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SMALL f-VECTORS OF 3-SPHERES AND OF 4-POLYTOPES

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Abstract. We present a new algorithmic approach that can be used to determine whether a given quadruple (f_0, f_1, f_2, f_3) is the f-vector of any convex 4-dimensional polytope, or more generally of a strongly regular cellular 3-sphere, that is, a regular cell complex homeomorphic to the 3-dimensional sphere such that any intersection of two faces (cells) is a face.

By implementing this approach, we classify the f-vectors of 4-polytopes in the range $f_0 + f_3 \leq 22$.

In particular, we prove that there are f-vectors of strongly regular cellular 3-spheres that are not f-vectors of any convex 4-polytopes. This answers a question that may be traced back to the works of Steinitz (1906/1922). In the range $f_0 + f_3 \leq 22$, there are exactly three such f-vectors with $f_0 \leq f_3$, namely $(10, 32, 33, 11)$, $(10, 33, 35, 12)$, and $(11, 35, 35, 11)$.

1. Introduction

In 1906, Ernst Steinitz [33] proved a remarkably simple and complete result: The set of all f-vectors (f_0, f_1, f_2) of 3-dimensional convex polytopes—where f_i denotes the number of i-dimensional faces—is given by all the integer points in a 2-dimensional polyhedral cone, whose boundary is given by the extremal cases of (f-vectors of) simple and of simplicial polytopes:

$$\mathcal{F}(\mathcal{P}^3) = \{(f_0, f_1, f_2) \in \mathbb{Z}^3 : f_0 - f_1 + f_2 = 2,\ f_2 \leq 2f_0 - 4,\ f_0 \leq 2f_2 - 4\}.$$

Steinitz's later work [34, 35] from 1922/1934 implies that the same characterization is valid also for the f-vectors of more general objects, such as regular cellular 2-spheres with the intersection property, or of interval-connected Eulerian lattices of length 4 (as described below).

The f-vectors of 4-dimensional polytopes, however, provide a much greater challenge. In his 1967 book, Grünbaum wrote:

> "*It would be rather interesting to find a characterization of those lattice points in $\mathbb{R}^4$ which are the f-vectors of 4-polytopes. This goal seems rather distant, however, in view of our inability to solve even such a small part of the problem as the lower bound conjecture for 4-polytopes.*" (Grünbaum [17, p. 191])

The lower bound conjecture was solved by Barnette in 1971/73 [3, 5], but the problem to characterize $\mathcal{F}(\mathcal{P}^4)$ remains wide open. Grünbaum himself initiated and started in [17, Sect. 10.4] a study of the 2-dimensional coordinate projections of the

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3-dimensional set $\mathcal{F}(\mathcal{P}^4) \subset \mathbb{R}^4$, which was eventually completed by Barnette and Reay [7] and Barnette [6]. A typical result in the series says that a pair (f_i, f_j) occurs in an f-vector if and only if it satisfies some simple linear or quadratic upper/lower bound inequalities and is not one of finitely-many "small" exceptions. For example, according to [17, Thm. 10.4.1] a pair (f_0, f_3) occurs for a 4-polytope if and only if the upper bound inequalities $f_3 \leq \frac{1}{2} f_0(f_0 - 3)$ and $f_0 \leq \frac{1}{2} f_3(f_3 - 3)$ are satisfied, with no exceptions in this case.

Any characterization of (a projection of) the set of f-vectors $\mathcal{F}(\mathcal{P}^4) \subset \mathbb{Z}^4$ contains a characterization of the extremal cases and a solution of the corresponding extremal problems. Some of these are visible in 2-dimensional coordinate projections. For example, the (f_0, f_3)-classification quoted above contains the upper bound theorem for 4-polytopes.

In view of Steinitz's results for dimension 3, one would also look at the set of f-vectors $\mathcal{F}(\mathcal{S}^3)$ of strongly regular cellular 3-spheres, that is, of regular cell complexes homeomorphic to the 3-dimensional sphere such that any intersection of two cells is a single cell; see Section 2 for definitions and details.

As a complete determination of $\mathcal{F}(\mathcal{P}^4)$ or of $\mathcal{F}(\mathcal{S}^3)$ seems out of reach, a natural approximation to these problems asks for a characterization of the closed convex cones with apex at the f-vector $f(\Delta_4) = (5, 10, 10, 5)$ of the 4-simplex that are generated by the f-vectors of 4-polytopes, respectively, of 3-spheres,

$$\operatorname{cone}(\mathcal{F}(\mathcal{P}^4)) \subseteq \operatorname{cone}(\mathcal{F}(\mathcal{S}^3)) \subset \mathbb{R}^4.$$

Equivalently, one asks for the linear inequalities that are valid for all f-vectors and tight at $f(\Delta_4) = (5, 10, 10, 5)$. For example, the inequalities $f_1 \geq 2f_0$ and $f_2 \geq 2f_3$ are of this form, satisfied with equality by simple, respectively, simplicial 4-polytopes. Thus, in particular, the f-vectors of simple and simplicial 4-polytopes are extremal in the coordinate projections to (f_0, f_1), respectively, (f_2, f_3).

It was noted in Ziegler [38] that a key parameter of an f-vector is the *fatness*

$$F(f_0, f_1, f_2, f_3) := \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}.$$

Though fatness is not defined for the (f-vector of a) simplex, every lower or upper bound on fatness corresponds to a linear inequality that is tight at the simplex. In [38] the second author also identified the two key problems that have prevented us up to now from determining $\operatorname{cone}(\mathcal{F}(\mathcal{P}^4))$ or $\operatorname{cone}(\mathcal{F}(\mathcal{S}^3))$:

- *Does fatness have an upper bound for 4-polytopes?*
  (It does not for 3-spheres, as proved by Eppstein, Kuperberg, and Ziegler [14].)
- *Is the fatness lower bound $F \geq 2.5$ valid for all 3-spheres?*
  (For 4-polytopes it follows from $g_2^{tor} \geq 0$, see Kalai [21].)

These are extremal problems on $\mathcal{F}(\mathcal{P}^4)$, respectively, $\mathcal{F}(\mathcal{S}^3)$ that cannot be solved by looking at the projections to only two coordinates. However, below we will suggest a different projection which displays fatness very clearly.

In this paper we are not directly dealing with the asymptotic questions. Rather we classify the f-vectors of "small" polytopes, and from this we derive new insights into what happens asymptotically. For this, we redefine "small" by measuring the *size* of an f-vector by

$$\operatorname{size}(f_0, f_1, f_2, f_3) := f_0 + f_3 - 10.$$

This is a linear quantity, with $\mathrm{size}(5, 10, 10, 5) = 0$ for the f-vector of the 4-simplex.

For the classification we have developed a new algorithmic approach, in order to determine for any given reasonably small (f_0, f_1, f_2, f_3), whether there is a 4-polytope with this f-vector.

We have implemented the algorithm and achieved a complete classification of the f-vectors of size up to 12. That is, for every vector (f_0, f_1, f_2, f_3) with $f_0 + f_3 \leq 22$ that satisfies the known necessary conditions on f-vectors of 4-polytopes, we have either constructed a 3-sphere or 4-polytope with this f-vector, or proved that none exists.

The results of our computations are detailed in Sections 4 and 5. As a main consequence of the enumeration, we obtain that the difference between strongly regular cellular 3-spheres and 4-polytopes is so substantial that it appears even at the level of f-vectors:

Theorem 1.1. *The set of f-vectors of 4-polytopes is a strict subset of the set of f-vectors of strongly regular cellular 3-spheres:*

$$\mathcal{F}(\mathcal{P}^4) \subsetneqq \mathcal{F}(\mathcal{S}^3).$$

Indeed, the sets differ in exactly five such f-vectors of $\mathrm{size}(P) = f_0 + f_3 - 10 \leq 12$, *namely*

- *of size* 11*:* $(10, 32, 33, 11)$, $(11, 33, 32, 10)$, *and*
- *of size* 12*:* $(10, 33, 35, 12)$, $(12, 35, 33, 10)$, $(11, 35, 35, 11)$.

For simplicial spheres, the question whether all f-vectors of $(d-1)$-spheres also occur for d-polytopes is—in view of the g-theorem for polytopes (cf. [37, Sect. 8.6])—equivalent to the g-conjecture for spheres. The answer is known to be "yes" for $d \leq 5$, but the g-conjecture for spheres remains open for larger d. However, in 1971, at the end of the paper in which he introduced the g-conjecture, McMullen had already voiced strong doubts:

> "*in every case in which the [g-]conjecture is known to be true, it also holds for the corresponding triangulated spheres.* (...) *However, there are fundamental differences between triangulated $(d-1)$-spheres and boundary complexes of simplicial d-polytopes.* (...) *We should therefore, perhaps, be wary of extending the conjecture to triangulated spheres.*" (McMullen [27, p. 569])

Our algorithm works in three steps, proceeding from combinatorial models via topological models to polytopes. It starts with an enumeration of the graphs that could be compatible with the given f-vector. It then looks at the possible combinatorial types of facets and enumerates their combinations into an entirely combinatorial model of polytopes, namely *interval-connected Eulerian lattices* of length 5. This new model will be described in Section 2, where we also prove that every such object corresponds to a regular cell-decomposition of a closed 3-manifold with the intersection property (Proposition 2.2). Thus the combinatorial types of regular cell-decompositions of the 3-sphere with the intersection property (which we refer to as strongly regular cellular 3-spheres, or simply as 3-spheres) form a subset of these Eulerian lattices. The class of combinatorial types of convex 4-polytopes is still more restrictive, as became clear, for example, in the revision and correction of Brückner's [12] work by Grünbaum and Sreedharan [18]: Not every diagram, and thus not every sphere, does correspond to a convex polytope.

In our search range of size$(f) \le 12$, all f-vectors of Eulerian lattices also appear as f-vectors of spheres. That is, while

$$\#\{f \in \mathcal{F}(\mathcal{S}^3) \setminus \mathcal{F}(\mathcal{P}^4) : \text{size}(f) \le 12\} = 5$$

we have

$$\#\{f \in \mathcal{F}(\mathcal{E}^5) \setminus \mathcal{F}(\mathcal{S}^3) : \text{size}(f) \le 12\} = 0.$$

So it may be that $\mathcal{F}(\mathcal{E}^5) = \mathcal{F}(\mathcal{S}^3)$, but the computations for size$(f) \le 12$ should not be counted as strong evidence, as indeed we did not encounter *any* manifolds that are not spheres in this range. Also, very natural higher-dimensional versions of $\mathcal{F}(\mathcal{E}^5) \overset{?}{=} \mathcal{F}(\mathcal{S}^3)$ turn out to be false. For example, simplicial 5-manifolds with negative g_3 appear in the enumerations of Lutz [25, pp. 56-58].

In Figure 1 we evaluate our classification results by looking at the f-vector set $\mathcal{F}(\mathcal{P}^4)$ in a particular projection, which is not a coordinate projection, and which has the virtue to show size (as first coordinate) and fatness (as "slope + 2") directly.

Let us note two more intriguing aspects of our enumeration results, which can also be seen in Figure 1:

Observations 1.2.

(i) The sets of "small" f-vectors $f = (f_0, f_1, f_2, f_3)$ of 3-spheres and of 4-polytopes differ in an essential way, which is detected by fatness:
- For size$(f) \le 10$, the f-vectors of 3-spheres and of 4-polytopes agree.
- For size$(f) \le 11$, the maximal fatness for 3-spheres is $4\frac{1}{11}$, for 4-polytopes it is 4.
- For size$(f) \le 12$, the maximal fatness for 3-spheres is $4\frac{1}{6}$, for 4-polytopes it is still 4.

(ii) In the range of "small" f-vectors of size$(f) \le 12$, the particularly "fat" 4-polytopes include the 2-*simple and* 2-*simplicial* polytopes in the sense of Grünbaum [17, Sect. 4.5]. The exceptionally fat 3-spheres are not 2-simple and 2-simplicial, but they still have f-vectors that are approximately symmetric, with $|f_0 - f_3|$ small.

For this we recall from Grünbaum [17, Sect. 4.5] that a 4-polytope P is 2-simple and 2-simplicial ("2s2s") if all 2-faces are triangles both for P and for its dual. The definition extends to 3-spheres and even to Eulerian lattices of length 5. Any such 2s2s object has a symmetric f-vector, with $f_0 = f_3$ and $f_1 = f_2$. The 2s2s property is detected by the flag vector, but not by the f-vector alone. Observation 1.2(ii) refers to the polytopes of fatness at least 4, which in the range size$(f) \le 12$, according to the classification of 2s2s 4-polytopes and 3-spheres of size at most 14 in Brinkmann and Ziegler [11, Thm. 2.1], are

- Werner's example W_9 with 9 vertices [36, Thm. 4.2.2],
- W_{10} as well as the hypersimplex $\Delta_4(2)$ and its dual with 10 vertices, and
- P_{11} by Paffenholz and Werner [30, Sect. 4.1].

(There may be additional polytopes with f-vector $(11, 34, 34, 11)$ and fatness 4, just like P_{11}, which are not 2s2s.) The pattern "2s2s polytopes are among the fattest examples" continues beyond the range size$(f) \le 12$ of our enumeration, where we find

- the 2s2s polytope W_{12}^{39} of Werner and Miyata [36, Tbl. 7.1 right] [28, Sect. 4.2] and
- the 2s2s sphere W_{12}^{40} with f-vector $(12, 40, 40, 12)$ constructed by Werner [36, Tbl. 7.1 left].

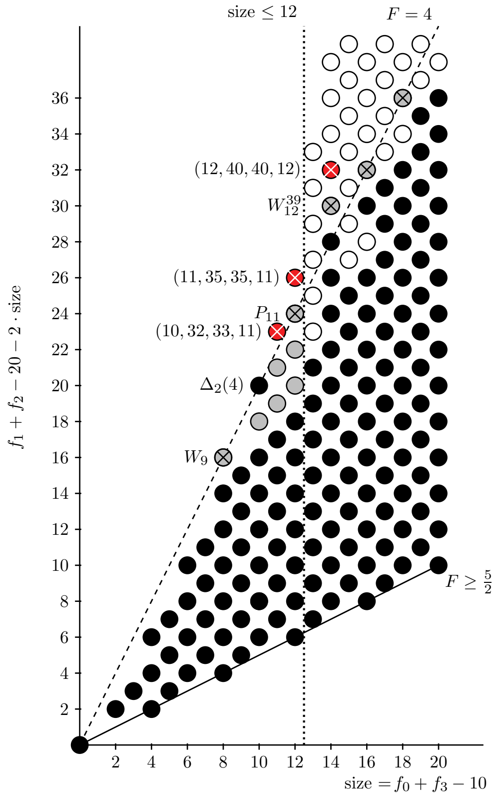

FIGURE 1. The size/fatness projection of the f-vector sets $\mathcal{F}(\mathcal{P}^4) \subset \mathcal{F}(\mathcal{S}^3)$.

This figure presents a particular 2-dimensional projection of $\mathcal{F}(\mathcal{P}^4) \subset \mathcal{F}(\mathcal{S}^3) \subset \mathbb{Z}^4$: The x-axis represents size $= f_0 + f_3 - 10$ of a 4-polytope or 3-sphere, while the y-axis represents $f_1 + f_2 - 20 - 2 \cdot \text{size}$, so the slope of a line through the origin is "fatness $-$ 2."

Black dots ● mark data points for which Höppner [19] had found polytopes.

Grey crossed dots ⊗ mark coordinates for which 2-simple 2-simplicial polytopes were found by Paffenholz and Werner [30] and Werner [36].

Grey dots ○ give additional data points where we now found polytopes.

Red dots ⊗ represent coordinates of points for which there are f-vectors of 3-spheres, but where we found no f-vectors of 4-polytopes; left of the dotted line this means that these do not exist.

The graph shown here is complete up to size 12, that is, to the left of the dotted vertical line.

White dots ○ appear only to the right of the dotted line: They mark locations where the existence of spheres or of polytopes has not been decided.

For this last example we had shown in [11] that it is non-polytopal and that it is the only 2s2s 3-sphere with such a flag vector. As a consequence, we established that the sets of flag vectors of 4-polytopes and 3-spheres differ [11, Theorem 1.1], but did not achieve a similar statement for sets of f-vectors. This is provided by Theorem 1.1. However, with our new algorithm presented here (plus massive computation) we also achieved a complete classification result for the f-vector $(12, 40, 40, 12)$.

Theorem 1.3. *There are* 4 *strongly regular cellular* 3*-spheres (all of them self-dual, one of them* 2*-simple* 2*-simplicial), but no* 4*-polytopes at all, with the f-vector* $(12, 40, 40, 12)$.

So altogether this paper provides six examples of f-vectors of 3-spheres that are not f-vectors of 4-polytopes, namely the five smallest ones listed in Theorem 1.1 and one more in Theorem 1.3. Of course one would now want to provide infinitely many examples, to show that the cones $\text{cone}(\mathcal{F}(\mathcal{P}^4)) \subseteq \text{cone}(\mathcal{F}(\mathcal{S}^3))$ do not coincide, and similar results for flag vectors and for their cones in higher dimensions. Our present methods do not seem to provide this.

2. Objects: Polytopes, spheres, and Eulerian lattices

There have been numerous substantial attempts to classify *all* 4-dimensional polytopes with some given parameters (e.g., f-vectors), or to classify the parameters that actually occur. They all depend on a hierarchy of combinatorial/topological/geometric models for convex polytopes of decreasing generality, which we use systematically in our algorithmic approach. For basics on convex polytopes, including diagrams/Schlegel diagrams, we refer to Grünbaum [17] and Ziegler [37]. For regular cell complexes, see Cooke and Finney [13] or Munkres [29]. Eulerian posets/lattices as combinatorial models arose from the work of Klee [22]; see Stanley [32, Chap. 3]. The less common objects we work with can be summarized as follows.

Definition 2.1.

- A finite graded lattice is *Eulerian* if any non-trivial interval has the same number of elements of odd and of even rank; it is *interval-connected* if the proper part of any interval of length at least 3 is connected.
- A cellular sphere (that is, a CW-complex homeomorphic to some S^d; cf. Munkres [29, § 38]) is *regular* if the attaching maps of the cells are homeomorphisms also on the boundary. The sphere has the *intersection property* if the intersection of any two cells is a single cell (which may be empty). A regular cellular sphere with the intersection property is also referred to as a *strongly regular* cellular sphere.

In this paper we concentrate entirely on the case of 4-dimensional polytopes, and correspondingly 3-spheres and Eulerian lattices of length 5. The interval connectivity for Eulerian lattices and the regularity and intersection property for cellular spheres are always assumed. We write:

- $\mathcal{P}^4$ for the set of combinatorial types of 4-polytopes;
- $\mathcal{S}^3$ for the set of combinatorial types of 3-spheres;
- $\mathcal{E}^5$ for the isomorphism types of length 5 Eulerian lattices.

The boundary complex of any 4-polytope is a 3-sphere (regular, cellular, with the intersection property); the face lattice of any such 3-sphere is an interval-connected Eulerian lattice of length 5.

Polytope theory has produced lots of examples to show that there are strict inclusions

$$\mathcal{P}^4 \subset \mathcal{S}^3 \subset \mathcal{E}^5$$

while there is no difference one dimension lower, by Steinitz's theorem. His theorem also yields that interval-connected Eulerian lattices form an *excellent* entirely combinatorial model for the topological/geometric structures we are studying.

Proposition 2.2. *Every interval-connected Eulerian lattice of length* $d+1 \leq 4$ *is the face lattice of a d-polytope. In particular,* $\mathcal{P}^3 = \mathcal{S}^2 = \mathcal{E}^4$.

Every interval-connected Eulerian lattice of length $d+1 = 5$ *is the face lattice of a (connected, closed) regular cellular* 3*-manifold with the intersection property.*

Proof sketch. For $d+1 \leq 3$ there is little to prove.

For $d+1 = 4$ an Eulerian lattice is the face lattice of a connected 2-manifold of Euler characteristic 2, so we have a sphere. The lattice property corresponds to what Steinitz calls "Bedingung des Nichtübergreifens" [35, S. 179], which is exactly the intersection property for a cellular 2-sphere. Steinitz's theorem [34, 35] yields that every such 2-sphere can be realized as a convex polytope.

For $d+1 = 5$ an Eulerian lattice is the face poset of a closed connected 3-manifold, whose cells and vertex links are polytopal by Steinitz's theorem. However, the fact that this manifold has Euler characteristic 0 yields no additional information about its type, by Poincaré duality. □

3. Enumeration Algorithm

Here we propose a new algorithm, which constructs, for a given vector (f_0, f_1, f_2, f_3), first the graphs and then the face lattices of all 3-manifolds with this f-vector, by using 0/1 integer programming in order to enumerate all families of facets that fit to this graph and all other constraints. The algorithm has the following outline.

Algorithm 3.1. `find_lattices(f)`
INPUT: A vector $(f_0, f_1, f_2, f_3) \in \mathbb{Z}^4$
OUTPUT: All Eulerian lattices of length 5 with this f-vector

(i) enumerate all graphs G on f_0 vertices and f_1 edges that are 4-connected;
(ii) for every graph G find all induced subgraphs that are planar and 3-connected;
(iii) construct for every graph G an integer program (IP) with binary variables corresponding to the possible facets and ridges (facets of the facets), and with constraints given by the f-vector, proper intersection, the Euler relation, and the graph;
(iv) enumerate all feasible solutions of this IP;
(v) check for every feasible solution whether it gives an Eulerian lattice.

Proposition 3.2. *Algorithm* 3.1 *enumerates all interval-connected Eulerian lattices of length* 5 *with* f*-vector* (f_0, f_1, f_2, f_3).

Proof. We rely on the interpretation of interval-connected length 5 Eulerian lattices as face lattices of cellular regular 3-manifolds with intersection property in Proposition 2.2. Since the graph of any such manifold is 4-connected, step (i) will not exclude any graph of some 3-manifold with f-vector (f_0, f_1, f_2, f_3).

Also, by Proposition 2.2, the graphs of interval-connected Eulerian lattices of length 4 (and thus of facets of cellular 4-manifolds) are exactly the planar and 3-connected graphs. Thus, with step (ii) we find a list $\mathcal{F}_G$ of all potential facets for a manifold with the given graph G.

From the list $\mathcal{F}_G$, we also get the list $\mathcal{R}_G$ of the potential ridges, simply from the faces of the facets. We now construct a 0/1-IP whose variables x_i represent the facets F_i, and the variables y_j the ridges R_j, such that all solutions correspond to pseudomanifolds formed by a subset of the facets in $\mathcal{F}_G$ and such that all face lattices of 3-manifolds with graph G and f-vector (f_0, f_1, f_2, f_3) are feasible solutions, with the constraints

$$\sum_i x_i = f_3, \tag{1}$$

$$\sum_j y_j = f_2, \tag{2}$$

$$2y_j - \sum_{F_i \,:\, R_j \text{ is a ridge of } F_i} x_i = 0 \qquad \text{for all ridges } R_j, \tag{3}$$

$$x_i, y_j \in \{0,1\}. \tag{4}$$

Condition (4) says that all variables are binary, which means that if a variable in the solution is 1 the corresponding face will selected. Equations (1) and (2) enforce that the total number of facets and ridges selected is f_3, respectively, f_2. Equation (3) ensures that the ridge R_j is used if and only if precisely two facets containing it as a ridge are selected. Similarly, we get constraints from the Euler relation for the intervals above the vertices and edges, such that all feasible solutions correspond to Eulerian posets. Moreover, for every edge we get an inequality forcing the number of faces containing it to be larger than zero. Finally, we get inequalities $x_i + x_j \leq 1$ for pairs of facets F_i, F_j if their intersection is not *proper* (i.e., that not both can appear in a 3-manifold simultaneously). Since the face lattice of any 3-manifold with the given f-vector and graph G satisfies the constraints of the IP, it will be in the set of feasible solutions of this IP. Therefore, with the last step we can complete the enumeration of all interval-connected Eulerian lattices with the given f-vector. □

We implemented Algorithm 3.1 in *sage* [31], using the *geng*-function of *nauty* [26] (which is a built-in function of *sage*) to enumerate all graphs on f_0 vertices, with f_1 edges, with minimal vertex degree at least 4, and being 2-connected (*nauty* cannot enumerate 4-connected graphs, so we had to relax to 2-connectedness, but this did not include too many extra graphs), and the MILP-library of *sage* to check the IPs for feasibility and to enumerate all their solutions. We enumerated all feasible solutions iteratively: Given a feasible solution, we store it and set the sum of the f_3 variables corresponding to the facets of this solution to be at most $f_3 - 1$. Thus, we excluded with an additional constraint precisely the solution we just found and optimized again. By iterating this until no feasible solution remained, we enumerated all feasible solutions of the original IP.

Finally, we had to check every solution to represent an interval-connected Eulerian lattice of length 5: By construction, we were looking at Eulerian posets. For each of these, interval-connectivity was easy to check, as was the intersection property: Both these properties were not completely built into our IP. Then we triangulated the corresponding manifold and used *sage* to calculate the Betti numbers, and thus verified that in all cases considered we were dealing with homology spheres. Then we used *BISTELLAR* by Lutz [24] to show that each of them was flip-equivalent to the boundary of the simplex, and thus a genuine sphere.

4. Enumeration and classification results

For the proof of Theorem 1.1, we started with the generation of all potential flag vectors bounded by $f_0 + f_3 \leq 22$, that is, all integer vectors $(f_0, f_1, f_2, f_3; f_{03}) \in \mathbb{Z}^5$ that satisfy all the linear and non-linear conditions on f-vectors and of flag vectors that were known to be valid for Eulerian lattices with the intersection property of length 5, as given by Barnette [4], Bayer [8], and Ling [23]. (See Bayer and Lee [9] and Höppner and Ziegler [20] for surveys.) Moreover, as we added to the f-vector information specific data about the combinatorial types of facets used, we could make use of constraints such as

$$f_{02} - 4f_2 + 3f_1 - 2f_0 \leq \binom{f_0}{2} - \frac{1}{2} \sum_{F \text{ facet},\ f_0(F) \geq 7} (m_i(F) + f_{02}(F) - 3f_2(F)) - \#\text{facets with 6 vertices} + \tfrac{1}{2}\#\text{pyramids over pentagon},$$

where $m_i(F)$ denotes the number of interior edges of a face F, proved in Brinkmann [10, Sect. 2.2.1], which sharpens an inequality by Bayer [8].

Moreover, we could (and did) assume that $f_0, f_3 \geq 9$, as the objects with up to 8 vertices and facets have been enumerated and analyzed in detail by Altshuler and Steinberg [1,2].

Furthermore, we ticked off on our candidate list all those vectors that are known to occur as f-vectors of 4-polytopes, for example, from the study of Höppner and Ziegler [20] or the enumeration of 2s2s-polytopes in Brinkmann and Ziegler [11].

For all remaining candidate vectors we enumerated all compatible Eulerian lattices by Algorithm `find_lattices(f)`, and then used the methods detailed in Brinkmann and Ziegler [11] in order to

- either use first numerical non-linear optimization techniques and then exact arithmetic sharpenings in order to find rational coordinates for at least one polytope with the given f-vector, or
- use biquadratic final polynomials for partial oriented matroids in order to prove that *all* spheres for the given f-vector are non-realizable.

The results are shown in Table 1: It lists, for each potential f-vector considered, the number of graphs to be checked (graphs on f_0 vertices, with f_1 edges, with minimal vertex degree at least 4, and being 2-connected), and the numbers

- $\#\mathcal{E}^5$ of Eulerian lattices of length 5,
- $\#\mathcal{S}^3$ of cellular 3-spheres,
- #np of non-polytopal 3-spheres among them, and
- $\#\mathcal{P}^4$ of convex 4-polytopes

with the given f-vector. In some instances for the last two quantitites we just give lower bounds, if we did not decide all cases. An asterisk $*$ marks objects where we have exact coordinates for at least one polytope and approximate (floating point) coordinates for the others. Blank spaces represent missing data (e.g., not enumerated/calculated). In particular, for $f_0 = 11$ we did not enumerate all f-vectors, but restricted ourselves to constructing polytopes.

Table 1: All potential f-vectors with $f_0, f_3 \geq 9$ and $f_0 + f_3 \leq 22$.

f-vector	# graphs	$\#\mathcal{E}^5$	$\#\mathcal{S}^3$	#np	$\#\mathcal{P}^4$	
$(9, m, m, 9)$	170	0	0	0	0	$m \leq 19$
$(9, 20, 20, 9)$	713	1	1	0	1	

Table 1, continued from previous page

f-vector	# graphs	$\#\mathcal{E}^5$	$\#\mathcal{S}^3$	#np	$\#\mathcal{P}^4$	
$(9, 21, 21, 9)$	1 754	0	0	0	0	
$(9, 22, 22, 9)$	2 770	129	129		≥ 54	
$(9, 23, 23, 9)$	3 129	211	211	≥ 2	$\geq 113^*$	
$(9, 24, 24, 9)$	2 723	118	118	≥ 2	$\geq 81^*$	
$(9, 25, 25, 9)$	1 917	7	7	0	7^*	
$(9, 26, 26, 9)$	1 154	1	1	0	1	W_9 [36, Thm. 4.2.2]
$(9, m, m, 9)$	1 132	0	0	0	0	$m \geq 27$
$(9, m, m+1, 10)$	2 673	0	0	0	0	$m \leq 21$
$(9, 22, 23, 10)$	2 770	12	12		$\geq 9^*$	
$(9, 23, 24, 10)$	3 129	398	398	≥ 1	$\geq 78^*$	
$(9, 24, 25, 10)$	2 723	904	904	≥ 7	$\geq 27^*$	
$(9, 25, 26, 10)$	1 917	524	524	≥ 15	$\geq 80^*$	
$(9, 26, 27, 10)$	1 154	67	67	≥ 2	$\geq 62^*$	
$(9, 27, 28, 10)$	610	0	0	0	0	
$(9, 28, 29, 10)$	294	0	0	0	0	
$(9, 29, 30, 10)$	133	0	0	0	0	
$(9, m, m+1, 10)$	95	0	0	0	0	$m \geq 30$
$(9, m, m+2, 11)$	5 443	0	0	0	0	$m \leq 22$
$(9, 23, 25, 11)$	3 129	66	66		≥ 34	
$(9, 24, 26, 11)$	2 723	1 188	1 188		≥ 105	
$(9, 25, 27, 11)$	1 917	2 650	2 650		≥ 52	
$(9, 26, 28, 11)$	1 154	1 344	1 344		≥ 1	
$(9, 27, 29, 11)$	610	125	125		≥ 60	
$(9, 28, 30, 11)$	294	3	3	1	2	
$(9, 29, 31, 11)$	133	0	0	0	0	
$(9, m, m+2, 11)$	103	0	0	0	0	$m \geq 30$
$(9, m, m+3, 12)$	5 443	0	0	0	0	$m \leq 22$
$(9, 23, 26, 12)$	3 129	3	3	0	3	
$(9, 24, 27, 12)$	2 723	335	335		≥ 129	
$(9, 25, 28, 12)$	1 917	3 275	3 275		≥ 171	
$(9, 26, 29, 12)$	1 154	5 928	5 928		≥ 276	
$(9, 27, 30, 12)$	610	2 171	2 171		≥ 516	
$(9, 28, 31, 12)$	294	113	113		≥ 33	
$(9, 29, 32, 12)$	133	0	0	0	0	
$(9, 30, 33, 12)$	59	0	0	0	0	
$(9, m, m+3, 12)$	44	0	0	0	0	$m \geq 31$
$(9, m, m+4, 13)$	8 536	0	0	0	0	$m \leq 23$
$(9, 24, 28, 13)$	2 723	33	33		$\geq 32^*$	
$(9, 25, 29, 13)$	1 917	1 223	1 223	≥ 1	$\geq 387^*$	
$(9, 26, 30, 13)$	1 154	7 677	7 677	≥ 3	≥ 309	
$(9, 27, 31, 13)$	610	9 773	9 773	≥ 32	≥ 13	
$(9, 28, 32, 13)$	294	2 136	2 136		≥ 439[1]	
$(9, 29, 33, 13)$	133	27	27	≥ 1	$\geq 9^*$	

[1]See Note added in proof.

Table 1, continued from previous page

f-vector	# graphs	$\#\mathcal{E}^5$	$\#\mathcal{S}^3$	#np	$\#\mathcal{P}^4$	
$(9,30,34,13)$	59	0	0	0	0	
$(9,m,m+4,13)$	44	0	0	0	0	$m \geq 31$
$(10,m,m,10)$	10 247	0	0	0	0	$m \leq 22$
$(10,23,23,10)$	35 219	4	4	0	4	
$(10,24,24,10)$	87 014	16	16		$\geq 2^*$	
$(10,25,25,10)$	152 369				≥ 296	pyramids
$(10,26,26,10)$	203 469	5 550	5 550	≥ 69	$\geq 2^*$	
$(10,27,27,10)$	217 596	5 561	5 561	≥ 204	$\geq 90^*$	
$(10,28,28,10)$	192 964	1 662	1 662	≥ 143	$\geq 13^*$	
$(10,29,29,10)$	145 773	128	128	≥ 2	$\geq 21^*$	
$(10,30,30,10)$	95 827	3	3	0	3	$\Delta_4(2)$, $\Delta_4(2)^*$,W_{10}
$(10,31,31,10)$	55 762	0	0	0	0	
$(10,m,m,10)$	53 718	0	0	0	0	$m \geq 32$
$(10,m,m+1,11)$	45 469	0	0	0	0	$m \leq 23$
$(10,24,25,11)$	87 014	6	6		$\geq 2^*$	
$(10,25,26,11)$	152 369	136	136		$\geq 10^*$	
$(10,26,27,11)$	203 469	6 794	6 794	≥ 11	≥ 633	
$(10,27,28,11)$	217 596	24 915	24 915		≥ 22	
$(10,28,29,11)$	192 964	30 355	30 355	≥ 1	≥ 159	
$(10,29,30,11)$	145 773	11 916	11 916		≥ 28	
$(10,30,31,11)$	95 827	1 441	1 441	≥ 61	≥ 1	
$(10,31,32,11)$	55 762	35	35	≥ 9	$\geq 20^*$	
$(10,32,33,11)$	29 199	2	**2**	**2**	**0**	
$(10,33,34,11)$	13 981	0	0	0	0	
$(10,34,35,11)$	6 202	0	0	0	0	
$(10,35,36,11)$	2 600	0	0	0	0	
$(10,m,m+1,11)$	1 736	0	0	0	0	$m \geq 36$
$(10,m,m+2,12)$	45 469	0	0	0	0	$m \leq 23$
$(10,24,26,12)$	87 014	2	2		≥ 1	
$(10,25,27,12)$	152 369	2	2		≥ 1	
$(10,26,28,12)$	203 469	1 051	1 051		≥ 178	
$(10,27,29,12)$	217 596	23 884	23 884		≥ 768	
$(10,28,30,12)$	192 964	91 727	91 727		≥ 455	
$(10,29,31,12)$	145 773	112 266	112 266		≥ 256	
$(10,30,32,12)$	95 827	47 141	47 141	≥ 13	≥ 1	
$(10,31,33,12)$	55 762	5 943	5 943	≥ 521	≥ 368	
$(10,32,34,12)$	29 199	225	225		≥ 7	
$(10,33,35,12)$	13 981	1	**1**	**1**	**0**	
$(10,34,36,12)$	6 202	0	0	0	0	
$(10,35,37,12)$	2 600	0	0	0	0	
$(10,m,m+2,12)$	1 736	0	0	0	0	$m \geq 36$
$(11,22,22,11)$	265	0	0	0	0	
$(11,23,23,11)$	10 391	0	0	0	0	
$(11,24,24,11)$	120 985	0	0	0	0	

Table 1, continued from previous page

f-vector	# graphs	$\#\mathcal{E}^5$	$\#\mathcal{S}^3$	#np	$\#\mathcal{P}^4$	
$(11, 25, 25, 11)$	696 184	0	0	0	0	
$(11, 26, 26, 11)$	2 504 998	21	21		≥ 1	
$(11, 27, 27, 11)$	6 383 318	322	322		≥ 1	
$(11, 28, 28, 11)$	12 417 723				≥ 2635	pyramids
$(11, 29, 29, 11)$	19 379 000				≥ 1	
$(11, 30, 30, 11)$	25 121 426				≥ 1	
$(11, 31, 31, 11)$	27 749 332				≥ 1	
$(11, 32, 32, 11)$	26 626 961				≥ 104	
$(11, 33, 33, 11)$	22 528 512				≥ 1	
$(11, 34, 34, 11)$	17 005 570	100	100	≥ 15	≥ 1	P_{11}
$(11, 35, 35, 11)$	11 561 155	2	**2**	**2**	**0**	
$(11, 36, 36, 11)$	7 134 337	0	0	0	0	

The spheres with the particular f-vectors $(10, 32, 33, 11)$, $(10, 33, 35, 12)$, and $(11, 35, 35, 11)$ of Theorem 1.1 will be presented and discussed in Section 5.

The proof of Theorem 1.3 follows the same pattern, with considerably higher computation times. Table 2 shows the results of the computation for the potential f-vectors $(12, m, m, 12)$ for large m: The numbers of graphs to check (graphs on f_0 vertices, with f_1 edges, with minimal vertex degree at least 4, 2-connected) and the numbers of strongly regular cellular 3-manifolds, strongly regular cellular 3-spheres, non-polytopal spheres among them, and 4-polytopes. Blank spaces represent missing data (e.g., not enumerated or calculated). For time reasons, and since there is a polytope, we did not enumerate the manifolds with f-vector $(12, 39, 39, 12)$. The results for larger m follow as any manifold with such an f-vector would be 2s2s, as verified in Brinkmann [10, Prop. 2.2.19], and these we have enumerated, see Brinkmann and Ziegler [11, Thm. 2.1].

Table 2: Results for the potential f-vectors $(12, 40, 40, 12)$ and $(12, 41, 41, 12)$.

f-vector	# graphs	$\#\mathcal{E}^5$	$\#\mathcal{S}^3$	#np	$\#\mathcal{P}^4$	
$(12, 39, 39, 12)$	4 078 410 035	≥ 1	≥ 1		≥ 1	W_{12}^{39}
$(12, 40, 40, 12)$	2 997 683 218	4	**4**	**4**	**0**	
$(12, 41, 41, 12)$	2 037 876 411	0	0	0	0	
$(12, m, m, 12)$	4 880 253 668	0	0	0	0	$m \geq 42$

5. Examples

According to Theorem 1.1 there are five f-vectors for which there is at least one 3-sphere but no 4-polytope. In this section we will present these 3-spheres. For each of these f-vectors,

- the fact that there are no other 3-spheres than those we present in the following depends on massive computation and does not seem to have a reasonably short or "compact" proof,
- the fact that the objects that we present are, indeed, spheres, can be verified in a variety of ways; in the following we present coordinates and images for a diagram (in the sense of polytope theory; see Ziegler [37, Lect. 5]),

- the fact that the spheres are not polytopal was verified on the computer with oriented matroid techniques; in principle, one can extract human-verifiable short proofs from the computation results; for this we give one example below.

Examples 5.1. There are two 3-spheres with f-vector $(10, 32, 33, 11)$:

- The sphere $(10^0_{32,33})$ is given by the facet list:

$$\begin{array}{ll} F_0 = \{v_0, v_2, v_4, v_5, v_9\} & F_6 = \{v_0, v_1, v_4, v_6, v_9\} \\ F_1 = \{v_0, v_2, v_4, v_6, v_8\} & F_7 = \{v_2, v_3, v_5, v_7, v_9\} \\ F_2 = \{v_1, v_3, v_6, v_7, v_9\} & F_8 = \{v_1, v_2, v_4, v_7, v_8\} \\ F_3 = \{v_1, v_3, v_4, v_6, v_8\} & F_9 = \{v_1, v_2, v_4, v_7, v_9\} \\ F_4 = \{v_0, v_2, v_5, v_7, v_8\} & F_{10} = \{v_0, v_3, v_5, v_6, v_8, v_9\} \\ F_5 = \{v_1, v_3, v_5, v_7, v_8\} & \end{array}$$

  It is non-polytopal, but it has diagrams based on each of the facets F_0, F_1, F_2, F_3, F_4, F_5, F_6, and F_7, but not based on one of F_8, F_9, or F_{10}. A diagram based on facet F_2 is given in Figure 2. This sphere cannot be realized by a fan, and thus it is not star-shaped in the sense of Ewald [15, Sect. III.5].

- The sphere $(10^1_{32,33})$ is given by the facet list:

$$\begin{array}{ll} F_0 = \{v_0, v_3, v_5, v_6, v_8\} & F_6 = \{v_0, v_2, v_4, v_6, v_8\} \\ F_1 = \{v_0, v_4, v_5, v_7, v_8\} & F_7 = \{v_2, v_4, v_6, v_7, v_9\} \\ F_2 = \{v_0, v_3, v_4, v_6, v_7\} & F_8 = \{v_2, v_4, v_5, v_8, v_9\} \\ F_3 = \{v_0, v_1, v_3, v_5, v_7\} & F_9 = \{v_1, v_4, v_5, v_7, v_9\} \\ F_4 = \{v_1, v_3, v_5, v_8, v_9\} & F_{10} = \{v_2, v_3, v_6, v_8, v_9\} \\ F_5 = \{v_1, v_3, v_6, v_7, v_9\} & \end{array}$$

  It is non-polytopal, but it has a diagram based on every facet and it can be represented by a fan. A diagram based on facet F_2 is given in Figure 3.

We did not manage to decide whether the second sphere $(10^1_{32,33})$ has a star-shaped embedding. An oriented matroid that would support such an embedding exists. (Clearly every star-shaped sphere can be represented by a fan. The converse is true for simplicial spheres, but not in general.)

The following is an example for a human-verifiable non-polytopality proof, for the first sphere in Examples 5.1. The non-existence proofs for diagrams use the same technique. For more details and more examples see Brinkmann [10].

$\underline{F_2}$
$v_0 \ = (906, 197, 915)$
$v_1 \ =$
$(228623/5810, 18, 986)$
$v_2 \ = (90, 942, 119)$
$v_3 \ = (983, 18, 10)$
$v_4 \ = (485, 502, 941)$
$v_5 \ = (448, 647, 296)$
$v_6 \ = (974, 18, 908)$
$v_7 \ = (18, 977, 14)$
$v_8 \ = (665, 333, 592)$
$v_9 \ = (983, 990, 985)$

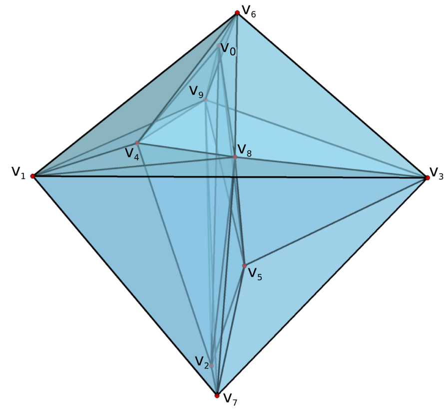

FIGURE 2. A diagram based on facet F_2 for the sphere $(10^0_{32,33})$ with f-vector $(10, 32, 33, 11)$.

$\underline{F_0}$
$v_0 \ = (11, 10, 26)$
$v_1 \ = (13, 16, 10)$
$v_2 \ = (9, 10, 11)$
$v_3 \ = (16, 8, 10)$
$v_4 \ = (9, 11, 14)$
$v_5 \ = (12, 24, 9)$
$v_6 \ = (11, 9, 11)$
$v_7 \ = (11, 14, 16)$
$v_8 \ = (5, 10, 8)$
$v_9 \ = (11, 13, 10)$

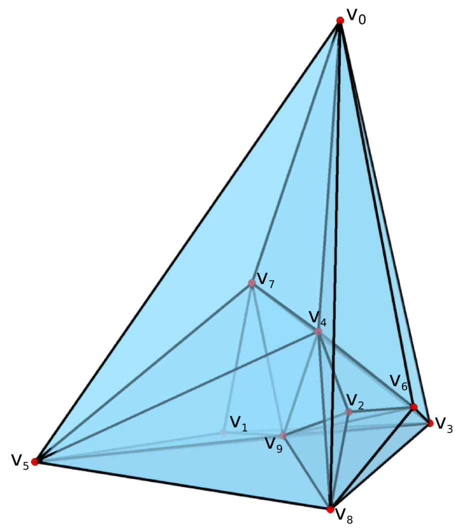

FIGURE 3. A diagram based on facet F_0 for the sphere $(10^1_{32,33})$ with f-vector $(10, 32, 33, 11)$.

Proposition 5.2. *The sphere* $(10^0_{32,33})$ *is non-polytopal.*

Proof. We will use a similar oriented matroid approach as in [11]. The following arguments may be verified with reference to the list of labeled facets displayed in Figure 4.

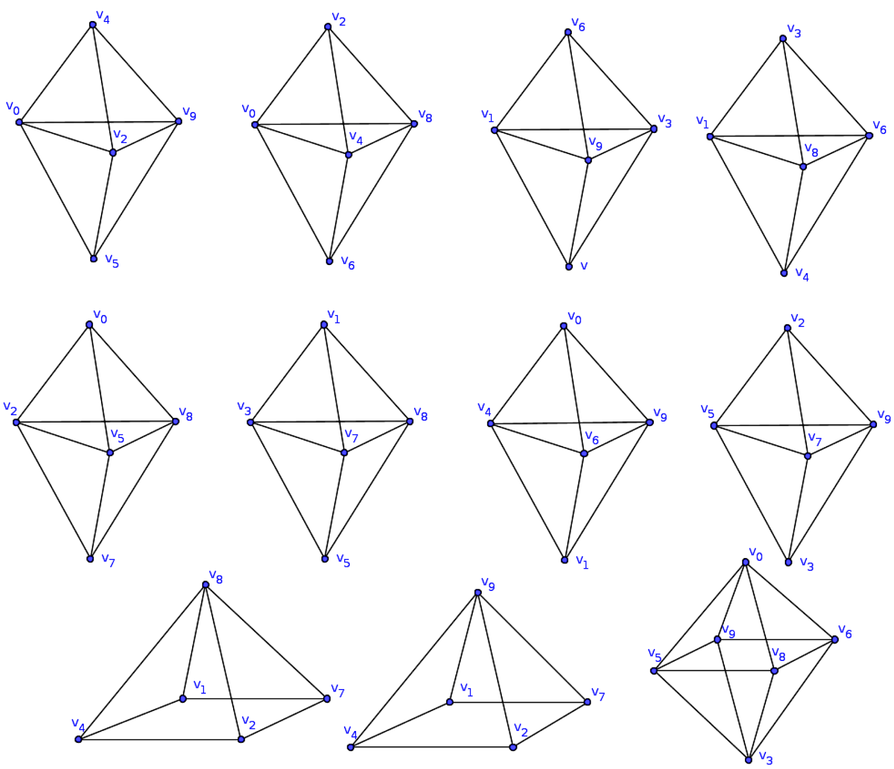

FIGURE 4. The facets of the sphere $(10^0_{32,33})$ from F_0 (top left) to F_{10} (bottom right).

With reference to facet F_0, we may choose $\chi(v_0\ ,v_2\ ,v_4\ ,v_9\ ,v_i\ ) = +1$ for all $v_i \notin F_0$. With this we can derive:

(5)
$$\chi(v_3, v_5, v_6, v_8, v_9) \overset{F_{10}}{=} 0,$$

(6)
$$\chi(v_0, v_1, v_2, v_4, v_9) = -1 \overset{F_9}{\Rightarrow} \chi(v_1, v_2, v_4, v_5, v_9) = 1 \overset{F_0}{\Rightarrow} \chi(v_2, v_4, v_5, v_8, v_9) = -1,$$

(7)
$$\chi(v_0, v_2, v_4, v_8, v_9) = -1 \overset{F_1}{\Rightarrow} \chi(v_0, v_1, v_2, v_4, v_8) = 1 \overset{F_8}{\Rightarrow} \chi(v_1, v_2, v_4, v_6, v_8) = -1$$
$$\overset{F_1}{\Rightarrow} \chi(v_2, v_4, v_6, v_8, v_9) = -1,$$

(8)
$$\chi(v_0, v_2, v_4, v_8, v_9) = -1 \overset{F_1}{\Rightarrow} \chi(v_0, v_2, v_4, v_7, v_8) = 1 \overset{F_8}{\Rightarrow} \chi(v_2, v_4, v_7, v_8, v_9) = 1,$$

(9)
$$\chi(v_0, v_2, v_4, v_8, v_9) = -1 \overset{F_1}{\Rightarrow} \chi(v_0, v_2, v_4, v_5, v_8) = 1 \overset{F_4}{\Rightarrow} \chi(v_0, v_2, v_5, v_6, v_8) = -1$$
$$\overset{F_1}{\Rightarrow} \chi(v_0, v_2, v_3, v_6, v_8) = -1 \overset{F_{10}}{\Rightarrow} \chi(v_0, v_1, v_3, v_6, v_8) = -1$$
$$\overset{F_3}{\Rightarrow} \chi(v_1, v_3, v_6, v_8, v_9) = -1 \overset{F_{10}}{\Rightarrow} \chi(v_2, v_3, v_6, v_8, v_9) = -1,$$

(10)
$$\chi(v_0, v_2, v_4, v_6, v_9) = -1 \overset{F_1}{\Rightarrow} \chi(v_0, v_1, v_2, v_4, v_6) = 1 \overset{F_6}{\Rightarrow} \chi(v_0, v_1, v_4, v_6, v_8) = 1$$
$$\overset{F_3}{\Rightarrow} \chi(v_1, v_4, v_6, v_8, v_9) = 1,$$

(11)
$$\chi(v_0, v_2, v_4, v_5, v_8) = 1 \overset{F_4}{\Rightarrow} \chi(v_0, v_2, v_3, v_5, v_8) = 1 \overset{F_{10}}{\Rightarrow} \chi(v_0, v_3, v_5, v_7, v_8) = 1$$
$$\overset{F_4}{\Rightarrow} \chi(v_0, v_1, v_5, v_7, v_8) = 1 \overset{F_5}{\Rightarrow} \chi(v_1, v_5, v_7, v_8, v_9) = 1,$$

(12)
$$\chi(v_0, v_1, v_2, v_4, v_6) \overset{(10)}{=} 1 \overset{F_6}{\Rightarrow} \chi(v_0, v_1, v_3, v_4, v_6) = 1 \overset{F_3}{\Rightarrow} \chi(v_1, v_3, v_4, v_6, v_9) = 1$$
$$\overset{F_2}{\Rightarrow} \chi(v_0, v_1, v_3, v_6, v_9) = 1 \overset{F_{10}}{\Rightarrow} \chi(v_0, v_3, v_6, v_7, v_9) = 1$$
$$\overset{F_2}{\Rightarrow} \chi(v_3, v_6, v_7, v_8, v_9) = -1,$$

(13)
$$\chi(v_0, v_1, v_3, v_6, v_9) \overset{(12)}{=} 1 \overset{F_6}{\Rightarrow} \chi(v_0, v_1, v_6, v_7, v_9) = -1 \overset{F_2}{\Rightarrow} \chi(v_1, v_6, v_7, v_8, v_9) = 1,$$

(14)
$$\chi(v_0, v_2, v_3, v_6, v_8) \overset{(9)}{=} -1 \overset{F_{10}}{\Rightarrow} \chi(v_0, v_3, v_4, v_6, v_8) = 1 \overset{F_3}{\Rightarrow} \chi(v_3, v_4, v_6, v_8, v_9) = 1,$$

(15)
$$\chi(v_0, v_2, v_4, v_7, v_8) \overset{(8)}{=} 1 \overset{F_4}{\Rightarrow} \chi(v_0, v_1, v_2, v_7, v_8) = -1 \overset{F_8}{\Rightarrow} \chi(v_1, v_2, v_5, v_7, v_8) = -1$$
$$\overset{F_4}{\Rightarrow} \chi(v_2, v_5, v_7, v_8, v_9) = -1,$$

(16)
$$\chi(v_0, v_1, v_2, v_7, v_8) \overset{(15)}{=} -1 \overset{F_8}{\Rightarrow} \chi(v_1, v_2, v_3, v_7, v_8) = -1 \overset{F_5}{\Rightarrow} \chi(v_1, v_3, v_7, v_8, v_9) = 1,$$

(17)
$$\chi(v_0, v_3, v_5, v_7, v_8) \overset{(11)}{=} 1 \overset{F_5}{\Rightarrow} \chi(v_3, v_5, v_7, v_8, v_9) = 1,$$

With these values for the partial chirotope, we can find some new values of χ using the Grassmann–Plücker relations:

$$\begin{aligned}
&\{\chi(v_7,v_8,v_9,v_1,v_3)\chi(v_7,v_8,v_9,v_5,v_6),\ \chi(v_7,v_8,v_9,v_1,v_5)\chi(v_7,v_8,v_9,v_3,v_6),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \chi(v_7,v_8,v_9,v_1,v_6)\chi(v_7,v_8,v_9,v_3,v_5)\}\\
&\overset{(16),(11),(12),(13),(17)}{=} \{1\cdot\chi(v_7,v_8,v_9,v_5,v_6), -1\cdot(-1), 1\cdot 1\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \chi(v_7,v_8,v_9,v_5,v_6) = -1, \qquad (18)
\end{aligned}$$

$$\begin{aligned}
&\{\chi(v_6,v_8,v_9,v_2,v_3)\chi(v_6,v_8,v_9,v_5,v_7),\ \chi(v_6,v_8,v_9,v_2,v_5)\chi(v_6,v_8,v_9,v_3,v_7),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \chi(v_6,v_8,v_9,v_2,v_7)\chi(v_6,v_8,v_9,v_3,v_5)\}\\
&\overset{(9),(18),(12),(5)}{=} \{(-1)\cdot 1, -\chi(v_6,v_8,v_9,v_2,v_5)\cdot 1, 0\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \chi(v_6,v_8,v_9,v_2,v_5) = -1, \qquad (19)
\end{aligned}$$

$$\begin{aligned}
&\{\chi(v_6,v_8,v_9,v_1,v_3)\chi(v_6,v_8,v_9,v_4,v_7), \chi(v_6,v_8,v_9,v_1,v_4)\chi(v_6,v_8,v_9,v_3,v_7),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \chi(v_6,v_8,v_9,v_1,v_7)\chi(v_6,v_8,v_9,v_3,v_4)\}\\
&\overset{(9),(10),(12),(13),(14)}{=} \{(-1)\cdot\chi(v_6,v_8,v_9,v_4,v_7), -1\cdot(-1), (-1)\cdot 1\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \chi(v_6,v_8,v_9,v_4,v_7) = -1, \qquad (20)
\end{aligned}$$

$$\begin{aligned}
&\{\chi(v_6,v_8,v_9,v_3,v_4)\chi(v_6 v_8,v_9,v_5,v_7), \chi(v_6,v_8,v_9,v_3,v_5)\chi(v_6,v_8,v_9,v_4,v_7),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \chi(v_6,v_8,v_9,v_3,v_7)\chi(v_6,v_8,v_9,v_4,v_5)\}\\
&\overset{(14),(18),(5),(12)}{=} \{1\cdot 1, 0, 1\cdot\chi(v_6,v_8,v_9,v_4,v_5)\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \chi(v_6,v_8,v_9,v_4,v_5) = -1, \qquad (21)
\end{aligned}$$

$$\begin{aligned}
&\{\chi(v_5,v_8,v_9,v_2,v_4)\chi(v_5,v_8,v_9,v_6,v_7), \chi(v_5,v_8,v_9,v_2,v_6)\chi(v_5,v_8,v_9,v_4,v_7),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \chi(v_5,v_8,v_9,v_2,v_7)\chi(v_5,v_8,v_9,v_4,v_6)\}\\
&\overset{(6),(18),(19),(15),(21)}{=} \{(-1)\cdot(-1), -1\cdot\chi(v_5,v_8,v_9,v_4,v_7), (-1)\cdot(-1)\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \chi(v_5,v_8,v_9,v_4,v_7) = 1. \qquad (22)
\end{aligned}$$

Finally, we get the Grassmann–Plücker relation

$$\begin{aligned}
&\{\chi(v_4,v_8,v_9,v_2,v_5)\chi(v_4,v_8,v_9,v_6,v_7),\ \chi(v_4,v_8,v_9,v_2,v_6)\chi(v_4,v_8,v_9,v_5,v_7),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \chi(v_4,v_8,v_9,v_2,v_7)\chi(v_4,v_8,v_9,v_5,v_6)\}\\
&\overset{(6),(20),(7),(22),(8),(21)}{=} \{1\cdot 1, -1\cdot(-1), (-1)\cdot(-1)\}, \qquad (23)
\end{aligned}$$

which is neither $\{0\}$, nor contains $\{-1,1\}$. Thus, the Grassmann–Plücker relations cannot be satisfied, so the sphere $(10^0_{32,33})$ does not support an oriented matroid. In particular, it is not polytopal. □

Example 5.3. There is exactly one 3-sphere with f-vector $(10,33,35,12)$. This sphere $(10_{33,35})$ is given by the facet list:

$F_0 = \{v_1, v_4, v_7, v_9\}$
$F_1 = \{v_2, v_4, v_7, v_9\}$
$F_2 = \{v_0, v_2, v_4, v_5, v_8\}$
$F_3 = \{v_0, v_2, v_4, v_6, v_9\}$
$F_4 = \{v_1, v_3, v_6, v_7, v_8\}$
$F_5 = \{v_1, v_3, v_4, v_6, v_9\}$
$F_6 = \{v_0, v_2, v_5, v_7, v_9\}$
$F_7 = \{v_1, v_3, v_5, v_7, v_9\}$
$F_8 = \{v_0, v_1, v_4, v_6, v_8\}$
$F_9 = \{v_1, v_2, v_4, v_7, v_8\}$
$F_{10} = \{v_2, v_3, v_5, v_7, v_8\}$
$F_{11} = \{v_0, v_3, v_5, v_6, v_8, v_9\}$

It is not polytopal. It has a diagram based on each of the facets F_2, F_3, F_4, F_5, F_6, F_7, F_8, and F_{10}, but not based on one of F_0, F_1, F_9, or F_{11}. A diagram based on facet F_2 is given in Figure 5. The sphere cannot be represented by a fan.

$\underline{F_2}$
$v_0 = (1306, 2451, 4264)$
$v_1 = (2471, 990, 1976)$
$v_2 = (2881, 3713, 856)$
$v_3 = (1412, 2367, 1947)$
$v_4 = (2812, 766, 2282)$
$v_5 = (1451, 2517, 2110)$
$v_6 = (1965, 1505, 2347)$
$v_7 = (1772, 2235, 976)$
$v_8 = (636, 941, 864)$
$v_9 = (1612, 2283, 2145)$

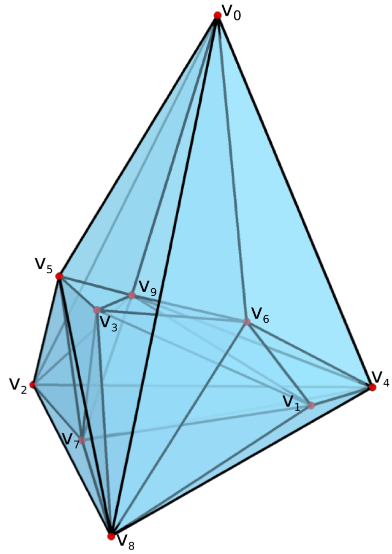

FIGURE 5. A diagram based on facet F_2 for the sphere $(10_{33,35})$ with f-vector $(10, 33, 35, 12)$.

Examples 5.4. There are exactly two 3-spheres with f-vector $(11, 35, 35, 11)$. They are dual to each other. These spheres (11^0_{35}) and (11^1_{35}) are given by facet lists:

(11^0_{35})
$F_0 = \{v_1, v_2, v_4, v_6, v_9\}$
$F_1 = \{v_3, v_5, v_7, v_8, v_9\}$
$F_2 = \{v_0, v_6, v_7, v_8, v_{10}\}$
$F_3 = \{v_2, v_3, v_4, v_8, v_9\}$
$F_4 = \{v_0, v_1, v_2, v_5, v_6, v_{10}\}$
$F_5 = \{v_3, v_4, v_7, v_8, v_{10}\}$
$F_6 = \{v_1, v_5, v_6, v_7, v_9\}$
$F_7 = \{v_0, v_1, v_2, v_4, v_8, v_{10}\}$
$F_8 = \{v_3, v_5, v_6, v_7, v_{10}\}$
$F_9 = \{v_0, v_2, v_6, v_7, v_8, v_9\}$
$F_{10} = \{v_1, v_3, v_4, v_5, v_9, v_{10}\}$

(11^1_{35})
$F_0 = \{v_2, v_4, v_7, v_9\}$
$F_1 = \{v_0, v_4, v_6, v_7, v_{10}\}$
$F_2 = \{v_0, v_3, v_4, v_7, v_9\}$
$F_3 = \{v_1, v_3, v_5, v_8, v_{10}\}$
$F_4 = \{v_0, v_3, v_5, v_7, v_{10}\}$
$F_5 = \{v_1, v_4, v_6, v_8, v_{10}\}$
$F_6 = \{v_0, v_2, v_4, v_6, v_8, v_9\}$
$F_7 = \{v_1, v_2, v_5, v_6, v_8, v_9\}$
$F_8 = \{v_1, v_2, v_3, v_5, v_7, v_9\}$
$F_9 = \{v_0, v_1, v_3, v_6, v_9, v_{10}\}$
$F_{10} = \{v_2, v_4, v_5, v_7, v_8, v_{10}\}$

Both spheres are not fan-like, hence they have no star-shaped embedding. Furthermore, the sphere (11^0_{35}) does not have a diagram with base F_6, F_9, or F_{10}; the sphere (11^1_{35}) has a diagram based on each of F_4 and F_6, but does not have a diagram with base F_0, F_1, F_3, F_5, F_9, or F_{10}. A diagram for (11^1_{35}) with base F_6 is given in Figure 6.

Similar details can be found in Brinkmann [10, Sect. 3.2.4] for the four self-dual 3-spheres of Theorem 1.3.

$\underline{F_6}$
$v_0 = (0, 0, 0)$
$v_1 = (1797, 1585, 512)$
$v_2 = (2009, 2395, 1622)$
$v_3 = (460, 1113, 648)$
$v_4 = (0, 0, 1000)$
$v_5 = (8565805/4137, 2055, 1316)$
$v_6 = (2850, 426, 139)$
$v_7 = (521, 1238, 853)$
$v_8 = (2946124555/1064794, 1020, 770)$
$v_9 = (423, 2580, 139)$
$v_{10} = (1161, 1055, 677)$

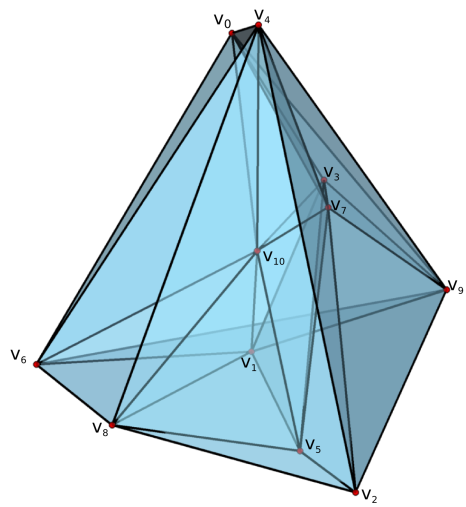

FIGURE 6. A diagram based on facet F_6 for the sphere (11^1_{35}) with f-vector $(11, 35, 35, 11)$.

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NOTE ADDED IN PROOF

After this work was completed, Moritz Firsching ([16]) achieved a complete classification of 4-polytopes and 3-spheres with nine vertices (paper in preparation). His computations yield more examples than documented in Table 1 for two specific f-vectors, namely 2224 spheres and 1829 polytopes for the f-vector $(9, 28, 32, 13)$ and 45 spheres and 26 polytopes for the f-vector $(9, 29, 33, 13)$. It would of course be desirable to have an independent computation and check for all the results of the present paper.

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