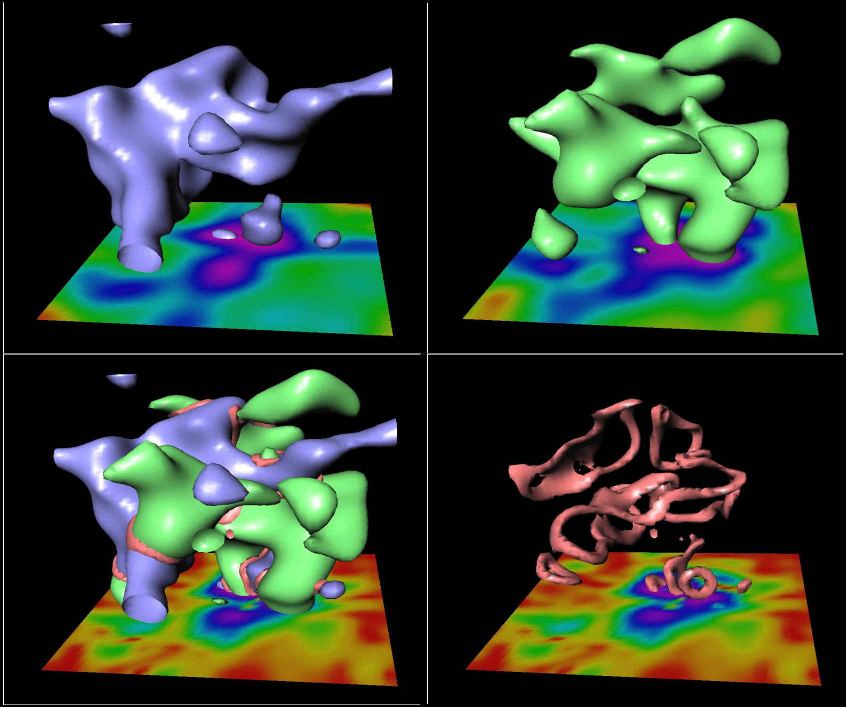

Random Fields and Random Geometry

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I: Gaussian fields and Kac-Rice formulae

Robert Adler
Electrical Engineering
Technion – Israel Institute of Technology

.

and

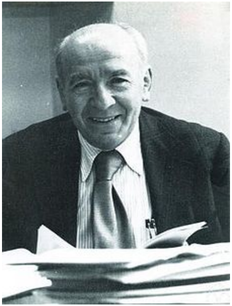
many, many others

October 25, 2011

I do not intend to cover all these slides in 75 minutes!

(Some of the material is for your later reference, and some for the afternoon tutorial.)

Our heroes



Marc Kac
1914–1984

Stephen O. Rice
1907–1986

Real roots of algebraic equations (Kac, 1943)

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Theorem: N_n = the number of real zeroes of f

$$\mathbb{E}\{N_n\} \;=\; \frac{4}{\pi}\int_0^1 \frac{[1 - n^2[x^2(1-x^2)/(1-x^{2n}]^2]^{1/2}}{1-x^2}\, dx$$

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Bound: For large n

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Zeroes of complex polynomials

$$f(z) \; = \; \xi_0 + a_1\xi_1 z + a_2\xi_2 z^2 + \cdots + a_{n-1}\xi_{n-1} z^{n-1}, \qquad z \in \mathbb{C}.$$

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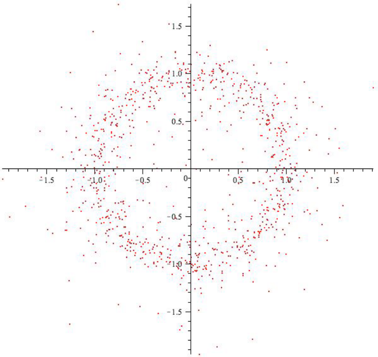

Zeros of 60 Hammersley random polynomials of degree 15

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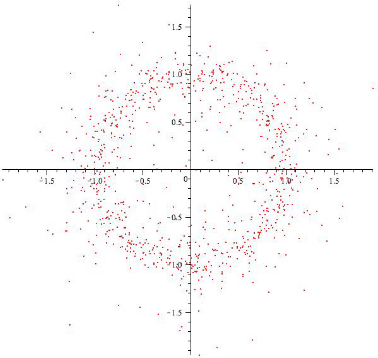

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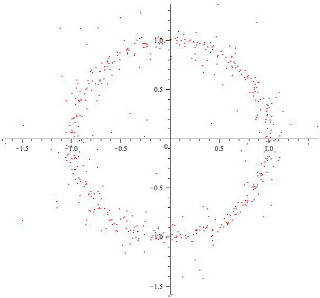

Zeros of 10 Hammersley random polynomials of degree 50

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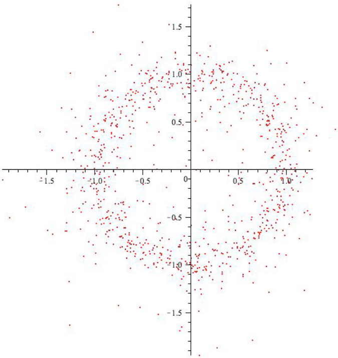

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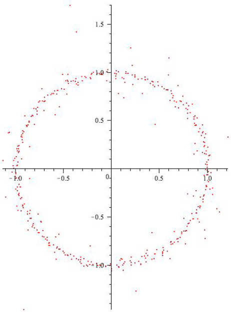
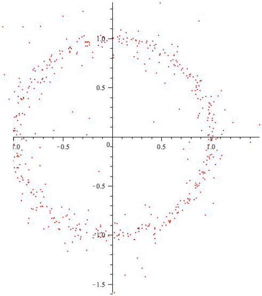


Zeros of 4 Hammersley random polynomials of degree 100



Zeros of 10 Hammersley random polynomials of degree 50

Shiffman

Thinking more generally:

1: $f : \mathbb{R}^N \to \mathbb{R}^N$, random

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4: More generally:

$$f:\ M \to N, \quad \dim(M) \neq \dim(N), \quad D \subset N$$

In this case, typically,

$$\dim\left(f^{-1}(D)\right) \ = \ \dim(M) - \dim(N) + \dim(D),$$

and it is not clear what the corresponding question is.

The original (non-specific) Rice formula

- $U_u \equiv U_u(f, T) \stackrel{\Delta}{=} \#\{t \in T : f(t) = u, \dot{f}(t) > 0\}$

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\mathbb{E}\{U_u\} &= \int_T \int_0^\infty y\, p_t(u,y)\, dydt \\
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where $p_t(x,y)$ is the joint density of $(f(t), \dot{f}(t))$,
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- $T = [0,T]$ is an interval. M is a general set (e.g. manifold)

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- Take a (positive) approximate delta function, δ_ε, supported on $[-\varepsilon, +\varepsilon]$, and $\int_{\mathbb{R}} \delta_\varepsilon(x)\, dx = 1$.

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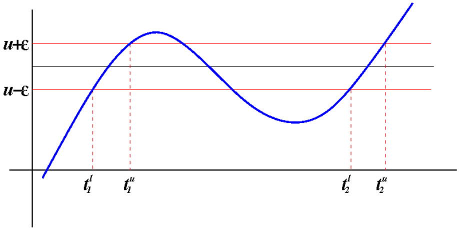

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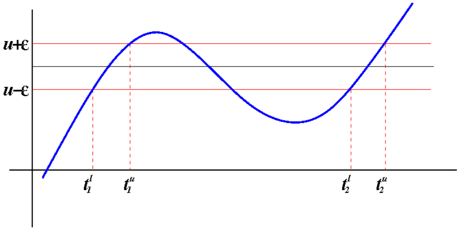

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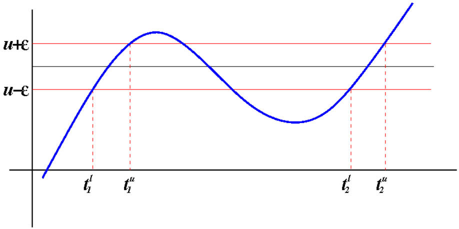

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- Take expectations, with some sleight of hand

$$\begin{aligned}
\mathbb{E}\{U_u(f,T)\} &= \mathbb{E}\left\{\lim_{\varepsilon\to 0}\int_T |\dot f(t)|\,\delta_\varepsilon(f(t)-u)\,1_{(0,\infty)}(f(t))\,dt\right\} \\
&= \int_T \lim_{\varepsilon\to 0}\mathbb{E}\left\{|\dot f(t)|\,\delta_\varepsilon(f(t)-u)\,1_{(0,\infty)}(f(t))\,dt\right\} \\
&= \int_T \lim_{\varepsilon\to 0}\int_{x=-\infty}^{\infty}\int_{y=0}^{\infty} |y|\,\delta_\varepsilon(x-u)\,p_t(x,y)\,dxdy\,dt \\
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- Mean, covariance, and variance functions

$$\begin{aligned}
\mu(t) &\stackrel{\Delta}{=} \mathbb{E}\{f(t)\} &&= \; 0 \\
C(s,t) &\stackrel{\Delta}{=} \{[f(s) - \mu(s)] \cdot [f(t) - \mu(t)]\} &&= \; \sum \varphi_j(s)\varphi_j(t) \\
\sigma^2(t) &\stackrel{\Delta}{=} C(t,t) &&= \; \sum \varphi_j^2(t)
\end{aligned}$$

Existence of Gaussian processes

Theorem

- Let M be a topological space.
- Let $C : M \times M$ be positive semi-definite.
- Then there exists a Gaussian process on $f : M \to \mathbb{R}$ with mean zero and covariance function C.
- Furthermore, f has a representation of the form $f(t) = \sum_j \xi_j \varphi_j(t)$, and if f is a.s. continuous then the sum converges uniformly, a.s., on compacts.

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Corollary

- If there is justice in the world (smoothness and summability)

$$\dot{f}(t) = \frac{\partial}{\partial t} f(t) = \sum_j \xi_j \dot{\varphi}_j(t),$$

and so if f is Gaussian, so is $\dot{f}$.

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- $\mathbb{E}\{\dot{f}(s)\dot{f}(t)\} \; = \; \mathbb{E}\{ \sum \xi_j \dot{\varphi}_j(s) \sum \xi_k \dot{\varphi}_k(t) \} \; = \; \sum \dot{\varphi}_j(s)\dot{\varphi}_j(t)$
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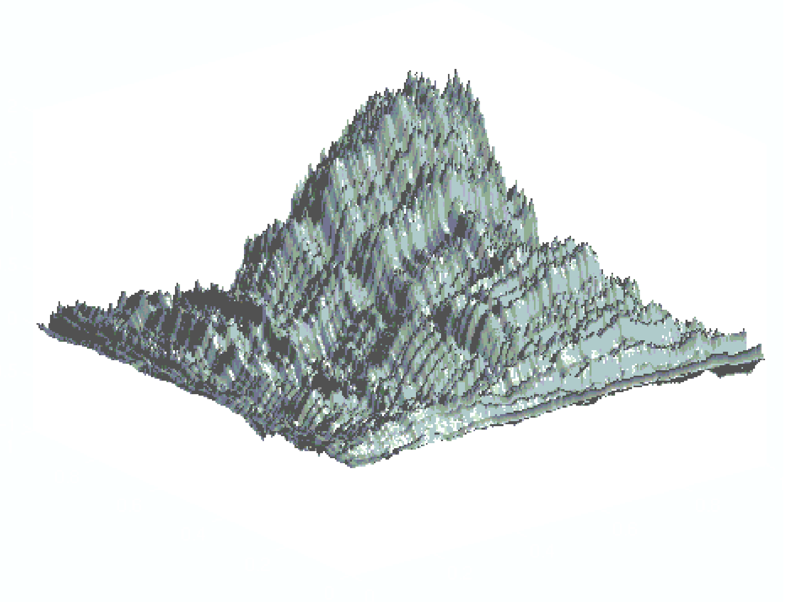
Example: The Brownian sheet on $\mathbb{R}^N$



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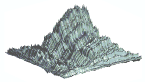

$$\begin{aligned} \mathbb{E}\{W(t)\} &= 0 \\ \mathbb{E}\{W(s)W(t)\} &= (s_1 \wedge t_1) \times \cdots \times (s_N \wedge t_N). \end{aligned}$$

where $t = (t_1, \ldots, t_N)$.

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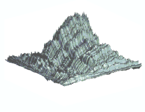

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where $t = (t_1, \ldots, t_N)$.

- Replace j by $j = (j_1, \ldots, j_N)$

$$
\varphi_j(t) = 2^{N/2} \prod_{i=1}^{N} \frac{2}{(2j_i + 1)\pi} \sin\left(\tfrac{1}{2}\,(2j_i + 1)\,\pi t_i \right)
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$\Rightarrow$

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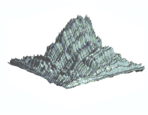

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- $N = 1$

  W is standard Brownian motion.

  The corresponding expansion is due to Lévy, and the corresponding RKHS is known as Cameron-Martin space.

Constant variance Gaussian processes

- We know that
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&= \tfrac{1}{2}\sum \frac{\partial}{\partial t}\varphi_j^2(t) \\
&= \tfrac{1}{2}\frac{\partial}{\partial t}\sum \varphi_j^2(t) \\
&= 0
\end{aligned}$$

Constant variance Gaussian processes

- We know that
  - $f(t) \;=\; \sum \xi_j \varphi_j(t)$
  - $\dot{f}(t) \;=\; \sum \xi_j \dot{\varphi}_j(t)$
  - $\mathbb{E}\{\dot{f}(s)\dot{f}(t)\} \;=\; \sum \dot{\varphi}_j(s)\dot{\varphi}_j(t)$
  - $\mathbb{E}\{f(s)\dot{f}(t)\} \;=\; \sum \varphi_j(s)\dot{\varphi}_j(t)$
- Thus, if

$$\sigma^2(t) \;=\; \mathbb{E}\{f^2(t)\} \;=\; \sum \varphi_j^2(t)$$

  is a constant, say 1, so that $\|\{\varphi_j(t)\}\|_{\ell_2} = 1$
- then

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\end{aligned}$$

- $\Rightarrow f(t)$ and its derivative $\dot{f}(t)$ are INDEPENDENT. (uncorrelated)

Constant variance Gaussian processes and Kac-Rice

- Generic Kac-Rice formula

$$\mathbb{E}\{U_u\} = \int_T p_t(u)\, \mathbb{E}\left\{|\dot{f}(t)|\, 1_{(0,\infty)}(\dot{f}(t)) \;\middle|\; f(t) = u\right\} dt$$

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- Notation

$$1 = \sigma^2(t), \qquad \lambda(t) \triangleq \mathbb{E}\{[\dot f(t)]^2\}$$

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- If $\lambda(t) \equiv \lambda$, we have the Rice formula

$$\mathbb{E}\{U_u\} \;=\; \frac{T\,\lambda^{1/2}}{2\pi}\, e^{-u^2/2}$$

Gaussian Kac-Rice with no simplification

$$\mathbb{E}\left\{U_u(T)\right\} = \int_T \lambda^{1/2}(t)\sigma^{-1}(t)[1-\mu^2(t)]^{1/2}\varphi\left(\frac{m(t)}{\sigma(t)}\right) \times \left[2\varphi(\eta(t)) + 2\eta(t)[2\Phi(\eta(t))-1]\right] dt$$

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m(t) &= \mathbb{E}\{f(t)\} \\
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\mu(t) &= \frac{\mathbb{E}\{[f(t)-m(t)]\cdot[\dot{f}(t)]\}}{\lambda^{1/2}(t)\sigma(t)} \\
\eta(t) &= \frac{\dot{m}(t) - \lambda^{1/2}(t)\mu(t)m(t)/\sigma(t)}{\lambda^{1/2}(t)[1-\mu^2(t)]^{1/2}}
\end{aligned}$$

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- Very important fact: Long term covariances do not appear in any of these formulae.

Some pertinent thoughts

- Real roots of Gaussian polynomials The original Kac result now makes sense:

$$
\begin{aligned}
f(t) &= \xi_0 + \xi_1 t + \cdots + \xi_{n-1} t^{n-1} \\
\mathbb{E}\{N_n\} &= \frac{4}{\pi} \int_0^1 \frac{[1 - n^2[x^2(1-x^2)/(1-x^{2n}]^2]^{1/2}}{1-x^2}\, dx
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- Downcrossings, crossings, critical points
  Critical points of different kinds are just zeroes of $\dot{f}$ with different side conditions. But now second derivatives appear, calculations will be harder. ($f(t)$, $f''(t)$ are not independent.)

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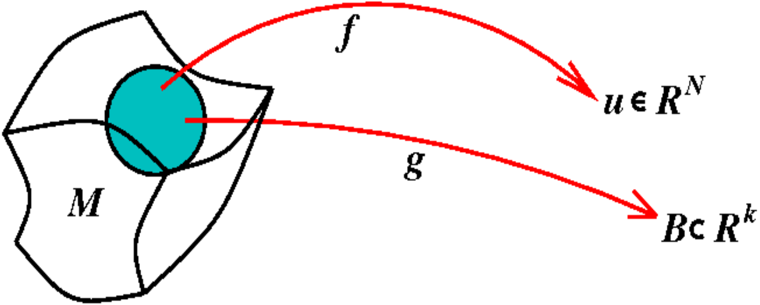
- Higher moments?
- Fixed points of vector valued processes?

The Kac-Rice "Metatheorem"



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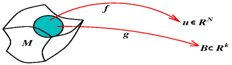

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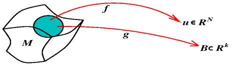

- The setup

$$\begin{aligned} f &= (f^1, \dots, f^N) \ : \ M \subset \mathbb{R}^N \to \mathbb{R}^N \\ g &= (g^1, \dots, g^K) \ : \ M \subset \mathbb{R}^N \to \mathbb{R}^K \end{aligned}$$

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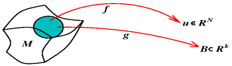

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- Number of points:

$$
\begin{aligned}
N_u \equiv N_u(M) &\equiv N_u(f, g : M, B) \\
&\triangleq \mathbb{E}\{\# \{t \in M : \; f(t) = u, \; g(t) \in B\}\}.
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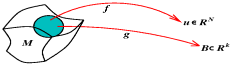

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- The "metatheorem", or generalised Kac-Rice

$$\begin{aligned} \mathbb{E}\{N_u\} &= \int_M \int_{\mathbb{R}^D} |\det \nabla y| \ 1_B(v) \, p_t(u, \nabla y, v) \, d(\nabla y) \, dv \, dt \\ &= \int_M \mathbb{E}\left\{ |\det \nabla f(t)| \ 1_B(g(t)) \,\Big|\, f(t) = u \right\} p_t(u) \, dt, \end{aligned}$$

$p_t(x, \nabla y, v)$ is the joint density of $(f_t, \nabla f_t, g_t)$

$$(\nabla f)(t) \ \equiv \ \nabla f(t) \equiv \left(f^i_j(t)\right)_{i,j=1,\ldots,N} \ \equiv \ \left(\frac{\partial f^i(t)}{\partial t_j}\right)_{i,j=1,\ldots,N}.$$

The original (non-specific) Rice formula: The proof

- Take a (positive) approximate delta function, δ_ε, supported on $[-\varepsilon, +\varepsilon]$, and $\int_{\mathbb{R}} \delta_\varepsilon(x)\, dx = 1$.

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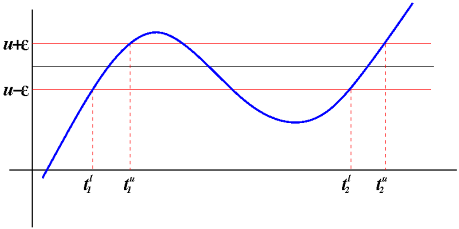

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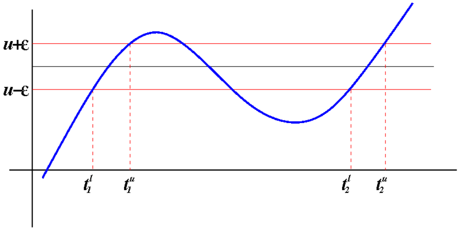

$$
\begin{aligned}
1 = \int_{\mathbb{R}} \delta_\varepsilon(x-u)\, dx &= \int_{t_1^\ell}^{t_1^u} |\dot f(t)|\, \delta_\varepsilon(f(t)-u)\, 1_{(0,\infty)}(f(t))\, dt \\
&= \lim_{\varepsilon \to 0} \int_{t_1^\ell}^{t_1^u} |\dot f(t)|\, \delta_\varepsilon(f(t)-u)\, 1_{(0,\infty)}(f(t))\, dt
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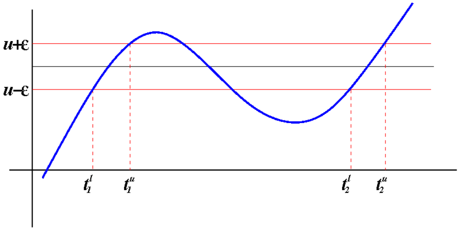

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$$

$$
U_u(f,T) \;=\; \lim_{\varepsilon \to 0} \int_T |\dot f(t)|\, \delta_\varepsilon(f(t)-u)\, 1_{(0,\infty)}(f(t))\, dt
$$

The Kac-Rice Conditions (the fine print)

.

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Let f, g, M and B be as above, with the additional assumption that the boundaries of M and B have finite $N-1$ and $K-1$ dimensional measures, respectively. Furthermore, assume that the following conditions are satisfied for some $u \in \mathbb{R}^N$:

(a) All components of f, ∇f, and g are a.s. continuous and have finite variances (over M).

(b) For all $t \in M$, the marginal densities $p_t(x)$ of $f(t)$ (implicitly assumed to exist) are continuous at $x = u$.

(c) The conditional densities $p_t(x|\nabla f(t), g(t))$ of $f(t)$ given $g(t)$ and $\nabla f(t)$ (implicitly assumed to exist) are bounded above and continuous at $x = u$, uniformly in $t \in M$.

(d) The conditional densities $p_t(z|f(t) = x)$ of $\det \nabla f(t)$ given $f(t) = x$, are continuous for z and x in neighbourhoods of 0 and u, respectively, uniformly in $t \in M$.

(e) The conditional densities $p_t(z|f(t) = x)$ of $g(t)$ given $f(t) = x$, are continuous for all z and for x in a neighbourhood u, uniformly in $t \in M$.

(f) The following moment condition holds:

$$\sup_{t\in M} \max_{1\le i,j\le N} \mathbb{E}\left\{\left|f^i_j(t)\right|^N\right\} < \infty.$$

(g) The moduli of continuity of each of the components of f, ∇f, and g satisfy

$$\mathbb{P}\{\,\omega(\eta) > \varepsilon\,\} = o\left(\eta^N\right), \qquad \text{as } \eta \downarrow 0,$$

for any $\varepsilon > 0$.

Higher (factorial) moments

- Factorial notation

$$(x)_k \triangleq x(x-1)\dots(x-k+1).$$

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- Kac-Rice (again?)

$$\begin{aligned}\mathbb{E}\{(N_u)_k\} &= \int_{M^k} \mathbb{E}\left\{ \prod_{j=1}^{k} |\det \nabla f(t_j)|\, 1_B(g(t_j)) \middle| \widetilde{f}(\widetilde{t}) = \tilde{u} \right\} p_{\widetilde{t}}(\tilde{u}) d\widetilde{t} \\ &= \int_{M^k} \int_{\mathbb{R}^{kD}} \prod_{j=1}^{k} |\det D_j|\; 1_B(v_j)\, p_{\widetilde{t}}(\tilde{u}, \tilde{D}, \tilde{v})\, d\tilde{D}\, d\tilde{v}\;\; d\widetilde{t},\end{aligned}$$

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\end{aligned}$$

$$\begin{aligned}
M^k &= \{\widetilde{t} = (t_1, \dots, t_k) : \; t_j \in M, \; 1 \le j \le k\} \\
\widetilde{f}(\widetilde{t}) &= (f(t_1), \dots, f(t_k)) : \; M^k \to \mathbb{R}^{Nk} \\
\widetilde{g}(\widetilde{t}) &= (g(t_1), \dots, g(t_k)) : \; M^k \to \mathbb{R}^{Kk} \\
D &= N(N+1)/2 + K
\end{aligned}$$

The Gaussian case: What can/can't be explicitly computed

- General mean and covariance functions

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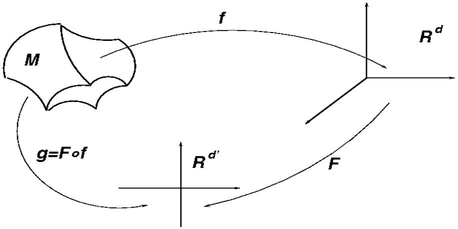
- $\mathbb{E}\left\{ \sum_{k=1}^{N}(-1)^k(\text{No. of critical points of index } k \text{ above } u)\right\}$

  Useful for Morse theory

The Gaussian-related case



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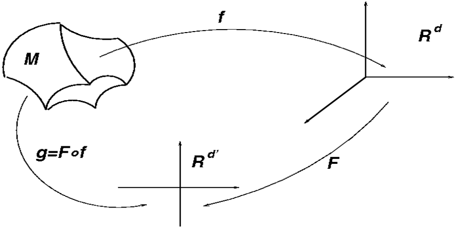

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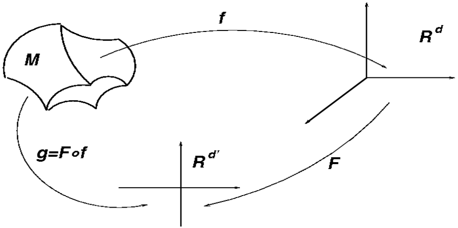

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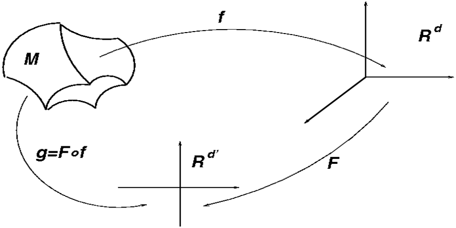
The Gaussian-related case



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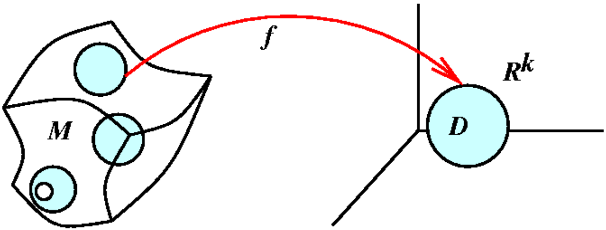
i.e. χ^2 fields with k degrees of freedom, T field with $k-1$ degrees of freedom, F field with n and m degrees of freedom.

The Gaussian Kinematic Formula (GKF)



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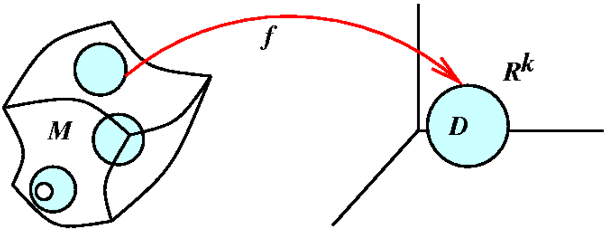

Jonathan's lecture

The perturbed-Gaussian case

- A physics approach

$$\varphi(x) \quad = \quad \varphi_G(x)\left[1 + \sum_{n=3}^{\infty} \mathrm{Tr}\left[\mathbb{E}_G\{h_n(X)\} \cdot h_n(x)\right]\right]$$

φ_G is iid Gaussian

$$h_n(x) \;\triangleq\; (-1)^n \frac{1}{\varphi_G(x)} \frac{\partial^n \varphi_G(x)}{\partial x^n}$$

are Hermite tensors of rank n with coefficients constructed from the moments $\mathbb{E}_G\{h_n(X)\}$

The perturbed-Gaussian case

- A physics approach

$$\varphi(x) \;=\; \varphi_G(x)\left[1+\sum_{n=3}^{\infty}\mathrm{Tr}\left[\mathbb{E}_G\{h_n(X)\}\cdot h_n(x)\right]\right]$$

φ_G is iid Gaussian

$$h_n(x) \;\triangleq\; (-1)^n \frac{1}{\varphi_G(x)}\frac{\partial^n \varphi_G(x)}{\partial x^n}$$

are Hermite tensors of rank n with coefficients constructed from the moments $\mathbb{E}_G\{h_n(X)\}$

- A statistical (Gaussian related) approach

$$f(t) \;=\; f_G(t) \;+\; \sum_{j=1}^{J} p_j \varepsilon_j f_j^{GR}(t)$$

Applications I: Exceedence probabilities via level crossings

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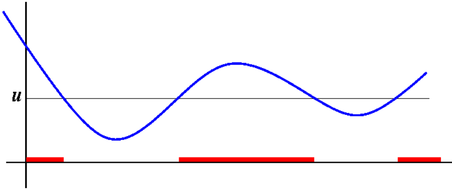

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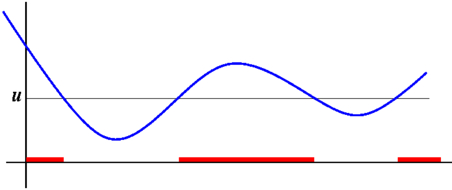

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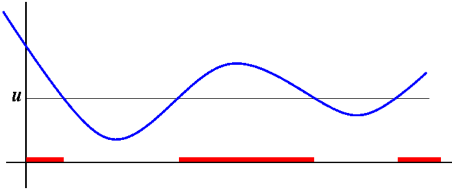

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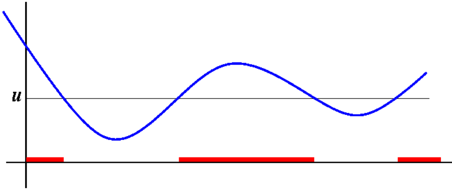

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- Note: Nothing is Gaussian here!
- Inequality is usually an approximation, for large u.

Applications II: Local maxima on the line

- Number of local maxima above the level u

$$M_u(T) = \#\left\{t \in [0, T] : \dot{f}(t) = 0,\ \ddot{f}(t) < 0,\ f(t) \geq u\right\}$$

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which holds in very wide generality.

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Λ the covariance matrix of first order derivatives of f
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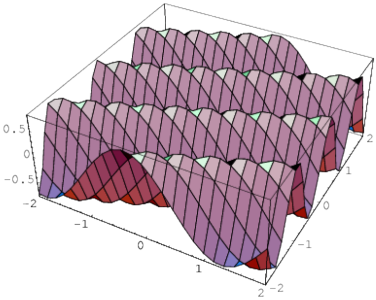
Applications IV: Longuet-Higgins and oceanography



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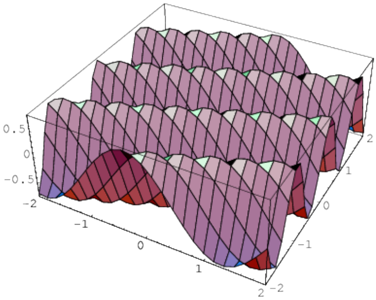

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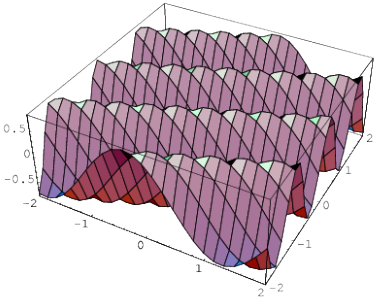

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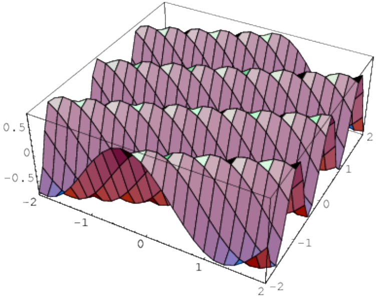

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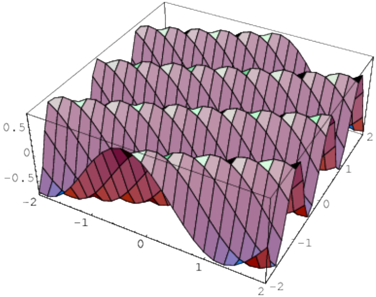

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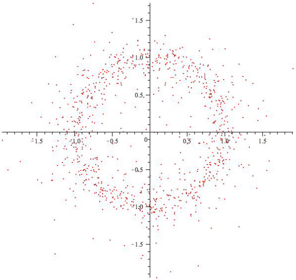
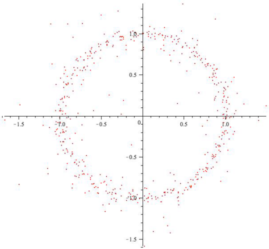
Applications V: Higher moments and complex polynomials

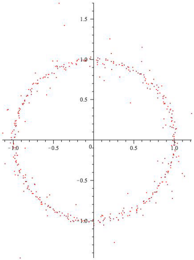
$$f(z) \;=\; \xi_0 + a_1\xi_1 z + a_2\xi_2 z^2 + \cdots + a_{n-1}\xi_{n-1} z^{n-1}, \qquad z \in \mathbb{C}.$$



Zeros of 60 Hammersley random polynomials of degree 15



Zeros of 10 Hammersley random polynomials of degree 50



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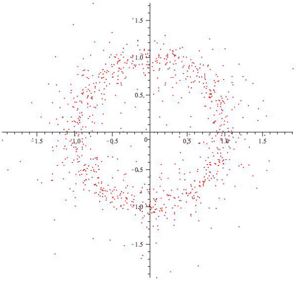

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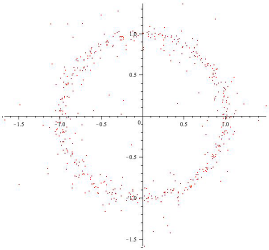

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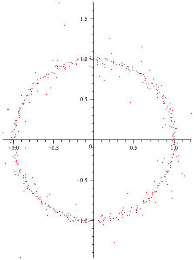

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- Means tell us where we expect the roots to be, but variances are needed to give concentration information.

Applications VI: Poisson limits and Slepian models

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Theorem

1: Sequences of increasingly rare events such as the existence of high level local maxima in N dimensions or level crossings in 1 dimension, looked at over long time periods or large regions so that a few of them still occur have an asymptotic Poisson distribution as long as dependence in time or space is not too strong.

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B is any ball

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- A a $n \times n$ matrix

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- (Some) random matrix problems are equivalent to random field problems, and vice versa

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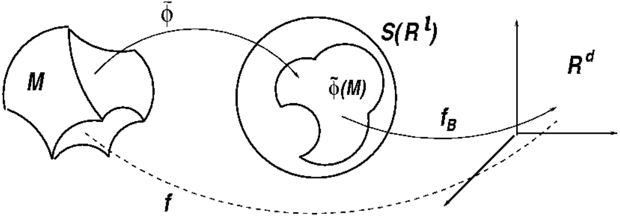

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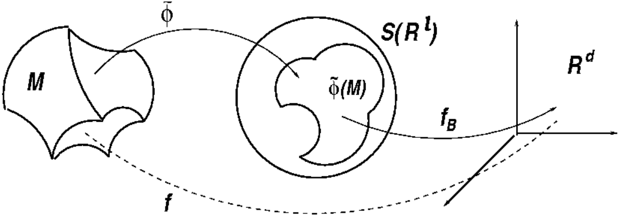

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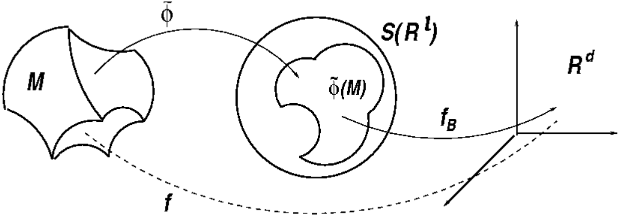

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\mathbb{E}\left\{\widetilde{f}(x)\widetilde{f}(y)\right\} &= \mathbb{E}\left\{f\left(\widetilde{\varphi}^{-1}(x)\right) f\left(\widetilde{\varphi}^{-1}(y)\right)\right\} \\
&= \sum \varphi_j\left(\widetilde{\varphi}^{-1}(x)\right) \varphi_j\left(\widetilde{\varphi}^{-1}(y)\right) \\
&= \sum x_j y_j \;=\; \langle x, y\rangle
\end{aligned}$$

The canonical Gaussian process on $S^{\ell-1}$

1: Has mean zero and covariance

$$\mathbb{E}\{f(s)f(s)\} = \langle s, t \rangle$$

for $s, t \in S^{\ell-1}$.

2: It can be realised as

$$f(t) = \sum_{j=1}^{\ell} t_j \xi_j.$$

3: It is stationary and isotropic since the covariance is function of only the (geodesic) distance between s and t.

Exceedence probabilities for canonical process: $M \subset S^{\ell-1}$

$$
\begin{aligned}
\mathbb{P}\Big\{ \sup_{t\in M} f_t \geq u \Big\} &= \int_0^\infty \mathbb{P}\Big\{ \sup_{t\in M} f_t \geq u \;\Big|\; |\xi| = r \Big\} \, \mathbb{P}_{|\xi|}(dr) \\
&= \int_0^\infty \mathbb{P}\Big\{ \sup_{t\in M} \langle \xi, t \rangle \geq u \;\Big|\; |\xi| = r \Big\} \, \mathbb{P}_{|\xi|}(dr) \\
&= \int_u^\infty \mathbb{P}\Big\{ \sup_{t\in M} \langle \xi, t \rangle \geq u \;\Big|\; |\xi| = r \Big\} \, \mathbb{P}_{|\xi|}(dr) \\
&= \int_u^\infty \mathbb{P}\Big\{ \sup_{t\in M} \langle \xi/r, t \rangle \geq u/r \;\Big|\; |\xi| = r \Big\} \, \mathbb{P}_{|\xi|}(dr) \\
&= \int_u^\infty \mathbb{P}\Big\{ \sup_{t\in M} \langle U, t \rangle \geq u/r \Big\} \, \mathbb{P}_{|\xi|}(dr)
\end{aligned}
$$

where U is uniform on $S^{\ell-1}$.

- We need

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- Working with tubes
  The tube of radius ρ around a closed set $M \in S^{\ell-1}$) is

$$\begin{aligned}
\mathrm{Tube}(M,\rho) &= \Big\{t\in S^{\ell-1} : \tau(t,M)\le\rho\Big\} \\
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- And so....

$$\mathbb{P}\Big\{ \sup_{t\in M} f_t \ge u \Big\} \quad = \quad \int_u^\infty \eta_l \left(\mathrm{Tube}(M, \cos^{-1}(u/r))\right) \mathbb{P}_{|\xi|}(dr)$$

and geometry has entered the picture, in a serious fashion!

Appendix II: Stationary and isotropic fields

- Definition: M has a group structure, $\mu(t) = const$ and $C(s, t) = C(s - t)$.

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- Spectral moments

$$\lambda_{i_1 \dots i_N} \triangleq \int_{\mathbb{R}^N} \lambda_1^{i_1} \cdots \lambda_N^{i_N}\, \nu(d\lambda)$$

  ν is symmetric $\Rightarrow$ odd ordered spectral moments are zero.

- Elementary considerations give

$$\mathbb{E}\left\{\frac{\partial^k f(s)}{\partial s_{i_1}\partial s_{i_1}\ldots\partial s_{i_k}}\;\frac{\partial^k f(t)}{\partial t_{i_1}\partial t_{i_1}\ldots\partial t_{i_k}}\right\} = \frac{\partial^{2k} C(s,t)}{\partial s_{i_1}\partial t_{i_1}\ldots\partial s_{i_k}\partial t_{i_k}}.$$

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- When f is stationary, and $\alpha, \beta, \gamma, \delta \in \{0, 1, 2, \dots\}$, then

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- Isotropy $(C(t) = C(\|t\|)$ $\Rightarrow \nu$ is spherically symmetric $\Rightarrow$

$$\mathbb{E}\{f_i(t) f_j(t)\} \;=\; -\mathbb{E}\{f(t) f_{ij}(t)\} \;=\; \lambda \delta_{ij}$$

Appendix III: Regularity of Gaussian processes

- The *canonical metric, d*

$$d(s,t) \stackrel{\Delta}{=} \left[\mathbb{E}\left\{\left(f(s) - f(t)\right)^2\right\}\right]^{\frac{1}{2}},$$

A ball of radius ε and centered at $t \in M$ is denoted by

$$B_d(t,\varepsilon) \stackrel{\Delta}{=} \{s \in M : \ d(s,t) \le \varepsilon\}\,.$$

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- Entropy Fix $\varepsilon > 0$ and let $N(M,d,\varepsilon) \equiv N(\varepsilon)$ denote the smallest number of d-balls of radius ε whose union covers M. Set

$$H(M,d,\varepsilon) \equiv H(\varepsilon) = \ln\left(N(\varepsilon)\right).$$

Then N and H are called the (metric) *entropy* and *log-entropy* functions for M (or f).

Dudley's theorem

Let f be a centered Gaussian field on a d-compact M Then there exists a universal K such that

$$\mathbb{E}\Big\{\sup_{t\in M} f_t\Big\} \;\le\; K\int_0^{\mathrm{diam}(M)} H^{1/2}(\varepsilon)\,d\varepsilon,$$

and

$$\mathbb{E}\left\{\omega_{f,d}(\delta)\right\} \le K\int_0^{\delta} H^{1/2}(\varepsilon)\,d\varepsilon,$$

where

$$\omega_{f,d}(\delta) \stackrel{\Delta}{=} \sup_{d(s,t)\le\delta} \left|f(t)-f(s)\right|, \quad \delta>0,$$

Furthermore, there exists a random $\eta\in(0,\infty)$ and a universal K such that

$$\omega_{f,d}(\delta) \;\le K\int_0^{\delta} H^{1/2}(\varepsilon)\,d\varepsilon,$$

for all $\delta<\eta$.

Special cases of the entropy result

- If f is also stationary

$$\begin{aligned} f \text{ is a.s. continuous on } M \quad &\Longleftrightarrow \quad f \text{ is a.s. bounded on } M \\ &\Longleftrightarrow \quad \int_0^\delta H^{1/2}(\varepsilon)\, d\varepsilon \ < \ \infty, \quad \forall \delta > 0 \end{aligned}$$

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- If $M \subset \mathbb{R}^N$, and

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p^2(u) \;\stackrel{\Delta}{=}\; \sup_{|s-t|\le u} \mathbb{E}\left\{|f_s - f_t|^2\right\},
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continuity & boundedness follow if, for some $\delta > 0$, either

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\int_0^\delta (-\ln u)^{\frac{1}{2}}\, dp(u) < \infty \quad \text{or} \quad \int_\delta^\infty p\left(e^{-u^2}\right)\, du < \infty.
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- A sufficient condition For some $0 < K < \infty$ and $\alpha, \eta > 0$,

$$\mathbb{E}\left\{|f_s - f_t|^2\right\} \leq \frac{K}{|\log|s-t||^{1+\alpha}},$$

for all s, t with $|s - t| < \eta$.

Appendix IV: Borell-Tsirelson inequality

- Finiteness theorem: $\|f\| \stackrel{\Delta}{=} \sup_{t\in M} f_t$

$$\mathbb{P}\{\|f\| < \infty\} = 1 \iff \mathbb{E}\{\|f\|\} < \infty,$$

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$$\mathbb{P}\{\|f\| - \mathbb{E}\{\|f\|\} > u\} \leq e^{-u^2/2\sigma_M^2}.$$

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- Asymptotics: For high levels u, the dominant behavior of *all* Gaussian exceedence probabilities is determined by $e^{-u^2/2\sigma_M^2}$.

Places to start reading and to find other references

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