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REGULARITY OF THE SOLUTION TO 1-D FRACTIONAL ORDER DIFFUSION EQUATIONS

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ABSTRACT. In this article we investigate the solution of the steady-state fractional diffusion equation on a bounded domain in $\mathbb{R}^1$. The diffusion operator investigated, motivated by physical considerations, is neither the Riemann-Liouville nor the Caputo fractional diffusion operator. We determine a closed form expression for the kernel of the fractional diffusion operator which, in most cases, determines the regularity of the solution. Next we establish that the Jacobi polynomials are pseudo eigenfunctions for the fractional diffusion operator. A spectral type approximation method for the solution of the steady-state fractional diffusion equation is then proposed and studied.

1. Introduction

In recent years the fractional derivative has received increased attention in modeling a variety of physical phenomena. Most often cited are applications in contaminant transport in ground water flow [3], viscoelasticity [19], turbulent flow [19, 24], and chaotic dynamics [31]. As interest in the fractional derivative has increased so has the approximation methods to solve equations involving fractional derivatives. Generally speaking (for the 1-D case), approximation methods which exist for integer order differential equations have been successfully adapted to the fractional order case. Specifically, to mention a few (a complete list is beyond the focus of this article), finite difference methods [6, 17, 21, 26, 27], finite element methods [10, 13, 18, 28], discontinuous Galerkin methods [30], mixed methods [4], spectral methods [5, 16, 20, 29, 32], enriched subspace methods [12]. To date most of the approximation schemes have focused on the one-sided fractional diffusion equation

$$\mathcal{L}_1^\alpha u(x) \ := \ -\mathbf{D}^\alpha u(x) \ = \ f(x), \ x \in (0,1), \quad u(0) = u(1) = 0\,, \tag{1.1}$$

for $1 < \alpha < 2$. (A formal definition of $\mathbf{D}^\alpha u(x)$ is given in the following section.)

Another interesting historical fact, a point of particular interest in this article, is the definition of the fractional derivative. Or more precisely stated, *definitions* of the fractional derivative. There has been a number of definitions of the fractional derivatives studied. Most relevant to our discussion are the Riemann-Liouville fractional derivative and the Caputo fractional derivative. We refer the reader to the monographs [2, 7, 14, 22, 23] for a detailed discussion of various fractional derivatives. Also, of particular note is the recent approach to modeling nonlocal

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diffusion problems using a linear integral operator introduced by Du, Gunzburger, Lehoucq, and Zhou (see [8]).

Motivated by our interest in physical applications, in the following section we present the Riemann-Liouville and Caputo fractional derivatives on a finite interval, which for the sake of specificity we take to be $I := (0, 1)$. (In the case where a function and its (integer) derivatives vanish at the endpoint of the interval the Riemann-Liouville and Caputo fractional derivatives agree.)

The motivation for this article was to investigate the regularity of the solution to the two-sided fractional diffusion equation

$$\mathcal{L}_r^\alpha u(x) \;:=\; -\left(r\mathbf{D}^\alpha u(x) \;+\; (1-r)\mathbf{D}^{\alpha *} u(x)\right) \;=\; f(x),\; x \in (0,1),\;\; u(0) = u(1) = 0 \tag{1.2}$$

for $1 < \alpha < 2$, and $0 < r < 1$, which we think is a more physical model of diffusion than (1.1). (In (1.2) diffusion occurs to both the left and right of any point in the domain.) A variational formulation of the solution to (1.2) was studied in [10], together with a finite element error analysis. The error analysis was based on assumptions on the regularity of the true solution u, which has been pointed out by a number of other authors, is not justified for a general right-hand side function f. In [13] Jin et al. presented a very nice analysis and discussion of the regularity of the solution to (1.1) for $\mathbf{D}^\alpha$ interpreted as the Riemann-Liouville fractional derivative and as the Caputo fractional derivative. In general, the solution of (1.1) has a singularity in the derivative at $x = 0$. Very helpful in studying the regularity of the solution to (1.1) is the existence of an explicit inverse to $\mathcal{L}_1^\alpha$ which satisfies $\left(\mathcal{L}_1^\alpha\right)^{-1} f(0) = 0$. We do not have an explicit inverse for $\mathcal{L}_r^\alpha$. Subsequently we have to think more generally about the operator $\mathcal{L}_r^\alpha$ and, in particular, the definition of $\mathbf{D}^\alpha$ in the context of diffusion problems.

In [9, Section 3] we considered the underlying model of fractional diffusion in the setting of the 1-D heat equation. We concluded that in the context of a diffusion operator the appropriate interpretation of the fractional derivative is neither the Riemann-Liouville definition nor the Caputo definition. Rather, for $1 < \alpha < 2$,

$$\mathbf{D}^\alpha u(x) \;:=\; D\,\mathbf{D}^{-(2-\alpha)}\,Du(x)\,. \tag{1.3}$$

The kernel of the operator $\mathcal{L}_r^\alpha$, $\ker(\mathcal{L}_r^\alpha)$ plays a key role in determining the regularity of the solution of (1.2). Thus the definition of $\mathbf{D}^\alpha$ is central in determining the regularity of the solution to (1.2). In Section 3 we discuss the regularity of the solution to (1.2), using the definition of $\mathbf{D}^\alpha$ given in (1.3). Somewhat of a surprise is that the regularity of the solution depends upon r. In order to numerically illustrate the regularity of the solution to (1.2) in Section 4 we present Finite Element Method (FEM) computations. The experimental rates of convergence of the FEM approximations are consistent with the regularity of the solution obtained in Section 3.

In Section 5 we establish that Jacobi polynomials are pseudo eigenfunctions for the fractional diffusion operator. Specifically (see Lemma 5.2) we show that

$$\mathcal{L}_r^\alpha \omega(x)\,\mathcal{G}_n(x) \;=\; \lambda_n\,\mathcal{G}_n^*(x)\,,$$

where $\mathcal{G}_n(x)$ and $\mathcal{G}_n^*(x)$ are Jacobi polynomials, $\omega(x)$ is the Jacobi weight, and λ_n the pseudo eigenvalue. Using this property we propose and study a spectral type approximation method for the solution of steady-state fractional diffusion equations. Two numerical examples are given to illustrate the performance of the method. Recently Mao, Chen, and Shen in [20] proposed and analyzed an analogous spectral type approximation scheme for $\mathcal{L}^{\alpha}_{1/2}u(x) = f(x)$. Recognizing that $\mathcal{L}^{\alpha}_{1/2}u(x) = f(x)$ corresponds to a Riesz fractional differential equation, they were able to use known properties (from [22]) of the action of the Riesz kernel on weighted Jacobi polynomials to build the spectral approximation method.

2. Notation and Properties

For u a function defined on (a, b), and $\sigma > 0$, we have that the left and right fractional integral operators are defined as:
Left Fractional Integral Operator:

$$ {}_aD_x^{-\sigma}u(x) := \frac{1}{\Gamma(\sigma)}\int_a^x (x-s)^{\sigma-1}\,u(s)\,ds\,. $$

Right Fractional Integral Operator:

$$ {}_xD_b^{-\sigma}u(x) := \frac{1}{\Gamma(\sigma)}\int_x^b (s-x)^{\sigma-1}\,u(s)\,ds\,. $$

Then, for $\mu > 0$, n the smallest integer greater than μ ($n-1 \le \mu < n$), $\sigma = n-\mu$, and D the derivative operator, the left and right Riemann-Liouville fractional differential operators are defined as:
Left Riemann-Liouville Fractional Differential Operator of order μ:

$$ {}_a^{RL}D_x^{\mu}u(x) := D^n\,{}_aD_x^{-\sigma}u(x) = \frac{1}{\Gamma(\sigma)}\frac{d^n}{dx^n}\int_a^x (x-s)^{\sigma-1}\,u(s)\,ds\,. $$

Right Riemann-Liouville Fractional Differential Operator of order μ:

$$ {}_x^{RL}D_b^{\mu}\,u(x) := (-D)^n\,{}_xD_b^{-\sigma}u(x) = \frac{(-1)^n}{\Gamma(\sigma)}\frac{d^n}{dx^n}\int_x^b (s-x)^{\sigma-1}\,u(s)\,ds\,. $$

The Riemann-Liouville and Caputo fractional differential operators differ in the location of the derivative operator.
Left Caputo Fractional Differential Operator of order μ:

$$ {}_a^{C}D_x^{\mu}u(x) := {}_aD_x^{-\sigma}\,D^n u(x) = \frac{1}{\Gamma(\sigma)}\int_a^x (x-s)^{\sigma-1}\,\frac{d^n}{ds^n}u(s)\,ds\,. $$

Right Caputo Fractional Differential Operator of order μ:

$$ {}_x^{C}D_b^{\mu}\,u(x) := (-1)^n\,{}_xD_b^{-\sigma}\,D^n u(x) = \frac{(-1)^n}{\Gamma(\sigma)}\int_x^b (s-x)^{\sigma-1}\,\frac{d^n}{ds^n}u(s)\,ds\,. $$

As our interest is in the solution of fractional diffusion equations on a bounded, connected subinterval of $\mathbb{R}$, without loss of generality, we restrict our attention to the unit interval $(0, 1)$.

For $s \ge 0$ let $H^s(0,1)$ denote the Sobolev space of order s on the interval $(0,1)$, and $\tilde{H}^s(0,1)$ the set of functions in $H^s(0,1)$ whose extension by 0 are in $H^s(\mathbb{R})$. Equivalently, for u defined on $(0,1)$ and $\tilde{u}$ its extension by zero, $\tilde{H}^s(0,1)$ is the closure of $C_0^{\infty}(0,1)$ with respect to the norm $\|u\|_{\tilde{H}^s(0,1)} := \|\tilde{u}\|_{H^s(\mathbb{R})}$. With respect to L^2 duality, for $s \ge 0$ we let $H^{-s}(0,1) := \left(\tilde{H}^s(0,1)\right)'$, the dual space of $\tilde{H}^s(0,1)$.

Useful below in establishing results about the kernel of the fractional diffusion operator is the hypergeometric function [15, 25].

Definition 1. The Gaussian three-parameter hypergeometric function ${}_2F_1(\cdot,\cdot;\cdot;\, x)$ for $|x| < 1$, is defined by an integral and series as follows:
(2.1)
$$ {}_2F_1(a,\, b;\, c;\, x) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 z^{b-1}(1-z)^{c-b-1}(1-zx)^{-a}\, dz = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\, x^n}{(c)_n\, n!}\,, $$

with convergence only if $Re(c) > Re(b) > 0$.

In (2.1) $(q)_n := \Gamma(q+n)/\Gamma(q)$ denotes the (rising) Pochhammer symbol.
From the series representation of ${}_2F_1(a,\, b;\, c;\, x)$ the following result is easy to see.

Proposition 1 (Interchange property)**.** *For $Re(c) > Re(b) > 0$ and $Re(c) > Re(a) > 0$, we have that*

$$ {}_2F_1(a,\, b;\, c;\, x) \;=\; {}_2F_1(b,\, a;\, c;\, x)\,. \tag{2.2} $$

For ease of notation, we use

$$ \mathbf{D}^{-\sigma} \;:=\; {}_0D_x^{-\sigma}\,, \quad \text{and} \quad \mathbf{D}^{-\sigma *} \;:=\; {}_xD_1^{-\sigma}\,. $$

3. Kernel of the operator $r\mathbf{D}^{\alpha} + (1-r)\mathbf{D}^{\alpha *}$

In this section we establish the kernel for the operator

$$ \mathcal{L}_r^{\alpha} \;=\; -\left(r\mathbf{D}^{\alpha} \,+\, (1-r)\mathbf{D}^{\alpha *}\right)\,, \tag{3.1} $$

where $1 < \alpha < 2$.

Important in the discussion is the precise definition of the operator (3.1). For our interest, arising from fractional advection-diffusion equations, the operator (3.1) is interpreted as
(3.2)
$$ \mathcal{L}_r^{\alpha} u \;=\; -\left(r\mathbf{D}^{\alpha} \,+\, (1-r)\mathbf{D}^{\alpha *}\right) u \;:=\; -\Big(rD\mathbf{D}^{-(2-\alpha)}D \,+\, (1-r)D\mathbf{D}^{-(2-\alpha)*}D\Big)\, u\,. $$

Remark. The definition given in (3.2) differs from the Riemann-Liouville definition for $\mathbf{D}^{\alpha}$, where both integer order derivatives occur after the fractional integral. These different interpretations represent different operators and hence they have different kernels. For example, $u = \mathit{constant}$ is in the kernel of the operator defined in (3.2). However, $u = \mathit{constant}$ is not in the kernel of (3.1) using the Riemann-Liouville definition of the fractional differential operators.

Lemma 3.1. *For $\alpha - 2 \le p,\, q < 0$, $k(x) := x^p\,(1-x)^q$, $K(x) := \int_0^x k(s)\, ds$, we have that $K(x) \in \ker(\mathcal{L}_r^{\alpha})$ if:*

$$ \text{(i)} \quad 3 - \alpha + p + q = 1\,, \quad \textit{and} \tag{3.3} $$
$$ \text{(ii)} \quad r\sin(\pi(-q)) = (1-r)\sin(\pi(-p))\,. \tag{3.4} $$

Proof. Using the definition of the fractional integral, we have

$$
\begin{aligned}
\mathbf{D}^{-(2-\alpha)}k(x) &= \frac{1}{\Gamma(2-\alpha)} \int_0^x (x-s)^{1-\alpha}\, s^p\, (1-s)^q\, ds \\
&= \frac{1}{\Gamma(2-\alpha)}\, x^{2-\alpha+p} \int_0^1 (1-z)^{1-\alpha}\, z^p\, (1\,-\,zx)^q\, dz \quad (\text{using } z = s/x) \\
&= \frac{1}{\Gamma(2-\alpha)}\, x^{2-\alpha+p} \frac{\Gamma(p+1)\,\Gamma(2-\alpha)}{\Gamma(3-\alpha+p)}\, {}_2F_1(-q,\, p+1;\, 3-\alpha+p;\, x) \\
&\qquad (\text{provided } 3-\alpha+p > p+1 > 0, \text{ which is true for } 1 < \alpha < 2) \\
&= \frac{\Gamma(p+1)}{\Gamma(3-\alpha+p)}\, x^{2-\alpha+p}\, {}_2F_1(p+1,\, -q;\, 3-\alpha+p;\, x) \\
&\qquad (\text{using Proposition 1, provided } 3-\alpha+p > -q > 0) \\
&= \frac{\Gamma(p+1)}{\Gamma(3-\alpha+p)}\, x^{2-\alpha+p} \\
&\qquad \cdot \frac{\Gamma(3-\alpha+p)}{\Gamma(-q)\,\Gamma(3-\alpha+p+q)} \\
&\qquad \cdot \int_0^1 (1-z)^{2-\alpha+p+q}\, z^{-q-1}\, (1\,-\,zx)^{-p-1}\, dz \\
&\qquad (\text{using } z = s/x) \\
&= \frac{\Gamma(p+1)}{\Gamma(-q)\,\Gamma(3-\alpha+p+q)}\, x^{2-\alpha+p}\, x^{-(2-\alpha+p)} \\
&\qquad \cdot \int_0^x (x-s)^{2-\alpha+p+q}\, s^{-q-1}\, (1\,-\,s)^{-p-1}\, ds \\
&= \frac{\Gamma(p+1)}{\Gamma(-q)\,\Gamma(3-\alpha+p+q)} \\
&\qquad \cdot \int_0^x (x-s)^{2-\alpha+p+q}\, s^{-q-1}\, (1\,-\,s)^{-p-1}\, ds \\
&= \frac{\Gamma(p+1)}{\Gamma(-q)}\, \mathbf{D}^{-(3-\alpha+p+q)} x^{-q-1}\, (1\,-\,x)^{-p-1}\,. \qquad (3.5)
\end{aligned}
$$

Next,

$$
\begin{aligned}
\mathbf{D}^{-(2-\alpha)*}k(x) &= \frac{1}{\Gamma(2-\alpha)} \int_x^1 (s-x)^{1-\alpha}\, s^p\, (1-s)^q\, ds \\
&= \frac{1}{\Gamma(2-\alpha)}\, (1-x)^{2-\alpha+q} \int_0^1 (1-z)^{1-\alpha}\, z^q\, (1 - z(1-x))^p\, dz \\
&\qquad \text{(using } z = (1-s)/(1-x)\text{)} \\
&= \frac{1}{\Gamma(2-\alpha)}\, (1-x)^{2-\alpha+q} \frac{\Gamma(q+1)\,\Gamma(2-\alpha)}{\Gamma(3-\alpha+q)} \\
&\qquad\qquad \cdot\, {}_2F_1(-p,\, q+1;\, 3-\alpha+q;\, (1-x)) \\
&\qquad \text{(provided } 3-\alpha+q > q+1 > 0\text{, which is true for } 1<\alpha<2\text{)} \\
&= \frac{\Gamma(q+1)}{\Gamma(3-\alpha+q)}\, (1-x)^{2-\alpha+q}\, {}_2F_1(q+1,\, -p;\, 3-\alpha+q;\, (1-x)) \\
&\qquad \text{(using Proposition 1, provided } 3-\alpha+q > -p > 0\text{)} \\
&= \frac{\Gamma(q+1)}{\Gamma(3-\alpha+q)}\, (1-x)^{2-\alpha+q} \\
&\qquad \cdot \frac{\Gamma(3-\alpha+q)}{\Gamma(-p)\,\Gamma(3-\alpha+p+q)} \\
&\qquad\qquad \cdot \int_0^1 (1-z)^{2-\alpha+p+q}\, z^{-p-1}\, (1 - z(1-x))^{-q-1}\, dz \\
&= \frac{\Gamma(q+1)}{\Gamma(-p)\,\Gamma(3-\alpha+p+q)}\, (1-x)^{2-\alpha+q}\, (1-x)^{-(2-\alpha+q)}\, \cdot \\
&\qquad \int_x^1 (s-x)^{2-\alpha+p+q}\, s^{-q-1}\, (1-s)^{-p-1}\, ds \\
&\qquad \text{(using } z = (1-s)/(1-x)\text{)} \\
&= \frac{\Gamma(q+1)}{\Gamma(-p)\,\Gamma(3-\alpha+p+q)} \int_x^1 (s-x)^{2-\alpha+p+q}\, s^{-q-1}\, (1-s)^{-p-1}\, ds \\
&= \frac{\Gamma(q+1)}{\Gamma(-p)}\, \mathbf{D}^{-(3-\alpha+p+q)*} x^{-q-1}\, (1-x)^{-p-1}\,. \qquad (3.6)
\end{aligned}
$$

Comparing (3.5) and (3.6), $r\, D\, D^{-(2-\alpha)}k(x) \;+\; (1-r)\, D\, D^{-(2-\alpha)*}k(x) \;=\; 0$ if

$$
\begin{aligned}
\text{(3.7)} \qquad \text{(i)} \quad 3-\alpha+p+q &= 1\,, \\
\text{(ii)} \quad r\frac{\Gamma(p+1)}{\Gamma(-q)} &= (1-r)\frac{\Gamma(q+1)}{\Gamma(-p)} \\
\iff r\,\Gamma(-p)\,\Gamma(1-(-p)) &= (1-r)\,\Gamma(-q)\,\Gamma(1-(-q)) \\
\iff r\frac{\pi}{\sin(\pi(-p))} &= (1-r)\frac{\pi}{\sin(\pi(-q))} \\
\text{(using } \Gamma(1-z)\,\Gamma(z) &= \pi/\sin(\pi z)\,,\ \text{valid for } z \neq 0,\, \pm1,\, \pm2, \ldots) \\
\text{(3.8)} \qquad \iff r\sin(\pi\,(-q)) &= (1-r)\sin(\pi\,(-p)).
\end{aligned}
$$

$\square$

For the case $\alpha = 1.6$ a plot of p vs. r satusftubg (3.7) and (3.8) is given in Figure 3.1.

Remark. For $0 \le r \le 1$, $1 < \alpha < 2$ given, p and q satisfying (3.3) and (3.4) can be equivalently expressed as $q = \alpha - 2 - p$ and $\alpha - 2 \le p \le 0$ satisfying $h(p) := r - \sin(\pi p)/(\sin(\pi(\alpha - p)) + \sin(\pi p)) = 0$. The existence and uniqueness of p and q satisfying (3.3) and (3.4) follows from noting that $h(\alpha - 2) = r - 1 \le 0$, $h(0) = r \ge 0$, and that $h'(p) > 0$.

Corollary 3.1. *The kernel of* $\mathcal{L}_r^\alpha(\cdot)$, $\ker(\mathcal{L}_r^\alpha)$, *is given by* $\ker(\mathcal{L}_r^\alpha) = \operatorname{span}\{1, K(x)\}$, *where* $K(x)$, *given in Lemma* 3.1, *may be written as*

$$K(x) = \int_0^x k(s)\, ds = \frac{1}{p+1} x^{p+1} \, {}_2F_1(-q, p+1\,; p+2\,; x).$$

Proof. From above it is clear that $\operatorname{span}\{1, K(x)\} \subset \ker(\mathcal{L}_r^\alpha)$. What remains is to show that $\dim(\ker(\mathcal{L}_r^\alpha)) = 2$.

With $z(x) = 1 + x$ and $f(x) = -r \frac{1}{\Gamma(2-\alpha)} x^{1-\alpha} + (1-r) \frac{1}{\Gamma(2-\alpha)} (1-x)^{1-\alpha}$, a straightforward calculation shows that $\mathcal{L}_r^\alpha z(x) = f(x)$ on I. As $K(1) \ne 0$ we can choose c_1 and c_2 such that $\hat{z}(x) := z(x) + c_1 1 + c_2 K(x)$ satisfies $\hat{z}(0) = \hat{z}(1) = 0$ and $\mathcal{L}_r^\alpha \hat{z}(x) = f(x)$.

Suppose there was another linearly independent function $s(x) \in \ker(\mathcal{L}_r^\alpha)$. Without loss of generality we may assume that $s(0) = s(1) = 0$. (If this was not the case we would form a linear combination of $s(x)$ with the other two linearly independent kernel function 1 and $K(x)$.) Then $\tilde{z}(x) := \hat{z}(x) + s(x)$ satisfies $\tilde{z}(0) = \tilde{z}(1) = 0$ and $\mathcal{L}_r^\alpha \tilde{z}(x) = f(x)$. However, the existence of $\tilde{z}(x) \ne \hat{z}(x)$ contradicts the uniqueness of the solution to $\mathcal{L}_r^\alpha u(x) = f(x)$, with $u(0) = u(1) = 0$, [10, Theorem 3.5]. $\square$

A plot of $K(x)$ for $\alpha = 1.6$ and $r = 0.2764$ (i.e., $p = 0.1, q = -0.3$) is given in Figure 3.2.

Example 3.1. The case $r = 1/2$. This corresponds to $\mathcal{L}_{1/2}^\alpha$. For $r = 1/2$, from (3.8), $p = q$. Then, using (3.7), we have $p = q = \alpha/2 - 1$.

Example 3.2. The case $r \to 1$. This corresponds to $\mathcal{L}_1^\alpha(u) = -D\mathbf{D}^{-(2-\alpha)}D(u)$. For this case the kernel is $\operatorname{span}\{1, x^{\alpha-1}\}$.
Now, from (3.8), as $r \to 1$ then

$$\sin(\pi(-q)) \to 0 \implies q \to 0\,.$$

Hence from (3.7) $p \to \alpha - 2 \implies K(x) = x^{\alpha-1}$.

Lemma 3.2. *For* $1 \le \alpha < 1.5$, $D\mathbf{D}^{-(2-\alpha)}D$ *maps from* $H^\alpha(I)$ *onto* $L^2(I)$.

Proof. We have that $D : H^\alpha(I) \longrightarrow H^{\alpha-1}(I)$. Now, for $1 < \alpha < 1.5$, then $0 < \alpha - 1 < 0.5$, hence $H^{\alpha-1}(I) = \tilde{H}^{\alpha-1}(I)$. As $D\mathbf{D}^{-(2-\alpha)} = {}_0^{RL}D_x^{\alpha-1}$, then from Theorem 3.1 [13] $D\mathbf{D}^{-(2-\alpha)} : \tilde{H}^{\alpha-1}(I) \longrightarrow L^2(I)$.

To establish that the mapping is onto, we have that for

$$f \in L^2(I), D\mathbf{D}^{-(2-\alpha)}D\, u = f,$$

where $u = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)\, ds$. $\square$

Corollary 3.2. *For* $1 \le \alpha < 1.5$, $r \in \mathbb{R}$, $\mathcal{L}_r^\alpha$ *maps from* $H^\alpha(I)$ <u>*into*</u> $L^2(I)$.

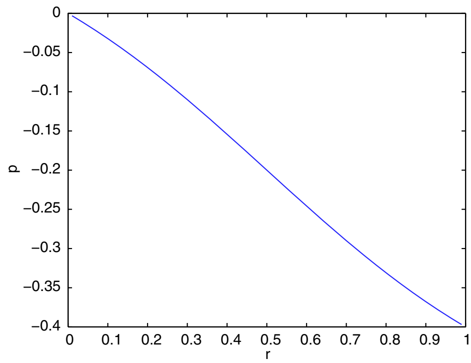

FIGURE 3.1. p values solving (3.7) and (3.8) for $\alpha = 1.6$.

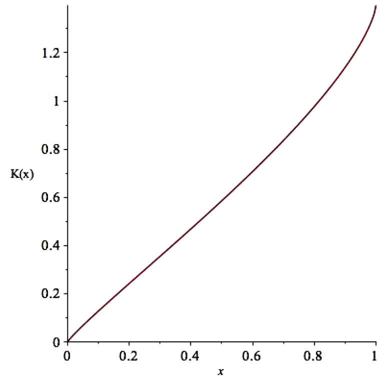

FIGURE 3.2. Plot of $K(x)$ for $\alpha = 1.6$ and $r = 0.2764$ (i.e., $p = -0.1$, $q = -0.3$).

Proof. An analogous argument to that given in the proof of Lemma 3.2 establishes that $D\mathbf{D}^{-(2-\alpha)*}D$ maps from $H^\alpha(I)$ onto $L^2(I)$. The stated result then follows. $\square$

In order to give a concise description of the range of $\mathcal{L}_r^\alpha$, with domain $H^\alpha(I)$, let

$$X^{(1-\alpha)} \;:=\; \{f \,:\, f(x) \,=\, cx^{1-\alpha},\, c \in \mathbb{R}\}$$

and

$$X^{(1-\alpha)*} \;:=\; \{f \,:\, f(x) \,=\, c(1-x)^{1-\alpha},\, c \in \mathbb{R}\}\,.$$

Lemma 3.3. *For $1 \le \alpha < 2$ $\mathcal{L}_r^\alpha$ maps from $H^\alpha(I)$ into $L^2(I) \oplus X^{(1-\alpha)} \oplus X^{(1-\alpha)*}$.*

Proof. The case for $1 \le \alpha < 1.5$ is covered by Corollary 3.2. For $f(x) \in H^\alpha(I)$, $\alpha \ge 1.5$, let $p(x)$ denote the Hermite cubic interpolating polynomial of $f(x)$. Namely,

$$\begin{aligned} p(x) &= (2x^3 - 3x^2 + 1)f(0) + (x^3 - 2x^2 + x)f'(0) \\ &\quad + (-2x^3 + 3x^2)f(1) + (x^3 - x^2)f'(1) \\ &= (-2(1-x)^3 + 3(1-x)^2)f(0) + (-(1-x)^3 + (1-x)^2)f'(0) \\ &\quad + (2(1-x)^3 - 3(1-x)^2 + 1)f(1) \\ &\quad + (-(1-x)^3 + 2(1-x)^2 - (1-x))f'(1)\,. \end{aligned}$$

Also, let $\tilde{f}(x) = f(x) - p(x) \in \tilde{H}^\alpha(I)$. From Theorem 2.1 of [13], $\mathcal{L}_r^\alpha \tilde{f}(x) \in L^2(I)$. Now,

$$\begin{aligned} \mathcal{L}_r^\alpha f(x) =& \mathcal{L}_r^\alpha \tilde{f}(x) + r\,(f(0))\,\mathbf{D}^\alpha 1 + r\,(f'(0))\,\mathbf{D}^\alpha x \\ &+ r\,(-3f(0) - 2f'(0) + 3f(1) - f'(1))\,\mathbf{D}^\alpha x^2 \\ &+ r\,(2f(0) + f'(0) - 2f(1) + f'(1))\,\mathbf{D}^\alpha x^3 + (1-r)\,(f(1))\,\mathbf{D}^{\alpha*} 1 \\ &+ (1-r)\,(-f'(1))\,\mathbf{D}^{\alpha*}(1-x) \\ &+ (1-r)\,(-3f(1) + 2f'(1) + 3f(0) + f'(0))\,\mathbf{D}^{\alpha*}(1-x)^2 \\ &+ (1-r)\,(2f(1) - f'(1) - 2f(0) - f'(0))\,\mathbf{D}^{\alpha*}(1-x)^3\,. \end{aligned}$$

As $\mathbf{D}^{\alpha}1 = \mathbf{D}^{\alpha*}1 = 0$; $\mathbf{D}^{\alpha}x \in X^{(1-\alpha)}$, $\mathbf{D}^{\alpha}(1-x) \in X^{(1-\alpha)*}$; $\mathbf{D}^{\alpha}x^2$, $\mathbf{D}^{\alpha}x^3$, $\mathbf{D}^{\alpha*}(1-x)^2$, $\mathbf{D}^{\alpha*}(1-x)^3 \in L^2(I)$, the stated result follows. □

4. Convergence of the finite element method approximation

In a finite element method (FEM) approximation to (1.2) the regularity of the solution u plays a fundamental role in the rate of convergence of the approximation u_h to u. In this section we present four numerical experiments and compare the numerical rate of convergence of the FEM approximation to that predicted theoretically.

From [10], with $X = \tilde{H}^{\alpha/2}(I)$, the weak formulation of (1.2) is: *Given* $f \in H^{-\alpha/2}(I)$ *determine* $u \in X$ *satisfying*

$$B(u\,,\,v) \;=\; \langle f\,,\,v\rangle \quad \forall v \in X\,, \tag{4.1}$$

where, $\langle \cdot\,,\,\cdot\rangle$ denotes the L^2 duality pairing between $H^{-\alpha/2}(I)$ and $\tilde{H}^{\alpha/2}(I)$, and $B(\cdot,\cdot) : X \times X \longrightarrow \mathbb{R}$ is defined by

$$\begin{aligned} B(w,v) \;:=&\; r\left(D^{-(2-\alpha)/2}Dw\,,\,D^{-(2-\alpha)/2*}Dv\right) \\ &+\;(1-r)\left(D^{-(2-\alpha)/2*}Dw\,,\,D^{-(2-\alpha)/2}Dv\right). \end{aligned} \tag{4.2}$$

For $0 = x_0 < x_1 < \cdots < x_N = 1$ denoting a quasi-uniform partition of $I := (0,1)$, $X_h \subset X$ denoting the space of continuous, piecewise polynomials of degree $\le k$ on the partition, the finite element approximation $u_h \in X_h$ to u is given by

$$B(u_h\,,\,v_h) \;=\; \langle f, \quad v_h\rangle \quad \forall v_h \in X_h\,. \tag{4.3}$$

Assuming that f is sufficiently regular such that the regularity of u is determined by the kernel of $\mathcal{L}_r^{\alpha}$, we have the following a priori error bounds, for $C > 0$ a constant and any $\epsilon > 0$ and $\delta > 0$, [10, Corollary 4.3].

$$\begin{aligned} \|u - u_h\|_{\tilde{H}^{\alpha/2}} &\le C \inf_{v_h \in X_h} \|u - v_h\|_{\tilde{H}^{\alpha/2}} \\ &\le C \begin{cases} h^{1/2-\epsilon}\|u\|_{H^{\alpha/2+1/2-\epsilon}}\,, & r = 1/2\,, \\ h^{\min\{p,q\}+3/2-\alpha/2-\epsilon}\|u\|_{H^{\min\{p,q\}+3/2-\epsilon}}\,, & r \neq 1/2\,, \end{cases} \end{aligned} \tag{4.4}$$

where p and q satisfy (3.7) and (3.8).

An application of the Aubin-Nitsche trick yields the following L^2 a priori error bounds, [10, Theorem 4.4].

$$\|u - u_h\| \;\le\; C \begin{cases} h^{1-2\epsilon}\|u\|_{H^{\alpha/2+1/2-\epsilon}}\,, & r = 1/2\,, \\ h^{2(\min\{p,q\}+3/2-\alpha/2)-2\epsilon}\|u\|_{H^{\min\{p,q\}+3/2-\epsilon}}\,, & r \neq 1/2\,. \end{cases} \tag{4.5}$$

For the Aubin-Nitsche trick the regularity of the associated adjoint problem is the same as that for u (assuming $f \in L^2(I)$). Hence the L^2 a priori error bound is simply twice that for $H^{\alpha/2}$.

For Examples 1 and 2 the true solution u was chosen to be $x + \mathrm{kerfun}(x)$, with $\mathrm{kerfun}(x) \in \ker(\mathcal{L}_r^{\alpha})$ chosen such that u satisfies $u(0) = u(1) = 0$. In Examples 3 and 4 the right-hand side $f(x)$ was chosen to be a constant. Results are reported for $\alpha = 1.4$ and $\alpha = 1.6$. Computations were also performed for $\alpha = 1.2$ and $\alpha = 1.8$ (not included) which exhibited similar behavior. The approximation space X_h used was the continuous, affine functions (i.e., $k = 1$) on a uniform partition of I. The $|u - u_h|_{H^{\alpha/2}}$ data presented in the tables denotes the Slobodetskii semi-norm,

$$|w|_{H^{\alpha/2}} := \left(\int_I \int_I \frac{|w(x) - w(y)|^2}{|x - y|^{1+\alpha}} dx\, dy \right)^{1/2} .$$

Example 1. With $\alpha = 1.4$, $r = 1/2$,

$$u(x) = x - C\, x^{\alpha/2}\, {}_2F_1(\alpha/2\,,\, 1 - \alpha/2\,;\, 1 + \alpha/2\,,\, x)\,, \tag{4.6}$$

where $C = \left({}_2F_1(\alpha/2\,,\, 1 - \alpha/2\,;\, 1 + \alpha/2\,,\, 1)\right)^{-1}$.
The corresponding right-hand side is

$$f(x) = \frac{-1}{2} \frac{1}{\Gamma(2 - \alpha)}\, x^{1-\alpha} + \frac{1}{2} \frac{1}{\Gamma(2 - \alpha)}\, (1 - x)^{1-\alpha} . \tag{4.7}$$

The numerical results are presented in Table 4.1.

TABLE 4.1. Example 1. Convergence rates for $\alpha = 1.4$ and $r = 1/2$.

h	$\lvert u - u_h \rvert_{H^{\alpha/2}(I)}$	Cvg. rate	$\lVert u - u_h \rVert_{L^2(I)}$	Cvg. rate
1/64	4.209E-02		8.402E-04	
1/128	2.962E-02	0.51	4.016E-04	1.07
1/256	2.088E-02	0.50	1.936E-04	1.05
1/512	1.475E-02	0.50	9.407E-05	1.04
1/1024	1.042E-02	0.50	4.598E-05	1.03
1/2048	7.364E-03	0.50	2.258E-05	1.03
Pred.		0.50		1.0

Example 2. With $\alpha = 1.4$, $p = -0.15$, $q = \alpha - p - 2$, $r = \sin(\pi p)/(\sin(\pi p) + \sin(\pi q))$,

$$u(x) = x - C\, x^{(p+1)}\, {}_2F_1(-q\,,\, p + 1\,;\, p + 2\,,\, x)\,, \tag{4.8}$$

where $C = \left({}_2F_1(-q\,,\, p + 1\,;\, p + 2\,,\, 1)\right)^{-1}$.
The corresponding right-hand side is

$$f(x) = -r \frac{1}{\Gamma(2 - \alpha)}\, x^{1-\alpha} + (1 - r) \frac{1}{\Gamma(2 - \alpha)}\, (1 - x)^{1-\alpha} . \tag{4.9}$$

The numerical results are presented in Table 4.2.

TABLE 4.2. Example 2. Convergence rates for $\alpha = 1.4$ and $r = 0.3149$.

h	$\lvert u - u_h \rvert_{H^{\alpha/2}(I)}$	Cvg. rate	$\lVert u - u_h \rVert_{L^2(I)}$	Cvg. rate
1/64	1.463E-01		1.609E-03	
1/128	1.146E-01	0.35	7.847E-04	1.04
1/256	8.990E-02	0.35	3.831E-04	1.03
1/512	7.052E-02	0.35	1.872E-04	1.03
1/1024	5.532E-02	0.35	9.157E-05	1.03
1/2048	4.340E-02	0.35	4.482E-05	1.03
Pred.		0.35		0.70

Example 3. With $\alpha = 1.6$, $r = 1/2$,

$$u(x) \;=\; x^{\alpha/2}(1-x)^{\alpha/2}\,. \tag{4.10}$$

The corresponding right-hand side is

$$f(x) \;=\; -\Gamma(1+\alpha)\,\cos(\pi\alpha/2)\,. \tag{4.11}$$

The numerical results are presented in Table 4.3.

TABLE 4.3. Example 3. Convergence rates for $\alpha = 1.6$ and $r = 0.5$.

h	$\lvert u - u_h\rvert_{H^{\alpha/2}(I)}$	Cvg. rate	$\lVert u - u_h\rVert_{L^2(I)}$	Cvg. rate
1/64	3.502E-02		6.559E-04	
1/128	2.461E-02	0.51	3.081E-04	1.09
1/256	1.734E-02	0.50	1.479E-04	1.06
1/512	1.224E-02	0.50	7.205E-05	1.04
1/1024	8.651E-03	0.50	3.542E-05	1.02
1/2048	6.115E-03	0.50	1.752E-05	1.02
Pred.		0.50		1.0

Example 4. With $\alpha = 1.6$, $p = 0.9$, $q = \alpha - p$, $r \;=\; \sin(\pi(p+1))/(\sin(\pi(p+1)) - \sin(\pi(\alpha - p)))$,

$$u(x) \;=\; x^{p}(1-x)^{q}\,. \tag{4.12}$$

The corresponding right-hand side is

$$f(x) \;=\; -(1-r)\,\Gamma(1+\alpha)\frac{\sin(\pi\alpha)}{\sin(\pi(\alpha-p))}\,. \tag{4.13}$$

The numerical results are presented in Table 4.4.

TABLE 4.4. Example 4. Convergence rates for $\alpha = 1.6$ and $r = 0.2764$.

h	$\lvert u - u_h\rvert_{H^{\alpha/2}(I)}$	Cvg. rate	$\lVert u - u_h\rVert_{L^2(I)}$	Cvg. rate
1/64	7.732E-02		7.083E-04	
1/128	5.827E-02	0.41	3.216E-04	1.14
1/256	4.402E-02	0.40	1.485E-04	1.12
1/512	3.331E-02	0.40	6.947E-05	1.10
1/1024	2.522E-02	0.40	3.289E-05	1.08
1/2048	1.910E-02	0.40	1.572E-05	1.06
Pred.		0.40		0.80

The numerical results are consistent with the theoretical predictions. Of particular note is that changing the convex combination of the adjoint operators in the definition of $\mathcal{L}_r^{\alpha}$, i.e., the factor r, changes the regularity of the solution, and hence the convergence rate of the FEM approximation.

5. Spectral type method for the solution of $\mathcal{L}_r^\alpha u = f$

In this section we discuss a "spectral type" approximation method for the numerical solution of $\mathcal{L}_r^\alpha u = f$. Recently Mao, Chen and Shen in [20] proposed and analyzed an analogous spectral type approximation scheme for the special case $\mathcal{L}_{1/2}^\alpha u = f$. Recognizing that $\mathcal{L}_{1/2}^\alpha u(x) = f(x)$ corresponds to a Riesz fractional differential equation, they were able to use known properties (from [22]) of the action of the Riesz kernel on weighted Jacobi polynomials to build the spectral approximation method.

Central to the method we propose is the following result.

Lemma 5.1. *For $1 < \alpha < 2$, $0 < \beta < \alpha$, and r satisfying*

$$r \;=\; \frac{\sin(\pi\beta)}{\sin(\pi(\alpha-\beta)) \,+\, \sin(\pi\beta)} \tag{5.1}$$

for $n = 0, 1, 2, \ldots$,

$$\mathcal{L}_r^\alpha \, x^\beta (1-x)^{\alpha-\beta} \, x^n = \sum_{j=0}^{n} a_{n,j} \, x^j \,, \qquad \textit{where} \tag{5.2}$$

$$\begin{aligned} a_{n,j} \;&= (-1)^{(n+1)} (1-r) \, \frac{\sin(\pi\,\alpha)}{\sin(\pi(\alpha-\beta))} \, \Gamma(1+\alpha-\beta) \\ &\quad \cdot \frac{(-1)^j \, \Gamma(1+\alpha+j)}{\Gamma(1+\alpha-\beta\,-\,n+j)\,\Gamma(1+n-j)\,\Gamma(j+1)} \,. \end{aligned}$$

Proof. For $0 < r < 1$, with $u(x) = x^\beta (1-x)^{(\alpha-\beta)} \, x^n$ using Maple (see [9]),

$$\begin{aligned} \mathbf{D}^{-(2-\alpha)} u(x) \;&=\; \frac{\Gamma(1+\beta\,+\,n)}{\Gamma(3-\alpha+\beta\,+\,n)} \, x^{n+2-\alpha+\beta} \\ &\qquad \cdot {}_2F_1(1+\beta+n\,,\, -\alpha+\beta\,; 3-\alpha+\beta\,+\,n\,;\, x)\,, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \mathbf{D}^{-(2-\alpha)*} u(x) &= \frac{\Gamma(-2+\alpha-\beta\,-\,n)}{\Gamma(-\beta\,-\,n)} \, x^{n+2-\alpha+\beta} \\ &\quad \cdot {}_2F_1(1+\beta+n\,,\, -\alpha+\beta\,; 3-\alpha+\beta\,+\,n\,;\, x) \\ &\quad + (-1)^n \, \Gamma(1+\alpha-\beta) \\ &\quad \times \sum_{k=0}^{n+2} \frac{(-1)^k \, \csc(\pi(\alpha-\beta)\,+\,k\pi)\, \sin(\pi\alpha\,+\,k\pi)\Gamma(-1+\alpha+k)}{\Gamma(-1+\alpha-\beta\,-n+k)\,\Gamma(3+n-k)\,\Gamma(k+1)} \, x^k. \end{aligned} \tag{5.4}$$

Using the identity

$$\Gamma(1-z) \;=\; \frac{\pi}{\sin(\pi z)} \, \frac{1}{\Gamma(z)}\,, \tag{5.5}$$

with $z = 1+\beta+n$, i.e., $1-z = -\beta-n$,

$$\begin{aligned} \Gamma(-\beta\,-n) &= \frac{\pi}{\sin(\pi(1+\beta\,+n))} \, \frac{1}{\Gamma(1+\beta\,+n)} \\ &= \frac{\pi}{\sin(\pi\beta)\,\cos(\pi(n+1))} \, \frac{1}{\Gamma(1+\beta\,+n)} \\ &= \frac{(-1)^{(n+1)}\,\pi}{\sin(\pi\beta)\,\Gamma(1+\beta\,+n)}\,. \end{aligned} \tag{5.6}$$

Again using (5.5) with $z = 3 - \alpha + \beta + n$,

$$\begin{aligned}\Gamma(-2+\alpha-\beta-n) &= \frac{\pi}{\sin(\pi(3-\alpha+\beta+n))}\,\frac{1}{\Gamma(3-\alpha+\beta+n)}\\ &= \frac{\pi}{\sin(-\pi(\alpha-\beta))\,\cos(\pi(n+3))}\,\frac{1}{\Gamma(3-\alpha+\beta+n)}\\ &= \frac{(-1)^{(n+4)}\,\pi}{\sin(\pi(\alpha-\beta))\,\Gamma(3-\alpha+\beta+n)}\,. \end{aligned} \tag{5.7}$$

Using (5.6) and (5.7)

$$\frac{\Gamma(-2+\alpha-\beta-n)}{\Gamma(-\beta-n)} = \frac{-\sin(\pi\beta)\,\Gamma(1+\beta+n)}{\sin(\pi(\alpha-\beta))\,\Gamma(3-\alpha+\beta+n)}\,. \tag{5.8}$$

The coefficient of $x^{n+2-\alpha+\beta}{}_2F_1(\cdot)$ in the linear combination

$$\left(r\,\mathbf{D}^{-(2-\alpha)} + (1-r)\,\mathbf{D}^{-(2-\alpha)*}\right)u(x)$$

is

$$\begin{aligned} &r\frac{\Gamma(1+\beta+n)}{\Gamma(3-\alpha+\beta+n)} + (1-r)\frac{\Gamma(-2+\alpha-\beta-n)}{\Gamma(-\beta-n)}\\ &= \frac{\Gamma(1+\beta+n)}{\Gamma(3-\alpha+\beta+n)}\left(r + (1-r)\frac{-\sin(\pi\beta)}{\sin(\pi(\alpha-\beta))}\right) \quad \text{(using (5.8))}\\ &= 0\,, \end{aligned}$$

provided r is given by (5.1).

Using standard trigonometric identities it is straightforward to show

$$\csc(\pi(\alpha-\beta) + k\pi)\,\sin(\pi\alpha + k\pi) = \frac{\sin(\pi\alpha)}{\sin(\pi(\alpha-\beta))}\,.$$

Thus,

$$\begin{aligned} &-D\,D\left(r\mathbf{D}^{-(2-\alpha)} + (1-r)\mathbf{D}^{-(2-\alpha)*}\right)u(x)\\ &\quad= (-1)^{(n+1)}(1-r)\,\Gamma(1+\alpha-\beta)\\ &\qquad\times\sum_{k=2}^{n+2}\frac{(-1)^k\,k\,(k-1)\,\frac{\sin(\pi\alpha)}{\sin(\pi(\alpha-\beta))}\,\Gamma(-1+\alpha+k)}{\Gamma(-1+\alpha-\beta-n+k)\,\Gamma(3+n-k)\,\Gamma(k+1)}\,x^{(k-2)} \end{aligned}$$

which, after reindexing, yields (5.2).

For the one-sided operators $\mathcal{L}_0^\alpha$ and $\mathcal{L}_1^\alpha$, corresponding to $r = 0$, $(\beta = 0)$ and $r = 1$, $(\beta = \alpha)$, we have [14]

$$\mathcal{L}_0^\alpha(1-x)^\alpha(1-x)^n = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)}(1-x)^n \quad \text{and} \quad \mathcal{L}_1^\alpha x^\alpha x^n = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)}x^n.$$

□

Jacobi polynomials play a key role in the approximation schemes. We briefly review their definition and properties central to the method [1,25].

Usual Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$, on $(-1\,,\,1)$.

Definition: $P_n^{(\alpha,\beta)}(x) := \sum_{m=0}^{n} p_{n,m}\,(x-1)^{(n-m)}(x+1)^m$, where

$$p_{n,m} := \frac{1}{2^n}\begin{pmatrix} n+\alpha \\ m \end{pmatrix}\begin{pmatrix} n+\beta \\ n-m \end{pmatrix}. \tag{5.9}$$

Orthogonality:

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\, P_j^{(\alpha,\beta)}(x)\, P_k^{(\alpha,\beta)}(x)\, dx \ = \ \begin{cases} 0, & k \neq j\,, \\ |||P_j^{(\alpha,\beta)}|||^2\,, & k = j\,, \end{cases}$$

(5.10)

$$\text{where} \quad |||P_j^{(\alpha,\beta)}||| \ = \ \left(\frac{2^{(\alpha+\beta+1)}}{(2j \,+\, \alpha \,+\, \beta \,+1)} \frac{\Gamma(j+\alpha+1)\,\Gamma(j+\beta+1)}{\Gamma(j+1)\,\Gamma(j+\alpha+\beta+1)} \right)^{1/2} .$$

Jacobi polynomials, $G_n(p,q,x)$, on $(0\,,\,1)$.
Definition: $G_n(p,q,x) \ := \ \sum_{j=0}^{n} g_{n,j}\, x^j$, where

$$g_{n,j} \ := \ (-1)^{(n-j)} \frac{\Gamma(q+n)}{\Gamma(p+2n)} \, \frac{\Gamma(n+1)}{\Gamma(j+1)\,\Gamma(n-j+1)} \, \frac{\Gamma(p+n+j)}{\Gamma(q+j)} \,. \tag{5.11}$$

Orthogonality:

$$\int_{0}^{1} x^{(q-1)}(1-x)^{(p-q)}\, G_j(p,q,x)\, G_k(p,q,x)\, dx \ = \ \begin{cases} 0, & k \neq j\,, \\ |||G_j^{(p,q)}|||^2\,, & k = j\,, \end{cases}$$

(5.12)

$$\text{where} \quad |||G_n^{(p,q)}||| \ = \ \left(\frac{\Gamma(n+1)\,\Gamma(n+q)\,\Gamma(n+p)\,\Gamma(n+p-q+1)}{(2n+p)\,\Gamma^2(2n+p)} \right)^{1/2} .$$

Note that $G_n(p,q,x) \ = \ \frac{\Gamma(n+1)\,\Gamma(n+p)}{\Gamma(2n+p)}\, P_n^{(p-q\,,\,q-1)}(2x-1)$.

The weighted $L^2(0,1)$ spaces, $L^2_\rho(0,1)$.
The weighted $L^2(0,1)$ spaces are convenient for analyzing the convergence of the spectral type methods presented below. For $\rho(x) > 0,\ x \in (0,1)$, let

$$L^2_\rho(0,1) \ := \ \{f(x) \ : \ \int_0^1 \rho(x)\, f(x)^2\, dx \ < \ \infty\}\,.$$

Associated with $L^2_\rho(0,1)$ is the inner product, $\langle \cdot, \cdot \rangle_\rho$, and norm, $\| \cdot \|_\rho$, defined by

$$\langle f\,,\,g \rangle_\rho := \int_0^1 \rho(x)\, f(x)\, g(x)\, dx \quad \text{and}$$
$$\|f\|_\rho := \ (\langle f\,,\,f \rangle_\rho)^{1/2} \ .$$

5.1. **Spectral type method approximation to $\mathcal{L}_r^\alpha u \ = \ f$.** For the general case $\mathcal{L}_r^\alpha u \ = \ f$, $r \neq 1/2$, the operator $\mathcal{L}_r^\alpha \cdot$ is not symmetric. Hence the singular behavior of the adjoint problem $(\mathcal{L}_r^\alpha)^* \cdot \ = \ \mathcal{L}_{1-r}^\alpha \cdot$ does not match that of $\mathcal{L}_r^\alpha \cdot$. In order to conveniently present the approximation method and its properties, in this section we use the following notation.

For $1 < \alpha < 2$ and r given, and β determined by (5.1),

$$
\begin{aligned}
\mathcal{L}_r^{\alpha} u &= r\, D^{\alpha} u \;+\; (1-r)\, D^{\alpha *} u, \\
\mathcal{L}_r^{\alpha *} u &= r\, D^{\alpha *} u \;+\; (1-r)\, D^{\alpha} u'; \\
\omega(x) &= x^{\beta}\, (1-x)^{\alpha-\beta}, \\
\omega^{*}(x) &= x^{\alpha-\beta}\, (1-x)^{\beta}; \\
\mathcal{G}_n(x) &= G_n(\alpha+1\,,\, \beta+1\,,\, x), \\
\mathcal{G}_n^{*}(x) &= G_n(\alpha+1\,,\, \alpha-\beta+1\,,\, x); \\
\lambda_n &= -(1-r)\frac{\sin(\pi\,\alpha)}{\sin(\pi(\alpha-\beta))}\,\frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)}, \\
\lambda_n^{*} &= -r\frac{\sin(\pi\,\alpha)}{\sin(\pi(\alpha-\beta))}\,\frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)}.
\end{aligned} \tag{5.13}
$$

From (5.12) we have the following orthogonality properties:

$$
\int_0^1 \omega(x)\, \mathcal{G}_j(x)\, \mathcal{G}_k(x)\, dx \;=\; 0\,, \quad k \neq j\,, \qquad \int_0^1 \omega^{*}(x)\, \mathcal{G}_j^{*}(x)\, \mathcal{G}_k^{*}(x)\, dx \;=\; 0\,, \quad k \neq j\,,
$$

and

$$
\begin{aligned}
\|\mathcal{G}_n\|_{\omega}^2 &= \|G_n(\alpha+1\,,\, \beta+1\,,\, x)\|_{\omega}^2 \\
&= \frac{\Gamma(n+1)\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)\,\Gamma(n+\alpha-\beta+1)}{(2n\,+\,\alpha\,+\,1)\;\Gamma^2(2n\,+\,\alpha\,+\,1)} \\
&= \|G_n(\alpha+1\,,\, \alpha-\beta+1\,,\, x)\|_{\omega^{*}}^2 \\
&= \|\mathcal{G}_n^{*}\|_{\omega^{*}}^2\,.
\end{aligned} \tag{5.14}
$$

Additionally, $\{\mathcal{G}_j(x)\}_{j=0}^{\infty}$ and $\{\mathcal{G}_j^{*}(x)\}_{j=0}^{\infty}$ are orthogonal basis for $L_{\omega}^2(0,1)$ and $L_{\omega^{*}}^2(0,1)$, respectively.

Using Stirling's formula we have that

$$
\lim_{n\to\infty} \frac{\Gamma(n+\mu)}{\Gamma(n)\, n^{\mu}} \;=\; 1\,, \text{ for } \mu \in \mathbb{R}. \tag{5.15}
$$

(5.16)
Thus $\lambda_n > 0$ for all $n \,=\, 0,1,2,\ldots$, and as $n \to \infty$ $\lambda_n \sin -(1-r)\frac{\sin(\pi\,\alpha)}{\sin(\pi(\alpha-\beta))}\,(n+1)^{\alpha}$.

Lemma 5.2. *For $n \,=\, 0,1,2,\ldots$,*

$$
\mathcal{L}_r^{\alpha}\, \omega(x)\, \mathcal{G}_n(x) = \;\lambda_n\, \mathcal{G}_n^{*}(x)\,, \tag{5.17}
$$

$$
\mathcal{L}_{1-r}^{\alpha}\, \omega^{*}(x)\, \mathcal{G}_n^{*}(x) = \;\lambda_n^{*}\, \mathcal{G}_n(x)\,. \tag{5.18}
$$

Proof. Up to a multiplicative constant, $\mathcal{G}_n(x)$ and $\mathcal{G}_n^{*}(x)$ are, respectively, determined by $(\mathcal{G}_n(x)\,,\, p(x))_{\omega} \;=\; 0$ and $(\mathcal{G}_n^{*}(x)\,,\, p(x))_{\omega^{*}} \;=\; 0$ for all $p(x) \in \mathcal{P}_{n-1}(x)$.

Let $p(x) \in \mathcal{P}_{n-1}(x)$. Then, from Lemma 5.1 there exists $\hat{p}(x) \in P_{n-1}(x)$ such that $\mathcal{L}_{1-r}^{\alpha} \omega^*(x)\, p(x) \;=\; \hat{p}(x)$. Then,

$$
\begin{aligned}
\left(\mathcal{L}_r^{\alpha}\left(\omega(x)\,\mathcal{G}_n(x)\right)\ ,\ p(x)\right)_{\omega^*} &= \int_0^1 \omega^*(x)\,\mathcal{L}_r^{\alpha}\left(\omega(x)\,\mathcal{G}_n(x)\right)\,p(x)\,dx \\
&= \int_0^1 \mathcal{L}_r^{\alpha}\left(\omega(x)\,\mathcal{G}_n(x)\right)\,\omega^*(x)\,p(x)\,dx \\
&= \int_0^1 \omega(x)\,\mathcal{G}_n(x)\,\mathcal{L}_{1-r}^{\alpha}\left(\omega^*(x)\,p(x)\right)\,dx \\
&= \int_0^1 \omega(x)\,\mathcal{G}_n(x)\,\hat{p}(x)\,dx \\
&= 0\,.
\end{aligned}
$$

Hence, $\mathcal{L}_r^{\alpha}\,\omega(x)\,\mathcal{G}_n(x) \;=\; C\,\mathcal{G}_n^*(x)$, for $C \in \mathbb{R}$.

As the coefficient of x^n in $\mathcal{G}_n(x)$ and $\mathcal{G}_n^*(x)$ is 1, then from Lemma 5.1,

$$
C \;=\; -(1-r)\frac{\sin(\pi\,\alpha)}{\sin(\pi(\alpha-\beta))}\,\frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)} \;=\; \lambda_n\,.
$$

An analogous argument to the above establishes (5.18). ☐

Remark. Note that $f(x) \in L^2_{\omega^*}(0,1)$ may be expressed as

$$
f(x) \;=\; \sum_{i=0}^{\infty} \frac{f_i^*}{\|\mathcal{G}_i^*\|^2_{\omega^*}}\,\mathcal{G}_i^*(x),
$$

where f_i^* is given by

$$
f_i^* \;:=\; \int_0^1 \omega^*(x)\,f(x)\,\mathcal{G}_i^*(x)\,dx\,. \tag{5.19}
$$

With f_i^* defined in (5.19), let

$$
u_N(x) \;=\; \omega(x)\sum_{i=0}^{N} c_i\,\mathcal{G}_i(x)\,,\ \text{where } c_i \;=\; \frac{1}{\lambda_i\,\|\mathcal{G}_i^*\|^2_{\omega^*}}\,f_i^*\,. \tag{5.20}
$$

Theorem 5.1. *Let $f(x) \in L^2_{\omega^*}(0,1)$ and $u_N(x)$ be as defined in* (5.20)*. Then,*

$$
u(x) \;:=\; \lim_{N\to\infty} u_N(x) \;=\; \omega(x)\sum_{j=0}^{\infty} c_j\,\mathcal{G}_j(x) \;\in\; L^2_{\omega^{-1}}(0,1).
$$

In addition, $\mathcal{L}_r^{\alpha} u(x) \;=\; f(x)$.

Proof. For $f_N(x) \;=\; \sum_{i=0}^{N}\frac{f_i^*}{\|\mathcal{G}_i^*\|^2_{\omega^*}}\,\mathcal{G}_i^*(x)$, we have that $f(x) \;=\; \lim_{N\to\infty} f_N(x)$, and $\{f_N(x)\}_{N=0}^{\infty}$ is a Cauchy sequence in $L^2_{\omega^*}(0,1)$. A straightforward calculation shows that $u_N(x) \in L^2_{\omega^{-1}}(0,1)$. Then (without loss of generality, assume $M > N$),

$$
\begin{aligned}
\|u_N(x) - u_M(x)\|^2_{\omega^{-1}} &= \left(\omega^{-1}(x)\, \omega(x) \sum_{j=N+1}^{M} c_j\, \mathcal{G}_j(x) \;,\; \omega(x) \sum_{j=N+1}^{M} c_j\, \mathcal{G}_j(x) \right) \\
&= \left(\omega(x) \sum_{j=N+1}^{M} \frac{f_j^*}{\lambda_j \, \|\mathcal{G}_j^*\|^2_{\omega^*}}\, \mathcal{G}_j(x) \;,\; \sum_{j=N+1}^{M} \frac{f_j^*}{\lambda_j \, \|\mathcal{G}_j^*\|^2_{\omega^*}}\, \mathcal{G}_j(x) \right) \\
&= \sum_{j=N+1}^{M} \frac{f_j^{*2}}{\lambda_j^2 \, \|\mathcal{G}_j^*\|^4_{\omega^*}} \|\mathcal{G}_j\|^2_{\omega} \\
&= \sum_{j=N+1}^{M} \frac{f_j^{*2}}{\lambda_j^2 \, \|\mathcal{G}_j^*\|^2_{\omega^*}} \qquad \text{(using (5.14))} \\
&\le C \left(\omega^*(x) \sum_{j=N+1}^{M} \frac{f_j^*}{\|\mathcal{G}_j^*\|^2_{\omega^*}}\, \mathcal{G}_j^*(x) \;,\; \sum_{j=N+1}^{M} \frac{f_j^*}{\|\mathcal{G}_j^*\|^2_{\omega^*}}\, \mathcal{G}_j^*(x) \right) \\
&\qquad \text{(using } \lambda_j\text{'s are bounded away from zero)} \\
&= C\, \|f_N(x) \;-\; f_M(x)\|^2_{\omega^*} .
\end{aligned}
$$

Hence $\{u_N(x)\}_{N=0}^{\infty}$ is a Cauchy sequence in $L^2_{\omega^{-1}}(0,1)$. As $L^2_{\omega^{-1}}(0,1)$ is complete [11], $u(x) := \lim_{N\to\infty} u_N(x) \in L^2_{\omega^{-1}}(0,1)$. Next, as $f_N(x) \to f(x)$ in $L^2_{\omega^*}(0,1)$, given $\epsilon > 0$ there exists $\tilde{N}$ such that for $N > \tilde{N}$, $\|f(x) - f_N(x)\|_{\omega^*} < \epsilon$. Then, for $N > \tilde{N}$, using Lemma 5.2,

$$
\begin{aligned}
\|f(x) \;-\; \mathcal{L}_r^{\alpha} u_N(x)\|_{\omega^*} &= \|f(x) \;-\; \mathcal{L}_r^{\alpha} \left(\omega(x) \sum_{j=0}^{N} \frac{f_j^*}{\lambda_j \, \|\mathcal{G}_j^*\|^2_{\omega^*}}\, \mathcal{G}_j(x) \right) \|_{\omega^*} \\
&= \; \|f(x) \;-\; \sum_{j=0}^{N} \frac{f_j^*}{\|\mathcal{G}_j^*\|^2_{\omega^*}}\, \mathcal{G}_j^*(x)\|_{\omega^*} \\
&= \; \|f(x) \;-\; f_N(x)\|_{\omega^*} \;<\; \epsilon .
\end{aligned}
$$

Hence, $f(x) \;=\; \mathcal{L}_r^{\alpha} u(x)$. □

5.1.1. *Invertibility of* $\mathcal{L}_r^{\alpha}\cdot$ *on* $L^2(0,1)$. We return to the question alluded to by Lemmas 3.2 and 3.3 in Section 3, namely the invertibility of $\mathcal{L}_r^{\alpha}\cdot$ on $L^2(0,1)$. Theorem 5.1, together with (5.20) and (5.19) gives an explicit inverse for $\mathcal{L}_r^{\alpha}\cdot$ on $L^2_{\omega^*}(0,1) \supset L^2(0,1)$. Hence we have the following.

Corollary 5.1. *For* $1 < \alpha < 2$, $0 < r < 1$, β *chosen such that* (5.1) *is satisfied,* ω *and* ω^* *as in* (5.13)*, given* $f \in L^2(0,1)$ *there exists a unique solution* $u \in L^2_{\omega^{-1}}(0,1)$ *such that* $\mathcal{L}_r^{\alpha} u \;=\; f$ *and* $u(0) = u(1) = 0$*. (For a solution to the nonhomogeneous boundary condition problem:* $\mathcal{L}_r^{\alpha} u_{nh} \;=\; f$ *subject to* $u_{nh}(0) = A$, $u_{nh}(1) = B$*, the homogeneous boundary condition for* u *is combined with a suitable function chosen from the kernel of* $\mathcal{L}_r^{\alpha}\cdot$ *(see Corollary* 3.1*).)* □

5.1.2. *A priori error estimate for* $u - u_N$*.* We have the following statement for the error between $u \;-\; u_N$.

Theorem 5.2. *For $f(x) \in L^2_{\omega^*}(0,1)$ and $u_N(x)$ given by* (5.20)*, there exists $C > 0$ such that*

$$\|u - u_N\|_{\omega^{-1}} \;\leq\; \frac{1}{\lambda_{N+1}} \|f\|_{\omega^*} \;\leq\; C\,(N+2)^{-\alpha}\,\|f\|_{\omega^*}. \tag{5.21}$$

Proof. With the definition of the $\|\cdot\|_{\omega^{-1}}$ norm, and (5.15)

$$\begin{aligned}
\|u - u_N\|^2_{\omega^{-1}} &= \int_0^1 \omega^{-1}(x) \Bigg(\omega(x) \sum_{i=0}^{\infty} \frac{f_i^*}{(\lambda_i\,\|\mathcal{G}_i^*\|^2_{\omega^*})}\,\mathcal{G}_i(x) \\
&\qquad\qquad - \omega(x) \sum_{i=0}^{N} \frac{f_i^*}{(\lambda_i\,\|\mathcal{G}_i^*\|^2_{\omega^*})}\,\mathcal{G}_i(x) \Bigg)^2 \, dx \\
&\leq \max_{N+1 \leq i} \left(\frac{1}{\lambda_i^2}\right) \int_0^1 \omega(x) \left(\sum_{i=N+1}^{\infty} \frac{f_i^*}{\|\mathcal{G}_i^*\|^2_{\omega^*}}\,\mathcal{G}_i(x) \right)^2 dx \\
&\leq \frac{1}{\lambda_{N+1}^2} \sum_{i=N+1}^{\infty} \frac{f_i^{*2}}{\|\mathcal{G}_i^*\|^4_{\omega^*}}\,\|\mathcal{G}_i\|^2_{\omega} \\
&= \frac{1}{\lambda_{N+1}^2} \sum_{i=N+1}^{\infty} \frac{f_i^{*2}}{\|\mathcal{G}_i^*\|^4_{\omega^*}}\,\|\mathcal{G}_i^*\|^2_{\omega^*} \qquad \text{(using (5.14))} \\
&\leq \left(\frac{-\sin(\pi\,(\alpha-\beta))}{(1-r)\,\sin(\pi\,\alpha)}\,\frac{\Gamma(N+2)}{\Gamma(N+2+\alpha)} \right)^2 \\
&\qquad \cdot \int_0^1 \omega^*(x) \sum_{i=0}^{\infty} \left(\frac{f_i^*}{\|\mathcal{G}_i^*\|^2_{\omega^*}}\,\mathcal{G}_i^*(x) \right)^2 dx \\
&= \left(\frac{-\sin(\pi\,(\alpha-\beta))}{(1-r)\,\sin(\pi\,\alpha)}\,\frac{\Gamma(N+2)}{\Gamma(N+2+\alpha)} \right)^2 \int_0^1 \omega^*(x)\,f(x)^2\,dx \\
&\leq \left(\frac{-\sin(\pi\,(\alpha-\beta))}{(1-r)\,\sin(\pi\,\alpha)}\,\frac{\Gamma(N+2)}{\Gamma(N+2+\alpha)} \right)^2 \|f\|^2_{\omega^*} \\
&\leq C\,(N+2)^{-2\,\alpha}\,\|f\|^2_{\omega^*}.
\end{aligned}$$

□

Corollary 5.2. *For $f(x) \in L^2_{\omega^*}(0,1)$ and $u_N(x)$ given by* (5.20)*, there exists $C > 0$ such that*

$$\|u - u_N\| \;\leq\; \frac{1}{\lambda_{N+1}} \|f\|_{\omega^*} \;\leq\; C\,(N+2)^{-\alpha}\,\|f\|_{\omega^*}\,. \tag{5.22}$$

As $\omega(x) = x^\beta(1-x)^{\alpha-\beta} < 1$, for $0 < x < 1$, then $\|u - u_N\| \leq \|u - u_N\|_{\omega^{-1}}$. Hence the bound (5.22) follows immediately from (5.21).

5.2. **Numerical Examples.** In this section we demonstrate the spectral type approximation methods discussed in Section 5.1 on Examples 1 and 2 presented in Section 4.

Example 1 (cont.). For this example $\alpha = 1.4$, $r = 1/2$ and $\beta = 0.7$. Hence we have from (5.13) that $\omega(x) = \omega^*(x) = x^\beta(1-x)^{\alpha-\beta} = x^{0.7}(1-x)^{0.7}$, $\mathcal{G}_n(x) =$

$\mathcal{G}_n^*(x)$, and from (5.19) and (5.20),

$$u_N(x) \;=\; x^{0.7}(1-x)^{0.7} \sum_{j=0}^{N} \frac{f_j^*}{\lambda_j \, \|\mathcal{G}_j\|_w^2} \, \mathcal{G}_j(x) \, .$$

Presented in Figure 5.1 is a plot of the true solution given in (4.6). Figure 5.2 contains a plot of the error, $u(x) - u_8(x)$, which exhibits a Gibbs type phenomena at the endpoints. Presented in Figure 5.3 is a plot of the L^2_ω and L^2 errors for the approximations. The convergence of the approximations is consistent with the theoretical results given in (5.21) and (5.22).

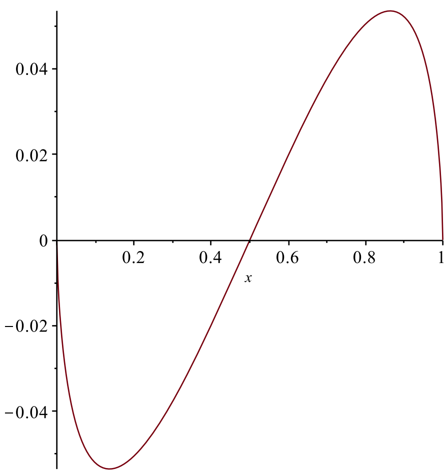

FIGURE 5.1. Solution of Example 1, $u(x)$ given in (4.6).

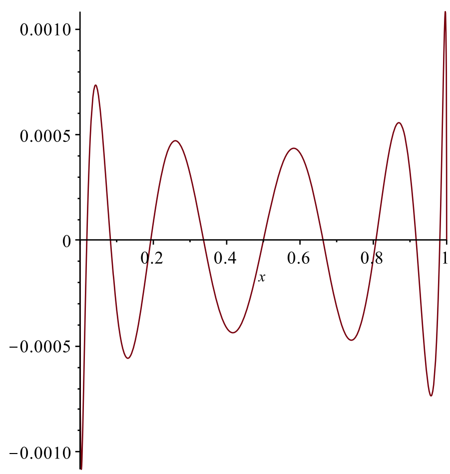

FIGURE 5.2. Plot of $u(x) - u_8(x)$ for Example 1.

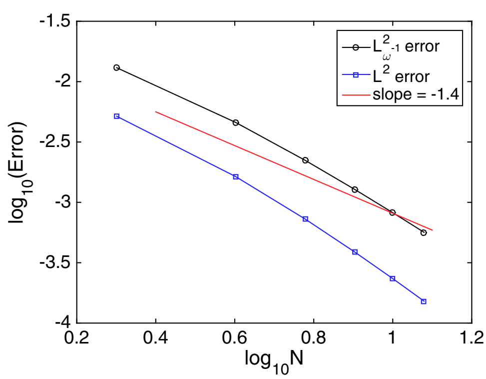

FIGURE 5.3. $L^2_{\omega^{-1}}$ and L^2 errors for Example 1.

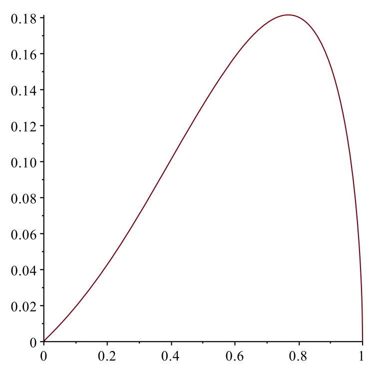

FIGURE 5.4. Solution of Example 2, $u(x)$ given in (4.8).

Example 2 (cont.). For this example $\alpha = 1.4$, $r = 0.3149$, $p = -0.15$ and $q = -0.45$. For these values the corresponding value for β is 0.85 (see (5.1)). From Section 5.1, (5.13), $\omega(x) = x^{\beta}(1-x)^{\alpha-\beta} = x^{0.85}(1-x)^{0.55}$, and from (5.19) and (5.20),

$$u_N(x) \;=\; x^{0.85}(1-x)^{0.55} \sum_{j=0}^{N} \frac{f_j^*}{\lambda_j \, \|\mathcal{G}_j^*\|_{w^*}^2} \, \mathcal{G}_j(x) \, .$$

Presented in Figure 5.4 is a plot of the true solution given in (4.8). Figure 5.5 contains a plot of the error, $u(x) - u_8(x)$. Presented in Figure 5.6 is a plot of the L^2_ω and L^2 errors for the approximations. The convergence of the approximations is consistent with the theoretical results given in (5.21) and (5.22).

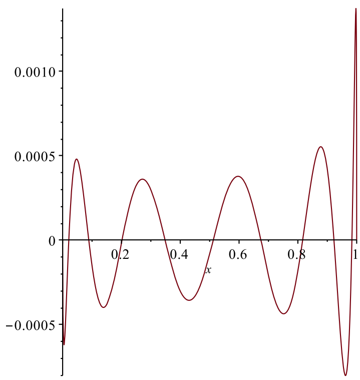

FIGURE 5.5. Plot of $u(x) - u_8(x)$ for Example 2.

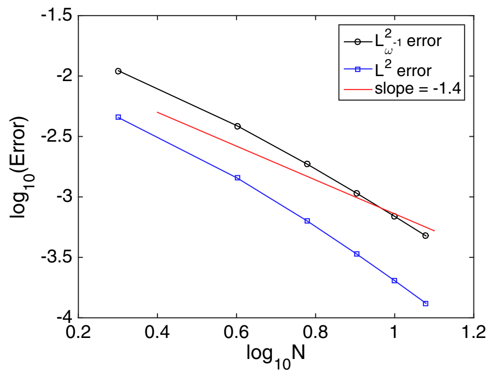

FIGURE 5.6. $L^2_{\omega^{-1}}$ and L^2 errors for Example 2.

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