ON THE ESSENTIAL MINIMUM OF FALTINGS' HEIGHT

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ABSTRACT. We study the essential minimum of the (stable) Faltings' height on the moduli space of elliptic curves. We prove that, in contrast to the Weil height on a projective space and the Néron-Tate height of an abelian variety, Faltings' height takes at least two values that are smaller than its essential minimum. We also provide upper and lower bounds for this quantity that allow us to compute it up to five decimal places. In addition, we give numerical evidence that there are at least four isolated values before the essential minimum.

One of the main ingredients in our analysis is a good approximation of the hyperbolic Green function associated to the cusp of the modular curve of level one. To establish this approximation, we make an intensive use of distortion theorems for univalent functions.

Our results have been motivated and guided by numerical experiments that are described in detail in the companion files.

1. INTRODUCTION

In this article, we study the essential minimum of the (stable) Faltings' height on the moduli space of elliptic curves. Our main result is that, in contrast to the Weil height on a projective space and the Néron-Tate height of an abelian variety, Faltings' height takes at least two values that are smaller than its essential minimum. Actually, our numerical experiments suggest that there are at least four such values: The first one is taken at the class of elliptic curves with j-invariant zero and the other three are taken at certain classes of elliptic curves whose j-invariant is a root of unity. We give a rigorous proof that there can be at most six classes of elliptic curves whose j-invariant is a root of unity and whose Faltings' height is smaller than the essential minimum.

We now proceed to describe our results more precisely. To recall the definition of Faltings' height, let $\mathbb{H} \coloneqq \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ be the upper half-plane, and let $\Delta \colon \mathbb{H} \to \mathbb{C}$ be the modular discriminant, normalized so that the product formula reads

$$\Delta(\tau) \coloneqq q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i \tau}.$$

Furthermore, consider the hyperbolic Green function $g_\infty \colon \mathbb{H} \to \mathbb{R}$, defined by

$$g_\infty(\tau) := -\log\left(\left(4\pi \operatorname{Im}(\tau)\right)^6 |\Delta(\tau)|\right).$$

This function is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Since Δ does not vanish on $\mathbb{H}$, the function g_∞ is finite and continuous.

Faltings' height is a numerical invariant attached to each abelian variety defined over a number field. To define it in the case of an elliptic curve E defined over a number field K, denote by $\Delta_{E/K}$ the minimal discriminant of E/K. Furthermore, for a given embedding $\sigma \colon K \to \mathbb{C}$, choose $\tau_\sigma \in \mathbb{H}$ such that

$$E_\sigma(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z} + \tau_\sigma \mathbb{Z}),$$

where E_σ is the elliptic curve over $\mathbb{C}$ obtained from E by base change using σ. Then the *Faltings' height* $\mathrm{h}_{\mathrm{F}}(E/K)$ *of* E/K can be defined as

$$(1.1) \qquad \mathrm{h}_{\mathrm{F}}(E/K) := \frac{1}{12[K:\mathbb{Q}]}\left(\log\left|N_{K/\mathbb{Q}}\left(\Delta_{E/K}\right)\right| + \sum_{\sigma\colon K\to\mathbb{C}} g_\infty(\tau_\sigma)\right);$$

see for example [Sil86, Proposition 1.1].[*] If L/K is a finite extension such that $E_L := E \otimes L$ is semistable, then it is not hard to check that $\mathrm{h}_{\mathrm{F}}(E_L/L) \le \mathrm{h}_{\mathrm{F}}(E/K)$. Moreover, the quantity $\mathrm{h}_{\mathrm{F}}(E_L/L)$ does not depend on the choice of L. In other words, on a given $\overline{\mathbb{Q}}$-isomorphism class of elliptic curves, Faltings' height attains its minimum at a semistable representative and its value does not depend on the choice of such semistable elliptic curve.

Faltings' height function h_{F} is the induced function

$$\mathrm{h}_{\mathrm{F}}\colon \overline{\mathbb{Q}} \to \mathbb{R}$$

that to a given algebraic number $\alpha \in \overline{\mathbb{Q}}$ attaches the real number $\mathrm{h}_{\mathrm{F}}(\alpha) := \mathrm{h}_{\mathrm{F}}(E_\alpha/L)$, where L is a number field containing α and E_α is a semistable elliptic curve defined over L with j-invariant equal to α.[†] Faltings showed that the function h_{F} behaves as a height on the moduli space of elliptic curves (e.g., it satisfies Northcott's property) and became a standard tool in diophantine geometry.

Our main results concern the essential minimum $\mu_{\mathrm{F}}^{\mathrm{ess}}$ of Faltings' height function, defined by

$$\mu_{\mathrm{F}}^{\mathrm{ess}} := \inf\left\{\theta \in \mathbb{R} : \text{ the set } \left\{\alpha \in \overline{\mathbb{Q}} : \mathrm{h}_{\mathrm{F}}(\alpha) \le \theta\right\} \text{ is infinite}\right\}.$$

Note that the set

$$(1.2) \qquad \left\{\mathrm{h}_{\mathrm{F}}(\alpha) : \alpha \in \overline{\mathbb{Q}}\right\} \setminus [\mu_{\mathrm{F}}^{\mathrm{ess}}, \infty)$$

is either finite, or formed by an increasing sequence converging to $\mu_{\mathrm{F}}^{\mathrm{ess}}$.

In the case of the Weil height on a projective space, the Néron-Tate height of an abelian variety, and the canonical height of a polarized dynamical system, the set corresponding to (1.2) is empty. Our first main result is that, in contrast, the set (1.2) contains at least two elements: The first minimum of h_{F} is $\mathrm{h}_{\mathrm{F}}(0)$, and the second $\mathrm{h}_{\mathrm{F}}(1)$.

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[*]We warn the reader that there are different normalizations of h_{F} in the literature, any two of them differing by an additive constant. In order to compare results by diverse authors, we have preferred a normalization different from the one in loc. cit.

[†]In the literature this function is also called the "stable Faltings' height" function.

Theorem 1. *We have*

(1.3)
$$\mathrm{h_F}(0) < \mathrm{h_F}(1) < \mu_{\mathrm{F}}^{\mathrm{ess}},$$

and there exists $\kappa > 0$ such that for every algebraic number $\alpha \neq 0, 1$ we have $\mathrm{h_F}(\alpha) \geq \mathrm{h_F}(1) + \kappa$.

Our numerical experiments suggest that in fact the set (1.2) contains at least four elements, and that its smallest elements, besides $\mathrm{h_F}(0)$ and $\mathrm{h_F}(1)$, are given by the values of $\mathrm{h_F}$ taken at the primitive roots of unity of orders 6 and 10. See the summary below, Section 8, and the companion files [BMRan] for precisions. Our second main result is that among the values of Faltings' height taken at roots of unity, these are the only ones that could belong to (1.2), with the possible exception of the values of $\mathrm{h_F}$ at the primitive roots of unity of orders 14, 15, and 22.

Theorem 2. *Let $n \geq 2$ be an integer different from 6, 10, 14, 15, and 22, and let ζ_n be a primitive root of unity of order n. Then $\mathrm{h_F}(\zeta_n) > \mu_{\mathrm{F}}^{\mathrm{ess}}$.*

The estimates used to prove Theorems 1 and 2 easily yield the following.

Corollary 1.1. *We have*

$$10^{-4} < \mathrm{h_F}(1) - \mathrm{h_F}(0) < \mu_{\mathrm{F}}^{\mathrm{ess}} - \mathrm{h_F}(0) < 2 \cdot 10^{-4} \ \ and$$
$$-0.748629 \leq \mathrm{h_F}(1) < \mu_{\mathrm{F}}^{\mathrm{ess}} \leq -0.748622,$$

and the set of values of $\mathrm{h_F}$ is dense in the interval $[-0.748622, \infty)$.

On the other hand, if α is an algebraic number whose Faltings' height is less than or equal to $\mu_{\mathrm{F}}^{\mathrm{ess}}$, then α is either an algebraic integer or of degree greater than or equal to 10520. Moreover, if the degree of α is at most 10, then α is an algebraic unit.

We did not try to get the best possible numerical estimates from the method we are using. We opted for weaker numerical estimates that are easier to verify.

The fact that the minimum value of $\mathrm{h_F}$ is

(1.4)
$$\mathrm{h_F}(0) = \frac{1}{12}g_\infty\left(e^{\pi i/3}\right) = -\frac{1}{2} \cdot \log\left(\frac{3}{(2\pi)^3}\Gamma\left(\frac{1}{3}\right)^6\right) = -0.748752485503338\ldots,$$

was observed by Deligne in [Del85, p. 29]. The inequality $\mathrm{h_F}(0) < \mu_{\mathrm{F}}^{\mathrm{ess}}$ has been observed independently by Löbrich [Löb15], showing that $\mu_{\mathrm{F}}^{\mathrm{ess}} - \mathrm{h_F}(0) \geq 4.601 \cdot 10^{-18}$. In [Zag93], Zagier studied a height function for which the set corresponding to (1.2) also contains at least two points. In [Doc01] and [Doc], Doche continued the study of such height function and determined an upper bound and a computer assisted lower bound for the corresponding essential minimum.

The following is a summary of what we have found in our numerical experiments, which have motivated and guided our results:

- *First four minima*:

$$\mathrm{h_F}(0) = -0.74875248\ldots, \mathrm{h_F}(1) = -0.74862817\ldots,$$
$$\mathrm{h_F}(\rho) = -0.74862517\ldots, \mathrm{h_F}(\xi) = -0.74862366\ldots,$$

where ρ is a primitive root of unity of order 6, a root of the polynomial $z^2 - z + 1$, and ξ is a primitive root of unity of order 10, a root of $z^4 - z^3 + z^2 - z + 1$.

- *Next known value*: $-0.74862330\ldots$, taken at the roots of the polynomial

$$z^8 - 2z^7 + 2z^6 - z^5 + z^4 - z^3 + z^2 - z + 1.$$

- *Bounds for the essential minimum*: $-0.74862345 \leq \mu_{\mathrm{F}}^{\mathrm{ess}} \leq -0.74862278$.
- *Density interval*: The values of $\mathrm{h_F}$ are dense in the interval $[-0.74862278, \infty)$.

See Section 8 and the companion files [BMRan] for further details. Note in particular that our numerical experiments locate the essential minimum $\mu_{\mathrm{F}}^{\mathrm{ess}}$ in an interval of length smaller than 10^{-6}. Furthermore, $\mathrm{h_F}$ takes exactly four values to the left of this interval and the values of $\mathrm{h_F}$ are dense to the right of this interval.

We remark that only the second chain of inequalities in Corollary 1.1 depends on the chosen normalization of Faltings' height.

The set of Faltings' heights of elliptic curves that are not necessarily semistable is a dense subset of $[\mathrm{h_F}(0), \infty)$, so by Theorem 1 it is strictly larger than the set of values of Faltings' stable height. Actually, even the set

$$\{\mathrm{h_F}(E/K) : K \text{ is a number field and } E/K \text{ is an elliptic curve such that } j(E) = 0\}$$

is dense in $[\mathrm{h_F}(0), \infty)$. This follows from the fact that the set of prime numbers p satisfying $p \equiv 1 \mod 9$ is infinite, and from the fact that for every such p and every integer $\ell \geq 1$ there is a number field K and an elliptic curve E/K such that $j(E) = 0$ and $h_F(E/K) = h_F(0) + \log p/(6\ell)$; see the proof of Theorem 1.3 in [Löb15].

We now proceed to explain the main ingredients of the proofs of Theorems 1 and 2, and simultaneously explain how the paper is organized. Our method is based on the interpretation of $\mathrm{h_F}$ as an Arakelov-theoretic height on the modular curve of level one, induced by the line bundle M_{12} of weight 12 modular forms, together with the Petersson metric $\|\cdot\|_{\mathrm{Pet}}$. The height $\mathrm{h_F}$ is computed by choosing a section of M_{12}. In Section 2 we review the Arakelov-theoretic interpretation of Faltings' height. We also collect the values at $e^{\pi i/3}$ of the classical Eisenstein series of weight 2, 4, and 6 and some of their derivatives, and compute $j'''(e^{\pi i/3})$ in terms of those values.

The proof of Theorem 1 is based on the following "minimax" procedure. Let s be a nonzero section of M_{12}. Then, for every $\alpha \in \overline{\mathbb{Q}}$ outside of the set $|\operatorname{div}(s)| = \{\alpha : s(\alpha) \in \{0, \infty\}\}$, we have $\mathrm{h_F}(\alpha) \geq \inf\left(-\log\|s\|_{\mathrm{Pet}}\right)$. Since $|\operatorname{div}(s)|$ is a finite set, this yields the lower bound

$$(1.5) \qquad\qquad \mu_{\mathrm{F}}^{\mathrm{ess}} \geq \inf\left(-\log\|s\|_{\mathrm{Pet}}\right);$$

cf. Proposition 2.3. Hence, to find a lower bound of $\mu_{\mathrm{F}}^{\mathrm{ess}}$ one is led to search for a section s maximizing the right-hand side of (1.5).

For instance, the choice $s = \Delta$ yields the lower bound $\mu_{\mathrm{F}}^{\mathrm{ess}} \geq \mathrm{h_F}(0)$, considering that $j\left(e^{\pi i/3}\right) = 0$ is an integer and that $g_\infty = -\log\|\Delta\|_{\mathrm{Pet}}$ reaches its minimum at $e^{\pi i/3}$. Since 0 is algebraic, a natural idea to improve this lower bound is to "penalize" the value $j = 0$ and look for a section of the form $s = j^a \cdot \Delta$, for some $a > 0$. The technical heart of the proof of Theorem 1 is to show that an appropriate choice of a yields the lower bound $\mathrm{h_F}(1) \leq \mu_{\mathrm{F}}^{\mathrm{ess}}$. This is the content of the following proposition.

Proposition A. *Let $g_{\mathrm{hyp}} \colon \mathbb{C} \to \mathbb{R}$ be the function defined by*

$$(1.6) \qquad\qquad g_\infty = g_{\mathrm{hyp}} \circ j.$$

Then we have that $0 < \partial_x g_{\mathrm{hyp}}(1) < 1$ and that the function $g_1 \colon \mathbb{C} \setminus \{0\} \to \mathbb{R}$ defined by

$$g_1(\zeta) := g_{\mathrm{hyp}}(\zeta) - \partial_x g_{\mathrm{hyp}}(1) \cdot \log |\zeta|,$$

attains its minimum value at, and only at, $\zeta = 1$.

See Remark 3.2 for an explanation of the choice $a = \partial_x g_{\mathrm{hyp}}(1)$. Once Proposition A is established, an infinitesimal version of the argument above yields the strict inequality $h_F(1) < \mu_F^{\mathrm{ess}}$. In Section 3 we show how to deduce Theorem 1 from Proposition A.

Our numerical experiments suggest that there are real numbers $a_1 > 0$ and $a_2 > 0$ such that the choice $s = j^{a_1}(j-1)^{a_2}\Delta$ leads to the more precise lower bound $h_F(e^{\pi i/3}) \leq \mu_F^{\mathrm{ess}}$, and ultimately to the strict inequality $h_F(e^{\pi i/3}) < \mu_F^{\mathrm{ess}}$. It is possible to prove this rigorously using the methods developed in this paper, but we do not do so here in order to keep this article at a reasonable length. We discuss further numerical experiments in Section 8 and in the companion files [BMRan].

The algorithm described above, which is applied here to Faltings' height, is valid for a general height (cf. Section 2.2 for a precise general formulation). In fact, this method was used in the aforementioned papers of Doche and Zagier and can be traced back to results on Mahler measures by Smyth [Sm81]. See [BGPS15, Theorem 3.7] for an application in the context of toric heights.

Another possible route to estimate μ_F^{ess} from below is to adapt to h_F the bounds on successive minima given by Zhang in [Zha95]. However, this approach yields a weaker lower bound of μ_F^{ess} than those given by Theorem 1. See Section 2.1 for further details.

One of the main ingredients in the proofs of Theorem 2 and Proposition A is an approximation of g_{hyp} and of its first and second derivatives, on a suitable neighborhood of the unit disk. Roughly speaking, we show that g_{hyp} is well approximated by the sum of a linear function and of an explicit function having a conic singularity at $\zeta = 0$. The following is a sample estimate in this direction, which is used in the proof of Theorem 2.

Proposition B. *Letting*

$$\gamma_0 := \frac{\sqrt{3}}{\pi}\Gamma\left(\frac{1}{3}\right)^2 \;\; \text{and} \;\; \gamma_1 := 3\log(192) - 6\log\left(\gamma_0^3 - \gamma_0^{-3}\right),$$

for every ζ in S^1 we have

$$\left| g_{\mathrm{hyp}}(\zeta) - \left(\gamma_1 - \frac{\mathrm{Re}(\zeta)}{13824}\right) \right| \leq 5 \cdot 10^{-7}.$$

The approximation of g_{hyp} is achieved in two independent steps. The first step is an approximation of the inverse function of j on a suitable neighborhood of the unit disk. This is done in Section 4. To this end we use the Koebe distortion theorem and several of its variants. Loosely speaking, this result gives a quantitative estimate on how well a given univalent function (that is, an injective and holomorphic function) is approximated by its linear part at a given point. We apply this theorem to the function induced by j on the quotient of a neighborhood of $e^{\pi i/3}$ in $\mathbb{H}$ by the stabilizer of this point in $\mathrm{SL}_2(\mathbb{Z})$, which is of order three. The computation of $j'''(e^{\pi i/3})$, alluded above, is important in the determination of the constants in the resulting approximations.

The second step is an approximation of the function g_∞ and of its first and second derivatives, on a suitable neighborhood of $e^{\pi i/3}$, and on a suitable coordinate. This is done in Section 5.

The proofs of Theorem 2, Corollary 1.1 and Proposition B are given in Section 6. After giving the proof of Proposition B in Section 6.1, we estimate the values of h_F at the roots of unity (Corollary 6.1).

Besides the approximation of g_{hyp} mentioned above, the main ingredient in the proof of Theorem 2 and Corollary 1.1 is a general method to estimate the essential minimum from above, which is based on the classical Fekete-Szegö theorem and an equidistribution result from [BGPRLS15]. See Section 6.2 for the precise formulation of the method. Here, we apply it to Faltings' height, but is also valid for other heights.

Since on a relatively large neighborhood of the unit disk the function g_{hyp} is very close to a function having radial symmetry, the integral of $\frac{1}{12}g_{hyp}$ against the Haar measure of S^1 gives a very good upper bound of μ_F^{ess}. However, this estimate is not sufficient for the proof of Theorem 2. We use instead a better upper bound that is obtained by integarting $\frac{1}{12}g_{hyp}$ against a certain translate of the Haar measure of S^1. This upper bound and the proofs of Theorem 2 and Corollary 1.1 are given in Section 6.2.

The proof of Proposition A is given in Section 7. The main part of the proof is divided in three cases, according to the proximity of ζ to the unit disk. By far, the most difficult case is the case where ζ is close the unit disk. To deal with this case we establish some convexity properties of g_{hyp} in this region, using the results of Sections 4 and 5.

Finally, in Section 8 we discuss numerical experiments around the determination of further isolated values of h_F and lower bounds of μ_F^{ess}. See also [BMRan] for a detailed presentation of these experiments.

2. MODULAR INGREDIENTS

2.1. Arakelov-theoretic interpretation of Faltings' height.
Let $\mathcal{X} := \mathbb{P}^1_{\mathbb{Z}}$ and consider the section $s_\infty\colon \mathrm{Spec}(\mathbb{Z}) \to \mathcal{X}$ given by $[1:0]$. We denote by D_∞ the divisor induced by this section and consider the line bundle $\mathcal{L} := O_{\mathcal{X}}(D_\infty)$. On the other hand, consider the modular curve $X := (\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})\cup\{\infty\}$ and the j-invariant $j\colon \mathbb{H} \to \mathbb{C}$, normalized by

$$j(\tau) := \frac{1}{q} + 744 + \cdots, \quad q = e^{2\pi i\tau}.$$

Every elliptic curve E over $\mathbb{C}$ has a Weierstrass equation of the form

$$(2.1) \qquad y^2 = x^3 - 27c_4x - 54c_6$$

with the notation of [Sil, pp. 46–48]. Then c_4 can be seen as a modular form of weight 4. The modular discriminant Δ is a modular form of weight 12 and we have the relation

$$(2.2) \qquad j = c_4^3/\Delta.$$

There is a holomorphic bijection

$$\iota\colon X \to \mathcal{X}(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$$

given by $\iota(\tau) = [c_4(\tau)^3 : \Delta(\tau)]$ and $\iota(\infty) = [1 : 0]$. This bijection identifies j with the absolute coordinate of $\mathbb{P}^1(\mathbb{C})$. Moreover, this choice of coordinates gives an isomorphism between the line bundle $\mathcal{L}(\mathbb{C})$ and the line bundle $M_{12}(\mathrm{SL}_2(\mathbb{Z})) \to X$ of weight 12 modular forms of level one, that identifies Δ with a canonical section of the former. Indeed, at the level of global sections, we have an isomorphism

$$w \colon H^0\left(\mathcal{X}(\mathbb{C}), \mathcal{L}(\mathbb{C})\right) = \{f \mid \mathrm{div}(f) + \infty \geq 0\} \longrightarrow M_{12}\left(\mathrm{SL}_2(\mathbb{Z})\right),$$

given by

$$(2.3) \qquad\qquad w(f) \coloneqq \Delta \cdot \iota^* f.$$

We recall that M_{12} carries the *Petersson metric*, defined in $M_{12}(\mathrm{SL}_2(\mathbb{Z}))$ for a section g by

$$\|g\|_{\mathrm{Pet}}(\tau) \coloneqq (4\pi \,\mathrm{Im}(\tau))^6 \, |g(\tau)|.$$

We endow $\mathcal{L}(\mathbb{C})$ with the metric for which (2.3) becomes an isometry, which we also denote by $\|\cdot\|_{\mathrm{Pet}}$. Let K be a number field and denote by K^0 (resp. K^∞) the set of nonarchimedean (resp. archimedean) places of K. For v in K^0 we denote by $\|\cdot\|_{\mathrm{can},v}$ the canonical metric on $\mathcal{L} \otimes K_v$. It is defined as follows. Let $\zeta = [\zeta_0 : \zeta_1]$ be the standard homogeneous coordinates of $\mathbb{P}^1$. Any nonzero section s of $\mathcal{L}$ can be identified canonically with a linear form $\ell_s(\zeta_0, \zeta_1)$. If $\zeta = [\zeta_0 : \zeta_1] \notin \mathrm{div}(s)$, then

$$\|s(\zeta)\|_{\mathrm{can},v} \coloneqq \frac{|\ell_s(\zeta_0, \zeta_1)|_v}{\max\{|\zeta_0|_v, |\zeta_1|_v\}}.$$

On every place v in K^∞ we denote by $\|\cdot\|_{\mathrm{Pet},v}$ the Petersson metric on $\mathcal{L} \otimes K_v$. Putting together all the metrics, $\mathcal{L} \otimes K$ becomes a metrized line bundle that we denote $\overline{\mathcal{L}}$.

Although the metrized line bundle $\overline{\mathcal{L}}$ is singular at $[1 : 0]$, it does induce a height function $h_{\overline{\mathcal{L}}}$ which is defined at every point ζ in $\mathcal{X}(\overline{\mathbb{Q}})$ different from $[1 : 0]$. To define $h_{\overline{\mathcal{L}}}(\zeta)$, let K be a number field that contains ζ. Then, the height $h_{\overline{\mathcal{L}}}(\zeta)$ of ζ is defined as

$$(2.4)$$
$$h_{\overline{\mathcal{L}}}(\zeta) \coloneqq \frac{\widehat{\deg}(\overline{\mathcal{L}}|_{D_\zeta})}{[K : \mathbb{Q}]} = \frac{1}{[K : \mathbb{Q}]} \left(\sum_{v \in K^0} -\log \|s(\zeta)\|_{\mathrm{can},v} + \sum_{v \in K^\infty} -\log \|s(\zeta)\|_{\mathrm{Pet},v} \right).$$

Lemma 2.1. *Let E/K be an elliptic curve and let L/K be a finite extension such that $E_L \coloneqq E \otimes L$ is semistable. Then, $h_{\mathrm{F}}(E_L/L) = \frac{1}{12} h_{\overline{\mathcal{L}}}(j(E))$.*

Proof. Let s_Δ be the section corresponding to Δ through (2.3). The corresponding linear form in the homogeneous coordinates $[c_4^3 : \Delta]$ is again Δ. Therefore

$$(2.5) \qquad -\log \|s_\Delta\|_{\mathrm{can},v} = -\log \frac{|\Delta|_v}{\max\{|c_4^3|_v, |\Delta|_v\}} = \log^+ |j|_v, \quad \text{for each } v \in L^0.$$

By the independence of $h_{\overline{\mathcal{L}}}(\zeta)$ on the choice of K, the fact that (2.3) is an isometry and equation (2.5), the assertion boils down to the equality

$$(2.6) \qquad\qquad -\log |\Delta_{E_L/L}|_v = \log^+ |j(E)|_v \quad \text{for each } v \in L^0.$$

For any place $v \in L^0$, we choose a minimal equation for the place v, having an associated quantity $c_4(E, v)$ as in [Sil, p. 46]. Then

$$|j(E)|_v = \frac{|c_4(E, v)^3|_v}{|\Delta_{E/L}|_v}.$$

Assume $|j(E)|_v > 1$. By hypothesis, E has split multiplicative reduction. Hence, $|c_4(E, v)|_v = 1$ (cf. [Sil, Proposition III.1.4]). The above relation implies (2.6).

Assume that $|j(E)|_v \leq 1$. Then, E has good reduction at v. Since $\Delta_{E/L}$ is minimal, we have that $|\Delta_{E/L}|_v = 1$ and both sides of (2.6) are zero. □

Now we compare the lower bounds of μ_F^{ess} in Theorem 1 with that obtained by Zhang's bounds on successive minima. Since the Petersson metric is singular, Theorem 5.2 in [Zha95] does not apply directly to our situation. We use instead the generalization by Bost and Freixas-i-Montplet [BFiM12, Theorem 3.5]. To state the lower bound, denote by $\mathrm{h}_{\overline{\mathcal{L}}}(\mathcal{X})$ the height of $\mathcal{X}$ with respect to $\overline{\mathcal{L}}$, by $\mu_{\overline{\mathcal{L}}}^{\mathrm{ess}}$ its essential minimum, and by ζ the Riemann zeta function. Combined with the computation of $\mathrm{h}_{\overline{\mathcal{L}}}(\mathcal{X})$ in [Küh01, Theorem 6.1], the lower bound reads

$$(2.7) \quad \mu_F^{\mathrm{ess}} = \frac{1}{12}\mu_{\overline{\mathcal{L}}}^{\mathrm{ess}} \geq \frac{1}{12} \cdot \frac{\mathrm{h}_{\overline{\mathcal{L}}}(\mathcal{X})}{2} = 6\left(\frac{1}{2}\zeta(-1) + \zeta'(-1)\right) = -1.2425268622\ldots,$$

which is weaker than the lower bound in Corollary 1.1, and cannot be used to deduce that $\mathrm{h}_F(0) < \mu_F^{\mathrm{ess}}$. Actually, these numerical estimates together with Corollary 1.1 imply the following.

Corollary 2.2. *Denoting by $\mu_{\overline{\mathcal{L}}}^{\mathrm{abs}}$ the infimum of $\mathrm{h}_{\overline{\mathcal{L}}}$ on $\overline{\mathbb{Q}}$, we have $\frac{1}{2}\mathrm{h}_{\overline{\mathcal{L}}}(\mathcal{X}) < \mu_{\overline{\mathcal{L}}}^{\mathrm{abs}} < \mu_{\overline{\mathcal{L}}}^{\mathrm{ess}}$.*

2.2. **Lower bounds through real sections.** We consider the graded semigroup

$$S_{\mathbb{Z}} = \coprod_{n \geq 0} \Gamma(\mathcal{X}, \mathcal{L}^{\otimes n}) \setminus \{0\}$$

with the tensor product as operation. We denote by $S_{\mathbb{R}}$ the corresponding semigroup with real coefficients. That is, any element of $s \in S_{\mathbb{R}}$, called a real global section, can be represented (nonuniquely) as

$$(2.8) \qquad s = s_1^{\otimes a_1} \otimes \cdots \otimes s_\ell^{\otimes a_\ell}, \quad s_1, \ldots, s_\ell \in S_{\mathbb{Z}}, a_1, \ldots, a_\ell > 0.$$

The *support* of the divisor of s is the set

$$|\operatorname{div}(s)| := \bigcup_k |\operatorname{div}(s_k)| \subset \mathcal{X}(\overline{\mathbb{Q}}),$$

and its *weight*

$$\frac{1}{12}(a_1 \deg(s_1) + \cdots + a_\ell \deg(s_\ell));$$

both are independent of the representation (2.8). We denote by $S_{\mathbb{R},1}$ the space of real global sections of weight one. Any real global section $s \in S_{\mathbb{R}}$ defines a *Green function*

$$\begin{array}{rcl} g_s \colon \quad \mathcal{X}(\mathbb{C}) & \longrightarrow & \mathbb{R} \cup \{\infty\} \\ x & \longmapsto & -\log \|s(x)\|_{\mathrm{Pet}}, \end{array}$$

where

$$\|s(x)\|_{\mathrm{Pet}} := \prod_{i=1}^{\ell} \|s_i(x)\|_{\mathrm{Pet}}^{a_i}.$$

The following is our main source of lower bounds of μ_F^{ess}.

Proposition 2.3. *Let $s \in S_{\mathbb{R},1}$ be a real global section of weight one and $x \in \mathcal{X}(\mathbb{C}) \setminus |\operatorname{div}(s)|$ an algebraic point not belonging to the support of the divisor of s. Then*

$$\mathrm{h}_{\mathrm{F}}(x) = \frac{1}{12}\,\mathrm{h}_{\overline{\mathcal{L}}}(x) \geq \inf_{y \in \mathcal{X}(\mathbb{C})} g_s(y) = -\log \sup_{y \in \mathcal{X}(\mathbb{C})} \|s(y)\|_{\mathrm{Pet}}.$$

In particular,

$$\mu_{\mathrm{F}}^{\mathrm{ess}} \geq \inf_{y \in \mathcal{X}(\mathbb{C})} g_s(y).$$

Proof. Choose a representation of s as in (2.8), and put $K := \mathbb{Q}(x)$. Let Σ be the set of embeddings of K in $\mathbb{C}$. Then by (2.4)

$$\mathrm{h}_{\mathrm{F}}(x) = \sum_{i=1}^{k} a_i \frac{1}{[K:\mathbb{Q}]} \left(\sum_{v \in K^0} -\log \|s_i(x)\|_{\mathrm{can},v} + \sum_{v \in K^\infty} -\log \|s_i(x)\|_{\mathrm{Pet},v} \right).$$

Since the sections s_i are global sections over the integer model $\mathcal{X}$, by the definition of the canonical metric we obtain that $\|s_i(x)\|_{\mathrm{can},v} \leq 1$. Therefore

$$\mathrm{h}_{\mathrm{F}}(x) \geq \sum_{i=1}^{k} a_i \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma \in \Sigma} -\log \|s_i(\sigma(x))\|_{\mathrm{Pet}}$$

$$= \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma \in \Sigma} g_s(\sigma(x))$$

$$\geq \inf_{y \in \mathcal{X}(\mathbb{C})} g_s(y).$$

The second statement follows directly from the first. $\qquad\square$

2.3. Review of low weight Eisenstein series.

Here, we recall the definition and special values of some classical Eisenstein series. Given $s \geq 0$, define $\sigma_s(n) = \sum_{d|n,d\geq 1} d^s$. For $\tau \in \mathbb{H}$ we put $q = e^{2\pi i \tau}$. Let

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \quad E_4(\tau) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_6(\tau) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n.$$

We also define

$$E_2^*(\tau) := E_2(\tau) - \frac{3}{\pi \operatorname{Im}(\tau)}.$$

The functions E_4 and E_6 are modular forms of level one and weights 4 and 6, respectively. The function E_2^* satisfies the relations

$$E_2^*\left(-\frac{1}{\tau}\right) = \tau^2 E_2^*(\tau), \quad E_2^*(\tau+1) = E_2^*(\tau) \text{ for all } \tau \in \mathbb{H},$$

but it is not holomorphic. On the other hand, E_2 is holomorphic (even at infinity) but it is not a classical modular form.

Ramanujan's identities (see, e.g., [Lan76, Theorem X.5.3]), imply the following relations

$$(2.9) \qquad E_2' = \frac{\pi i}{6}(E_2^2 - E_4), \quad E_4' = \frac{2\pi i}{3}(E_2 E_4 - E_6).$$

Lemma 2.4. *Letting $\rho := e^{\pi i/3}$, we have*

(1) $E_2(\rho) = \frac{2\sqrt{3}}{\pi}$, $\quad E_2'(\rho) = \frac{2i}{\pi}$, $\quad E_2''(\rho) = -\frac{4}{\sqrt{3}\pi} - \frac{\pi^2}{9}E_6(\rho)$;

(2) $E_6(\rho) = \frac{3^3}{2^9} \cdot \frac{\Gamma(1/3)^{18}}{\pi^{12}}$;

(3) $j'''(\rho) = -i\pi^3 \cdot 2^{10} \cdot 3 \cdot E_6(\rho)$;

(4) $\Delta(\rho) = -\frac{3^3}{2^{24}} \cdot \frac{\Gamma(1/3)^{36}}{\pi^{24}}$.

Proof. A proof of statements (2) and (3) can be found in [Wus14, p. 777]. We proceed to justify statement (1). Since E_2^* is weakly modular of weight two, and ρ is fixed by $\tau \mapsto \frac{1}{1-\tau}$, we have that $E_2^*(\rho) = 0$, implying $E_2(\rho) = \frac{2\sqrt{3}}{\pi}$. Similarly, using that E_4 is modular of weight four, we have that $E_4(\rho) = 0$. Then, using (2.9), we obtain

$$E_2'(\rho) = \frac{\pi i}{6}E_2^2(\rho) = \frac{\pi i}{6}\left(\frac{2\sqrt{3}}{\pi}\right)^2 = \frac{2i}{\pi}.$$

Using (2.9) again, we have that $E_2'' = \frac{\pi i}{6}(2E_2 \cdot E_2' - E_4')$ and $E_4'(\rho) = -\frac{2\pi i}{3}E_6(\rho)$. Then,

$$E_2''(\rho) = \frac{\pi i}{6}\left(2 \cdot \frac{2\sqrt{3}}{\pi} \cdot \frac{2i}{\pi} + \frac{2\pi i}{3}E_6(\rho)\right) = -\frac{4}{\sqrt{3}\pi} - \frac{\pi^2}{9}E_6(\rho),$$

proving claim (1). Finally, statement (4) follows from the other statements and the identity $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$ [Lan76, p. 9 and X.§4, Theorem 4.1]. $\qquad\square$

We record here a result of Masser on the zeroes and real values of E_2^*, which is shown in the proof of [Mas75, Lemma 3.2].

Lemma 2.5. *The function E_2^* vanishes at, and only at, the $\mathrm{SL}_2(\mathbb{Z})$-orbits of i and ρ. Moreover, we have that $\mathrm{Im}\left(E_2^*(z)\right) = 0$ if and only if $\mathrm{Re}(z) \in \frac{1}{2}\mathbb{Z}$.*

We denote by ∂ the holomorphic derivative. That is, for a given complex variable $\tau = x + iy \in \mathbb{C}$, we have $\partial = \frac{1}{2}(\partial_x - i\partial_y)$.

Lemma 2.6. *The following identities hold:*

$$(2.10) \quad g_\infty(\tau) \;=\; 2\pi\,\mathrm{Im}(\tau) - 6\log(\mathrm{Im}(\tau)) - 6\log(4\pi) - 24\sum_{r=1}^{\infty}\log|1 - q(\tau)^r|.$$

$$(2.11) \quad \partial g_\infty \;=\; -\pi i E_2^*.$$

Proof. Equation (2.10) is a direct consequence of the product formula for the modular discriminant. We then deduce

$$
\begin{aligned}
(2.12) \qquad \partial g_\infty(\tau) \;&=\; \frac{3i}{\mathrm{Im}(\tau)} - \pi i + 24\pi i\sum_{r=1}^{\infty} r\frac{q(\tau)^r}{1 - q(\tau)^r} \\
&=\; \frac{3i}{\mathrm{Im}(\tau)} - \pi i + 24\pi i\sum_{r=1}^{\infty} r\sum_{s=1}^{\infty} q(\tau)^{rs} \\
&=\; \frac{3i}{\mathrm{Im}(\tau)} - \pi i + 24\pi i\sum_{r=1}^{\infty}\sigma_1(r)q(\tau)^r \\
&=\; -\pi i E_2^*(\tau).
\end{aligned}
$$

$\qquad\square$

3. First and second minima of Faltings' height

In this section we prove Theorem 1 assuming Proposition A. The proof is in Section 3.2, after we give in Section 3.1 a proof of (1.4) and of the fact that the minimum value of h_F is $h_F(0)$.

In what follows we use the following formula of h_F. First, for each prime number p fix an extension $|\cdot|_p$ to $\overline{\mathbb{Q}}$ of the p-adic norm on $\mathbb{Q}$. Furthermore, consider the action of the Galois group $\mathrm{GAL}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ on $\overline{\mathbb{Q}}$ and for α in $\overline{\mathbb{Q}}$ denote by $\mathcal{O}(\alpha)$ the orbit of α. Then, choosing s in (2.4) as the section corresponding to $\Delta \in M_{12}\left(SL_2(\mathbb{Z})\right)$ through (2.3), we have by Lemma 2.1,

$$(3.1) \quad h_F(\alpha) = \frac{1}{12}\left(\frac{1}{\#\mathcal{O}(\alpha)}\sum_{\alpha' \in \mathcal{O}(\alpha)} g_{\mathrm{hyp}}(\alpha') + \frac{1}{\#\mathcal{O}(\alpha)}\sum_{p \text{ prime}}\sum_{\alpha' \in \mathcal{O}(\alpha)} \log^+ |\alpha'|_p\right).$$

Throughout this section we set $\rho := e^{\pi i/3}$ and denote by

$$(3.2) \qquad T := \left\{\tau \in \mathbb{H} : |\mathrm{Re}(\tau)| \le \frac{1}{2}, |\tau| \ge 1\right\}$$

the closure of the standard fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$.

3.1. **Minimum value of Faltings' height.** In this section we prove (1.4) and the fact that the minimum value of h_F is $h_F(0)$.

The first equality in (1.4) is a direct consequence of (3.1) and $j(\rho) = 0$ and the second one is a direct consequence of Lemma 2.4, (4). To show that the minimum value of h_F is $h_F(0)$, consider the lower bound

$$\inf\left\{h_F(\alpha) : \alpha \in \overline{\mathbb{Q}}\right\} \ge \frac{1}{12}\inf\left\{g_{\mathrm{hyp}}(\zeta) : \zeta \in \mathbb{C}\right\},$$

which follows trivially from (3.1). Since by (3.1) we also have $h_F(0) = \frac{1}{12}g_{\mathrm{hyp}}(0)$, the following lemma implies that the minimum value of h_F is $h_F(0)$.

Lemma 3.1. *For every τ in T, we have $g_\infty(\tau) \ge g_\infty\left(\frac{1}{2} + i\,\mathrm{Im}(\tau)\right)$, with equality if and only if $\mathrm{Re}(\tau) = \frac{1}{2}$. Moreover, the function $t \mapsto g_\infty\left(\frac{1}{2} + it\right)$ is strictly increasing on $\left[\frac{\sqrt{3}}{2}, +\infty\right)$. In particular, the function g_{hyp} attains its minimum value at, and only at, $\zeta = 0$.*

Proof. To prove the first statement, fix $\tau \in T$ and define a 1-periodic, smooth function $l\colon \mathbb{R} \to \mathbb{R}$ by $l(s) := g_\infty(s + i\,\mathrm{Im}(\tau))$. Since g_∞ is real valued, using (2.11) we have that

$$l'(s) = 2\,\mathrm{Re}\left(\partial g_\infty(s + i\,\mathrm{Im}(\tau))\right) = 2\pi\,\mathrm{Im}\left(E_2^*(s + i\,\mathrm{Im}(\tau))\right).$$

Hence, by Lemma 2.5, we conclude that the maximum and minimum values of $l(\cdot)$ are attained at $s \in \left\{0, \frac{1}{2}\right\}$. Then, the desired inequality $l(0) > l(\frac{1}{2})$ is equivalent to $|\Delta(i\,\mathrm{Im}(\tau))| \le |\Delta(\frac{1}{2} + i\,\mathrm{Im}(\tau))|$, and this is clear from the product formula for Δ.

To prove the second statement, note that the function $h\colon (0, +\infty) \to \mathbb{R}$ defined by $h(t) := g_\infty\left(\frac{1}{2} + it\right)$ satisfies

$$h'(t) = -2\,\mathrm{Im}\left(\partial g_\infty\left(\frac{1}{2} + it\right)\right) = 2\pi\,\mathrm{Re}\left(E_2^*\left(\frac{1}{2} + it\right)\right) = 2\pi E_2^*\left(\frac{1}{2} + it\right).$$

The last equality easily follows from the definition of E_2^*. In particular, h' is continuous and $\lim_{t \to +\infty} h'(t) = 2\pi$. The desired statement follows from the fact

that $h'(t)$ does not vanish on $\left(\frac{\sqrt{3}}{2}, +\infty\right)$, because the function E_2^* vanishes only at the orbits of i and ρ, cf. Lemma 2.5.   □

3.2. Second minimum of Faltings' height.
In this section we prove Theorem 1 assuming Proposition A. We postpone the proof of Proposition A to Section 7.

From the product formula for the modular discriminant we deduce the asymptotic expansion

$$(3.3) \qquad g_\infty(\tau) = -\log|q| - 6\log(-\log|q|) + O(1), \quad \mathrm{Im}(\tau) \to \infty, \quad q = e^{2\pi i \tau}.$$

Since $j(\tau) = \frac{1}{q} + O(1)$ when $\mathrm{Im}(\tau) \to \infty$, we infer from the definition of g_{hyp} in (1.6) the asymptotic expansion

$$(3.4) \qquad g_{\mathrm{hyp}}(z) = \log|z| - 6\log(\log|z|) + O(1) \text{ as } |z| \to \infty.$$

On the other hand, the function g_{hyp} is invariant under complex conjugation. More precisely,

$$(3.5) \qquad g_{\mathrm{hyp}}(\overline{z}) = g_{\mathrm{hyp}}(z) \quad \text{ for all } z \in \mathbb{C}.$$

Indeed, choose $\tau \in \mathbb{H}$ with $j(\tau) = z$. Since the coefficients in the q-expansion of j and Δ are real, we have the identities

$$(3.6) \qquad \overline{j(\tau)} = j(-\overline{\tau}), \quad \overline{\Delta(\tau)} = \Delta(-\overline{\tau}).$$

Then,

$$g_{\mathrm{hyp}}(\overline{z}) = g_{\mathrm{hyp}} \circ j(-\overline{\tau}) = g_\infty(-\overline{\tau}).$$

Since $\mathrm{Im}(-\overline{\tau}) = \mathrm{Im}(\tau)$ and $|\Delta(-\overline{\tau})| = |\overline{\Delta(\tau)}| = |\Delta(\tau)|$, we have that

$$g_\infty(-\overline{\tau}) = g_\infty(\tau) = g_{\mathrm{hyp}} \circ j(\tau) = g_{\mathrm{hyp}}(z),$$

justifying (3.5).

Proof of Theorem 1, assuming Proposition A. By (3.1) and Lemma 3.1 we have

$$\mathrm{h}_{\mathrm{F}}(1) = \frac{1}{12} g_{\mathrm{hyp}}(1) > \frac{1}{12} g_{\mathrm{hyp}}(0) = \mathrm{h}_{\mathrm{F}}(0).$$

Thus, to prove the theorem it is enough to show that there is $\kappa > 0$ such that for every algebraic number $\alpha \neq 0, 1$ we have $\mathrm{h}_{\mathrm{F}}(\alpha) \geq \mathrm{h}_{\mathrm{F}}(1) + \kappa$. To do this, we essentially apply, for a sufficiently small $\varepsilon > 0$, Proposition 2.3 with $s = (j - 1)^\varepsilon j^{\partial_x g_{\mathrm{hyp}}(1)} \Delta$.

By Proposition A, we have $1 - \partial_x g_{\mathrm{hyp}}(1) > 0$. For each ε in $(0, 1 - \partial_x g_{\mathrm{hyp}}(1))$, let $G_\varepsilon \colon \mathbb{C} \setminus \{0, 1\} \to \mathbb{R}$ be defined by

$$G_\varepsilon(z) := g_1(z) - \varepsilon \log|z - 1| = g_{\mathrm{hyp}}(z) - \partial_x g_{\mathrm{hyp}}(1) \cdot \log|z| - \varepsilon \log|z - 1|,$$

and for each prime number p, let $G_{\varepsilon,p} \colon \mathbb{C}_p \setminus \{0, 1\} \to \mathbb{R}$ be defined by

$$G_{\varepsilon,p}(z) := \log^+|z|_p - \partial_x g_{\mathrm{hyp}}(1) \cdot \log|z|_p - \varepsilon \log|z - 1|_p.$$

Since $\partial_x g_{\mathrm{hyp}}(1) + \varepsilon < 1$ and $\partial_x g_{\mathrm{hyp}}(1) > 0$, the function $G_{\varepsilon,p}$ is nonnegative. Then by (3.1) and by the product formula, for every α in $\overline{\mathbb{Q}} \setminus \{0, 1\}$ we have

$$12\,\mathrm{h}_{\mathrm{F}}(\alpha) = \frac{1}{\#\mathcal{O}(\alpha)} \sum_{\alpha' \in \mathcal{O}(\alpha)} G_\varepsilon(\alpha') + \frac{1}{\#\mathcal{O}(\alpha)} \sum_{p \text{ prime}} \sum_{\alpha' \in \mathcal{O}(\alpha)} G_{\varepsilon,p}(\alpha').$$

Since for each prime p the function $G_{\varepsilon,p}$ is nonnegative, to prove the theorem it is enough to show that

$$(3.7) \qquad \inf_{\mathbb{C}\setminus\{0,1\}} G_\varepsilon > g_{\mathrm{hyp}}(1).$$

Using the asymptotic of g_{hyp} given by (3.4), it follows that there are $\varepsilon_0 > 0$ and $R_0 > 0$ such that for each ε in $(0, \varepsilon_0)$ and each z in $\mathbb{C}$ satisfying $|z| > R_0$, we have

$$G_\varepsilon(z) \ge g_{\mathrm{hyp}}(1) + 1.$$

By Proposition A, there is $\varepsilon \in (0, \varepsilon_0)$ such that for some $\delta > 0$ and every z satisfying $|z - 1| \ge 1/2$ and $|z| \le R_0$, we have

$$G_\varepsilon(z) \ge g_{\mathrm{hyp}}(1) + \delta.$$

Finally, using Proposition A again, for each z in $\mathbb{C}$ satisfying $|z - 1| \le 1/2$, we have

$$G_\varepsilon(z) = g_1(z) - \varepsilon \log|z - 1| \ge g_{\mathrm{hyp}}(1) + \varepsilon \log 2.$$

This completes the proof of (3.7) and of the theorem. $\qquad\square$

Remark 3.2. For any given $\zeta \in \mathbb{C}$, we write $\zeta = x + iy$ the real and imaginary parts. For a real number a, set $c(a) := \inf_{\zeta \in \mathbb{C}} g_{\mathrm{hyp}}(\zeta) - a \log|\zeta|$ and note that by Proposition 2.3 with $s = j^a \Delta$,

$$(3.8) \qquad \mu_{\mathrm{F}}^{\mathrm{ess}} \ge c(a)/12.$$

Using Proposition A, we see that for any choice of a we have $c(a) \le \inf_{|\zeta|=1} g_{\mathrm{hyp}}(\zeta) = g_{\mathrm{hyp}}(1)$. Hence, the bound $\mu_{\mathrm{F}}^{\mathrm{ess}} \ge g_{\mathrm{hyp}}(1)/12 = \mathrm{h}_{\mathrm{F}}(1)$ is the best we can hope for using (3.8). In order to identify the value of a such that $c(a) = g_{\mathrm{hyp}}(1)/12$, we impose that $\zeta = 1$ is a critical point of $g_{\mathrm{hyp}}(\cdot) - a \log|\cdot|$, thus finding that necessarily $a = \partial_x g_{\mathrm{hyp}}(1)$.

4. DISTORTION ESTIMATES

In this section we estimate the inverse of j on a suitable neighborhood of the unit disk. After recalling the Koebe distortion theorem and some of its variants below, we explain the set up in Section 4.1 and then we proceed to the estimates in Section 4.2.

Theorem 4.1. *Let $\mathbb{D}$ be the open unit disk, and let $f_0 \colon \mathbb{D} \to \mathbb{C}$ be an univalent (i.e., holomorphic and injective) function such that $f_0(0) = 0$ and $f_0'(0) = 1$. Then for every $w \in \mathbb{D}$,*

(1)

$$\frac{|w|}{(1 + |w|)^2} \le |f_0(w)| \le \frac{|w|}{(1 - |w|)^2}, \qquad \frac{1 - |w|}{(1 + |w|)^3} \le |f_0'(w)| \le \frac{1 + |w|}{(1 - |w|)^3};$$

(2)

$$\left| w \frac{f_0'}{f_0}(w) \right| \le \frac{1 + |w|}{1 - |w|}, \qquad \left| w \frac{f_0''}{f_0'}(w) - \frac{2|w|^2}{1 - |w|^2} \right| \le \frac{4|w|}{1 - |w|^2};$$

(3)

$$|f_0(w) - w| \le \frac{|w|^2(2 - |w|)}{(1 - |w|)^2}, \qquad \left| w \frac{f_0'}{f_0}(w) - 1 \right| \le 2|w| \frac{(1 + |w|)^2}{(1 - |w|)^3}.$$

Proof. Parts (1) and (2) are proved in [Pom75, Lemma 1.3 and Theorem 1.6]. Part (3) is undoubtedly well-known but we provide a proof due to the lack of a suitable reference. Write $f_0(w) = \sum_{n=1}^{\infty} a_n w^n$. Then, $a_1 = 1$ and de Branges' theorem ensures that $|a_n| \leq n$ for all n; see for example [Pom75]. Hence,

$$|f_0(w) - w| = \left| \sum_{n=2}^{\infty} a_n w^n \right| \leq \sum_{n=2}^{\infty} n|w|^n = \frac{|w|^2(2 - |w|)}{(1 - |w|)^2}.$$

Similarly,

$$|wf_0'(w) - f_0(w)| = \left| \sum_{n=2}^{\infty} a_n(n-1)w^n \right| \leq |w|^2 \sum_{n=2}^{\infty} (n-1)n|w|^{n-2} = \frac{2|w|^2}{(1 - |w|)^3}.$$

This estimate, combined with part (1) finishes the proof. □

4.1. **Set up.** Since the j-invariant is injective when restricted to a fundamental domain, we aim to use Theorem 4.1 to deduce an approximation of it by a rational function on a neighborhood of $\rho := \frac{1+i\sqrt{3}}{2}$.

In order to transport the situation to a disk, consider the function $\psi \colon \mathbb{D} \to \mathbb{H}$ defined by

(4.1) $$\psi(w) := \frac{\overline{\rho}w + \rho}{w + 1},$$

and let $j_\mathbb{D} \colon \mathbb{D} \to \mathbb{C}$ be the function defined by

(4.2) $$j_\mathbb{D} := j \circ \psi.$$

Consider the following fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$,

$$T_0 := \left\{ \tau \in \mathbb{H} : 0 \leq \mathrm{Re}(\tau) \leq \frac{1}{2}, \quad |\tau - 1| > 1 \right\} \setminus \{it : 0 < t \leq 1\}.$$
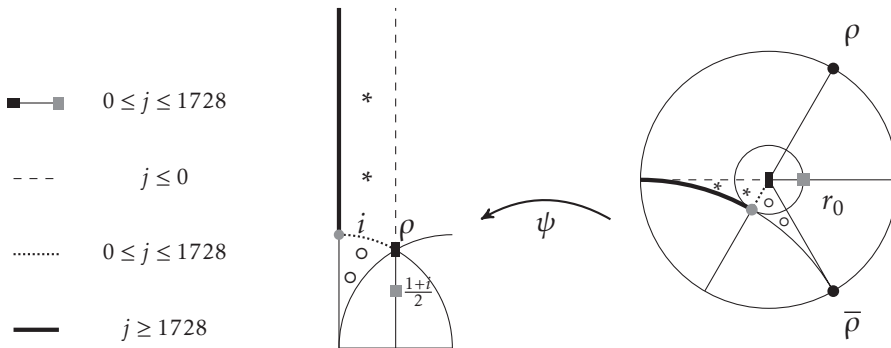

FIGURE 4.1. Action of ψ on the fundamental region.

Lemma 4.2. *Define*

$$r_0 := 2 - \sqrt{3}, \quad B(0, r_0) := \{w \in \mathbb{C} : |w| < r_0\}$$

and

$$B^* := \{z \in B(0, r_0) : \arg z \in [\pi, 5\pi/3]\}.$$

Then, we have that $\psi(B^*) \subseteq T_0$.

Proof. Let $\varphi\colon \mathbb{H} \to \mathbb{D}$ be the inverse of the function ψ, which is given by

$$\varphi(\tau) := -\frac{\tau - \rho}{\tau - \overline{\rho}}.$$

We show the equivalent assertion $B^* \subseteq \varphi(T_0)$. Since φ is a conformal mapping, it is enough to study the image of the boundary in $\mathbb{H}$ of T_0, which is the union of the three sets

$$L_0 := \{it : t > 0\}, \quad L_1 := \left\{\frac{1}{2} + it : t \geq \frac{\sqrt{3}}{2}\right\},$$

$$C := \left\{\tau \in \mathbb{H} : |\tau - 1| = 1, 0 \leq \operatorname{Re}(\tau) \leq \frac{1}{2}\right\}.$$

Since φ is a Möbius transformation, the three sets are sent into line or circle segments. Noting that $\varphi(\overline{\rho}) = \infty$, we find

$$\varphi(L_1) = (-1, 0], \quad \varphi(C) = \{t\overline{\rho} : 0 \leq t < 1\}.$$

Let R be the circle that passes through the points $\{\overline{\rho}, \varphi(i), -1\}$. Then, $\varphi(L_0)$ is the open arc of R that contains $\varphi(i)$ and has extreme points $\overline{\rho}$ and -1.

A calculation shows that $\varphi(i) = -r_0\rho$. We conclude the proof by observing that $\arg(\overline{\rho}) = \frac{5\pi}{3}$. $\qquad\square$

Note that $\psi(0) = \rho$, and that $j_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{C}$ is invariant under the rotation $z \mapsto -\rho z$. It follows that there is a holomorphic function $f\colon \mathbb{D} \to \mathbb{C}$ such that for every w in $\mathbb{D}$ we have

$$(4.3) \qquad\qquad j_{\mathbb{D}}(w) = f(w^3).$$

Lemma 4.3. *Let $r_0 = 2 - \sqrt{3}$. The function f defined in (4.3) is univalent on $B\left(0, r_0^3\right)$. In addition, we have that $f'(0) = i\frac{\sqrt{3}}{2} \cdot j'''(\rho) = \left(\frac{\sqrt{3}}{\pi}\Gamma\left(\frac{1}{3}\right)^2\right)^9$. In particular, $f'(0)$ is a real number and*

$$237698 \leq f'(0) \leq 237699.$$

Proof. We first show that f is injective. Let $w_1, w_2 \in B(0, r_0^3)$ be such that $f(w_1) = f(w_2)$. Choose $z_1, z_2 \in B(0, r_0)$ such that

$$z_i^3 = w_i, \quad \arg z_i \in [\pi, 5\pi/3), \quad i = 1, 2.$$

Then, we have that $j_{\mathbb{D}}(z_1) = j_{\mathbb{D}}(z_2)$, implying

$$j\left(\psi(z_1)\right) = j\left(\psi(z_2)\right).$$

Since Lemma 4.2 ensures that $\psi(z_1), \psi(z_2) \in T_0$ and the j-invariant is injective on any fundamental domain, we conclude $\psi(z_1) = \psi(z_2)$, whence $z_1 = z_2$, showing that $w_1 = w_2$.

Using Lemma 2.4 and $\psi'(0) = -i\sqrt{3}$, we find that

$$(4.4) \qquad f'(0) = \frac{1}{6}j_{\mathbb{D}}'''(0) = \frac{1}{6}j'''(\rho)\psi'(0)^3 = \frac{1}{6}\left(-2^{10}3\pi^3 i E_6(\rho)\right)\left(-i\sqrt{3}\right)^3$$

$$= 2^9 3\sqrt{3}\pi^3 E_6(\rho) = \left(\frac{\sqrt{3}}{\pi}\Gamma\left(\frac{1}{3}\right)^2\right)^9 = 237698.411625786\ldots \qquad\square$$

Next we apply the distortions statements in Theorem 4.1 several times to $f|_{B(0,r_0^3)}$. To normalize this function, throughout the rest of this section we put

$$\varepsilon_1 := \left(r_0^3 f'(0) \right)^{-1},$$

and let $f_0 : \mathbb{D} \to \mathbb{C}$ be the function defined by

$$(4.5) \qquad\qquad f_0(z) := \varepsilon_1 f(r_0^3 z).$$

It is univalent and satisfies $f_0(0) = 0$ and $f_0'(0) = 1$.

Lemma 4.4. *For every z in $\mathbb{D}$, we have the inequalities*

$$\left| \frac{f'}{f}(r_0^3 z) r_0^3 z \right| \le \frac{1 + |z|}{1 - |z|}, \quad \left| \frac{f'}{f}(r_0^3 z) r_0^3 z - 1 \right| \le 2|z| \frac{(1 + |z|)^2}{(1 - |z|)^3}$$

and

$$\left| \frac{f''}{f}(r_0^3 z) r_0^6 z^2 \right| \le \frac{2|z|(2 + |z|)}{(1 - |z|)^2}.$$

Proof. Note that the function f_0 defined by (4.3) satisfies

$$\frac{f_0'}{f_0}(z) = r_0^3 \frac{f'}{f}(r_0^3 z) \text{ and } \frac{f_0''}{f_0'}(z) = r_0^3 \frac{f''}{f'}(r_0^3 z).$$

Hence, the first and second asserted inequalities are a direct consequence of Theorem 4.1, (2), and (3). On the other hand, we can use Theorem 4.1, (2), again to obtain

$$\left| r_0^3 z \frac{f''}{f'}(r_0^3 z) \right| \le \left| z \frac{f_0''}{f_0'}(z) - \frac{2|z|^2}{1 - |z|^2} \right| + \frac{2|z|^2}{1 - |z|^2} \le \frac{2|z|(2 + |z|)}{1 - |z|^2}.$$

Then,

$$\left| \frac{f''}{f}(r_0^3 z) r_0^6 z^2 \right| = \left| r_0^3 z \frac{f''}{f'}(r_0^3 z) \cdot \frac{f'}{f}(r_0^3 z) r_0^3 z \right| \le \frac{2(2 + |z|)|z|}{(1 - |z|)^2},$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. Approximating the inverse of j on a neighborhood of the unit disk.
Here, we provide estimates on $j_{\mathbb{D}}^{-1}$ on a neighborhood of the unit circle. To this end, for each α in $\left(0, \frac{1}{4\varepsilon_1} \right)$ denote by $\kappa(\alpha)$ the smallest solution x of

$$1 + x = \alpha \left(1 + (1 + x)\varepsilon_1 \right)^2,$$

which is given explicitly by

$$\kappa(\alpha) = \frac{1}{2\varepsilon_1^2} \cdot \left(\frac{1}{\alpha} - 2\varepsilon_1(1 + \varepsilon_1) - \sqrt{\frac{1}{\alpha} \left(\frac{1}{\alpha} - 4\varepsilon_1 \right)} \right),$$

and put

$$r_+(\alpha) := \left(\frac{1 + \kappa(\alpha)}{f'(0)} \right)^{1/3}, \text{ and } r_-(\alpha) := (1 - 4\alpha\varepsilon_1)^{1/3} r_+(\alpha).$$

In the rest of this section we denote

$$\kappa_1 := \kappa(1), \quad r_1^+ := r_+(1), \quad r_1^- := r_-(1).$$

Noting that

$$(4.6) \qquad\qquad \kappa_1 = 2\varepsilon_1 \left(\frac{2 + \varepsilon_1}{1 - 2\varepsilon_1 - 2\varepsilon_1^2 + \sqrt{1 - 4\varepsilon_1}} \right),$$

and that by Lemma 4.3,

$$(4.7) \qquad \frac{1}{4573} \leq \varepsilon_1 \leq \frac{1}{4572},$$

we have

$$(4.8) \qquad 2\varepsilon_1 \leq \kappa_1 \leq 2\varepsilon_1(1 + 3\varepsilon_1) \leq \frac{1}{2284}.$$

Lemma 4.5. *Let $w \in B(0, r_0)$ and put $\zeta = j_{\mathbb{D}}(w)$. Then, we have that $r_-(|\zeta|) \leq |w| \leq r_+(|\zeta|)$. In particular, if $|j_{\mathbb{D}}(w)| = 1$, then $r_1^- \leq |w| \leq r_1^+$.*

Proof. Applying Theorem 4.1, (1), to f_0, it follows that for each z in $\mathbb{D}$ satisfying $|z| = (1 + \kappa(|\zeta|))\,\varepsilon_1$, we have by the definition of $\kappa(\cdot)$

$$\left| f\left(r_0^3 z\right) \right| = \frac{|f_0(z)|}{\varepsilon_1} \geq \frac{1}{\varepsilon_1} \cdot \frac{|z|}{(1 + |z|)^2} = \frac{1 + \kappa(|\zeta|)}{(1 + (1 + \kappa(|\zeta|))\,\varepsilon_1)^2} = |\zeta|.$$

Hence, the domain bounded by the Jordan curve $f\left(\partial B\left(0, \frac{1 + \kappa(|\zeta|)}{f'(0)}\right)\right)$ contains $B(0, |\zeta|)$. Since $\zeta = f(w^3)$, it follows that w^3 is in $B\left(0, \frac{1 + \kappa(|\zeta|)}{f'(0)}\right)$. This proves the second desired inequality.

To prove the first inequality, we apply Theorem 4.1, (1), to f_0 and $z = \left(\frac{w}{r_0}\right)^3$. The inequality we have just proved implies $|z| \leq (1 + \kappa(|\zeta|))\,\varepsilon_1$. Hence, we obtain

$$|\zeta| = \left| f\left(w^3\right) \right| = \frac{1}{\varepsilon_1}\left| f_0\left(\left(\frac{w}{r_0}\right)^3\right) \right| \leq \frac{1}{\varepsilon_1} \cdot \frac{|z|}{(1 - |z|)^2} \leq \frac{f'(0)|w|^3}{(1 - (1 + \kappa(|\zeta|))\,\varepsilon_1)^2}.$$

Then, by the definition of $\kappa(\cdot)$,

$$|w|^3 \geq |\zeta| \frac{(1 - (1 + \kappa(|\zeta|))\,\varepsilon_1)^2}{f'(0)} = (1 - 4|\zeta|\varepsilon_1) \cdot \frac{1 + \kappa(|\zeta|)}{f'(0)}.$$

This proves the first inequality, and completes the proof of the lemma. $\qquad \square$

Lemma 4.6. *Let $\zeta \in S^1$ and let $w \in B(0, r_0^3)$ be such that $j_{\mathbb{D}}(w) = \zeta$. Then, we have that:*

(1) $|\zeta - f'(0)w^3| \leq \frac{\varepsilon_1(1 + \kappa_1)^2(2 - \varepsilon_1(1 + \kappa_1))}{(1 - \varepsilon_1(1 + \kappa_1))^2} \leq \frac{1}{2283}$;

(2) $\left| \log(1 - |w|^2) - \log\left(1 - f'(0)^{-\frac{2}{3}}\right) \right| \leq \frac{2}{3} f'(0)^{-\frac{2}{3}} \kappa_1 \left(1 - (r_1^+)^2\right)^{-1} \leq 7.7 \cdot 10^{-8}$;

(3) $185 \leq |j'_{\mathbb{D}}(w)| \leq 186.054$.

Proof. Set $z = \frac{w^3}{r_0^3}$. From the definitions and using part (3) of Theorem 4.1 we have that

$$|\zeta - f'(0)w^3| = \frac{1}{\varepsilon_1}|f_0(z) - z| \leq \frac{|z|^2(2 - |z|)}{\varepsilon_1(1 - |z|)^2}.$$

By Lemma 4.5, we have that $|z| \leq (1 + \kappa_1)\varepsilon_1$. This estimate, (4.8), and Lemma 4.3 justify the first assertion.

In view of Lemma 4.5, we have that

$$\left| \log(1 - |w|^2) - \log(1 - f'(0)^{-\frac{2}{3}}) \right| \leq (|w|^2 - f'(0)^{-\frac{2}{3}}) \cdot \frac{1}{1 - (r_1^+)^2}$$

$$\leq f'(0)^{-\frac{2}{3}} \left((1 + \kappa_1)^{\frac{2}{3}} - 1 \right) \cdot \frac{1}{1 - (r_1^+)^2}$$

$$\leq \frac{2}{3} f'(0)^{-\frac{2}{3}} \frac{\kappa_1}{1 - (r_1^+)^2}.$$

The second assertion is obtained by evaluating this last quantity.

Using the definition of $j_{\mathbb{D}}$, we have

$$j_{\mathbb{D}}'(w) = 3w^2 f'(w^3) = 3w_1^2 f'(0) f_0'(z).$$

Then, Theorem 4.1, (1), implies

$$\frac{1 - |z|}{(1 + |z|)^3} \leq \left| \frac{j_{\mathbb{D}}'(w)}{3w^2 f'(0)} \right| \leq \frac{1 + |z|}{(1 - |z|)^3}.$$

Lemma 4.5, ensures that $r_1^- \leq |w| \leq r_1^+$. Using Lemma 4.3 and (4.6), we find

$$|j_{\mathbb{D}}'(w)| \leq 3(r_1^+)^2 f'(0) \cdot \frac{1 + \frac{(r_1^+)^3}{r_0^3}}{\left(1 - \frac{(r_1^+)^3}{r_0^3}\right)^3} = \frac{3 f'(0)^{1/3}(1 + \kappa_1)^{2/3}}{(1 - 4\varepsilon_1)(1 - (1 + \kappa_1)^2 \varepsilon_1^2)} \leq 186.$$

A similar reasoning leads to the lower bound $|j_{\mathbb{D}}'(w)| \geq 185$. □

5. Approximating g_∞ on a neighborhood of the locus $|j| \leq 1$

The aim of this section is to provide an approximation of g_∞ and of its first and second derivatives, on a suitable neighborhood of the locus $|j| \leq 1$. This is stated as Proposition 5.1 below. It is convenient to express this approximation in terms of the function $g_{\mathbb{D}} \colon \mathbb{D} \to \mathbb{R}$ given by

$$(5.1) \qquad\qquad g_{\mathbb{D}} := g_\infty \circ \psi,$$

where ψ is defined in (4.1). The approximation is also stated in terms of the derivative $f'(0) = i\frac{\sqrt{3}}{2} \cdot j'''(\rho)$, computed in Lemma 4.3, and of the holomorphic function $\widehat{h} \colon \mathbb{D} \to \mathbb{C}$ defined by

$$\widehat{h}(w) := \frac{\Delta \circ \psi(w)}{(1 + w)^{12}}.$$

Note that for every w in $\mathbb{D}$ we have

$$\operatorname{Im}(\psi(w)) = \frac{\sqrt{3}}{2} \cdot \frac{1 - |w|^2}{|1 + w|^2}$$

and

$$(5.2) \qquad g_{\mathbb{D}}(w) = -\log(1728\pi^6) - 6\log(1 - |w|^2) - \log|\widehat{h}(w)|.$$

Proposition 5.1. *For every w in $\mathbb{D}$ satisfying $|w| \leq 1 - \frac{\pi}{2\sqrt{3}}$, we have*

$$(5.3) \quad \left| g_{\mathbb{D}}(w) - \left(g_{\mathbb{D}}(0) - 6\log(1 - |w|^2) - \frac{f'(0)}{13824}\operatorname{Re}(w^3) \right) \right| \leq 6^3 |w|^6,$$

$$(5.4) \quad \left| (\log \widehat{h})'(w)w - \frac{3 \cdot f'(0)}{13824}w^3 \right| \leq 6^4 |w|^6,$$

$$(5.5) \quad \left| (\log \widehat{h})''(w)w^2 - \frac{6 \cdot f'(0)}{13824}w^3 \right| \leq 5 \cdot 6^4 |w|^6.$$

The proof of this proposition boils down to an estimate of sixth order derivative of the holomorphic function $\log \widehat{h}$ (Lemma 5.5). This is done in Section 5.2, after establishing some properties of the function $\widehat{h}$ in Section 5.1.

5.1. **Some properties of $\widehat{h}$.** Recall that a holomorphic function is *real*, if it is defined on a connected domain that is invariant under complex conjugation and if the function commutes with complex conjugation.

Lemma 5.2. *The functions $j_{\mathbb{D}}$ and $\widehat{h}$ are both real.*

Proof. A routine calculation shows that $-\overline{\psi(w)} = \psi(\overline{w}) - 1$. Using (3.6), we obtain

$$\overline{j_{\mathbb{D}}(w)} = \overline{j \circ \psi(w)} = \overline{j\left(-\overline{\psi(w)}\right)} = j(\psi(\overline{w}) - 1) = j \circ \psi(\overline{w}) = j_{\mathbb{D}}(\overline{w}).$$

By the same argument, we have that

$$\overline{\widehat{h}(w)} = \frac{\Delta\left(-\overline{\psi(w)}\right)}{(1 + \overline{w})^{12}} = \frac{\Delta\left(\psi(\overline{w}) - 1\right)}{(1 + \overline{w})^{12}} = \widehat{h}(\overline{w}). \qquad \square$$

Lemma 5.3. *There is a holomorphic function $h\colon \mathbb{D} \to \mathbb{C}$, such that for each w in $\mathbb{D}$ we have $h(w^3) = \widehat{h}(w)$.*

Proof. It is enough to show that h is invariant under the rotation $w \mapsto -\rho w$. Fix w in $\mathbb{D}$ and put $\tau := \psi(w)$. Noting that $\psi(-\rho w) = \frac{\tau - 1}{\tau}$, and using that Δ is a modular form of weight 12, we have

$$\widehat{h}(-\rho w) = \frac{\Delta \circ \psi(-\rho w)}{(1 - \rho w)^{12}} = \frac{\Delta\left(\frac{\tau - 1}{\tau}\right)}{(1 - \rho w)^{12}} = \Delta(\tau)\left(\frac{\tau}{1 - \rho w}\right)^{12}$$

$$= \Delta \circ \psi(w)\left(\frac{\rho}{1 + w}\right)^{12} = \widehat{h}(w). \qquad \square$$

Lemma 5.4. *We have*
  (1) $(-\log \widehat{h})'(0) = (-\log \widehat{h})''(0) = (-\log \widehat{h})^{(4)}(0) = (-\log \widehat{h})^{(5)}(0) = 0$,
  (2) $\frac{(-\log \widehat{h})'''(0)}{3!} = -i\frac{\sqrt{3}}{2} \cdot \frac{j'''(\rho)}{13824} = -\frac{f'(0)}{13824}$.

Proof. The function $\widehat{h}$ does not vanish on $\mathbb{D}$, hence we can choose a branch of the logarithm such that $K_0(w) := -\log \widehat{h}(w)$ is holomorphic. Lemma 5.3, ensures that there is a holomorphic function $K_1(w)$ such that $K_0(w) = K_1(w^3)$. This justifies the first part of the assertion. On the other hand, we have that

$$K_0'(w) = \frac{12}{1 + w} - \left(\frac{\Delta'}{\Delta} \circ \psi(w)\right) \cdot \psi'(w).$$

Note that by (2.11) and the equation $\partial g_\infty(\tau) = \frac{3i}{\operatorname{Im}(\tau)} - \frac{1}{2} \cdot \frac{\Delta'(\tau)}{\Delta(\tau)}$,

$$\frac{\Delta'}{\Delta} = 2\pi i E_2 \quad \text{and} \quad \psi'(w) = -\frac{i\sqrt{3}}{(1+w)^2}.$$

We conclude that

$$(1+w)^2 K_0'(w) = 2\sqrt{3}\pi \left(\frac{2\sqrt{3}}{\pi}(1+w) - E_2 \circ \psi(w) \right).$$

Taking holomorphic derivative, we get

$$2(1+w)K_0'(w) + (1+w)^2 K_0''(w) = 2\sqrt{3}\pi \left(\frac{2\sqrt{3}}{\pi} - E_2' \circ \psi(w) \cdot \psi'(w) \right),$$

implying

$$2(1+w)^3 K_0'(w) + (1+w)^4 K_0''(w) = 6\pi \left(\frac{2}{\pi}(1+w)^2 + iE_2' \circ \psi(w) \right).$$

Taking holomorphic derivative once more, we get

$$6(1+w)^2 K_0'(w) + 2(1+w)^3 K_0''(w) + 4(1+w)^3 K_0''(w) + (1+w)^4 K_0'''(w)$$
$$= 6\pi \left(\frac{4}{\pi}(1+w) + \frac{\sqrt{3}}{(1+w)^2} E_2'' \circ \psi(w) \right).$$

Setting $w = 0$, we obtain

$$K_0'''(0) = 24 + 6\sqrt{3}\pi E_2''(\rho).$$

Then, using Lemma 2.4, (1) and (2), and Lemma 4.3, we conclude the proof. $\square$

5.2. Approximating $g_\mathbb{D}$. In this section we give the proof of Proposition 5.1. The following is the main ingredient.

Lemma 5.5. *For every w in $\mathbb{D}$ satisfying $|w| \leq 1 - \frac{\pi}{2\sqrt{3}}$, we have*

$$\frac{\left| (\log \widehat{h})^{(6)}(w) \right|}{6!} \leq 6^3.$$

The proof of this lemma is given after the following one.

Lemma 5.6. *For every integer $n \geq 4$, we have*

$$n^5 \sigma_1(n) \leq \left(3^5 \cdot 4 \right) 14^{n-3}.$$

Proof. Since $\sigma_1(n) \leq \frac{n(n+1)}{2}$, it is enough to prove that for every $n \geq 4$ we have

$$n^6(n+1) \leq \left(3^5 \cdot 8 \right) 14^{n-3}.$$

We proceed by induction. The case $n = 4$ can be readily verified. For the induction step, just note that for each $n \geq 4$ we have

$$\frac{(n+1)^5(n+2)}{n^5(n+1)} \leq \frac{5^5 \cdot 6}{4^5 \cdot 5} \leq 14.   \square$$

Proof of Lemma 5.5. Let $\ell\colon \mathbb{H} \to \mathbb{C}$ be the function defined by

$$\ell(\tau) := \left(\frac{\Delta'}{\Delta}\right)(\tau) = 2\pi i - 48\pi i \sum_{n=1}^{\infty} \sigma_1(n)q^n, \quad q = q(\tau) = e^{2\pi i \tau}.$$

Note that for every integer $k \geq 1$ we have

$$\ell^{(k)} = -48\pi i \sum_{n=1}^{\infty} (2\pi i n)^k \sigma_1(n) q^n$$

and

$$\psi^{(k)}(w) = (-i\sqrt{3})k!\left(-\frac{1}{1+w}\right)^{k+1}.$$

Defining the complex polynomial

$$P(\zeta) := 1 - 30\zeta + 300\zeta^2 - 1200\zeta^3 + 1800\zeta^4 - 720\zeta^5,$$

and using the formula

$$
\begin{aligned}
(\ell \circ \psi \cdot \psi')^{(5)} = \; & \ell^{(5)} \circ \psi \cdot (\psi')^6 + 15\ell^{(4)} \circ \psi \cdot (\psi')^4 \psi'' \\
& + 45\ell''' \circ \psi \cdot (\psi')^2(\psi'')^2 + 20\ell''' \circ \psi \cdot (\psi')^3 \psi''' \\
& + 60\ell'' \circ \psi \cdot \psi'\psi''\psi''' + 15\ell'' \circ \psi \cdot (\psi'')^3 + 15\ell'' \circ \psi \cdot (\psi')^2 \psi^{(4)} \\
& + 10\ell' \circ \psi \cdot (\psi''')^2 + 15\ell' \circ \psi \cdot \psi''\psi^{(4)} + 6\ell' \circ \psi \cdot \psi'\psi^{(5)} \\
& + \ell \circ \psi \cdot \psi^{(6)},
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{(\log \widehat{h})^{(6)}(w)}{6!} &= \frac{1}{6!}\left(\frac{\widehat{h}'}{\widehat{h}}\right)^{(5)}(w) \\
&= \frac{2}{(1+w)^6} + \frac{(\ell \circ \psi \cdot \psi')^{(5)}(w)}{6!} \\
&= \frac{2}{(1+w)^6} - \frac{2\sqrt{3}\pi}{(1+w)^7} \\
&\quad - \frac{2^5 3^2 \pi^6}{5(1+w)^{12}} \sum_{n=1}^{\infty} P\left(\frac{1+w}{2\sqrt{3}\pi n}\right) n^5 \sigma_1(n)(q \circ \psi(w))^n.
\end{aligned}
$$

(5.6)

Fix w_0 in $\mathbb{D}$ satisfying $|w_0| \leq 1 - \frac{\pi}{2\sqrt{3}}$. Since $g_{\mathbb{D}}$ is real and invariant under the rotation $w \mapsto -\rho w$, it is enough to consider the case where $\arg(w_0)$ is in $\left[\frac{\pi}{2}, \frac{\pi}{3}\right]$. This last condition implies that $\psi(w_0)$ is in T, so, if we put $q_0 := \exp\left(-\sqrt{3}\pi\right)$, then $|q \circ \psi(w_0)| \leq q_0$. On the other hand, noting that

$$P(\zeta) = (1 - 10\zeta)^3 - 200(1 - 10\zeta)(1 + \zeta)\zeta^3 - 2720\zeta^5,$$

$$\left|\frac{1+w_0}{2\sqrt{3}\pi}\right| \leq \frac{1}{9}, \quad \left|1 - 10\left(\frac{1+w_0}{2\sqrt{3}\pi}\right)\right| \leq \frac{1}{6}, \quad \left|1 - 10\left(\frac{1+w_0}{4\sqrt{3}\pi}\right)\right| \leq \frac{7}{12},$$

and that for every integer $n \geq 3$ we have

$$\left|1 - 10\left(\frac{1+w_0}{2\sqrt{3}\pi n}\right)\right| \leq 1,$$

we obtain

$$(5.7) \qquad \left| P\left(\frac{1+w_0}{2\sqrt{3}\pi} \right) \right| \leq \frac{2}{19}, \quad \left| P\left(\frac{1+w_0}{4\sqrt{3}\pi} \right) \right| \leq \frac{2}{9},$$

and for every integer $n \geq 3$,

$$\left| P\left(\frac{1+w_0}{2\sqrt{3}\pi n} \right) \right| \leq \frac{91}{90}.$$

Together with Lemma 5.6,

$$(5.8) \qquad |q \circ \psi(w_0)| \leq q_0 \text{ and } q_0^{-1} = \exp\left(\sqrt{3}\pi \right) > 6^3,$$

the last inequality implies

$$\left| \sum_{n=3}^{\infty} P\left(\frac{1+w}{2\sqrt{3}\pi n} \right) n^5 \sigma_1(n)(q \circ \psi(w_0))^n \right| \leq \frac{91}{90} \sum_{n=3}^{\infty} n^5 \sigma_1(n) q_0^n$$

$$\leq \frac{91}{90} \cdot 3^5 \cdot 4q_0^3 \frac{14}{13} = \frac{2^2 3^3 7^2}{5} q_0^3 \leq \frac{7^2}{2^7 3^6 \cdot 5} \leq \frac{5}{2^6 3^6}.$$

Combined with (5.6), (5.7), (5.8), and the inequality $\frac{\pi}{2} > 15^{\frac{1}{6}}$, this implies

$$\left| \frac{(\log \widehat{h})^{(6)}(w_0)}{6!} \right| \leq \frac{2(2\sqrt{3})^6}{\pi^6} + \frac{2\sqrt{3}\pi(2\sqrt{3})^7}{\pi^7}$$

$$+ \frac{2^5 3^2 \pi^6 (2\sqrt{3})^{12}}{5\pi^{12}} \left(\frac{2}{19} q_0 + \frac{2}{9} \cdot 2^5 \cdot 3q_0^2 + \frac{5}{2^6 3^6} \right)$$

$$\leq \frac{1728}{\pi^6} \left(14 + \frac{2^{11} 3^5}{5} \left(\frac{1}{2^2 3^3 \cdot 19} + \frac{1}{3^7} + \frac{5}{2^6 3^6} \right) \right).$$

$$= 6^3 \cdot \frac{8}{\pi^6} \left(14 + \frac{2^9 3^2}{5 \cdot 19} + \frac{2^{11}}{3^2 \cdot 5} + \frac{2^5}{3} \right).$$

$$\leq 6^3 \frac{8}{\pi^6} \left(14 + 49 + 46 + 11 \right)$$

$$= 6^3 \left(\frac{2^6 \cdot 15}{\pi^6} \right)$$

$$\leq 6^3. \qquad \qquad \square$$

Proof of Proposition 5.1. Inequality (5.3) is equivalent to the assertion

$$(5.9) \qquad \left| -\log|\widehat{h}(w)| + \log|\Delta(\rho)| + \frac{f'(0)}{13824} \operatorname{Re}(w^3) \right| \leq 6^3 |w|^6 \text{ for all } |w| \leq 1 - \frac{\pi}{2\sqrt{3}}.$$

Fix w with $0 < |w| \leq 1 - \frac{\pi}{2\sqrt{3}}$ and $\zeta \in \mathbb{C}$ satisfying $|\zeta| = 1$. The function $\widehat{h}$ does not vanish on $\mathbb{D}$, hence we can choose a branch of the logarithm such that $K_0(w) := -\log \widehat{h}(w)$ is holomorphic. Let $\alpha \colon (-\varepsilon, 1 + \varepsilon) \to \mathbb{R}$ be given by $\alpha(t) = \operatorname{Re}(\zeta K_0(tw))$. The function α is well defined and smooth if $\varepsilon > 0$ is small enough. We have that $\alpha^{(n)}(t) = \operatorname{Re}\left(\zeta K_0^{(n)}(tw)w^n \right)$ for all $n \geq 0$.

Using Lemma 5.4, (1), and the sixth order Taylor expansion of α we have that

$$(5.10) \qquad \operatorname{Re}(\zeta K_0(w)) = \alpha(1) = \alpha(0) + \frac{1}{3!}\alpha'''(0) + \frac{1}{6!}\alpha^{(6)}(t^*),$$

for some $0 \le t^* \le 1$. Then by Lemma 5.5,

$$\frac{1}{6!}|\alpha^{(6)}(t^*)| \le \frac{1}{6!}|K_0^{(6)}(t^*w)| \cdot |w|^6 = \left| \frac{(\log \widehat{h})^{(6)}(t^*w)}{6!} \right| \cdot |w|^6 \le 6^3 |w|^6.$$

By Lemma 2.4, (4), the quantity $\Delta(\rho)$ is a nonzero real number. Then, we have that

$$K_0(0) = -\log |\Delta(\rho)|, \quad \alpha(0) = \mathrm{Re}(-\zeta \log |\Delta(\rho)|).$$

On the other hand, $\alpha'''(0) = \mathrm{Re}\left(\zeta K_0'''(0)w^3\right)$, so Lemma 5.4 implies (5.9).

Since $\mathrm{Re}\left[\zeta(\log \widehat{h})''(w)w^2\right] = -\alpha''(1)$, the same argument, applied to all possible ζ and to the fourth order Taylor expansion of $-\alpha''(\cdot)$, allows us to prove (5.5). Similarly, the identity $\mathrm{Re}\left[\zeta(\log \widehat{h})'(w)w\right] = -\alpha'(1)$ enables us to use the fifth order Taylor expansion of $-\alpha'(\cdot)$ in order to prove (5.4). $\qquad\square$

6. Numerical estimates

In this section we prove Theorem 2, Corollary 1.1, and Proposition B. The proof of Proposition B is given in Section 6.1, where we also estimate the values of h_F taken at the roots of unity (Corollary 6.1). The proofs of Theorem 2 and Corollary 1.1 are given in Section 6.2. The main ingredient, besides those developed in the previous sections, is a general way to find upper bounds for the essential minimum (Proposition 6.2). This leads us to make a numerical estimate of the integral g_{hyp} over a certain translate of the unit circle.

In the rest of this section we denote by μ (resp. ϕ) the classical Möbius (resp. Euler's totient) function.

6.1. Approximating g_{hyp} on the unit circle.
In this section we combine the distortion estimates in Section 4 with the estimates from Section 5 to prove Proposition B. As a consequence, we obtain approximations of values of h_F at roots of unity (Corollaries 6.1).

Proof of Proposition B. Note that by (1.4) and Lemma 4.3 we have

$$(6.1) \qquad \gamma_1 = g_{\mathbb{D}}(0) - 6 \log \left(1 - f'(0)^{-\frac{2}{3}}\right).$$

So, if $w \in B(0, r_0)$ is such that $j_{\mathbb{D}}(w) = \zeta$, then by Lemma 4.5, Lemma 4.6, (1) and (2), and (5.3), we have

$$\left| g_{\mathrm{hyp}}(\zeta) - \left(\gamma_1 - \frac{\mathrm{Re}(\zeta)}{13824}\right) \right|$$

$$= \left| g_{\mathbb{D}}(w) - \left(g_{\mathbb{D}}(0) - 6\log\left(1 - f'(0)^{-\frac{2}{3}}\right) - \frac{\mathrm{Re}(\zeta)}{13824}\right) \right|$$

$$\le 6^3 (r_1^+)^6 + 6 \left| \log(1 - |w|^2) - \log\left(1 - f'(0)^{-\frac{2}{3}}\right) \right| + \frac{1}{13824} \left| \mathrm{Re}(\zeta) - \mathrm{Re}(w^3) f'(0) \right|$$

$$\le 6^3 \cdot \left(\frac{1 + \kappa_1}{f'(0)} \right)^2 + 6 \cdot 7.7 \cdot 10^{-8} + \frac{1}{13824 \cdot 2283}$$

$$\le 5 \cdot 10^{-7}.$$

$\qquad\square$

Corollary 6.1. *For every integer $n \geq 1$ and every primitive root of unity ζ_n of order n, we have*

$$-0.7486222078 - \frac{1}{165888} \cdot \frac{\mu(n)}{\phi(n)} \leq h_F(\zeta_n) \leq -0.7486221244 - \frac{1}{165888} \cdot \frac{\mu(n)}{\phi(n)}.$$

In particular, $-0.748628236 \leq h_F(1) \leq -0.748628152$.

Proof. Since ζ_n is an algebraic integer, by (3.1) we have

$$h_F(\zeta_n) = \frac{1}{12 \cdot \phi(n)} \sum_{\zeta \in \mathcal{O}(\zeta_n)} g_{hyp}(\zeta).$$

Thus, in view of the identity

$$\sum_{\zeta \in \mathcal{O}(\zeta_n)} \zeta = \mu(n),$$

the corollary is a direct consequence of Proposition B. $\qquad\square$

6.2. **Estimating the essential minimum from above.** We use the following criterion to estimate the essential minimum μ_F^{ess} from above. It is stated for the height h_F and the section $s = \Delta$, but it is clearly valid for general heights and sections. The proof is based on the classical Fekete-Szegö theorem and an equidistribution result shown in [BGPRLS15].

Proposition 6.2. *Let K be a compact subset of $\mathbb{C}$ that is invariant under complex conjugation and whose logarithmic capacity is equal to 1, and denote by ρ_K its equilibrium measure. Then there is a sequence of pairwise distinct algebraic integers $(p_l)_{l \geq 1}$ such that*

$$\lim_{l \to \infty} h_F(p_l) = \frac{1}{12} \int g_{hyp} \, d\rho_K.$$

In particular, $\mu_F^{ess} \leq \frac{1}{12} \int g_{hyp} \, d\rho_K$.

Proof. Denote by $\mathfrak{M}_{\mathbb{Q}} = \{\text{prime numbers}\} \cup \{\infty\}$ the set of places of $\mathbb{Q}$ and for each prime number p denote by $\mathbb{C}_p$ the completion of $(\overline{\mathbb{Q}}, |\cdot|_p)$. Furthermore, for a point ζ in $\mathbb{C}$, denote by δ_ζ the Dirac mass at ζ.

By the Fekete-Szegö theorem there is a sequence of pairwise distinct algebraic integers $(p_l)_{l \geq 1}$ such that for each $l \geq 1$ the set $\mathcal{O}(p_l)$ is contained in

$$\{\zeta \in \mathbb{C} : \text{ there is } \zeta_0 \in K \text{ such that } |\zeta - \zeta_0| \leq 1/l\};$$

see [FS55]. Note that by (3.1), for each $l \geq 1$ we have

$$h_F(p_l) = \frac{1}{12 \# \mathcal{O}(p_l)} \sum_{q \in \mathcal{O}(p_l)} g_{hyp}(q).$$

On the other hand, applying [BGPRLS15, Proposition 7.4] to the closed bounded adelic set formed by $E_\infty = K$ and for every prime number p by the unit ball in $\mathbb{C}_p$, we have that the measure

$$\frac{1}{\# \mathcal{O}(p_l)} \sum_{q \in \mathcal{O}(p_l)} \delta_q$$

converges to ρ_K in the weak* topology as $l \to \infty$. Since the function g_{hyp} is continuous, this implies the proposition. $\qquad\square$

To obtain a numerical upper bound of μ_F^{ess}, we apply the previous criterion with K equal to a translate of the unit circle by a real number a. Our numerical experiments, described in Section 8 and in [BMRan], suggest that the best choice for the center is $a = 0.205$, which is what we use in the proof of Corollary 1.1. First we give a formula for the corresponding integral.

Lemma 6.3. *For a given $t \in [0,1]$ and $a \in (0,2)$, let $w_a(t) \in B(0, r_0)$ be the only complex number with argument in $[\pi, \frac{5}{3}\pi)$ such that $j_{\mathbb{D}}(w_a(t)) = a + e^{2\pi i t}$ (cf. Figure 4.1). Similarly, let $s_a \in (0, r_0)$ be the only element such that $j_{\mathbb{D}}(s_a) = a$. Then,*

(6.2)
$$\int_0^1 g_{\mathrm{hyp}}\left(a + e^{2\pi i t}\right) \, \mathrm{d}t = -\log(1728\pi^6) - \log|\widehat{h}(s_a)| - 6 \int_0^1 \log\left(1 - |w_a(t)|^2\right) \, \mathrm{d}t.$$

Proof. Recall the identities $j_{\mathbb{D}}(w) = f(w^3)$ (4.3) and $\widehat{h}(w) = h(w^3)$ (Lemma 5.3). Since f is univalent (Lemma 4.3), the inverse function f^{-1} is holomorphic and well defined on the image of f. Also, we have the relations

(6.3)
$$w_a(t)^3 = f^{-1}(a + e^{2\pi i t}), \quad s_a^3 = f^{-1}(a).$$

In particular, we see that $w_a(\cdot)$ is a continuous function. Since $|w_a(t)| \le r_0 < 1$ for all $t \in [0,1]$, we deduce that the integral in the right-hand side of (6.2) is well defined. By (5.2) and (6.3), the left-hand side of (6.2) is equal to

$$-\log(1728\pi^6) - 6 \int_0^1 \log\left(1 - \left|f^{-1}\left(a + e^{2\pi i t}\right)\right|^{2/3}\right) \, \mathrm{d}t$$
$$- \int_0^1 \log\left|h \circ f^{-1}\left(a + e^{2\pi i t}\right)\right| \, \mathrm{d}t.$$

By Cauchy's formula,

$$\int_0^1 \log\left|h \circ f^{-1}\left(a + e^{2\pi i t}\right)\right| \, \mathrm{d}t \;\; = \;\; \mathrm{Re}\left(\frac{1}{2\pi i} \int_{\partial B(a,1)} \frac{\log h \circ f^{-1}(w)}{w - a} \, \mathrm{d}w\right)$$
$$= \;\; \log\left|h \circ f^{-1}(a)\right|.$$

Using (6.3) we conclude the proof. $\qquad\qquad\square$

The following lemma is used to prove the last assertion of Corollary 1.1.

Lemma 6.4. *Let α be a nonzero algebraic number, denote by d its degree, by a the leading coefficient of the minimal polynomial in $\mathbb{Z}[x]$ of α and by b the constant coefficient. Then, we have that*

(6.4)
$$\frac{1}{d}\log|a| \le \frac{12(\mathrm{h}_F(\alpha) - \mathrm{h}_F(1))}{1 - \partial_x g_{\mathrm{hyp}}(1)}, \quad \frac{1}{d}\log|b| \le \frac{12(\mathrm{h}_F(\alpha) - \mathrm{h}_F(1))}{\partial_x g_{\mathrm{hyp}}(1)}.$$

Proof. Put $\omega := 1 - \partial_x g_{\mathrm{hyp}}(1)$. By (3.1), the product formula, and the fact that b is a nonzero integer, we have

$$12\,\mathrm{h_F}(\alpha) - \frac{1}{d} \sum_{\alpha' \in \mathcal{O}(\alpha)} g_1(\alpha') = \frac{1}{d} \sum_{p \text{ prime}} \sum_{\alpha' \in \mathcal{O}(\alpha)} \log\max\left\{|\alpha'|_p^{\omega-1}, |\alpha'|_p^{\omega}\right\}$$

$$\geq \frac{\omega}{d} \sum_{p \text{ prime}} \sum_{\alpha' \in \mathcal{O}(\alpha)} \log\max\{1, |\alpha'|_p\}$$

$$= \frac{\omega}{d} \sum_{p \text{ prime}} -\log|a|_p$$

$$= \frac{\omega}{d} \log|a|.$$

Thus, the first inequality in (6.4) follows from the following consequence of Proposition A,

$$\frac{1}{d} \sum_{\alpha' \in \mathcal{O}(\alpha)} g_1(\alpha') \geq g_1(1) = 12\,\mathrm{h_F}(1).$$

The second inequality in (6.4) follows from a similar argument. Namely,

$$12\,\mathrm{h_F}(\alpha) - \frac{1}{d} \sum_{\alpha' \in \mathcal{O}(\alpha)} g_1(\alpha') = \frac{1}{d} \sum_{p \text{ prime}} \sum_{\alpha' \in \mathcal{O}(\alpha)} \log\max\left\{|\alpha'|_p^{\omega-1}, |\alpha'|_p^{\omega}\right\}.$$

$$\geq \frac{\omega-1}{d} \sum_{p \text{ prime}} \sum_{\alpha' \in \mathcal{O}(\alpha)} \log\min\{1, |\alpha'|_p\}.$$

$$= \frac{\omega-1}{d} \sum_{p \text{ prime}} \log|b|_p.$$

$$= \frac{1-\omega}{d} \log|b|. \qquad \square$$

Proof of Corollary 1.1. The first inequality is a direct consequence of (1.4) and Corollary 6.1 and the second and the fifth from Theorem 1. Furthermore, the fourth inequality follows from Corollary 6.1.

To prove the first statement and the upper bound of $\mu_{\mathrm{F}}^{\mathrm{ess}}$, for each $a \geq 0.205$ we use Proposition 6.2 with K equal to $C_a := \{\zeta \in \mathbb{C} : |\zeta - a| = 1\}$. Lemmas 4.5 and 6.3, the estimate (5.3) and the formula $g_{\mathbb{D}}(0) = -\log(1728\pi^6) - \log|\Delta(\rho)|$ imply that for each a in $(0, 2)$ we have

$$(6.5) \quad \int_0^1 g_{\mathrm{hyp}}\left(a + e^{2\pi i t}\right) \mathrm{d}t$$

$$\leq g_{\mathbb{D}}(0) - \frac{f'(0)}{13824} r_-(a)^3 + 6^3 r_+(a)^6 - 6 \int_0^1 \log\left(1 - r_+\left(|a + e^{2\pi i t}|\right)^2\right) \mathrm{d}t.$$

Taking $a = 0.205$, note that the numbers $r_-(0.205)$ and $r_+(0.205)$ can be computed to high precision. Similarly, the function $t \mapsto \log\left(1 - r_+\left(|0.205 + e^{2\pi i t}|\right)^2\right)$ is an explicit composition of sums, products, logarithms, sines, cosines and square roots, hence can be computed to high precision too (e.g., up to an error absolutely bounded by 10^{-15} in SAGE). By Proposition 6.2 with $K = C_{0.205}$ and (6.5) with $a = 0.205$,

a numerical estimate gives

$$(6.6) \qquad \mu_{\mathrm{F}}^{\mathrm{ess}} \leq \frac{1}{12} \int_0^1 g_{\mathrm{hyp}} \left(0.205 + e^{2\pi i t} \right) \mathrm{d}t \leq -0.7486227509.$$

The first assertion follows from this last estimate and from Proposition 6.2, by observing that the function

$$a \mapsto \int_0^1 g_{\mathrm{hyp}} \left(a + e^{2\pi i t} \right) \mathrm{d}t$$

is continuous and converges to ∞ as $a \to \infty$ by the asymptotic (3.3).

On the other hand, the third inequality of the corollary follows from (6.6) and (1.4).

To prove the previous last statement of the corollary, let α be an algebraic number that is not an algebraic integer and whose Faltings' height is less than or equal to $\mu_{\mathrm{F}}^{\mathrm{ess}}$. Then (6.6), Corollary 6.1, Proposition 7.1 and a numerical estimate imply that the right-hand side of the first inequality in (6.4) with $h_{\mathrm{F}}(\alpha)$ replaced by $\mu_{\mathrm{F}}^{\mathrm{ess}}$ is less than or equal to $1/15177$. On the other hand, our assumption that α is not an algebraic integer implies that the number a as in the statement of Lemma 6.4 satisfies $|a| \geq 2$. Thus, by (6.4) and our hypothesis $h_{\mathrm{F}}(\alpha) \leq \mu_{\mathrm{F}}^{\mathrm{ess}}$, the degree d of α satisfies $d \geq \log 2 \cdot 15177 > 10519$.

Now assume that α is an algebraic number of degree at most 10 such that $h_{\mathrm{F}}(\alpha) \leq \mu_{\mathrm{F}}^{\mathrm{ess}}$. By the previous considerations, α is an algebraic integer. Then, using the second inequality in (6.4), combined with the estimates (6.6), Corollary 6.1, Proposition 7.1 and a numerical estimate, the parameter b in the statement of Lemma 6.4 can be bounded from above as

$$|b| \leq \exp \left(10 \cdot 1032 \cdot 12 \cdot (\mu_{\mathrm{F}}^{\mathrm{ess}} - h_F(1)) \right) \leq 1.98.$$

Since $|b|$ is a positive integer, we conclude that $|b| = 1$ and that α is an algebraic unit. This completes the proof of the last statement and of the corollary. □

Proof of Theorem 2. The hypotheses on n imply either $\mu(n) \in \{0, 1\}$ or $\phi(n) \geq 12$. In all the cases, $\frac{\mu(n)}{\phi(n)} \geq -\frac{1}{12}$. Using Corollary 6.1, a numerical estimate gives

$$h_{\mathrm{F}}(\zeta_n) \geq -0.748622711.$$

Then the theorem follows from (6.6). □

7. Proof of Proposition A

First, we establish the following numerical estimate of $\partial_x g_{\mathrm{hyp}}(1)$ implying the inequalities $0 < \partial_x g_{\mathrm{hyp}}(1) < 1$ (Proposition 7.1). These estimates are also used below to show convexity properties of g_{hyp}. The proof of the remaining part of Proposition A is divided in three cases, according to the proximity to the unit disk.

Throughout the rest of this section we use the functions $j_{\mathbb{D}}$, f and $g_{\mathbb{D}}$, defined in (4.2), (4.3), and (5.1), respectively.

Proposition 7.1. *We have* $\frac{1}{1032} \leq \partial_x g_{\mathrm{hyp}}(1) \leq \frac{1}{1025}$.

Proof. Let r_1 be the only number in $(0, r_0)$ such that $j_{\mathbb{D}}(r_1) = 1$. Since $j_{\mathbb{D}}$ is real (Lemma 5.2), f is also real. Together with the fact that f is univalent and that $f'(0) > 0$ (Lemma 4.3), we conclude that $j'_{\mathbb{D}}(r_1) = 3r_1^2 f'(r_1^3) > 0$.

On the other hand, (3.5) implies that $\partial_y g_{\mathrm{hyp}}(1) = 0$. Hence,

$$(7.1) \qquad \partial_x g_{\mathrm{hyp}}(1) = 2\operatorname{Re}\left(\partial g_{\mathrm{hyp}}(1)\right) = \frac{2\operatorname{Re}\left(\partial g_{\mathbb{D}}(r_1)\right)}{j'_{\mathbb{D}}(r_1)}.$$

Using (5.2) and (5.4), we obtain

$$\begin{aligned} 2\operatorname{Re}\left(\partial g_{\mathbb{D}}(r_1)\right) &= \frac{12 r_1}{1 - r_1^2} - \operatorname{Re}\left((\log \widehat{h})'(r_1)\right) \\ &= \frac{12 r_1}{1 - r_1^2} - 3 \cdot \frac{f'(0)}{13824} r_1^2 + 4 \cdot E, \end{aligned}$$

where $|E| \le 6^4 r_1^5$. Moreover, Lemma 4.5 ensures that $r_1^- \le r_1 \le r_1^+$, leading to

$$2\operatorname{Re}\left(\partial g_{\mathbb{D}}(r_1)\right) \le \frac{12 r_1^+}{1 - (r_1^+)^2} - 3 \cdot \frac{f'(0)}{13824}(r_1^-)^2 + 4 \cdot 6^4 (r_1^+)^5,$$

$$2\operatorname{Re}\left(\partial g_{\mathbb{D}}(r_1)\right) \ge \frac{12 r_1^-}{1 - (r_1^-)^2} - 3 \cdot \frac{f'(0)}{13824}(r_1^+)^2 - 4 \cdot 6^4 (r_1^+)^5.$$

These estimates, together with (7.1) and Lemma 4.6, (3), prove the claim. $\qquad\square$

To complete the proof of Proposition A, let T be defined by (3.2), fix ζ in $\mathbb{C}$, and let τ in T be such that $j(\tau) = \zeta$. There are three cases, according to the location of τ.

We also use the following estimate several times:

$$(7.2) \qquad g_{\mathrm{hyp}}(1) \le -8.89835372,$$

which is a direct consequence of Corollary 6.1, and the formula $g_{\mathrm{hyp}}(1) = 12\, h_{\mathrm{F}}(1)$; cf. (3.1).

Case 1 ($\operatorname{Im}(\tau) \ge 1$).

Lemma 7.2. *For τ in $\mathbb{H}$ satisfying $\operatorname{Im}(\tau) \ge 1$, we have*

$$\left|g_\infty(\tau) - (2\pi \operatorname{Im}(\tau) - 6\log(\operatorname{Im}(\tau)) - 6\log(4\pi))\right| \le \frac{24}{\exp(2\pi) - 2}.$$

Proof. Let τ in $\mathbb{H}$ be such that $\operatorname{Im}(\tau) \ge 1$, and note that $q = e^{2\pi i \tau}$ satisfies $|q| \le \exp(-2\pi)$. This implies that for every integer $r \ge 1$ we have

$$|\log(1 - q^r)| \le \frac{\exp(2\pi)}{\exp(2\pi) - 1}|q|^r \quad \text{and} \quad \left|\sum_{r=1}^{\infty} \log(1 - q^r)\right| \le \frac{1}{\exp(2\pi) - 2}.$$

Then the desired estimate is obtained by applying the definition of g_∞. $\qquad\square$

Lemma 7.3. *For every τ in $\mathbb{H}$ satisfying $\operatorname{Im}(\tau) \ge 1$, we have*

$$|j(\tau)| \le 4\exp(2\pi \operatorname{Im}(\tau)).$$

Proof. Let $\widehat{j} \colon \mathbb{D} \to \overline{\mathbb{C}}$ be the holomorphic function such that $\widehat{j}(0) = \infty$, and such that for every τ in $\mathbb{H}$ we have $\widehat{j}(q(\tau)) = j(\tau)$. Since this function is univalent on $B(0, \exp(-2\pi))$, and the derivative of $\widehat{j}^{-1}$ at $q = 0$ is equal to 1, by the Koebe one quarter theorem [Pom75, Corollary 1.4, p. 22] for every τ in $\mathbb{H}$ satisfying $\operatorname{Im}(\tau) \ge 1$, we have

$$|j(\tau)|^{-1} = |\widehat{j}(q(\tau))|^{-1} \ge \frac{1}{4}|q(\tau)| = \frac{1}{4}\exp(-2\pi \operatorname{Re}(\tau)). \qquad\square$$

We now proceed to the proof of Proposition A in the case where τ satisfies $\text{Im}(\tau) \geq 1$. First note that by Proposition 7.1, (7.2), and Lemma 7.2, we have

$$g_\infty(\tau) - g_{\text{hyp}}(1) \geq 2\pi \text{Im}(\tau) - 6 \log(\text{Im}(\tau)) - 6 \log(4\pi) - g_{\text{hyp}}(1) - \frac{24}{\exp(2\pi) - 2}$$

$$\geq 2\pi \text{Im}(\tau) - 6 \log(\text{Im}(\tau)) - 6.25$$

$$\geq 0.03 \text{Im}(\tau)$$

$$\geq 0.02 + \partial_x g_{\text{hyp}}(1)(10 \text{Im}(\tau))$$

$$\geq 0.02 + \partial_x g_{\text{hyp}}(1)(2\pi \text{Im}(\tau) + \log 4).$$

Combined with Lemma 7.3 and the definition of g_1, this implies

$$g_1(\zeta) = g_1(j(\tau)) \geq g_1(1) + 0.02.$$

This proves Proposition A in the case where $\text{Im}(\tau) \geq 1$.

Case 2 $(\frac{1}{\pi} \log(19) \leq \text{Im}(\tau) \leq 1)$.

Lemma 7.4. *For each τ in T satisfying $\text{Im}(\tau) \leq 1$, we have $|j(\tau)| \leq 1728$.*

Proof. Note that the image of $\{\tau \in T : \text{Im}(\tau) \leq 1\}$ is a Jordan domain bounded by the curve $j(\{\tau \in T : \text{Im}(\tau) = 1\})$. So, it is enough to prove the inequality in the case $\text{Im}(\tau) = 1$. Using that the coefficients in the q-expansion of j are positive, for every x in $\mathbb{R}$ we have $|j(x+i)| \leq j(i) = 1728$, finishing the proof of the lemma. $\square$

Lemma 7.5. *For each τ in T satisfying $\frac{1}{\pi} \log(19) \leq \text{Im}(\tau) \leq 1$, we have*

$$g_\infty(\tau) \geq 2 \log(19) - 6 \log(4 \log(19)) - 24 \log \left(\frac{19^2 + 1}{19^2} \right).$$

Proof. Lemma 3.1, combined with (2.10), imply that for each τ in T satisfying $\frac{1}{\pi} \log(19) \leq \text{Im}(\tau) \leq 1$, we have

$$g_\infty(\tau) \geq g_\infty \left(\frac{1}{2} + i \frac{1}{\pi} \log(19) \right)$$

$$= 2 \log(19) - 6 \log(4 \log(19)) - 24 \sum_{n=1}^\infty \log \left(1 - \left(-\frac{1}{19^2} \right)^n \right).$$

Then, we are reduced to showing that

$$(7.3) \qquad \sum_{n=2}^\infty \log \left(1 - \left(-\frac{1}{19^2} \right)^n \right) \leq 0.$$

Setting $s = \frac{1}{19^2}$, the arithmetic-geometric mean implies that for each $r \geq 1$,

$$\left(1 - (-s)^{2r} \right) \left(1 - (-s)^{2r+1} \right) = \left(1 - s^{2r} \right) \left(1 + s^{2r+1} \right) \leq \left(1 - \frac{s^{2r} - s^{2r+1}}{2} \right)^2.$$

Since $0 < s < 1$, the last quantity is strictly less than 1. Hence,

$$\prod_{r=1}^\infty \left(1 - (-s)^{2r} \right) \left(1 - (-s)^{2r+1} \right) < 1,$$

justifying (7.3). $\square$

We now proceed to the proof of Proposition A in the case where τ satisfies $\frac{1}{\pi} \log(19) \leq \mathrm{Im}(\tau) \leq 1$. Proposition 7.1, (7.2), and Lemmas 7.4 and 7.5 imply that

$$g_1(\zeta) - g_1(1) = g_1 \circ j(\tau) - g_{\mathrm{hyp}}(1) = g_\infty(\tau) - \partial_x g_{\mathrm{hyp}}(1) \log|j(\tau)| - g_{\mathrm{hyp}}(1)$$

is bounded from below by

$$2\log(19) - 6\log(4\log(19)) - 24\log\left(\frac{19^2 + 1}{19^2}\right) - \frac{1}{1025}\log 1728 + 8.9835372 \geq 10^{-3},$$

finishing the proof of Proposition A in this case.

Case 3 ($\mathrm{Im}(\tau) \leq \frac{1}{\pi}\log(19)$).

Lemma 7.6. *Let $\psi \colon \mathbb{D} \to \mathbb{H}$ be as defined in (4.1). Then for every τ in T satisfying $\mathrm{Im}(\tau) \leq \frac{1}{\pi}\log(19)$, we have*

$$\left|\psi^{-1}(\tau)\right| \leq 1 - \frac{\pi}{2\sqrt{3}}.$$

Proof. Put $I := \frac{1}{\pi}\log(19)$. Since the image by j of the set $\{\tau \in T : \mathrm{Im}(\tau) = I\}$ is a Jordan curve, it is enough to prove the lemma in the case where $\mathrm{Im}(\tau) = I$. By symmetry, it is enough to consider the case where $\mathrm{Re}(\tau) \geq 0$.

Put

$$\tau_2 = \sqrt{1 - I^2} + iI \text{ and } \tau_2' = \frac{1}{2} + iI.$$

Note that the image by ψ^{-1} of the line $\{\tau \in \mathbb{H} : \mathrm{Im}(\tau) = I\}$ is a circle that is tangent to the unit circle at $w = -1$, and that is contained in the left half-plane. Thus, the image by ψ^{-1} of the segment $[\tau_2, \tau_2']$ is the arc of this circle that is contained in the angular sector bounded by the rays $\{t < 0\}$ and $\{-t\rho : t > 0\}$. It follows that for each τ in the segment $[\tau_2, \tau_2']$, we have

$$\left|\psi^{-1}(\tau)\right| \leq \left|\psi^{-1}(\tau_2)\right| = \frac{1 - 2\sqrt{1 - I^2}}{2 + \sqrt{3}I - \sqrt{1 - I^2}} \leq 1 - \frac{\pi}{2\sqrt{3}}. \qquad \square$$

Lemma 7.7. *For every w in $\mathbb{D}$ such that $0 < |w| \leq 1 - \frac{\pi}{2\sqrt{3}}$, we have $g_1 \circ j_{\mathbb{D}}(w) \geq g_1 \circ j_{\mathbb{D}}(|w|)$, with equality if and only if $w^3 = |w|^3$.*

The proof of this lemma is given after the following lemma.

Lemma 7.8. *Let $J \colon B(0, r_0) \setminus \{0\} \to \mathbb{C}$ be defined by $J(w) := \left(\frac{j_{\mathbb{D}}'}{j_{\mathbb{D}}}\right)(w) \cdot w$. Then for every w in $B\left(0, 1 - \frac{\pi}{2\sqrt{3}}\right) \setminus \{0\}$ we have $|J'(w)| \leq 4000|w|^2$.*

Proof. Using $J(w) = 3w^3 \left(\frac{f'}{f}\right)(w^3)$ and applying the first two inequalities in Lemma 4.4 with $z = w^3/r_0^3$, we obtain

$$|J(w)| \leq 3\frac{1 + |z|}{1 - |z|} \quad \text{and} \quad \left|\frac{J(w)}{3} - 1\right| \leq |w|^3 \left(\frac{2}{r_0^3} \cdot \frac{(1 + |z|)^2}{(1 - |z|)^3}\right).$$

Since these upper bounds are increasing in $|z|$, a numerical estimate with $|z|$ replaced by $\left(1 - \frac{\pi}{2\sqrt{3}}\right)^3 / r_0^3$ gives

$$|J(w)| \leq 4 \text{ and } |J(w) - 3| \leq 400|w|^3.$$

Using these inequalities and the third inequality in Lemma 4.4, we have

$$|J'(w) \cdot w| = \left| J(w)\,(3 - J(w)) - 9w^6 \left(\frac{f''}{f}\right)(w^3) \right| \le |w|^3 \left(1600 + \frac{18}{r_0^3} \cdot \frac{2 + |z|}{(1 - |z|)^2}\right).$$

The desired inequality follows by observing that the upper bound is increasing in $|z|$ and by estimating it with $|z|$ replaced by $\left(1 - \frac{\pi}{2\sqrt{3}}\right)^3 / r_0^3$. $\qquad\square$

Proof of Lemma 7.7. By Theorem 4.1, (1), applied to f_0 defined in (4.5) and $z = \left(\frac{w}{r_0}\right)^3$, we have

$$\log|j_{\mathbb{D}}(w)| - \log|j_{\mathbb{D}}(|w|)| = \log\left(\frac{|f(w^3)|}{f(|w|^3)}\right) \le 2\log\left(\frac{1 + r_0^{-3}|w|^3}{1 - r_0^{-3}|w|^3}\right) \le 6r_0^{-3}|w|^3.$$

Here, we have used the elementary inequality

$$0 \le x \le \left(\frac{1 - \frac{\pi}{2\sqrt{3}}}{2 - \sqrt{3}}\right)^3 \Rightarrow \log\left(\frac{1 + x}{1 - x}\right) \le 3x.$$

Assume first that w satisfies $\mathrm{Re}\left(\frac{w^3}{|w|^3}\right) \le \frac{1}{2}$. Then, by Lemma 4.3, Proposition 7.1, and Proposition 5.1, we have

$$g_1 \circ j_{\mathbb{D}}(w) - g_1 \circ j_{\mathbb{D}}(|w|)$$
$$\ge \frac{f'(0)}{13824}|w|^3 \left(1 - \mathrm{Re}\left(\frac{w^3}{|w|^3}\right)\right) - 2 \cdot 6^3 |w|^6 - \left(\frac{6}{1025}r_0^{-3}\right)|w|^3$$
$$\ge |w|^3 \left(\frac{237698}{13824} \cdot \frac{1}{2} - 2 \cdot 6^3 \cdot \left(1 - \frac{\pi}{2\sqrt{3}}\right)^3 - \frac{6r_0^{-3}}{1025}\right)$$
$$\ge |w|^3$$
$$> 0.$$

Now we assume $\mathrm{Re}\left(\frac{w^3}{|w|^3}\right) > \frac{1}{2}$, put $r := |w|$ and $\theta := \arg(w)$, and let $H \colon \mathbb{R} \to \mathbb{R}$ be the function defined by $H(\widetilde{\theta}) := g_1 \circ j_{\mathbb{D}}\left(r \exp\left(i\widetilde{\theta}\right)\right)$. Using the function J defined in Lemma 7.8, we have

$$H''(\theta) = \mathrm{Re}\left[(\log\widehat{h})''(r\exp(i\theta))r^2\exp(2i\theta) + (\log\widehat{h})'(r\exp(i\theta))r\exp(i\theta) \right.$$
$$\left. + \partial_x g_{\mathrm{hyp}}(1) \cdot J'(r\exp(i\theta)) \cdot r\exp(i\theta)\right].$$

Combining Proposition 7.1, Proposition 5.1, (5.4), and (5.5), and Lemmas 4.3 and 7.8, and using $\mathrm{Re}(\exp(3i\theta)) \ge \frac{1}{2}$, we have

$$H''(\theta) \ge \frac{9f'(0)}{13824}r^3 \mathrm{Re}(\exp(3i\theta)) - 5 \cdot 6^4 r^6 - 6^4 r^6 - 4r^3$$
$$\ge r^3\left(\frac{9 \cdot 237698}{13824} \cdot \frac{1}{2} - 6^5\left(1 - \frac{\pi}{2\sqrt{3}}\right)^3 - 4\right)$$
$$\ge r^3.$$

This proves that, if we denote by θ_0 the unique number in $\left[0, \frac{\pi}{3}\right]$ such that

$$\mathrm{Re}(\exp(3i\theta_0)) = \frac{1}{2},$$

then H is strictly convex on $[-\theta_0, \theta_0]$. Moreover, Lemma 5.2 implies that H is even, hence it attains its minimum on $[-\theta_0, \theta_0]$ at, and only, at $\theta = 0$. This completes the proof of the lemma. □

Lemma 7.9. *The restriction V of $g_1 \circ j_{\mathbb{D}}$ to $\left(0, 1 - \frac{\pi}{2\sqrt{3}}\right]$ is strictly convex. Moreover, if r_1 is the only number in $\left(0, 1 - \frac{\pi}{2\sqrt{3}}\right)$ such that $j_{\mathbb{D}}(r_1) = 1$; cf. Figure 4.1, then V attains its minimum at, and only at, $r = r_1$.*

Proof. Since $j_{\mathbb{D}}$ is real (Lemma 5.2), f is also real. Together with the fact that f is univalent and that $f(0) = 0$ and $f'(0) > 0$ (Lemma 4.3), we conclude for each r in $(0, r_0)$ we have $J(r) = 3r^3 f'(r^3)/f(r^3) > 0$.

By (5.5) in Proposition 5.1, Proposition 7.1, and Lemmas 4.3 and 7.8, we have

$$V''(r) = \frac{12(1+r^2)}{(1-r^2)^2} - (\log \widehat{h})''(r) + \partial_x g_{\mathrm{hyp}}(1) \operatorname{Re}\left(\frac{J(r)}{r^2}\right) - \partial_x g_{\mathrm{hyp}}(1) \operatorname{Re}\left(\frac{J'(r)}{r}\right)$$

$$\geq \frac{12(1+r^2)}{(1-r^2)^2} - \frac{6f'(0)}{13824}r - 5 \cdot 6^4 \cdot r^4 - \frac{1}{1025}4000r$$

$$\geq 12 - \left(1 - \frac{\pi}{2\sqrt{3}}\right)\left(\frac{6 \cdot 237698}{13824} + 5 \cdot 6^4 \cdot \left(1 - \frac{\pi}{2\sqrt{3}}\right)^3 + 4\right)$$

$$\geq 1.$$

This proves that V is strictly convex on $\left(0, 1 - \frac{\pi}{2\sqrt{3}}\right)$.

To finish the proof, it is enough to show that $r = r_1$ is a critical point of V. Indeed,

$$V'(r) = \partial_x g_{\mathbb{D}}(r) - \partial_x g_{\mathrm{hyp}}(1) \operatorname{Re}\left(\left(\frac{j'_{\mathbb{D}}}{j_{\mathbb{D}}}\right)(r)\right).$$

The relation $g_{\mathbb{D}} = g_{\mathrm{hyp}} \circ j_{\mathbb{D}}$ implies $\partial_x g_{\mathbb{D}} = (\partial_x g_{\mathrm{hyp}}) \circ j_{\mathbb{D}} \cdot \partial_x j_{\mathbb{D}}$. Since $j_{\mathbb{D}}(r_1) = 1$ and $j_{\mathbb{D}}$ is real, we see that $V'(r_1) = 0$. This completes the proof of the lemma. □

We now proceed to the proof of Proposition A in the remaining case where τ satisfies $\operatorname{Im}(\tau) \leq \frac{1}{\pi} \log(19)$. Lemma 7.6 ensures that $w := \psi^{-1}(\tau)$ satisfies $|w| \leq 1 - \frac{\pi}{2\sqrt{3}}$. Then, combining Lemmas 7.7 and 7.9, we have

$$g_1(\zeta) = g_1 \circ j(\tau) = g_1 \circ j_{\mathbb{D}}(w) \geq g_1 \circ j_{\mathbb{D}}(|w|) \geq g_1 \circ j_{\mathbb{D}}(r_1) = g_1(1),$$

with equality if and only if $j(\tau) = 1$. This finishes the proof of Proposition A.

8. NUMERICAL EXPERIMENTS

In this section we briefly describe our numerical experiments that give us two more minima of the stable Faltings' height, both of which are larger than $h_F(0)$ and $h_F(1)$. We use the procedure described Section 2.1 to find a lower bound of μ_F^{ess}, with a carefully chosen family of sections. The minima of h_F that we find are attained at the common support of these sections. See [BMRan] for the SAGE source code we use in our experiments and further details.

Recall the metrized line bundle $\overline{\mathcal{L}} = (M_{12}, \|\cdot\|_{\mathrm{Pet}})$ of weight 12 modular forms with the Petersson metric defined in Section 2.1. We have that $\mathcal{L} \simeq O_{\mathcal{X}}(D_\infty)$. The sections of $\mathcal{L}^{\otimes n}$ are in one-to-one correspondence with the space of homogeneous polynomials of degree n with integral coefficients in the variables X, Y, where $[X : Y]$ are homogeneous coordinates of $\mathbb{P}^1$ and the point at infinity is $[1 : 0]$.

We start with the section $s_0 = \Delta$. Using the notation introduced above, this is the section Y that has a zero at infinity. Then $s_0^{1/12}$ has weight one and $g_{s_0^{1/12}}(\zeta) = \frac{1}{12} g_{\mathrm{hyp}}(\zeta)$, so by Lemma 3.1 we have

$$\inf_{\zeta \in \mathcal{X}(\mathbb{C})} g_{s_0^{1/12}}(\zeta) = g_{s_0^{1/12}}(0) = -0.74875248\ldots,$$

which proves that the minimum value of Faltings' height is $h_0 := \mathrm{h_F}(0) = g_{s_0^{1/12}}(0)$.

We next define

$$s_1 := X, \, a_{1,1} := \frac{1}{12} \partial_x g_{\mathrm{hyp}}(1) \text{ and } s_{a_{1,1}} := s_1^{a_{1,1}} s_0^{1/12 - a_{1,1}} = X^{a_{1,1}} Y^{1/12 - a_{1,1}},$$

so that $12 g_{s_{a_{1,1}}} = g_1$ is the function defined in Proposition A. By this proposition we know that $g_{s_{a_{1,1}}}$ attains its minimum at the point 1, so

$$h_1 := \mathrm{h_F}(1) = g_{s_{a_{1,1}}}(1) = \inf_{\zeta \in \mathcal{X}(\mathbb{C})} g_{s_{a_{1,1}}}(\zeta) = -0.74862817\ldots.$$

To check that this value is the second minimum of $\mathrm{h_F}$, write

$$s_2 := X - Y$$

and consider sections of the form

$$s_{a_{2,1}, a_{2,2}} := s_1^{a_{2,1}} s_2^{a_{2,2}} s_0^{1/12 - a_{2,1} - a_{2,2}}.$$

We compute numerically

$$\sup_{a_{2,1}, a_{2,2}} \inf_{\zeta \in \mathcal{X}(\mathbb{C})} g_{s_{a_{2,1}, a_{2,2}}}(\zeta) = -0.74862517\ldots,$$

which is a lower bound of $\mathrm{h_F}$ on $\overline{\mathbb{Q}} \setminus \{0, 1\}$. Since this number is larger than h_1, this proves that h_1 is the second minimum of $\mathrm{h_F}$ on $\overline{\mathbb{Q}}$. The experimental values of the coefficients are

$$a_{2,1} = 0.0000808846, \qquad a_{2,2} = 0.000006017184$$

and the new minimum is attained at the points

$$0.50004865 + i.86601467 \text{ and } 0.50004865 + i0.86601467.$$

The numbers above are very close to the solutions ρ and $\overline{\rho}$ of the equation $z^2 - z + 1 = 0$, which are the primitive roots of unity of order 6. We write

$$h_2 := \mathrm{h_F}(\rho) = -0.74862517\ldots \text{ and } s_3 := X^2 - XY + Y^2$$

and consider sections of the form

$$s_{a_{3,1}, a_{3,2}, a_{3,3}} := s_1^{a_{3,1}} s_2^{a_{3,2}} s_3^{a_{3,3}} s_0^{1/12 - a_{3,1} - a_{3,2} - 2a_{3,3}}.$$

Numerically we obtain

$$\sup_{a_{3,1}, a_{3,2}, a_{3,3}} \inf_{\zeta \in \mathcal{X}(\mathbb{C})} g_{s_{a_{3,1}, a_{3,2}, a_{3,3}}}(\zeta) = -0.74862386\ldots,$$

which is a lower bound of $\mathrm{h_F}$ on $\overline{\mathbb{Q}} \setminus \{0, 1, \rho, \overline{\rho}\}$. Since this number is larger than h_2, this shows that h_2 is the third minimum of $\mathrm{h_F}$ on $\overline{\mathbb{Q}}$. The coefficients we obtain are

$$a_{3,1} = 0.00007979626, \quad a_{3,2} = 0.000004433084, \quad a_{3,3} = 0.000002454098.$$

Testing other roots of unity, we found that if ξ is a primitive root of unity of order 10, then

$$h_3 := \mathrm{h_F}(\xi) = -0.74862366\ldots$$

is close to the next possible value of Faltings' height. The corresponding cyclotomic polynomial is $z^4 - z^3 + z^2 - z + 1$, so we write

$$s_4 := X^4 - X^3 Y + X^2 Y^2 - X Y^3 + Y^4$$

and consider sections of the form

$$s_{a_{4,1}, a_{4,2}, a_{4,3}, a_{4,4}} := s_1^{a_{4,1}} s_2^{a_{4,2}} s_3^{a_{4,3}} s_4^{a_{4,4}} s_0^{1/12 - a_{4,1} - a_{4,2} - 2a_{4,3} - 4a_{4,4}}.$$

Numerically we obtain

$$\sup_{a_{4,1}, a_{4,2}, a_{4,3}, a_{4,4}} \inf_{\zeta \in \mathcal{X}(\mathbb{C})} g_{s_{a_{4,1}, a_{4,2}, a_{4,3}, a_{4,4}}} = -0.74862360\ldots$$

which gives us a new lower bound for $\mu_{\mathrm{F}}^{\mathrm{ess}}$ and shows that h_3 is the fourth minimum of Faltings' height. The corresponding coefficients are

$$a_{4,1} = 0.000078055985, \qquad a_{4,2} = 0.000003803298,$$
$$a_{4,3} = 0.000002385096, \qquad a_{4,4} = 0.000000865203.$$

Acknowledgments

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