shows that when the penalty parameter is chosen small enough and tends to 0, the velocity divergence decreases and tends also to 0. Additionally, we observe that the L^2-norm of the velocity divergence vanishes roughly as $\mathcal{O}(\varepsilon\delta t)$ with ε sufficiently small. Finally, Figure 8 illustrates the L^2-norm of the velocity divergence as a function of the time step δt with $\varepsilon = 10^{-6}$. We notice that the velocity divergence is approximately of order $\mathcal{O}(\varepsilon\delta t)$ with a penalty parameter ε

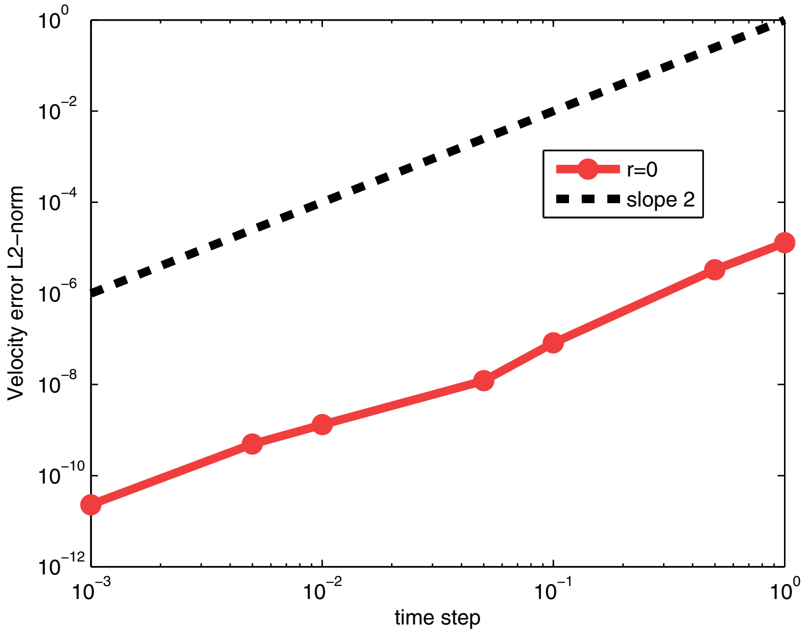

FIGURE 3. Stokes problem - Error on the velocity in L^2-norm vs δt at $T = 2$, mesh size $1/h = 128$, $\varepsilon = 10^{-10}$ and $r = 0$.

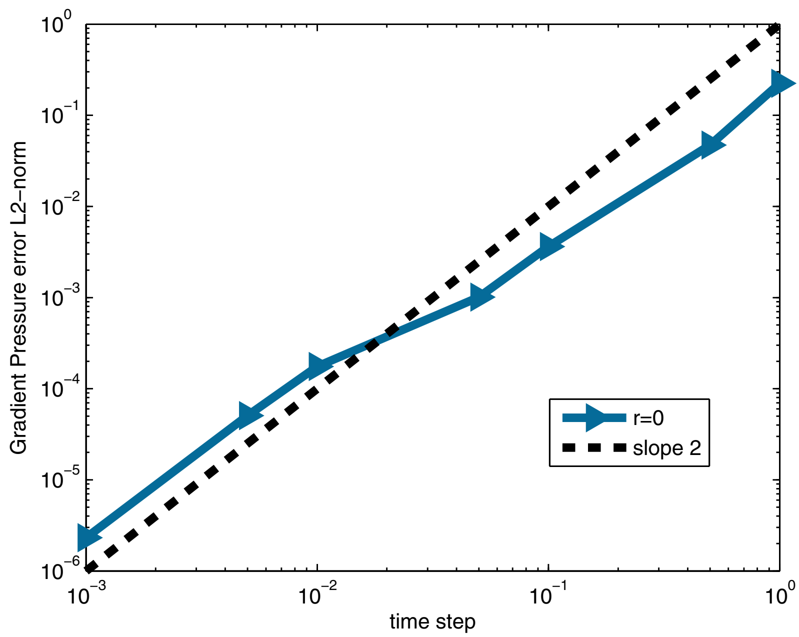

FIGURE 4. Stokes problem - Error on the gradient pressure in L^2-norm vs δt at $T = 2$, mesh size $1/h = 128$, $\varepsilon = 10^{-10}$ and $r = 0$.

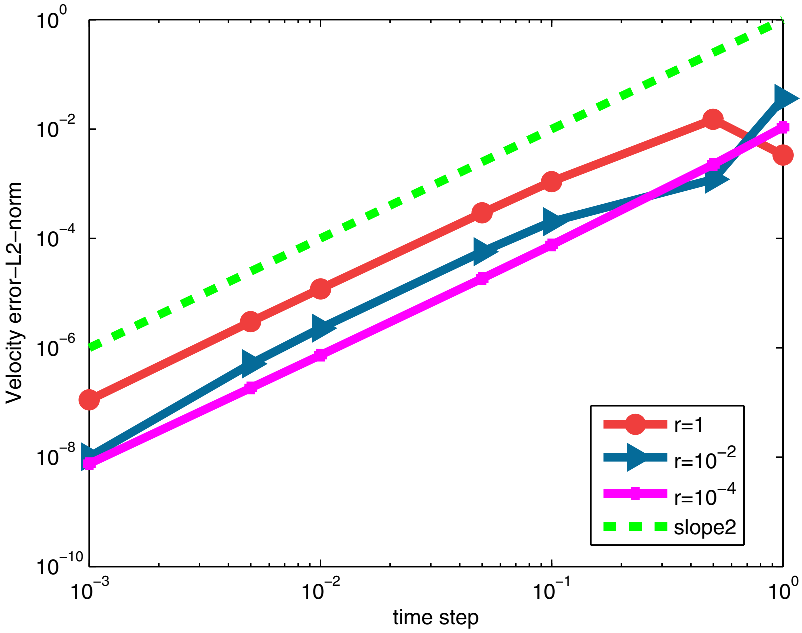

FIGURE 5. Stokes problem - Error on the velocity in L^2-norm vs δt at $T = 2$, mesh size $1/h = 128$ and $\varepsilon = 10^{-10}$.

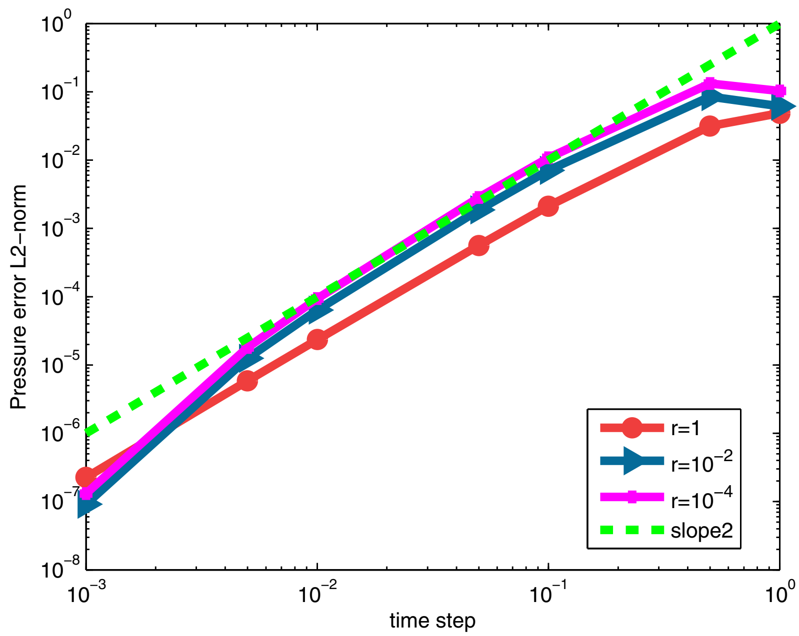

FIGURE 6. Stokes problem - Error on the pressure in L^2-norm vs δt at $T = 2$, mesh size $1/h = 128$ and $\varepsilon = 10^{-10}$.

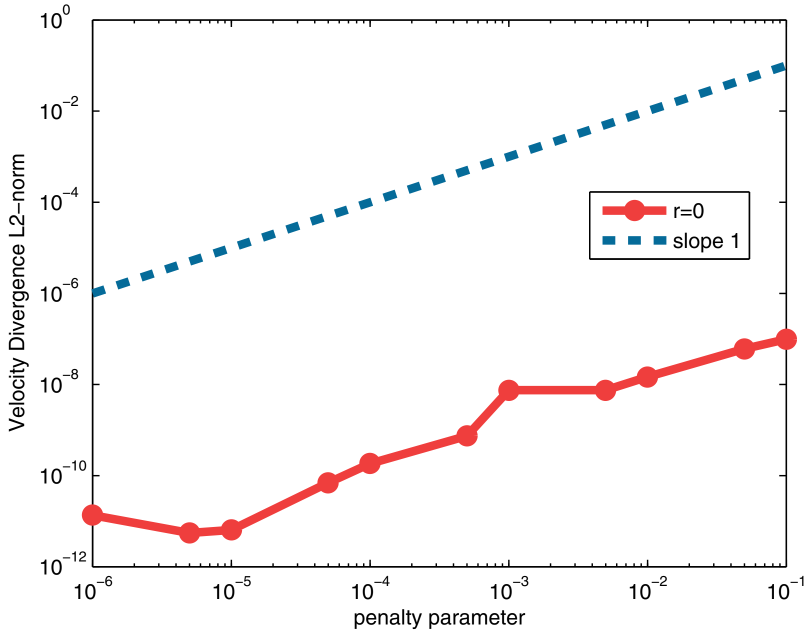

FIGURE 7. Stokes problem - Velocity divergence L^2-norm vs ε at T=2, mesh size $1/h = 128$ and $r = 0$.

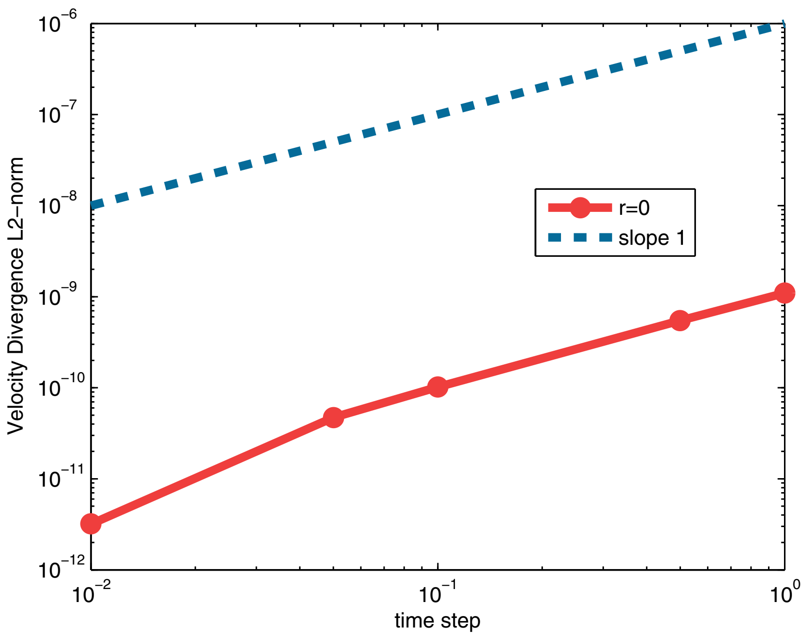

FIGURE 8. Stokes problem - Velocity divergence L^2-norm vs δt at T=2, mesh size $1/h = 128$, ε=10^{-6} and $r = 0$.

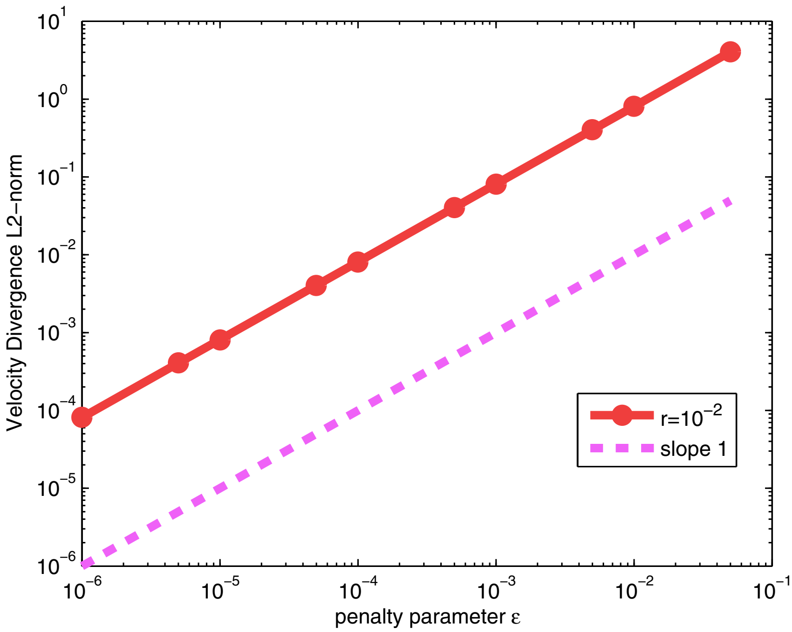

FIGURE 9. Stokes problem - Velocity divergence L^2-norm versus ε at T=2, mesh size $1/h = 128$ and $r = 10^{-2}$.

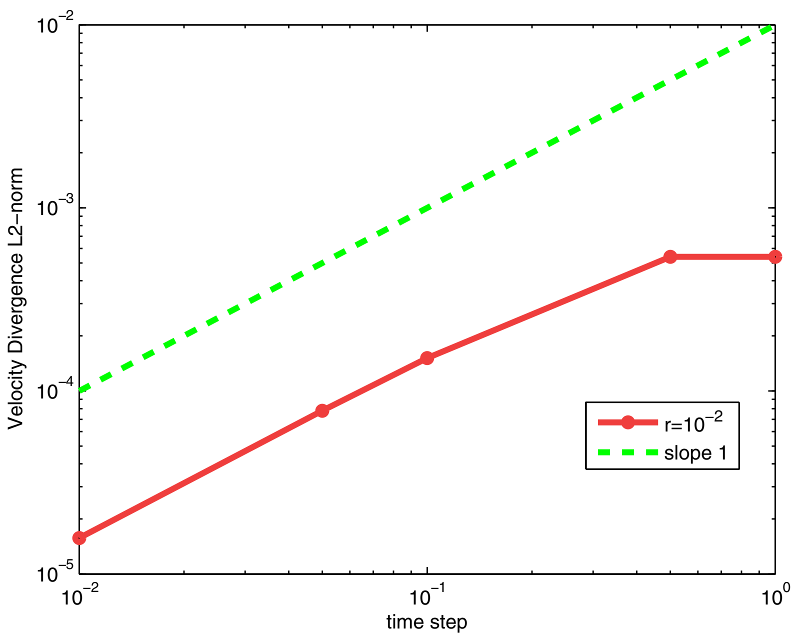

FIGURE 10. Stokes problem - Velocity divergence L^2-norm versus time step at T=2, mesh size $1/h = 128$ and ε=10^{-6}.

small enough. We repeat in Figure 9 and Figure 10 the same tests with an augmentation parameter r equal to 10^{-2}. Again, we observe that the L^2-norm of the velocity divergence vanishes as $\mathcal{O}(\varepsilon\delta t)$.

6. Concluding remarks

In this article, we have analyzed the second-order vector penalty-projection method for the incompressible Stokes problem with Dirichlet conditions enforced on the entire boundary. Our conclusions are twofold.

First, we have shown the stability of the scheme using BDF2 to discretize in time. Moreover, we have shown that, while the Dirichlet boundary conditions imposed on the velocity degenerate into a nonrealistic Neumann boundary condition for the pressure in the case of the usual projection methods [27], the second-order vector penalty-projection method leads to optimal error estimates since it preserves the original Dirichlet conditions. Consequently, the pressure approximation is no longer plagued by an artificial Neumann boundary condition. As a result, the VPP method provides optimal temporal convergence of order 2 theoretically as well as numerically; more precisely, the vector penalty-projection method yields $\mathcal{O}(\delta t^2)$ accuracy for both the velocity and the pressure in the norm of $l^\infty(\mathbf{L}^2(\Omega))$ and $l^2(L^2(\Omega))$, respectively. The counterpart in this method is that the divergence of the velocity at each time step is not exactly zero, as for the projection methods (at least in the semi-discrete setting in time), since the VPP velocity correction step is proved to be an approximate divergence-free projection [1, 7]. However, it is not really a drawback since the velocity divergence is in practice of order $\mathcal{O}(\varepsilon\delta t)$ with a penalty parameter ε taken as small as desired up to machine precision.

Second, we have shown that this family of methods opens the way to the splitting methods with an order of time convergence greater than 2 since the splitting error for velocity and pressure varies as $\mathcal{O}(\varepsilon)$ which can be made negligible with respect to the consistency error of higher-order schemes when ε is chosen sufficiently small.

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Aix-Marseille Université, Institut de Mathématiques de Marseille (I2M) - CNRS UMR7373, Centrale Marseille, 13453 Marseille cedex 13 - France
Email address: `philippe.angot@univ-amu.fr`

Aix-Marseille Université, Institut de Mathématiques de Marseille (I2M) - CNRS UMR7373, Centrale Marseille, 13453 Marseille cedex 13 - France
Email address: `rima.cheaytou@gmail.com`