Packing, Curvature, and Tangling

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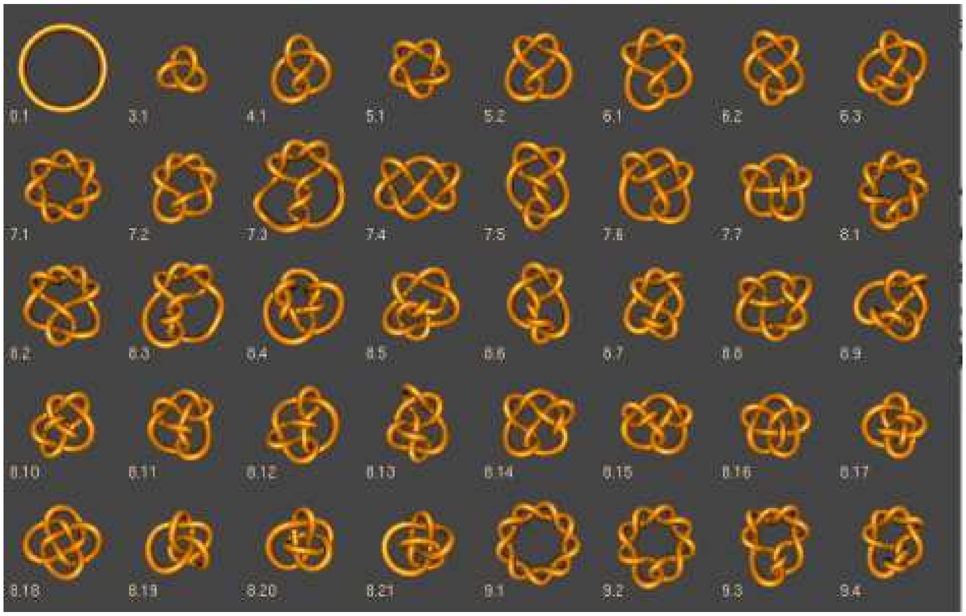

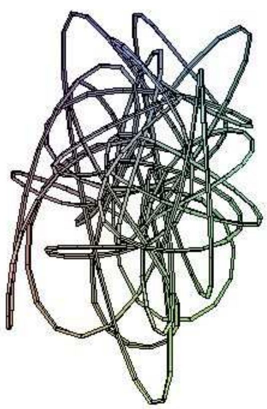

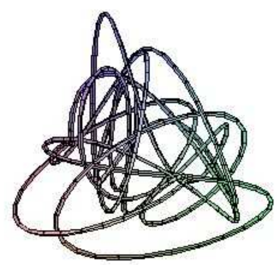

Figure 1: "Classical" knot theory seeks perfect classification of knots. What can we say about knots that are very complicated? What kind of knots can be produced by a particular physical system?

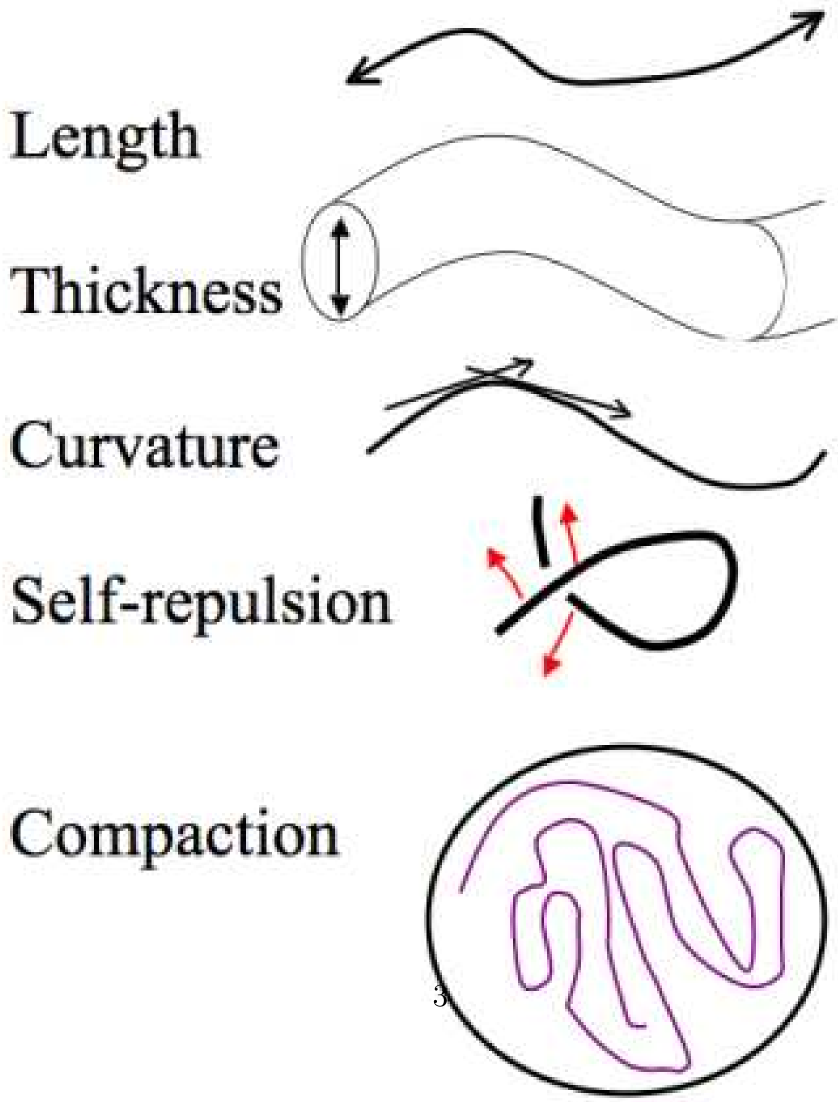

Figure 2: Study geometric (≈ physical) properties.

- How are these geometric properties of curves related to each other?
- How are they related to topological complexity or other measures of entanglement?

Need to define *rope-length* of a curve.

Definition 1. *K smooth knot in $\mathbb{R}^3$.*
· *Consider disks of radius r normal to K, centered at points of K.*
· *For r small, disks are pairwise disjoint and form a tubular neighborhood of K.*
· *Define $R(K)$,* the thickness radius *of K, = sup of such "good" radii.*
The ropelength *of K is*

$$E_L(K) = \frac{\text{total arclength of } K}{R(K)} \ .$$

Theorem 1 (Earlier result)**.** *If K is a smooth knot in $\mathbb{R}^3$,*
· *R = thickness radius,*
· *L = arclength of K, then*

crossing number of $K \leq$ (constant) $\times (\frac{L}{R})^{4/3}$.

(short thick rope $\implies$ simple knots)

Theorem 2 (Today's topic). *If K is a smooth knot in $\mathbb{R}^3$,*
· R = thickness radius,
· L = arclength of K, and
· κ = total curvature of K, then

crossing number of $K \leq \; 4 \, \frac{L}{R} \, \kappa$.

Similar proofs $\implies$
writhe, Möbius Energy, Normal Energy, and Symmetric Energy each is bounded by (constant)× (rope-length)×(total curvature).

Understanding the questions ...

Consider how crossing number can grow with rope-length.

We can use rope to see knots and links
with bounded total curvature
but large crossing numbers.

The theorem says (if total curvature is bounded), the crossing numbers can only grow in proportion to the ropelength.

There are families knots in which the crossing numbers grow as fast as the (4/3) power of $\frac{L}{R}$.

Our theorem says that such families must have unbounded total curvature: If the total curvature is bounded, then the rate of growth of crossings with ropelength can only be linear.

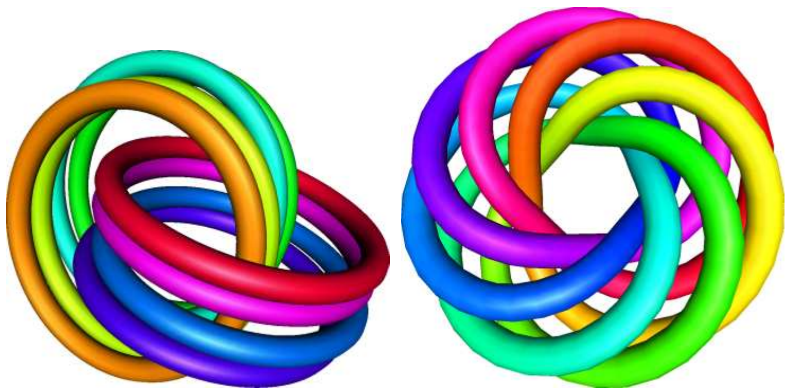

Figure 3: Knots and links whose crossing numbers grow like the (4/3) power of ropelength.

PLAN: Show crossing number $\leq$ "packing of the curve" $\leq$ curvature.

(For RHS) Find properties of a curve that will *force* it to have a lot of curvature.

The total curvature of smooth closed curve in $\mathbb{R}^3$ must be at least 2π (Fenchel).

Nontrivial knot $\implies$ (Fary-Milnor) total curvature $> 4\pi$.

Successive composition or other satellite constructions $\implies$ bridge number $\implies$ curvature.

But topological complexity alone (as high crossing-number) is not enough.

Lemmas: Packing of a curve $\implies$ lots of curvature:

Lemma 2.1 (Packing $\implies$ curvature)**.** *If a long smooth curve A with arclength L is contained in a solid ball of radius ρ, then*
total curvature $\geq$ proportional to L/ρ.

Lemma 2.2 (Oscillating $\implies$ curvature)**.** *If A connects concentric spheres of radii $a \geq 2$ and $b \geq a + 1$, by running from the inner sphere to the outer sphere and back again, then*
total curvature $\geq$ proportional to $1/\sqrt{a}$.

Meanwhile (for LHS) , bound crossing number in terms of how the curve is packed in space.

Define the *average crossing number* of K.

Definition 2. *Let K be a smooth knot. From almost every direction, if we project K into a plane, the projection is regular, in particular there are only finitely many crossings. We can average this crossing-number over all directions of projection (i.e. over the almost-all set of directions that give regular projections). This* average crossing number *is denoted* acn(K).

Certainly, the minimum crossing-number of the knot-type, cr[K], satisfies cr[K] $\leq$ acn(K).

The average crossing number of a knot can be [FHW] expressed as an integral over the knot, similar to Gauss's double integral formula for the linking number of two loops. Specifically,

$$\mathrm{acn(K)} = \frac{1}{4\pi}\int_{\mathrm{x}\in\mathrm{K}}\int_{\mathrm{y}\in\mathrm{K}} \frac{|<\mathrm{T_x}\,,\mathrm{T_y}\,,\mathrm{x}-\mathrm{y}>|}{|\mathrm{x}-\mathrm{y}|^3} \quad ,$$

where T_x, T_y are the unit tangents at x, y and $< u, v, w >$ is the triple scalar product $(u \times v) \cdot w$ of the three vectors u, v, w.

We need one more lemma, perhaps the most complicated to prove.

In this next lemma, call the curve Y instead of A, to help clarify how the lemmas will be used later: We will prove this lemma by applying the previous lemmas to subarcs A of Y.

Suppose Y is a smooth curve in $\mathbb{R}^3$, and x_0 is a point some finite distance from Y. The integral

$$\int_{y \in Y} \frac{1}{|y - x_0|^2} \tag{1}$$

can be thought of as measuring the "illumination" of x_0 by Y.

Lemma 2.3 (Illumination and curvature)**.** *Suppose Y is a smooth curve in $\mathbb{R}^3$, and x_0 is a point such that $\forall y \in Y$, $|y - x_0| \geq 2$. Then the illumination of x_0 by Y is bounded by the total curvature of Y.*

More precisely,

$$\int_{y \in Y} \frac{1}{|y - x_0|^2} \leq c_1 + c_2\, \kappa(Y)\ , \tag{2}$$

where c_1, c_2 are universal constants independent of Y.

The "illumination lemma" is perhaps the most complicated part of the proof. Before proving it, here are four special cases. The general argument does not reduce to these special cases – the examples to give an intuitive sense of why the proposition might be true (the first four), and some of the issues one needs to confront in building a proof (the fifth).

A spiral to show the lemma is sharp

Let Y be the polar coordinates curve $r = 3 - 1/\theta$, $\theta = 1 \ldots \Theta$. As Θ increases, the illumnination (of $x_0 =$ the origin) is aymptotic to $\frac{1}{3}\kappa(Y)$.

Y is a ray

Suppose Y is a straight line, starting at a point 2 units from x_0 and aiming radially away from x_0. Then the line integral is just $\int_2^\infty 1/s^2\, ds = 1/2$. □

Y is a straight line

Suppose Y is a straight line, infinite in both directions, and tangent to the sphere of radius 2 centered at x_0. Then

$$\int_{y \in Y} \frac{1}{|y - x_0|^2} = \int_{-\infty}^{\infty} \frac{1}{4 + s^2}\, ds \quad = \quad \frac{\pi}{2}\,.$$

If the line is a finite segment, or the minimum distance from Y to x_0 is > 2, then the integral is $< \pi/2$. □

Y is a certain kind of polygon

Suppose Y' is a polygonal path (or closed curve) consisting of e edges (of possibly varying lengths), such that each pair of consecutive edges meets at a right angle. Form a smooth curve Y by replacing the corners of Y' with small quarter-circles. Then, by the second special case, each edge of Y contributes $< \pi/2$ to the illumination integral, so

$$\int_{y\in Y} \frac{1}{|y-x_0|^2} < e\,\frac{\pi}{2} = \kappa(Y) \quad \text{has endpoints}$$

$$\kappa(Y) + \pi/2 \quad \text{closed.}$$

Y is a monotone arc

Suppose Y is a smooth curve, starting at a point y_0 with $|y_0 - x_0| = 2$, with the property that the distance function $|y - x_0|$ is monotone increasing on Y.

For $n = 2, 3, \ldots$, let $B[n]$ denote the round ball of radius n centered at x_0, and let $S[n, n+1]$ denote the spherical shell with radii n and $n+1$. By our assumption of monotonicity, each intersection $Y \cap S[n, n+2]$ is a connected arc, which we denote Y_n. Then

$$\int_{y\in Y} \frac{1}{|y-x_0|^2} = \sum_{n=2}^{\infty} \int_{y\in Y_n} \frac{1}{|y-x_0|^2}$$

$$\leq \sum_{n=2}^{\infty} \frac{\ell(Y \cap S[n, n+2])}{n^2}\ .$$

We would like to bound each of the numbers $\ell(Y \cap S[n, n+2])$, in terms of total curvature of Y, somehow using the packing lemma.

That lemma gives upper bounds for the lengths $\ell(Y \cap B[n])$ in terms of total curvature, but doesn't explicitly bound the amounts in given shells. We get around this problem by bounding (not the illumination integral from Y itself, but rather) the illumination integral for a hypothetical curve Y^* that is packed around x_0 in such a way as to make the illumination integral as large as possible subject to the constraints. (In this intuitive discussion of the special case, we will continue with the image of a "hypothetical curve". In the actual proof of Lemma 2.3, we will be more rigorous.)

Outline proof of Theorem

Rescale the knot so the thickness radius
$R(K) = 1$. This has no effect on the total curvature or on the average crossing number, and simplifies the ratio $E_L(K)$ to just the length, L. We want to show

$$\mathrm{acn(K)} \leq \mathrm{c} \cdot \mathrm{L} \cdot \kappa(\mathrm{K}) \,, \tag{3}$$

where c is some coefficient that works for all knots.

Omit the $\frac{1}{4\pi}$ (the first contribution to "c") and write the integral as a sum of two terms:

$$\mathrm{Near}(K) = \int_{x\in K} \int_{\mathrm{arc(x,y)} \leq \pi} \frac{| < T_x \, , T_y \, , x - y > |}{|x-y|^3} \ ,$$

and

$$\mathrm{Far}(K) = \int_{x\in K} \int_{\mathrm{arc(x,y)} \geq \pi} \frac{| < T_x \, , T_y \, , x - y > |}{|x-y|^3} \ .$$

Analyze these contributions separately, and find bounds of the form

$$\mathrm{Near}(K) \leq b_1 \ L$$

$$\mathrm{Far}(K) \leq c_1\ L + c_2\ L\ \kappa(K)\ .$$

where the coefficients are independent of K.

In each case, we bound the inner integral and then multiply by L to bound the double integral.

Combining Near and Far, we get a bound for any smooth curve K of the form

$$\mathrm{acn}(K) \leq aL + bL\kappa(K).$$

But if K is a *closed* curve, then by Fenchel's theorem, $\kappa(K) \geq 2\pi$. Thus letting $c = b + \frac{a}{2\pi}$, we have

$$\mathrm{acn}(K) \leq c\ L\ \kappa(K)\ .$$

□

Bounding Near(K)

We shall show that the inner integral is uniformly bounded, independent of K.

For any smooth curve with thickness radius R, it is shown in LSDR that the curvature at each point is at most $1/R$. So in the present situation, we know that the curvature of K is everywhere ≤ 1.

Let $\theta \to x(\theta)$ be a unit speed parametrization of K. So $x'(\theta) = T_x$ and $|x''(\theta)| \leq 1$. We are studying points y for which $\mathrm{arc}(x, y) \leq \pi$, so we can take for the parameter set the interval $[0, \pi]$, with our starting point $x = x(0)$ and $y = y(\theta)$ for some $\theta \in [0, \pi]$. Using the same parameter set, let $\theta \to p(\theta)$ be an arclength preserving parametrization of the unit semi-circle. Since the curvature of K is everywhere bounded by the curvature of the unit circle, Schur's theorem tells us that for each θ,

$$|x(\theta) - x(0)| \geq |p(\theta) - p(0)| \ , \tag{4}$$

That is,

$$|y - x| \geq \sqrt{2 - 2\cos\theta} \ . \tag{5}$$

Thus

$$\frac{| < T_x \, , T_y \, , x - y > |}{|x - y|^3} \leq \frac{| < T_x \, , T_y \, , \frac{x-y}{|x-y|} > |}{2 - 2\cos\theta} \tag{6}$$

$$= \frac{| < T_x \, , T_y \, , \frac{x-y}{\theta} > |}{2 - 2\cos\theta} \ \frac{\theta}{|x - y|} \tag{7}$$

Using Schur's theorem again, we have $|x-y| \geq |p(\theta)-p(0)| = \sqrt{2 - 2\cos\theta}$. The function $\frac{\theta}{\sqrt{2-2\cos\theta}}$ is increasing on $[0, \pi]$, with maximum value $\pi/2$. So

$$\frac{| < T_x \, , T_y \, , x - y > |}{|x - y|^3} \leq \frac{\pi}{2} \ \frac{| < T_x \, , T_y \, , \frac{x-y}{\theta} > |}{2 - 2\cos\theta} \ . \tag{8}$$

The vectors T_y and $\frac{x-y}{\theta}$ are each first-order (in terms of θ) close to T_x. Specifically, we have for T_y,

$$T_y = T_x + \int_{t=0}^{\theta} x''(t) \ . \tag{9}$$

Since $|x''| \leq 1$, this says we can write T_y as $T_x + V$, where $|V| \leq \theta$. Meanwhile, (**??**) says we can write $\frac{|x-y|}{\theta}$ as $T_x + W$, where $|W| \leq \frac{1}{2}\theta$. Thus

$$T_x \times T_y = T_x \times V \ ,$$

a vector perpendicular to T_x with length $\leq \theta$. When we take the dot product of this vector with $T_x + W$, we just get the dot product with W, so a number whose size is at most $\frac{1}{2}\theta^2$.

We now have

$$\frac{| < T_x\, , T_y\, , x-y > |}{|x-y|^3} \leq \frac{\pi}{4} \ \frac{\theta^2}{2 - 2\cos\theta} \ \leq \frac{\pi}{4}\left(\frac{\pi}{2}\right)^2 \ , \tag{10}$$

so the inner integral is bounded by $b_1 = (2\pi) \cdot (\frac{\pi}{4}) \cdot (\frac{\pi}{2})^2$, since the points y run from (what we might denote as) $x - \pi$ to $x + \pi$.

Multiply this bound for the inner integral by L to bound the double integral. □

Bounding Far(K)

As in the previous case, we bound the inner integral,

$$\int_{\text{arc(x,y)} \geq \pi} \frac{| < T_x\, , T_y\, , x-y > |}{|x-y|^3} \ ,$$

then multiply by L to bound the double integral.

Write the integrand as the triple scalar product of three unit vectors, divided by $|x-y|^2$. Since the numerator has magnitude at most 1, it suffices to bound

$$\int_{\text{arc(x,y)} \geq \pi} \frac{1}{|x-y|^2} \ .$$

For any smooth curve with thickness radius R, it is shown in LSDR that points x, y with $\operatorname{arc}(x,y) \geq \pi R$ must have $|x-y| \geq 2R$. So in our situation, when $\operatorname{arc}(x,y) \geq \pi$, we know $|x-y| \geq 2$.

Fix x and let $Y = \{y \in K \mid \operatorname{arc}(x,y) \geq \pi\}$.

By the illumination lemma,

$$\int_Y \frac{1}{|y-x|^2} \leq c_1 + c_2\kappa(Y) \leq c_1 + c_2\kappa(X)\ .$$

Thus

$$\operatorname{Far}(K) \leq c_1 L + c_2 \kappa(X) L\ .$$

□
