# Knot Theory and Physics 

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#### Abstract

This article is an introduction to relationships between knot theory and theoretical physics. We give an exposition of the theory of polynomial invariants of knots and links, the Witten functional integral formulation of knot and link invariants, and the beginnings of topological quantum field theory, and show how the theory of knots is related to a number of key issues in mathematical physics, including loop quantum gravity and quantum information theory.


## 1 Introduction

This article is an introduction to some of the relationships between knot theory and theoretical physics. Knots themselves are macroscopic physical phenomena in three-dimensional space, occuring in rope, vines, telephone cords, polymer chains, DNA, certain species of eel and many other places in the natural and man-made world. The study of topological invariants of knots leads to relationships with statistical mechanics and quantum physics. This is a remarkable and deep situation where the study of a certain (topological) aspects of the macroscopic world is entwined with theories developed for the subtleties of the microscopic world. The present article is an introduction to the mathematical side of these connections, with some hints and references to the related physics.

We begin with a short introduction to knots, links, braids and the bracket polynomial invariant of knots and links. The article then discusses Vassiliev invariants of knots and links, and how these invariants are naturally related to Lie algebras and to Witten's gauge theoretic approach. This part of the article is an introduction to how Vassiliev invariants in knot theory arise naturally in the context of Witten's functional integral.

The article is divided into 5 sections beyond the introduction. Section 2 is a quick introduction to the topology of knots and links. Section 3 discusses Vassiliev invariants and invariants of rigid vertex graphs. Section 4 introduces the basic formalism and shows how Witten's functional integral is related directly to Vassiliev invariants. Section 5 discusses the loop transform and loop quantum gravity in this context. Section 6 is an introduction to topological quantum field theory, and to the use of these techniques in producing unitary representations of the braid group, a topic of intense interest in quantum information theory.

Acknowledgement. It gives the author pleasure to thank the National Science Foundation for support of this research under NSF Grant DMS-0245588. Much of this effort was sponsored by the Defense Advanced Research Projects Agency (DARPA) and Air Force Research Laboratory, Air Force Materiel Command, USAF, under agreement F30602-01-2-05022. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright annotations thereon. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Defense Advanced Research Projects Agency, the Air Force Research Laboratory, or the U.S. Government. (Copyright 2005.)

## 2 Knots, Braids and Bracket Polynomial

The purpose of this section is to give a quick introduction to the diagrammatic theory of knots, links and braids. A knot is an embedding of a circle in three-dimensional space, taken up to ambient isotopy.


Figure 1 - A knot diagram.


Figure 2 - The Reidemeister Moves.
That is, two knots are regarded as equivalent if one embedding can be obtained from the other through a continuous family of embeddings of circles
in three-space. A link is an embedding of a disjoiint collection of circles, taken up to ambient isotopy. Figure 1 illustrates a diagramm for a knot. The diagram is regarded both as a schematic picture of the knot, and as a plane graph with extra structure at the nodes (indicating how the curve of the knot passes over or under itself by standard pictorial conventions).


Figure 3-Braid Generators.
Ambient isotopy is mathematically the same as the equivalence relation generated on diagrams by the Reidemeister moves. These moves are illustrated in Figure 2. Each move is performed on a local part of the diagram that is topologically identical to the part of the diagram illustrated in this figure (these figures are representative examples of the types of Reidemeister moves) without changing the rest of the diagram. The Reidemeister
moves are useful in doing combinatorial topology with knots and links, notaby in working out the behaviour of knot invariants. A knot invariant is a function defined from knots and links to some other mathematical object (such as groups or polynomials or numbers) such that equivalent diagrams are mapped to equivalent objects (isomorphic groups, identical polynomials, identical numbers).


Figure 4 - Closing Braids to form knots and links.


Figure 5 - Borromean Rings as a Braid Closure.
Another significant structure related to knots and links is the Artin Braid Group. A braid is an embedding of a collection of strands that have their ends top and bottom row points in two rows of points that are set one above the other with respect to a choice of vertical. The strands are not individually knotted and they are disjoint from one another. See Figures 3, 4 and 5 for illustrations of braids and moves on braids. Braids can be multiplied by attaching the bottom row of one braid to the top row of the other braid. Taken up to ambient isotopy, fixing the endpoints, the braids form a group under this notion of multiplication. In Figure 3 we illustrate the form of the basic generators of the braid group, and the form of the relations among these generators. Figure 4 illustrates how to close a braid by attaching the top strands to the bottom strands by a collection of parallel arcs. A key theorem of Alexander states that every knot or link can be represented as a closed braid. Thus the theory of braids is critical to the theory of knots and links. Figure 5 illustrates the famous Borrowmean Rings (a link of three unknotted loops such that any two of the loops are unlinked) as the closure of a braid.

We now discuss a significant example of an invariant of knots and links, the bracket polynomial. The bracket polynomial can be normalized to produce an invariant of all the Reidemeister moves. This normalized invariant is known as the Jones polynomial [7]. The Jones polynomial was originally discovered by a different method than the one given here.

The bracket polynomial, $\langle K>=<K>(A)$, assigns to each unoriented link diagram $K$ a Laurent polynomial in the variable $A$, such that

1. If $K$ and $K^{\prime}$ are regularly isotopic diagrams, then $\langle K\rangle=\left\langle K^{\prime}\right\rangle$.
2. If $K \amalg O$ denotes the disjoint union of $K$ with an extra unknotted and unlinked component $O$ (also called 'loop' or 'simple closed curve' or 'Jordan curve'), then

$$
<K \amalg O>=\delta<K>
$$

where

$$
\delta=-A^{2}-A^{-2}
$$

3. $\langle K>$ satisfies the following formulas

$$
\begin{aligned}
& \left.<\chi>=A<\asymp>+A^{-1}<\right)(> \\
& \left.<\bar{\chi}>=A^{-1}<\asymp>+A<\right)(>
\end{aligned}
$$

where the small diagrams represent parts of larger diagrams that are identical except at the site indicated in the bracket. We take the convention that the letter chi, $\chi$, denotes a crossing where the curved line is crossing over the straight segment. The barred letter denotes the switch of this crossing, where the curved line is undercrossing the straight segment.

In computing the bracket, one finds the following behaviour under Reidemeister move I:

$$
<\gamma>=-A^{3}<\smile>
$$

and

$$
<\bar{\gamma}>=-A^{-3}<\smile>
$$

where $\gamma$ denotes a curl of positive type as indicated in Figure 6, and $\bar{\gamma}$ indicates a curl of negative type, as also seen in this figure. The type of a
curl is the sign of the crossing when we orient it locally. Our convention of signs is also given in Figure 6. Note that the type of a curl does not depend on the orientation we choose. The small arcs on the right hand side of these formulas indicate the removal of the curl from the corresponding diagram.

The bracket is invariant under regular isotopy and can be normalized to an invariant of ambient isotopy by the definition

$$
f_{K}(A)=\left(-A^{3}\right)^{-w(K)}<K>(A)
$$

where we chose an orientation for $K$, and where $w(K)$ is the sum of the crossing signs of the oriented link $K . w(K)$ is called the writhe of $K$. The convention for crossing signs is shown in Figure 6.


Figure 6 - Crossing Signs and Curls
The State Summation. In order to obtain a closed formula for the bracket, we now describe it as a state summation. Let $K$ be any unoriented link diagram. Define a state, $S$, of $K$ to be a choice of smoothing for each crossing of $K$. There are two choices for smoothing a given crossing, and thus there are $2^{N}$ states of a diagram with $N$ crossings. In a state we label each smoothing with $A$ or $A^{-1}$ as in the expansion formula for the bracket. The label is called a vertex weight of the state. There are two evaluations related to a state. The first one is the product of the vertex weights, denoted

$$
<K \mid S>
$$

The second evaluation is the number of loops in the state $S$, denoted

$$
\|S\| .
$$

Define the state summation, $\langle K\rangle$, by the formula

$$
<K>=\sum_{S}<K \mid S>\delta^{\|S\|-1}
$$

It follows from this definition that $\langle K\rangle$ satisfies the equations

$$
\begin{gathered}
\left.<\chi>=A<\asymp>+A^{-1}<\right)(>, \\
<K \amalg O>=\delta<K>, \\
<O>=1
\end{gathered}
$$

The first equation expresses the fact that the entire set of states of a given diagram is the union, with respect to a given crossing, of those states with an $A$-type smoothing and those with an $A^{-1}$-type smoothing at that crossing. The second and the third equation are clear from the formula defining the state summation. Hence this state summation produces the bracket polynomial as we have described it at the beginning of the section.

Remark. By a change of variables one obtains the original Jones polynomial, $V_{K}(t)$, for oriented knots and links from the normalized bracket:

$$
V_{K}(t)=f_{K}\left(t^{-\frac{1}{4}}\right) .
$$

Remark. The bracket polynomial provides a connection between knot theory and physics, in that the state summation expression for it exhibits it as a generalized partition function defined on the knot diagram. Partition functions are ubiquitous in statistical mechanics, where they express the summation over all states of the physical system of probability weighting functions for the individual states. Such physical partition functions contain large amounts of information about the corresponding physical system. Some of this information is directly present in the properties of the function, such as the location of critical points and phase transition. Some of the information can be obtained by differentiating the partition function, or performing other mathematical operations on it.

In fact, by defining a generalization of the bracket polynomial, defined on knot diagrams but not invariant under the Reidemeister moves, we can capture significant partition functions that are physically meaningful. There is not room in this survey to detail how this generalization can be used to express the Potts model for planar graphical configurations, and how it expresses the relationship between the Potts model and the Temperley-Lieb algebra in diagrammatic form. There is much more in this connection with statistical mechanics in that the local weights in a partition function are often expressed in terms of solutions to a matrix equation called the Yang-Baxter equation, that turns out to fit perfectly invariance under the third Reidemeister move. As a result, there are many ways to define partition functions of knot diagrams that give rise to invariants of knots and links. The subject is intertwined with the algebraic structure of Hopf algebras and quantum groups, useful for producing systematic solutions to the Yang-Baxter equation. In fact Hopf algebras are deeply connected with the problem of constructing invariants of three-dimensional manifolds in relation to invariants of knots. We have chosen, in this survey paper, to not discuss the details of these approaches, but rather to proceed to Vassiliev invariants and the relationships with Witten's functional integral. The reader is referred to $[8,9,10,7,17]$ for more information about relationships of knot theory with statistical mechanics, Hopf algebras and quantum groups. For topology, the key point is that Lie algebras can be used to construct invariants of knots and links. This is shown nowhere more clearly than in the theory of Vassiliev invariants that we take up in the next section.

## 3 Vassiliev Invariants and Invariants of Rigid Vertex Graphs

In this section we study the combinatorial topology of Vassiliev invariants. As we shall see, by the end of this section, Vassiliev invariants are directly conncected with Lie algebras, and representations of Lie algebras can be used to construct them. This aspect of link invariants is one of the most fundamental for connections with physics. Just as symmetry considerations in physics lead to a fundamental relationship with Lie algebras, topological invariance leads to a fundamental relationship of the theory of knots and
links with Lie algebras.
If $V(K)$ is a (Laurent polynomial valued, or more generally - commutative ring valued) invariant of knots, then it can be naturally extended to an invariant of rigid vertex graphs by defining the invariant of graphs in terms of the knot invariant via an 'unfolding of the vertex. That is, we can regard the vertex as a 'black box" and replace it by any tangle of our choice. Rigid vertex motions of the graph preserve the contents of the black box, and hence implicate ambient isotopies of the link obtained by replacing the black box by its contents. Invariants of knots and links that are evaluated on these replacements are then automatically rigid vertex invariants of the corresponding graphs. If we set up a collection of multiple replacements at the vertices with standard conventions for the insertions of the tangles, then a summation over all possible replacements can lead to a graph invariant with new coefficients corresponding to the different replacements. In this way each invariant of knots and links implicates a large collection of graph invariants.

The simplest tangle replacements for a 4 -valent vertex are the two crossings, positive and negative, and the oriented smoothing. Let $\mathrm{V}(\mathrm{K})$ be any invariant of knots and links. Extend V to the category of rigid vertex embeddings of 4 -valent graphs by the formula

$$
V\left(K_{*}\right)=a V\left(K_{+}\right)+b V\left(K_{-}\right)+c V\left(K_{0}\right)
$$

where $K_{+}$denotes a knot diagram $K$ with a specific choice of positive crossing, $K_{-}$denotes a diagram identical to the first with the positive crossing replaced by a negative crossing and $K_{*}$ denotes a diagram identical to the first with the positive crossing replaced by a graphical node.

There is a rich class of graph invariants that can be studied in this manner. The Vassiliev Invariants [4] constitute the important special case of these graph invariants where $a=+1, b=-1$ and $c=0$. Thus $V(G)$ is a Vassiliev invariant if

$$
V\left(K_{*}\right)=V\left(K_{+}\right)-V\left(K_{-}\right) .
$$

Call this formula the exchange identity for the Vassiliev invariant $V$. See Figure 7.


Figure 7 - Exchange Identity for Vassiliev Invariants
$V$ is said to be of finite type $k$ if $V(G)=0$ whenever $|G|>k$ where $|G|$ denotes the number of (4-valent) nodes in the graph $G$. The notion of finite type is of extraordinary significance in studying these invariants. One reason for this is the following basic Lemma.

Lemma. If a graph $G$ has exactly $k$ nodes, then the value of a Vassiliev invariant $v_{k}$ of type $k$ on $G, v_{k}(G)$, is independent of the embedding of $G$.

Proof. Omitted. //
The upshot of this Lemma is that Vassiliev invariants of type $k$ are intimately involved with certain abstract evaluations of graphs with $k$ nodes. In fact, there are restrictions (the four-term relations) on these evaluations demanded by the topology and it follows from results of Kontsevich [4] that such abstract evaluations actually determine the invariants. The knot invariants derived from classical Lie algebras are all built from Vassiliev invariants of finite type. All of this is directly related to Witten's functional integral [19].

In the next few figures we illustrate some of these main points. In Figure 8 we show how one associates a so-called chord diagram to represent the abstract graph associated with an embedded graph. The chord diagram is a circle with arcs connecting those points on the circle that are welded to form the corresponding graph. In Figure 9 we illustrate how the four-term relation is a consequence of topological invariance.


Figure 8 - Chord Diagrams


Figure 9 - The Four Term Relation from Topology


Figure 10 - The Four Term Relation from Categorical Lie Algebra


Figure 11 - Calculating Lie Algebra Weights

In Figure 10 we show how the four term relation is a consequence of the abstract pattern of the commutator identity for a matrix Lie algebra. That is, we show how a diagrammatic version of the formula

$$
T^{a} T^{b}-T^{b} T^{a}=f_{c}^{a b} T^{c}
$$

fits directly with the four-term relation. The formula we have quoted here states that the commutator of the matrices $T^{a}$ and $T^{b}$ is equal to a sum of the matrices $T^{c}$ with coefficients (the structure coefficients of the Lie algebra) $f_{c}^{a b}$. Such a relation is the most concrete way to define a matrix Lie algebra. There are other levels of abstraction that can be employed here. The same diagrammatic can be interpreted directly in terms of the Jacobi identity that defines a Lie algebra. We shall content ourselves with this matrix point of view here, and add that it is assumed here that the structure coefficients are invariant under cyclic permutation, as assumption that is not needed in the general case The four term relation is directly related to a categorical generalisation of Lie algebras.

Figure 11 illustrates how the weights are assigned to the chord diagrams in the Lie algebra case - by inserting Lie algebra matrices into the circle and taking a trace of a sum of matrix products. The relationship between Vassiliev invariants and Lie algebras has been known since Bar-Natan's thesis. See also [11]. In [4] the reader will find a good account of Kontsevich's Theorem, showing how Lie algebra weight systems, and in fact any weight system satsfying the four-term-relation, can be used to construct knot invariants. Conceptually, the ideas in back of the Kontsevich Theorem are directly related to Witten's approach to knot invariants via quantum field theory. We give an exposition of this approach in the next section of this article.

Example. Let

$$
P_{K}(t)=f_{K}\left(e^{t}\right),\left(A=e^{t}\right)
$$

where $f_{K}(A)$ is the normalized bracket polynomial invariant discussed in the last section. Then $P_{K}(t)$ is expressed as a power series in $t$ with coefficients $v_{n}(K), n=0,1,2, \cdots$ that are invariants of the knot or link $K$. It is not hard to show that these coefficent invariants (extended to graphs so that the Vassiliev exchange identity is satisfied) are Vassiliev invariants of finite type. In fact, most of the so-called polynomial invariants of knots and links (relatives of the bracket and Jones polynonmials) give rise to Vassiliev invariants in just this way. Thus Vassiliev invariants of finite type are ubiquitous in this area of knot theory. One can think of Vassiliev invariants as building blocks for the other invariants, or that these invariants are sources of Vassiliev invariants.

## 4 Vassiliev Invariants and Witten's Functional Integral

In [19] Edward Witten proposed a formulation of a class of 3-manifold invariants as generalized Feynman integrals taking the form $Z(M)$ where

$$
Z(M)=\int D A e^{(i k / 4 \pi) S(M, A)}
$$

Here $M$ denotes a 3 -manifold without boundary and $A$ is a gauge field (also called a gauge potential or gauge connection) defined on $M$. The gauge field is a one-form on a trivial $G$-bundle over $M$ with values in a representation of
the Lie algebra of $G$. The group $G$ corresponding to this Lie algebra is said to be the gauge group. In this integral the action $S(M, A)$ is taken to be the integral over $M$ of the trace of the Chern-Simons three-form $A \wedge d A+$ $(2 / 3) A \wedge A \wedge A$. (The product is the wedge product of differential forms.)
$Z(M)$ integrates over all gauge fields modulo gauge equivalence.
The formalism and internal logic of Witten's integral supports the existence of a large class of topological invariants of 3-manifolds and associated invariants of knots and links in these manifolds.

The invariants associated with this integral have been given rigorous combinatorial descriptions but questions and conjectures arising from the integral formulation are still outstanding. Specific conjectures about this integral take the form of just how it implicates invariants of links and 3-manifolds, and how these invariants behave in certain limits of the coupling constant $k$ in the integral. Many conjectures of this sort can be verified through the combinatorial models. On the other hand, the really outstanding conjecture about the integral is that it exists! At the present time there is no measure theory or generalization of measure theory that supports it in full generality. Here is a formal structure of great beauty. It is also a structure whose consequences can be verified by a remarkable variety of alternative means.

The formalism of the Witten integral implicates invariants of knots and links corresponding to each classical Lie algebra. In order to see this, we need to introduce the Wilson loop. The Wilson loop is an exponentiated version of integrating the gauge field along a loop $K$ in three space that we take to be an embedding (knot) or a curve with transversal self-intersections. For this discussion, the Wilson loop will be denoted by the notation

$$
W_{K}(A)
$$

to denote the dependence on the loop $K$ and the field $A$. It is usually indicated by the symbolism $\operatorname{tr}\left(P e^{\oint_{K} A}\right)$. Thus

$$
W_{K}(A)=\operatorname{tr}\left(P e^{\oint_{K} A}\right)
$$

Here the $P$ denotes path ordered integration - we are integrating and exponentiating matrix valued functions, and so must keep track of the order
of the operations. The symbol $t r$ denotes the trace of the resulting matrix. This Wilson loop integration exists by normal means and does not require functional integration.

With the help of the Wilson loop functional on knots and links, Witten writes down a functional integral for link invariants in a 3-manifold $M$ :

$$
\begin{aligned}
Z(M, K) & =\int D A e^{(i k / 4 \pi) S(M, A)} \operatorname{tr}\left(P e^{\oint_{K} A}\right) \\
& =\int D A e^{(i k / 4 \pi) S} W_{K}(A)
\end{aligned}
$$

Here $S(M, A)$ is the Chern-Simons Lagrangian, as in the previous discussion. We abbreviate $S(M, A)$ as $S$ and write $W_{K}(A)$ for the Wilson loop. Unless otherwise mentioned, the manifold $M$ will be the three-dimensional sphere $S^{3}$

An analysis of the formalism of this functional integral reveals quite a bit about its role in knot theory. One can determine how the Witten integral behaves under a small deformation of the loop $K$.

## Theorem.

1. Let $Z(K)=Z\left(S^{3}, K\right)$ and let $\delta Z(K)$ denote the change of $Z(K)$ under an infinitesimal change in the loop K. Then

$$
\delta Z(K)=(4 \pi i / k) \int d A e^{(i k / 4 \pi) S}[V o l] T_{a} T_{a} W_{K}(A)
$$

where $V o l=\epsilon_{r s t} d x^{r} d x^{s} d x^{t}$.
The sum is taken over repeated indices, and the insertion is taken of the matrices $T_{a} T_{a}$ at the chosen point $x$ on the loop $K$ that is regarded as the center of the deformation. The volume element $V o l=\epsilon_{r s t} d x_{r} d x_{s} d x_{t}$ is taken with regard to the infinitesimal directions of the loop deformation from this point on the original loop.
2. The same formula applies, with a different interpretation, to the case where $x$ is a double point of transversal self intersection of a loop K, and the deformation consists in shifting one of the crossing segments perpendicularly to the plane of intersection so that the self-intersection point disappears. In this case, one $T_{a}$ is inserted into each of the transversal crossing segments so that $T_{a} T_{a} W_{K}(A)$ denotes a Wilson loop with a self intersection at $x$ and insertions of $T_{a}$ at $x+\epsilon_{1}$ and $x+\epsilon_{2}$ where $\epsilon_{1}$ and $\epsilon_{2}$ denote small displacements along the two arcs of $K$ that intersect at $x$. In this case, the volume form is nonzero, with two directions coming from the plane of movement of one arc, and the perpendicular direction is the direction of the other arc.

Remark. One shows that the result of a topological variation has an analytic expression that is zero if the topological variation does not create a local volume. Thus we have shown that the intergral of $e^{(i k / 4 \pi) S(A)} W_{K}(A)$ is topologically invariant as long as the curve $K$ is moved by the local equivalent of regular isotopy.

In the case of switching a crossing the key point is to write the crossing switch as a composition of first moving a segment to obtain a transversal intersection of the diagram with itself, and then to continue the motion to complete the switch. Up to the choice of our conventions for constants, the switching formula is, as shown below (See Figure 12).

$$
\begin{aligned}
Z\left(K_{+}\right)-Z\left(K_{-}\right) & =(4 \pi i / k) \int D A e^{(i k / 4 \pi) S} T_{a} T_{a}<K_{* *} \mid A> \\
& =(4 \pi i / k) Z\left(T^{a} T^{a} K_{* *}\right)
\end{aligned}
$$

where $K_{* *}$ denotes the result of replacing the crossing by a self-touching crossing. We distinguish this from adding a graphical node at this crossing by using the double star notation.


Figure 12 - The Difference Formula

A key point is to notice that the Lie algebra insertion for this difference is exactly what is done (in chord diagrams) to make the weight systems for Vassiliev invariants (without the framing compensation). Thus the formalism of the Witten functional integral takes one directly to these weight systems in the case of the classical Lie algebras. In this way the functional integral is central to the structure of the Vassiliev invariants.

## 5 The Loop Transform and Quantum Gravity

Suppose that $\psi(A)$ is a (complex valued) function defined on gauge fields. Then we define formally the loop transform $\widehat{\psi}(K)$, a function on embedded loops in three-dimensional space, by the formula

$$
\widehat{\psi}(K)=\int D A \psi(A) W_{K}(A)
$$

In other words, we take $\widehat{\psi}(K)$ to be the functional equivalence class of $\psi(A) W_{K}(A)$ in the sense of the previous section.

$$
\widehat{\psi}(K) \sim \psi(A) W_{K}(A)
$$

If $\Delta$ is a differential operator defined on $\psi(A)$, then we can use this integral transform to shift the effect of $\Delta$ to an operator on loops via integration by parts:

$$
\begin{gathered}
\widehat{\Delta \psi}(K)=\int D A \Delta \psi(A) W_{K}(A) \\
=-\int D A \psi(A) \Delta W_{K}(A) .
\end{gathered}
$$

Again, this is a statement about the equivalence classes:

$$
\widehat{\Delta \psi}(K) \sim(\Delta \psi(A)) W_{K}(A) \sim-\psi(A)\left(\Delta W_{K}(A)\right)
$$

When $\Delta$ is applied to the Wilson loop the result can be an understandable geometric or topological operation. One cane illustrate this situation with operators $G$ and $H$.

$$
\begin{gathered}
G=-F_{i j}^{a} d x^{i} \delta / \delta A_{j}^{a}(x) \\
H=-\epsilon_{\text {ars }} F_{i j}^{a} \delta / \delta A_{i}^{s} \delta / \delta A_{j}^{r}
\end{gathered}
$$

with summation over the repeated indices. Each of these operators has the property that its action on the Wilson loop has a geometric or topological interpretation. One has

$$
\widehat{G \psi}(K)=\delta \widehat{\psi}(K)
$$

where this variation refers to the effect of varying $K$. As we saw in the previous section, this means that if $\widehat{\psi}(K)$ is a topological invariant of knots and links, then $\widehat{G \psi}(K)=0$ for all embedded loops $K$. This condition is a transform analogue of the equation $G \psi(A)=0$. This equation is the differential analogue of an invariant of knots and links. It may happen that $\delta \widehat{\psi}(K)$ is not strictly zero, as in the case of our framed knot invariants. For example with

$$
\psi(A)=e^{(i k / 4 \pi) \int \operatorname{tr}(A \wedge d A+(2 / 3) A \wedge A \wedge A)}
$$

we conclude that $\widehat{G \psi}(K)$ is zero for flat deformations (in the sense of the previous section) of the loop $K$, but can be non-zero in the presence of a twist or curl. In this sense the loop transform provides a subtle variation on the strict condition $G \psi(A)=0$.

In [3] and other publications by Ashtekar, Rovelli, Smolin and their colleagues, the loop transform is used to study a reformulation and quantization of Einstein gravity. The differential geometric gravity theory of Einstein is reformulated in terms of a background gauge connection and in the quantization, the Hilbert space consists in functions $\psi(A)$ that are required to satisfy the constraints

$$
G \psi=0
$$

and

$$
H \psi=0
$$

Thus we see that $\widehat{G}(K)$ can be partially zero in the sense of producing a framed knot invariant, and that $\widehat{H}(K)$ is zero for non-self intersecting loops. This means that the loop transforms of $G$ and $H$ can be used to investigate a subtle variation of the original scheme for the quantization of gravity. This program is being actively pursued by a number of researchers. The Vassiliev invariants arising from a topologically invariant loop transform are of significance to this theory.

## 6 Braiding, Topological Quantum Field Theory and Quantum Computing

The purpose of this section is to discuss in a very general way how braiding is related to topological quantum field theory and to the enterprise [6] of using this sort of theory as a model for anyonic quantum computation. The ideas in the subject of topological quantum field theory are well expressed in the book [2] by Michael Atiyah and the paper [19] by Edward Witten. The simplest case of this idea is C. N. Yang's original interpretation of the YangBaxter Equation. Yang articulated a quantum field theory in one dimension of space and one dimension of time in which the $R$-matrix giving the scattering ampitudes for an interaction of two particles whose (let us say) spins
corresponded to the matrix indices so that $R_{a b}^{c d}$ is the amplitude for particles of $\operatorname{spin} a$ and spin $b$ to interact and produce particles of $\operatorname{spin} c$ and $d$. Since these interactions are between particles in a line, one takes the convention that the particle with spin $a$ is to the left of the particle with spin $b$, and the particle with spin $c$ is to the left of the particle with spin $d$. If one follows the concatenation of such interactions, then there is an underlying permutation that is obtained by following strands from the bottom to the top of the diagram (thinking of time as moving up the page). Yang designed the Yang-Baxter equation for $R$ so that the amplitudes for a composite process depend only on the underlying permutation corresponding to the process and not on the individual sequences of interactions.

In taking over the Yang-Baxter equation for topological purposes, we can use the same intepretation, but think of the diagrams with their under- and over-crossings as modeling events in a spacetime with two dimensions of space and one dimension of time. The extra spatial dimension is taken in displacing the woven strands perpendicular to the page, and allows the use of braiding operators $R$ and $R^{-1}$ as scattering matrices. Taking this picture to heart, one can add other particle properties to the idealized theory. In particular one can add fusion and creation vertices where in fusion two particles interact to become a single particle and in creation one particle changes (decays) into two particles. Matrix elements corresponding to trivalent vertices can represent these interactions. See Figure 13.


Figure 13 -Creation and Fusion
Once one introduces trivalent vertices for fusion and creation, there is the question how these interactions will behave in respect to the braiding
operators. There will be a matrix expression for the compositions of braiding and fusion or creation as indicated in Figure 15. Here we will restrict ourselves to showing the diagrammatics with the intent of giving the reader a flavor of these structures. It is natural to assume that braiding intertwines with creation as shown in Figure 16 (similarly with fusion). This intertwining identity is clearly the sort of thing that a topologist will love, since it indicates that the diagrams can be interpreted as embeddings of graphs in three-dimensional space. Figure 14 illustrates the Yang-Baxter equation. The intertwining identity is an assumption like the Yang-Baxter equation itself, that simplifies the mathematical structure of the model.


Figure 14 - YangBaxterEquation


Figure 15-Braiding


Figure 16 - Intertwining

It is to be expected that there will be an operator that expresses the recoupling of vertex interactions as shown in Figure 17 and labeled by $Q$. The actual formalism of such an operator will parallel the mathematics of recoupling for angular momentum. See for example [9]. If one just considers the abstract structure of recoupling then one sees that for trees with four branches (each with a single root) there is a cycle of length five as shown in Figure 17. One can start with any pattern of three vertex interactions and go through a sequence of five recouplings that bring one back to the same tree from which one started. It is a natural simplifying axiom to assume that this composition is the identity mapping. This axiom is called the pentagon identity.


Figure 17 - Recoupling


Figure 18 - Pentagon Identity

Finally there is a hexagonal cycle of interactions between braiding, recoupling and the intertwining identity as shown in Figure 19. One says that the interactions satisfy the hexagon identity if this composition is the identity.


Figure 19 - Hexagon Identity

A three-dimensional topological quantum field theory is an algebra of interactions that satisfies the Yang-Baxter equation, the intertwining identity, the pentagon identity and the hexagon identity. There is not room in this summary to detail the way that these properties fit into the topology of knots and three-dimensional manifolds, but a sketch is in order. For the case of topological quantum field theory related to the group $S U(2)$ there is a construction based entirely on the combinatorial topology of the bracket polynomial (See Section 2 of this article.). See [10, 9] for more information on this approach.

It turns out that the algebraic properties of a topological quantum field theory give it enough power to rigourously model three manifold invariants described by the Witten integral. This is done by regarding the threemanifold as a union of two handlebodies with boundary an orientable surface $S_{g}$ of genus $g$. The surface is divided up into trinions as illustrated in Figure 20. A trinion is a surface with boundary that is topologically equivalent to a sphere with three punctures. In Figure 20 we illustrate two trinions, the second shown as a neighborhood of a trivalent vertex, and a surface of genus three that is decomposed into three trinions. It turns out that there is a way to associate a vector space $V\left(S_{g}\right)$ to a surface with a trinion decomposition, defined in terms of the associated topological quantum field theory, such that the isomorpism class of the vector space $V\left(S_{g}\right)$ does not depend upon the choice of decomposition. This independence is guaranteed by the braiding, hexagon and pentagon identities in such a way that one can associate a well-defined vector $\left|M_{\epsilon}\right\rangle$ in $V\left(S_{g}\right)$ whenenver $M$ is a three manifold whose boundary is $S_{g}$. Furthermore, if a closed three-manifold $M^{3}$ is decomposed along a surface $S_{g}$ into the union of $M_{-}$and $M_{+}$where these parts are otherwise disjoint three-manifolds with boundary $S_{g}$, then the inner product $I(M)=\left\langle M_{-} \mid M_{+}\right\rangle$is, up to normalization, an invariant of the three-manifold $M_{3}$. With the definition of topological quantum field theory given above, knots and links can be incorporated as well, so that one obtains a source of invariants $I\left(M^{3}, K\right)$ of knots and links in orientable three-manifolds.


Figure 20 - Decomposition of a Surface into Trinions
The invariant $I\left(M^{3}, K\right)$ can be formally compared with the Witten integral

$$
Z\left(M^{3}, K\right)=\int D A e^{(i k / 4 \pi) S(M, A)} W_{K}(A)
$$

It can be shown that up to limits of the heuristics, $Z(M, K)$ and $I\left(M^{3}, K\right)$ are essentially equivalent for appropriate choice of gauge groups.

This point of view, leads to more abstract formulations of topological quantum field theories as ways to associate vector spaces and linear transformations to manifolds and cobordisms of manifolds. (A cobordism of surfaces is a three manifold whose boundary consists in these surfaces.)

As the reader can see, a three-dimensional $T Q F T$ is, at base, a highly simplified theory of point particle interactions in $2+1$ dimensional spacetime. It can be used to articulate invariants of knots and links and invariants of three manifolds. The reader interested in the $S U(2)$ case of this structure and its implications for invariants of knots and three manifolds can consult $[9,10,5]$. One expects that physical situations involving $2+1$ spacetime will be approximated by such an idealized theory. It is thought for example, that aspects of the quantum Hall effect will be related to topological quantum field theory [18]. One can imagine a physics where the geometrical space is two dimensional and the braiding of particles corresponds to their interactions through circulating around one another in the plane. Anyons are particles that do not just change their wave-functions by a sign under interchange, but rather by a complex phase or even a linear combination of states. It is hoped that TQFT models will describe applicable physics. One can think about the possible applications of anyons to quantum computing. The $T Q F T^{\prime} s$ then provide a class of anyonic models where the braiding is essential to the physics and to the quantum computation.


Figure 21 - A More Complex Braiding Operator

The key point in the application and relationship of $T Q F T$ and quantum information theory is, in our opinion, contained in the structure illustrated in Figure 21. There we show a more complex braiding operator, based on the composition of recoupling with the elementary braiding at a vertex. (This structure is implicit in the Hexagon identity of Figure 19.) The new braiding operator is a source of unitary representations of braid group in situations (which exist mathematically) where the recoupling transformations are themselves unitary. This kind of pattern is utilized in the work of Freedman and collaborators [6] and in the case of classical angular momentum formalism has been dubbed a "spin-network quantum simlator" by Rasetti and collaborators [15]. In [14] we show how certain natural deformations [9] of Penrose spin networks [16] can be used to produce such the Freedman-Kitaev model for anyonic topological quantum computation. It is legitimate to speculate that networks of this kind are present in physical reality.

Quantum computing can be regarded as a study of the structure of the preparation, evolution and measurement of quantum systems. In the quantum computation model, an evolution is a composition of unitary transformations (usually finite dimensional over the complex numbers). The unitary transformations are applied to an initial state vector that has been prepared prior to this process. Measurements are projections to elements of an orthonormal basis of the space upon which the evolution is applied. The result of measuring a state $|\psi\rangle$, written in the given basis, is probabilistic. The probability of obtaining a given basis element from the measurement is equal to the absolute square of the coefficient of that basis element in the state being measured.

It is remarkable that the above lines constitute an essential summary of quantum theory. All applications of quantum theory involve filling in details of unitary evolutions and specifics of preparations and measurements. Such unitary evolutions can be seen as approximated arbitrarily closely by representations of the Artin braid group. The key to the anyonic models of quantum computation via topological quantum field theory, or via deformed spin networks is that all unitary evolutions can be approximated by a single coherent method for producing representations of the braid group. This beautiful mathematical fact points to a deep role for topology in the structure of quantum physics.

The future of knots, links and braids in relation to physics will be very exciting. There is no question that unitary representations of the braid group and quantum invariants of knots and links play a fundamental role in the mathematical structure of quantum mechanics, and we hope that time will show us the full meaning of this relationship.

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