

**A BRIEF INTRODUCTION TO HILBERT SPACE FRAME
THEORY AND ITS APPLICATIONS
AMS SHORT COURSE: JOINT MATHEMATICS MEETINGS
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ABSTRACT. This is a short introduction to Hilbert space frame theory and its applications for those outside the area who want to enter the subject. We will emphasize frame theory in \mathbb{R}^N since it is the easiest way to get into the subject. The sections on **The Basics of Hilbert Space Theory** and **The Basics of Operator Theory** are a review and we will assume the participants have read these sections before arriving.

1. READING LIST

For a more complete treatment of frame theory we recommend the books of Han/Kornelson/Larson/Weber [58], Christensen [41], the book of Casazza and Kutyniok [32], the tutorials of Casazza [23, 24] and the Memoir of Han and Larson [59]. For a complete treatment of frame theory in time-frequency analysis we recommend the book of Gröchenig [54]. For an introduction to frame theory and filter banks plus applications to engineering we recommend Kovačević and Chebira [63]. Also, a wealth of information can be found at the **Frame Research Center** website:

<http://www.framerc.org/>

2. THE BASICS OF HILBERT SPACE THEORY

Given a positive integer N , we denote by \mathbb{H}^N the real or complex Hilbert space of dimension N . This is either \mathbb{R}^N or \mathbb{C}^N with the inner product given by: for $x = (a_1, a_2, \dots, a_N)$ and $y = (b_1, b_2, \dots, b_N)$

$$\langle x, y \rangle = \sum_{i=1}^N a_i \bar{b}_i.$$

The **norm of x** is

$$\|x\|^2 = \langle x, x \rangle.$$

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For $x, y \in \mathbb{H}^N$, $\|x - y\|$ is the distance from the vector x to the vector y . For future reference, note that in the real case,

$$\begin{aligned}\|x - y\|^2 &= \langle x - y, x - y \rangle \\ &= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 - 2\langle x, y \rangle + \|y\|^2.\end{aligned}$$

In the complex case:

$$\langle x, y \rangle + \langle y, x \rangle = \langle x, y \rangle + \overline{\langle x, y \rangle} = 2\operatorname{Re} \langle x, y \rangle,$$

where $\operatorname{Re} c$ denotes the real part of the complex number c . We will concentrate on finite dimensional Hilbert spaces since it is the easiest way to get started on the subject of frames. Most of these results hold for infinite dimensional Hilbert spaces and at the end we will look at the infinite dimensional case.

The next lemma contains a **standard trick** for calculations.

Lemma 2.1. *If $x \in \mathbb{H}^N$ and $\langle x, y \rangle = 0$ for all $y \in \mathbb{H}^N$ then $x = 0$.*

Proof. Letting $y = x$ we have

$$0 = \langle x, y \rangle = \langle x, x \rangle = \|x\|^2,$$

and so $x = 0$. □

Definition 2.2. *A set of vectors $\{e_i\}_{i=1}^M$ in \mathbb{H}^N is called:*

(1) **linearly independent** if for any scalars $\{a_i\}_{i=1}^M$,

$$\sum_{i=1}^M a_i e_i = 0 \quad \Rightarrow \quad a_i = 0, \text{ for all } i = 1, 2, \dots, M.$$

Note that this requires $e_i \neq 0$ for all $i = 1, 2, \dots, M$.

(2) **complete** (or a **spanning set**) if

$$\overline{\operatorname{span}}\{e_i\}_{i=1}^M = \mathbb{H}^N.$$

(3) **orthogonal** if for all $i \neq j$, $\langle e_i, e_j \rangle = 0$.

(4) **orthonormal** if it is orthogonal and unit norm.

(5) an **orthonormal basis** if it is complete and orthonormal.

The following is immediate from the definitions.

Proposition 2.3. *If $\{e_i\}_{i=1}^N$ is an orthonormal basis for \mathbb{H}^N , then for every $x \in \mathbb{H}$ we have*

$$x = \sum_{i=1}^N \langle x, e_i \rangle e_i.$$

The following is an immediate deduction from the previous proposition.

Proposition 2.4 (Parseval's Identity). *If $\{e_i\}_{i=1}^N$ is an orthonormal basis for \mathbb{H}^N , then for every $x \in \mathbb{H}^N$, we have*

$$\|x\|^2 = \sum_{i=1}^N |\langle x, e_i \rangle|^2.$$

The basic identities for Hilbert space are contained in the next proposition.

Proposition 2.5. *Let $x, y \in \mathbb{H}^N$.*

(1) **Cauchy-Schwarz Inequality:**

$$|\langle x, y \rangle| \leq \|x\| \|y\|,$$

with equality if and only if $x = cy$ for some constant c .

(2) **Triangle Inequality:**

$$\|x + y\| \leq \|x\| + \|y\|.$$

(3) **Polarization Identity:** *Assuming \mathbb{H}^N is real,*

$$\langle x, y \rangle = \frac{1}{4} [\|x + y\|^2 - \|x - y\|^2].$$

(4) **Pythagorean Theorem:** *Given pairwise orthogonal vectors $\{x_i\}_{i=1}^M$,*

$$\left\| \sum_{i=1}^M x_i \right\|^2 = \sum_{i=1}^M \|x_i\|^2.$$

Proof. (1) The inequality is trivial if $y = 0$. If $y \neq 0$, by dividing through the inequality by $\|y\|$, we may assume $\|y\| = 1$. Now we compute:

$$\begin{aligned} 0 &< \|x - \langle x, y \rangle y\|^2 \\ &= \|x\|^2 - 2\langle x, y \rangle \langle x, y \rangle + |\langle x, y \rangle|^2 \|y\|^2 \\ &= \|x\|^2 - |\langle x, y \rangle|^2 \\ &= \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2. \end{aligned}$$

Note that the first inequality would be equality if $x = cy$.

(2) Applying (1) at the appropriate step:

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + 2\operatorname{Re} \langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

(3) We compute assuming \mathbb{H}^N is a real Hilbert space:

$$\begin{aligned} \|x + y\|^2 - \|x - y\|^2 &= \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 - (\|x\|^2 - 2\langle x, y \rangle + \|y\|^2) \\ &= 4\langle x, y \rangle. \end{aligned}$$

(4) Since $\langle x_i, x_j \rangle = 0$ for all $i \neq j$, we have

$$\begin{aligned} \left\| \sum_{i=1}^M x_i \right\|^2 &= \left\langle \sum_{i=1}^M x_i, \sum_{j=1}^M x_j \right\rangle \\ &= \sum_{i,j=1}^M \langle x_i, x_j \rangle \\ &= \sum_{i=1}^M \langle x_i, x_i \rangle \\ &= \sum_{i=1}^M \|x_i\|^2. \end{aligned} \quad \square$$

We now look at subspaces of the Hilbert space.

Definition 2.6. Let W, V be subspaces of \mathbb{H}^N .

(1) A vector $x \in \mathbb{H}^N$ is **orthogonal** to W , denoted $x \perp W$ if

$$\langle x, y \rangle = 0 \text{ for all } y \in W.$$

The **orthogonal complement** of W is

$$W^\perp = \{x \in \mathbb{H} : x \perp W\}.$$

(2) The subspaces W, V are **orthogonal**, denoted $W \perp V$ if $W \subset V^\perp$, that is,

$$\langle x, y \rangle = 0 \text{ for all } x \in W, y \in V.$$

A simple calculation shows that $W^{\perp\perp} = W$. Fundamental to Hilbert space theory are **orthogonal projections** as defined next.

Definition 2.7. An operator $P : \mathbb{H}^N \rightarrow \mathbb{H}^N$ is called a **projection** if $P^2 = P$. It is an **orthogonal projection** if P is also self-adjoint.

For any subspace $W \subset \mathbb{H}^N$, there is an orthogonal projection of \mathbb{H} onto W called the **nearest point projection**. One way to define it is to pick any orthonormal basis $\{e_i\}_{i=1}^K$ for W and define

$$Px = \sum_{i=1}^K \langle x, e_i \rangle e_i.$$

Note that for all $j = 1, 2, \dots, K$ we have

$$\langle Px, e_j \rangle = \left\langle \sum_{i=1}^K \langle x, e_i \rangle e_i, e_j \right\rangle = \sum_{i=1}^K \langle x, e_i \rangle \langle e_i, e_j \rangle = \langle x, e_j \rangle.$$

We need to check that this operator is well defined, that is, we must show it is independent of the choice of basis.

Lemma 2.8. *If $\{e_i\}_{i=1}^K$ and $\{g_i\}_{i=1}^K$ are orthonormal bases for W , then*

$$\sum_{i=1}^K \langle x, e_i \rangle e_i = \sum_{i=1}^K \langle x, g_i \rangle g_i.$$

Proof. We compute:

$$\begin{aligned} \sum_{i=1}^K \langle x, e_i \rangle e_i &= \sum_{i=1}^K \left\langle x, \sum_{j=1}^K \langle e_i, g_j \rangle g_j \right\rangle e_i \\ &= \left(\sum_{i,j=1}^K \overline{\langle e_i, g_j \rangle} \langle x, g_j \rangle \right) e_i \\ &= \sum_{i=1}^K \left\langle \sum_{j=1}^K \langle x, g_j \rangle g_j, e_i \right\rangle e_i \\ &= \sum_{j=1}^K \langle x, g_j \rangle g_j. \quad \square \end{aligned}$$

In this case, $Id - P$ is also an orthogonal projection onto W^\perp . This projection maps each vector in \mathbb{H}^N onto the unique nearest vector to x in W . In particular, $\|Px\| \leq \|x\|$ for all $x \in \mathbb{H}$.

Proposition 2.9. *Let P be an orthogonal projection onto a subspace W . Then*

$$\|x - Px\| \leq \|x - y\|, \text{ for all } y \in W.$$

Proof. Choose an orthonormal basis $\{e_i\}_{i=1}^K$ for W . Then

$$\begin{aligned} \operatorname{Re} \langle x, y \rangle &\leq |\langle x, y \rangle| \\ &= \left| \sum_{i=1}^K \langle x, e_i \rangle \langle y, e_i \rangle \right| \\ &= \left| \sum_{i=1}^K \langle Px, e_i \rangle \langle y, e_i \rangle \right| \\ &\leq \left(\sum_{i=1}^K |\langle Px, e_i \rangle|^2 \right)^{1/2} \left(\sum_{i=1}^K |\langle y, e_i \rangle|^2 \right)^{1/2} \\ &\leq \|Px\| \|y\| \\ &\leq \frac{1}{2} (\|Px\|^2 + \|y\|^2) \end{aligned}$$

Therefore,

$$\begin{aligned}
\|x - y\|^2 &= \|x\|^2 + \|y\|^2 - 2\operatorname{Re} \langle x, y \rangle \\
&\geq \|x\|^2 + \|y\|^2 - (\|Px\|^2 + \|y\|^2) \\
&= \|x\|^2 - \|Px\|^2 \\
&= \|x\|^2 + \|Px\|^2 - 2\|Px\|^2 \\
&= \|x\|^2 + \|Px\|^2 - 2\langle Px, Px \rangle \\
&= \|x\|^2 + \|Px\|^2 - 2\langle Px, x \rangle \\
&= \|x - Px\|^2. \quad \square
\end{aligned}$$

Another important concept is taking orthogonal sums of subspaces of a Hilbert space.

Definition 2.10. If $\{W_i\}_{i \in I}$ (the index set I is allowed to be infinite) are subspaces of a Hilbert space \mathbb{H}^N , their **orthogonal direct sum** is

$$\left(\sum_{i \in I} \oplus W_i \right)_{\ell_2} = \{(x_i)_{i \in I} : x_i \in W_i, \}$$

and inner product defined by

$$\langle (x_i)_{i \in I}, (y_i)_{i \in I} \rangle = \sum_{i \in I} \langle x_i, y_i \rangle.$$

It follows that

$$\|(x_i)_{i \in I}\|^2 = \sum_{i \in I} \|x_i\|^2.$$

3. THE BASICS OF OPERATOR THEORY

Definition 3.1. A linear operator $T : \mathbb{H}^N \rightarrow \mathbb{H}^K$ between Hilbert spaces \mathbb{H}^N and \mathbb{H}^K satisfies:

$$T(ax + by) = aT(x) + bT(y) \text{ for all } x, y \in \mathbb{H}^N \text{ and scalars } a, b.$$

The **norm** of the operator is

$$\begin{aligned}
\|T\| &= \sup_{\|x\|=1} \|Tx\| \\
&= \sup_{\|x\| \leq 1} \|Tx\| \\
&= \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.
\end{aligned}$$

Therefore, for all $x \in \mathbb{H}^N$ we have: $\|Tx\| \leq \|T\|\|x\|$. It follows that if $T : \mathbb{H}^{N_1} \rightarrow \mathbb{H}^{N_2}$ and $S : \mathbb{H}^{N_2} \rightarrow \mathbb{H}^{N_3}$ then

$$\|STx\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\|$$

and therefore

$$\|ST\| \leq \|S\|\|T\|.$$

A useful alternative method for calculating $\|T\|$ is contained in the following lemma:

Lemma 3.2. *If $T : \mathbb{H}^N \rightarrow \mathbb{H}^K$ is an operator, then*

$$\|T\| = \sup\{|\langle Tx, y \rangle| : \|x\| = \|y\| = 1\}.$$

Proof. We compute for $\|x\| = 1 = \|y\|$,

$$|\langle Tx, y \rangle| \leq \|Tx\|\|y\| \leq \|T\|\|x\| = \|T\|.$$

Hence,

$$\|T\| \geq \sup\{|\langle Tx, y \rangle| : \|x\| = \|y\| = 1\}.$$

Conversely,

$$\begin{aligned} & \sup\{|\langle Tx, y \rangle| : \|x\| = \|y\| = 1\} \\ & \geq \sup\left\{\left|\left\langle Tx, \frac{Tx}{\|Tx\|} \right\rangle\right| : \|x\| = 1 \text{ and } Tx \neq 0\right\} \\ & = \sup_{\|x\|=1} \frac{\|Tx\|^2}{\|Tx\|} \\ & = \sup_{\|x\|=1} \|Tx\| = \|T\|. \quad \square \end{aligned}$$

A basic calculation for identifying operators is contained in the following lemma.

Lemma 3.3. *If $S, T : \mathbb{H}^N \rightarrow \mathbb{H}^K$ are operators satisfying*

$$\langle Tx, y \rangle = 0, \text{ for all } x, y \in \mathbb{H}^N,$$

then $T = 0$. Hence, if

$$\langle Tx, y \rangle = \langle Sx, y \rangle \text{ for all } x, y \in \mathbb{H}^N,$$

then $S = T$.

Proof. Given $x \in \mathbb{H}^N$, by letting $y = Tx$ we obtain:

$$0 = \langle Tx, Tx \rangle = \|Tx\|^2 = 0,$$

and so $Tx = 0$ and $T = 0$. □

Linear operators on finite spaces can always be represented as a matrix, so we can work with linear operators and their matrices interchangeably.

Definition 3.4. Let $T : \mathbb{H}^N \rightarrow \mathbb{H}^K$ be a linear operator, let $\{e_i\}_{i=1}^N$ be an orthonormal basis for \mathbb{H}^N and let $\{g_j\}_{j=1}^K$ be an orthonormal basis for \mathbb{H}^K . The **matrix representation** of T (with respect to these orthonormal bases) is $T = [a_{ij}]_{1 \leq i \leq N, 1 \leq j \leq K}$ where

$$a_{ij} = \langle Te_i, g_j \rangle.$$

Definition 3.5. Let $T : \mathbb{H}^N \rightarrow \mathbb{H}^K$ be a linear operator.

(1) The **kernel of \mathbf{T}** is

$$\ker T = \{x \in \mathbb{H}^N : Tx = 0\}.$$

The **range of \mathbf{T}** (or **image of \mathbf{T}**) is

$$\text{ran } T = \{Tx : x \in \mathbb{H}^N\}.$$

The **rank of \mathbf{T}** , denoted $\text{rank } T$ is the dimension of the $\text{ran } T$. Notice that

$$N = \dim \ker T + \text{rank } T.$$

(2) T is **injective** if $\ker T = \{0\}$, and **surjective** if $\text{ran } T = \mathbb{H}^K$. It is **bijective** if it is both injective and surjective.

(3) The **adjoint operator** $T^* : \mathbb{H}^K \rightarrow \mathbb{H}^N$ is defined by:

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \text{ for all } x \in \mathbb{H}^N, y \in \mathbb{H}^K.$$

Note that $T^{**} = T$ and $(S + T)^* = S^* + T^*$.

(4) T is **bounded** if $\|T\| < \infty$.

(5) T is **invertible** if T is bounded, injective and onto and so has an inverse operator T^{-1} which is also bounded.

Moreover, the next lemma shows that if T is injective then T^*T is injective and thus if T is injective, then T^*T has an inverse.

There are important relationships between the kernel of T and the range of T^* and vice versa.

Lemma 3.6. Let $T : \mathbb{H}^N \rightarrow \mathbb{H}^K$. Then

- (1) $\ker T = [\text{ran } T^*]^\perp$.
- (2) $[\ker T]^\perp = \text{ran } T^*$.
- (3) $\ker T^* = [\text{ran } T]^\perp$.
- (4) $[\ker T^*]^\perp = \text{ran } T$.

Proof. (1) If $Tx = 0$ then for all $y \in \mathbb{H}^K$, then

$$\langle x, T^*y \rangle = \langle Tx, y \rangle = 0.$$

So

$$\ker T \subset [T^*(\mathbb{H}^K)]^\perp = [\text{ran } T^*]^\perp.$$

On the other hand, if $x \perp T^*(\mathbb{H}^K)$, then for all $y \in \mathbb{H}^K$

$$0 = \langle x, T^*y \rangle = \langle Tx, y \rangle.$$

That is, $Tx \perp \mathbb{H}^K$, and so $Tx = 0$ and hence

$$[T^*(\mathbb{H}^K)]^\perp \subset \ker T.$$

(2) Observe from (1) that:

$$[\ker T]^\perp = [T^*(\mathbb{H}^K)]^{\perp\perp} = T^*(\mathbb{H}^K) = \text{ran } T^*.$$

The relations (3) and (4) follow by replacing T by T^* in (1) and (2). \square

Lemma 3.7. *For operators $T : \mathbb{H}^{N_1} \rightarrow \mathbb{H}^{N_2}$ and $S : \mathbb{H}^{N_2} \rightarrow \mathbb{H}^{N_3}$, we have:*

- (1) $(ST)^* = T^*S^*$.
- (2) $\|T^*\| = \|T\|$.
- (3) $\|T^*T\| = \|T\|^2$.

Proof. (1) We compute:

$$\begin{aligned} \langle x, (ST)^*y \rangle &= \langle STx, y \rangle \\ &= \langle Tx, S^*y \rangle \\ &= \langle x, T^*S^*y \rangle. \end{aligned}$$

(2) We use Lemma 3.2.

$$\begin{aligned} \|T\| &= \sup\{|\langle Tx, y \rangle| : \|x\| = \|y\| = 1\} \\ &= \sup\{|\langle x, T^*y \rangle| : \|x\| = \|y\| = 1\} \\ &= \|T^*\|. \end{aligned}$$

(3) We know that $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$. On the other hand, for all $\|x\| = 1$ via Cauchy-Schwarz,

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle \leq \|T^*Tx\|\|x\| \leq \|T^*T\|\|x\| = \|T^*T\|.$$

Hence, $\|T\|^2 \leq \|T^*T\|$. \square

Definition 3.8. *Let $T : \mathbb{H}^N \rightarrow \mathbb{H}^K$ be an injective linear operator. The Moore-Penrose inverse of \mathbf{T} is T^\dagger where*

$$T^\dagger = (T^*T)^{-1}T^*.$$

If T is an injective linear operator then $T^\dagger T = I$.

Definition 3.9. *A linear operator $T : \mathbb{H}^N \rightarrow \mathbb{H}^K$ is called:*

- (1) **self-adjoint**, if $\mathbb{H}^N = \mathbb{H}^K$ and $T = T^*$.
- (2) **normal**, if $\mathbb{H}^N = \mathbb{H}^K$ and $T^*T = TT^*$.
- (3) **an isometry**, if $\|Tx\| = \|x\|$ for all $x \in \mathbb{H}^N$.
- (4) **a partial isometry** if T restricted to $[\ker T]^\perp$ is an isometry.

(5) **positive**, if $\mathbb{H}^N = \mathbb{H}^K$, T is self-adjoint, and

$$\langle Tx, x \rangle \geq 0 \text{ for all } x \in \mathbb{H}.$$

In this case we write $T \geq 0$.

(6) **unitary**, if $TT^* = T^*T = Id$.

It follows that TT^* and T^*T are self-adjoint for any operator T . A fundamental example of a self-adjoint, positive, invertible operator for us is to take vectors $\{\varphi_i\}_{i=1}^M$ in \mathbb{H}^N and define the operator:

$$Sx = \sum_{i=1}^M \langle x, \varphi_i \rangle \varphi_i.$$

In this case we have:

$$\langle Sx, x \rangle = \sum_{i=1}^M \langle x, \varphi_i \rangle \langle \varphi_i, x \rangle = \sum_{i=1}^M |\langle x, \varphi_i \rangle|^2.$$

Given positive operators S, T , $S + T$ is a positive operator but ST may not be. But we can define inequalities for positive operators:

Definition 3.10. If S, T are positive operators on \mathbb{H}^N , we write $S \leq T$ if $T - S \geq 0$.

Proposition 3.11. Let $T : \mathbb{H}^N \rightarrow \mathbb{H}^K$ be a linear operator.

- (1) If $\mathbb{H}^N = \mathbb{H}^K$, the following are equivalent:
 - (a) T is self-adjoint.
 - (b) $\langle Tx, y \rangle = \langle x, Ty \rangle$, for all $x, y \in \mathbb{H}^N$.
- (2) The following are equivalent:
 - (a) T is an isometry.
 - (b) $T^*T = Id$.
 - (c) $\langle Tx, Ty \rangle = \langle x, y \rangle$, for all $x, y \in \mathbb{H}^N$.
- (3) The following are equivalent:
 - (a) T is unitary.
 - (b) T and T^* are isometries.
 - (c) T is an isometry and T^* is injective.
 - (d) T is a surjective isometry.
 - (e) T is bijective and $T^{-1} = T^*$.
- (4) If U is any unitary operator, then $\|T\| = \|TU\| = \|UT\|$.

Proof. (1) We compute,

$$\langle Tx, y \rangle = \langle x, T^*y \rangle = \langle x, Ty \rangle,$$

for all x, y if and only if $T = T^*$.

(2) (a) \Rightarrow (b): By assumption, for any $x \in \mathbb{H}^N$, $\|Tx\| = \|x\|$. Now

$$\langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = \|Tx\|^2 = \|x\|^2 = \langle Ix, x \rangle.$$

Borrowing from the future theorem 3.20, if $\{e_i\}_{i=1}^N$ is an orthonormal basis of eigenvectors for T^*T with respective eigenvalues $\{\lambda_i\}_{i=1}^N$ then for all $j = 1, 2, \dots, N$

$$1 = \langle e_j, e_j \rangle = \langle T^*T e_j, e_j \rangle = \langle \lambda_j e_j, e_j \rangle = \lambda_j.$$

We conclude that $T^*T = Id$.

(b) \Rightarrow (c): For any $x, y \in \mathbb{H}^N$ we have

$$\langle Tx, Ty \rangle = \langle T^*Tx, y \rangle = \langle x, y \rangle.$$

(c) \Rightarrow (a): (c) implies,

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx \rangle \\ &= \langle x, x \rangle \\ &= \|x\|^2. \end{aligned}$$

(3) (a) \Leftrightarrow (b): This is immediate by the definitions and (2).

(b) \Rightarrow (c): By (2, b) T^* is injective.

(c) \Rightarrow (d): We observe:

$$\mathbb{H}^K = \{0\}^\perp = [\ker T^*]^\perp = T(\mathbb{H}^N).$$

(d) \Rightarrow (e): Since T is bijective by assumption, $S = T^{-1}$ exists. We compute using (2,b):

$$T^* = T^*I = T^*(TS) = (T^*T)S = IS = S.$$

(e) \Rightarrow (a): Immediate by the definitions.

(4) Since U is an isometry by (3), the result is now immediate from Lemma 3.2. \square

An operator $T : \mathbb{H}^N \rightarrow \mathbb{H}^K$ is a **Hilbert space isomorphism** if for all $x, y \in \mathbb{H}^N$ we have

$$\langle Tx, Ty \rangle = \langle x, y \rangle.$$

It follows that such a T is a unitary operator.

Proposition 3.12. *Every two N -dimensional Hilbert spaces are Hilbert space isomorphic.*

Proof. Let $\{e_i\}_{i=1}^N$ be an orthonormal basis of \mathbb{H}_1^N and let $\{g_i\}_{i=1}^N$ be an orthonormal basis of \mathbb{H}_2^N . The operator T given by: $Te_i = g_i$ for all $i = 1, 2, \dots, N$ is clearly a Hilbert space isomorphism. \square

Diagonalization is fundamental to frame theory.

Definition 3.13. *Let $T : \mathbb{H}^N \rightarrow \mathbb{H}^N$ be a linear operator. A nonzero vector $x \in \mathbb{H}^N$ is an **eigenvector** of T with **eigenvalue** λ if*

$$Tx = \lambda x.$$

*The operator T is **diagonalizable** if there exists an orthonormal basis for \mathbb{H}^N consisting of eigenvectors for T .*

A fundamental result is:

Proposition 3.14. *For any operator $T : \mathbb{H}^N \rightarrow \mathbb{H}^K$, then T^*T and TT^* have the same nonzero eigenvalues.*

Proof. If $T^*Tx = \lambda x$ with $\lambda \neq 0$ and $x \neq 0$, then $Tx \neq 0$ and

$$TT^*(Tx) = T(T^*Tx) = T(\lambda x) = \lambda Tx. \quad \square$$

We will see in the next section that T^*T may have zero eigenvalues even if TT^* does not have zero eigenvalues.

Further restrictions on T gives more information on the eigenvalues. The following result is a corollary of Proposition 3.11.

Corollary 3.15. *Let $T : \mathbb{H}^N \rightarrow \mathbb{H}^N$ be a linear operator.*

- (1) *If T is unitary, then its eigenvalues have modulus one.*
- (2) *If T is self-adjoint, then its eigenvalues are real.*
- (3) *If T is positive, then its eigenvalues are nonnegative.*

We can classify the diagonalizable operators. In the infinite dimensional setting this is called the **spectral theorem**. We will do a special case which is more instructive than the general case. We need a series of lemmas.

Lemma 3.16. *If T is a self-adjoint operator on a complex Hilbert space \mathbb{H}^N , then*

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x \rangle|.$$

Proof. To begin, set

$$M := \sup_{\|x\|=1} \{|\langle Tx, x \rangle|\}.$$

We will first show that $M \leq \|T\|$: For any $y \in \mathbb{H}^N$, we have

$$|\langle Tx, y \rangle| \leq \|Tx\| \|y\| \leq \|T\| \|x\| \|y\|,$$

and thus if $\|x\| = 1$, then $|\langle Tx, x \rangle| \leq \|T\|$. Taking the supremum over all $x \in \mathbb{H}^N$ such that $\|x\| = 1$, we get $M \leq \|T\|$.

Next we will show that $\|T\| \leq M$: For any $x, y \in \mathbb{H}^N$, we have

$$\begin{aligned} 4\operatorname{Re} \langle Tx, y \rangle &= (\langle Tx, x \rangle + 2\operatorname{Re} \langle Tx, y \rangle + \langle Ty, y \rangle) \\ &\quad - (\langle Tx, x \rangle - 2\operatorname{Re} \langle Tx, y \rangle + \langle Ty, y \rangle) \\ &= \langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle \end{aligned}$$

where the fact that T is self-adjoint was used in the second equality. Hence,

$$4\operatorname{Re} \langle Tx, y \rangle \leq M(\|x+y\|^2 + \|x-y\|^2) = 2M(\|x\|^2 + \|y\|^2).$$

Note that there exists a $\theta \in [0, 2\pi)$ such that $e^{i\theta} \langle Tx, y \rangle = |\langle Tx, y \rangle|$. Now, replace y with $e^{-i\theta}y$ to obtain:

$$4|\langle Tx, y \rangle| \leq 2M(\|x\|^2 + \|y\|^2).$$

Finally, if $\|x\| = 1$ and $y = Tx/\|Tx\|$, then

$$\|Tx\| = \left\langle Tx, \frac{Tx}{\|Tx\|} \right\rangle \leq M.$$

Combining both steps gives $\|T\| = M$ as desired. \square

Lemma 3.17. *If T is normal then $\|Tx\| = \|T^*x\|$ for all $x \in \mathbb{H}^N$.*

Proof. We compute:

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx \rangle \\ &= \langle T^*Tx, x \rangle \\ &= \langle TT^*x, x \rangle \\ &= \langle T^*x, T^*x \rangle \\ &= \|T^*x\|^2. \end{aligned} \quad \square$$

Lemma 3.18. *If T is normal and*

$$Tx = \lambda x$$

for some $x \in \mathbb{H}^N$ and scalar λ , then

$$T^*x = \bar{\lambda}x.$$

Proof. If T is normal, then $T - \lambda \cdot Id$ is normal. Applying Lemma 3.17 gives

$$\begin{aligned} Tx = \lambda x &\Leftrightarrow \|(T - \lambda \cdot Id)x\| = 0 \\ &\Leftrightarrow \|(T - \lambda \cdot Id)^*x\| = 0 \\ &\Leftrightarrow \|T^*x - \bar{\lambda}x\| = 0 \\ &\Leftrightarrow T^*x = \bar{\lambda}x. \end{aligned} \quad \square$$

Finally, we will need:

Lemma 3.19. *If T is normal and $Tx = \lambda x$, then*

$$T([x]^\perp) \subset [x]^\perp.$$

Proof. If $y \perp x$, then

$$\langle Ty, x \rangle = \langle y, T^*x \rangle = \langle y, \bar{\lambda}x \rangle = 0.$$

Hence, $Ty \perp x$. That is, $T([x]^\perp) \subset [x]^\perp$. \square

Now we are ready to prove the main theorem on diagonalization.

Theorem 3.20. *If T is a self-adjoint linear operator a complex Hilbert space \mathbb{H}^N , then the following are equivalent.*

(1) *T is diagonalizable.*

(2) *There exists an orthonormal basis $\{e_i\}_{i=1}^N$ of eigenvectors for T with respective eigenvalues $\{\lambda_i\}_{i=1}^N$ such that*

$$Tx = \sum_{i=1}^N \lambda_i \langle x, e_i \rangle e_i \text{ for all } x \in \mathbb{H}^N,$$

and

$$\|T\| = \max_{1 \leq i \leq N} |\lambda_i|.$$

Proof. (1) \Rightarrow (2): By Lemma 3.16, we can choose $x \in \mathbb{H}^N$ with $\|x\| = 1$ and $\|T\| = \langle Tx, x \rangle$. Now we compute:

$$\begin{aligned} \|(T - \|T\| \cdot Id)x\|^2 &= \langle Tx - \|T\|x, Tx - \|T\|x \rangle \\ &= \langle Tx, Tx \rangle + \|T\|^2 - 2\|T\|\langle Tx, x \rangle \\ &= 2\|T\|(\|T\| - \langle Tx, x \rangle) \\ &= 0. \end{aligned}$$

It follows that $\lambda_1 = \|T\|$ is an eigenvalue for T with eigenvector $e_1 = x$. If we restrict the operator T to $[e_1]^\perp$, we have by Lemma 3.19 that

$$T|_{[e_1]^\perp} : [e_1]^\perp \rightarrow [e_1]^\perp,$$

and is still a normal operator. So we can repeat the above argument to get a second orthogonal eigenvector of norm one and eigenvalue of the form:

$$\lambda_2 = \|T|_{[e_1]^\perp}\|, \quad e_2 \in [e_1]^\perp.$$

Continuing, we get a sequence of orthonormal vectors $\{e_i\}_{i=1}^N$ and eigenvalues $\{\lambda_i\}_{i=1}^N$ so that

$$\lambda_j = \|T|_{[\text{span}\{e_i\}_{i=1}^{j-1}]^\perp}\|.$$

(2) \Rightarrow (1): This is immediate. \square

Remark 3.21. *There is a slightly more general result available. Namely, (1) and (2) above are equivalent to T being a normal operator. Proofs of this result are less instructive than the above and much deeper, so we leave it to the reader to look it up in [74]. The problem here is that Lemma 3.16 fails for real Hilbert spaces. For example, if T is a 90 degree rotation on \mathbb{R}^2 , then $TT^* = T^*T = Id$, but $\langle Tx, x \rangle = 0$ for all x .*

If P is a projection from \mathbb{H}^N , we have

$$\langle Px, x \rangle = \langle P^2x, x \rangle = \langle Px, Px \rangle = \|Px\|^2,$$

and so P is a positive operator. Moreover, if $P : \mathbb{H}^N \rightarrow W$, $\dim W = m$, then P has eigenvalue 1 with multiplicity m and eigenvalue 0 of multiplicity $N - m$.

Definition 3.22. *Let $T : \mathbb{H}^N \rightarrow \mathbb{H}^N$ be an invertible positive operator with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N > 0$. The **condition number** is $\frac{\lambda_1}{\lambda_N}$.*

Positive operators have **powers** of the operator well-defined. The proof of the following corollary is elementary.

Corollary 3.23. *Let T be an invertible positive operator on \mathbb{H}^N with eigenvectors $\{e_i\}_{i=1}^N$ and respective eigenvalues $\{\lambda_i\}_{i=1}^N$. For any $a \in \mathbb{R}$ we define the operator T^a by*

$$T^a x = \sum_{i=1}^N \lambda_i^a \langle x, e_i \rangle e_i, \text{ for all } x \in \mathbb{H}^N.$$

Then T^a is a positive operator, $T^a T^b = T^{a+b}$ for all real numbers a, b . In particular, T^{-1} and $T^{-1/2}$ are positive operators.

Proposition 3.24. *Let T be a linear operator on \mathbb{H}^N and let $\{e_i\}_{i=1}^N$ and $\{g_i\}_{i=1}^N$ be orthonormal bases for \mathbb{H}^N . Then*

$$\sum_{i=1}^M \langle T e_i, e_i \rangle = \sum_{i=1}^M \langle T g_i, g_i \rangle.$$

Proof. We compute:

$$\begin{aligned} \sum_{i=1}^M \langle T e_i, e_i \rangle &= \sum_{i=1}^M \left\langle \sum_{j=1}^N \langle T e_i, g_j \rangle g_j, e_i \right\rangle \\ &= \sum_{i=1}^M \sum_{j=1}^N \langle T e_i, g_j \rangle \langle g_j, e_i \rangle \\ &= \sum_{i=1}^M \sum_{j=1}^N \langle e_i, T^* g_j \rangle \langle g_j, e_i \rangle \\ &= \sum_{j=1}^N \left\langle \sum_{i=1}^M \langle g_j, e_i \rangle e_i, T^* g_j \right\rangle \\ &= \sum_{j=1}^N \langle g_j, T^* g_j \rangle \\ &= \sum_{j=1}^N \langle T g_j, g_j \rangle. \end{aligned} \quad \square$$

Now we can define trace and have it well defined.

Definition 3.25. *Let $T : \mathbb{H}^N \rightarrow \mathbb{H}^N$ be an operator. The **trace of T** is*

$$\text{Tr } T = \sum_{i=1}^N \langle T e_i, e_i \rangle,$$

where $\{e_i\}_{i=1}^N$ is any orthonormal basis of \mathbb{H}^N . And as we saw above, the Trace is independent of the choice of orthonormal basis.

Corollary 3.26. *Let T be a diagonalizable operator on \mathbb{H}^N with eigenvalues $\{\lambda_i\}_{i=1}^N$. Then*

$$\text{Tr } T = \sum_{i=1}^N \lambda_i.$$

Proof. Let $\{e_i\}_{i=1}^N$ be an orthonormal basis of eigenvectors for T with respective eigenvalues $\{\lambda_i\}_{i=1}^N$. Then

$$\text{Tr } T = \sum_{i=1}^N \langle T e_i, e_i \rangle = \sum_{i=1}^M \langle \lambda_i e_i, e_i \rangle = \sum_{i=1}^N \lambda_i. \quad \square$$

4. HILBERT SPACE FRAMES

Hilbert space frames were introduced by Duffin and Schaeffer in 1952 [48] to address some deep questions in non-harmonic Fourier series. The idea was to weaken Parseval's identity.

Parseval's Identity 4.1. *If $\{\varphi_i\}_{i=1}^M$ is an orthonormal sequence in a Hilbert space \mathbb{H}^N , then for all $x \in \mathbb{H}^N$ we have*

$$\sum_{i=1}^M |\langle x, \varphi_i \rangle|^2 = \|x\|^2.$$

We do not need to have an orthonormal sequence to have equality in Parseval's identity. For example, if $\{e_i\}_{i=1}^N$ and $\{g_i\}_{i=1}^N$ are orthonormal bases for a Hilbert space \mathbb{H}^N then

$$\left\{ \frac{1}{\sqrt{2}} e_1, \frac{1}{\sqrt{2}} g_1, \frac{1}{\sqrt{2}} e_2, \frac{1}{\sqrt{2}} g_2, \dots, \frac{1}{\sqrt{2}} e_N, \frac{1}{\sqrt{2}} g_N \right\}_{i=1}^N$$

satisfies Parseval's identity.

Definition 4.2. *A family of vectors $\{\varphi_i\}_{i=1}^M$ is a **frame** for a Hilbert space \mathbb{H}^N if there are constants $0 < A \leq B < \infty$ so that for all $x \in \mathbb{H}^N$*

$$(4.1) \quad A\|x\|^2 \leq \sum_{i=1}^M |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2.$$

The constants A and B are called the **lower** (respectively, **upper**) **frame bounds** for the frame. The largest lower frame bound and the smallest upper frame bound are called the **optimal frame bounds**. If $A = B$ this is an **A -tight frame** and if $A = B = 1$ this is a **Parseval frame**. If $\|\varphi_i\| = \|\varphi_j\|$ for all $i, j \in I$, this is an **equal norm frame** and if $\|\varphi_i\| = 1$ for all $i \in I$

this is a **unit norm** frame. If there exists a constant c so that $|\langle \varphi_i, \varphi_j \rangle| = c$ for all $i \neq j$, then the frame is called an **equiangular frame**. The values $\{\langle x, \varphi_i \rangle\}_{i=1}^M$ are called the **frame coefficients** of the vector x with respect to the frame. If we have just the right hand inequality in inequality (4.1) we call $\{\varphi_i\}_{i \in I}$ a **B -Bessel** sequence. The frame is **bounded** if $\min_{1 \leq i \leq M} \|\varphi_i\| > 0$. It follows from the left-hand-side of inequality (4.1), that a frame must span the Hilbert space and so $M \geq N$. In the finite dimensional case, spanning is equivalent to being a frame.

Proposition 4.3. $\Phi = \{\varphi_i\}_{i=1}^M$ is a frame for \mathbb{H}^N if and only if $\text{span } \Phi = \mathbb{H}^N$.

Proof. For the right hand inequality,

$$\sum_{i=1}^M |\langle x, \varphi_i \rangle|^2 \leq \sum_{i=1}^M \|x\|^2 \|\varphi_i\|^2 \leq B \|x\|^2,$$

where

$$B = \sum_{i=1}^M \|\varphi_i\|^2.$$

For the left hand inequality, we proceed by contradiction. Suppose we can find a sequence $\{x_n\}_{n=1}^\infty$ with $\|x_n\| = 1$ so that

$$\sum_{i=1}^M |\langle x_n, \varphi_i \rangle|^2 \leq \frac{1}{n},$$

and thus there is a norm convergent subsequence $\{x_{n_j}\}_{j=1}^\infty$ of $\{x_n\}_{n=1}^\infty$, say $x_{n_j} \rightarrow x$. Then

$$\sum_{i=1}^M |\langle x, \varphi_i \rangle|^2 = \lim_{j \rightarrow \infty} \sum_{i=1}^M |\langle x_{n_j}, \varphi_i \rangle|^2 = 0.$$

That is, $x \perp \varphi_i$ for all $i = 1, 2, \dots, M$ and so Φ does not span \mathbb{H}^N . \square

Note that spanning does not imply that a sequence is a frame when the space is infinite dimensional. For example, suppose $\{e_i\}_{i=1}^\infty$ is an orthonormal basis for an infinite dimensional Hilbert space \mathbb{H} . Then the sequence $\{e_i/i\}_{i=1}^\infty$ spans the space, but is not frame since the lower frame bound is zero.

Notice that there are no restrictions put on the frame vectors. For example, if $\{e_i\}_{i=1}^N$ is an orthonormal basis for \mathbb{H}^N , then $\{e_1, 0, e_2, 0, e_3, 0, \dots, e_N, 0\}$ is a Parseval frame for \mathbb{H}^N . Also,

$$\left\{ e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \dots, \frac{e_N}{\sqrt{N}}, \dots, \frac{e_N}{\sqrt{N}} \right\}$$

is a Parseval frame for \mathbb{H}^N . That is, zeros are allowed and so is redundancy.

The smallest redundant family in \mathbb{R}^2 has 3 vectors and can be chosen to be a unit norm, tight, and equiangular frame called the **Mercedes Benz Frame**. To obtain the frame, choose 3 equally spaced points on the unit circle, each 120 degrees apart. The vectors to these points from the origin is the frame.

4.1. Frame Operators. If $\{\varphi_i\}_{i=1}^M$ is a frame for \mathbb{H}^N with frame bounds A, B , define the **analysis operator** of the frame $T : \mathbb{H}^N \rightarrow \ell_2^M$ to be

$$Tx = \sum_{i=1}^M \langle x, \varphi_i \rangle e_i = \{\langle x, \varphi_i \rangle\}_{i=1}^M, \quad \text{for all } x \in \mathbb{H}^N,$$

where $\{e_i\}_{i=1}^M$ is the natural orthonormal basis of ℓ_2^M . It follows that

$$\|Tx\|^2 = \sum_{i=1}^M |\langle x, \varphi_i \rangle|^2,$$

so $\|T\|^2$ is the optimal Bessel bound of the frame.

The adjoint of the analysis operator is the **synthesis operator** which is given by

$$T^* e_i = \varphi_i.$$

Note that the matrix representation of the synthesis operator of a frame $\{\varphi_i\}_{i=1}^M$ is the $N \times M$ matrix with the frame vectors as its columns.

$$T^* = \begin{bmatrix} | & | & \cdots & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_M \\ | & | & \cdots & | \end{bmatrix}$$

In practice, we often work with the matrix representation of the synthesis operator with respect to the eigenbasis of the frame operator (as defined below). It will be shown later that the rows and columns of this matrix representation must satisfy very specific properties that proves useful in constructing frames. See Proposition 4.29 and Proposition 4.30.

Theorem 4.4. *Let $\{\varphi_i\}_{i=1}^M$ be a family of vectors in a Hilbert space \mathbb{H}^N . The following are equivalent:*

- (1) $\{\varphi_i\}_{i=1}^M$ is a frame for \mathbb{H} .
- (2) The operator T^* is bounded, linear, and surjective.
- (3) The operator T bounded, linear, and injective.

Moreover, $\{\varphi_i\}_{i=1}^M$ is a Parseval frame if and only if the synthesis operator is a quotient map (that is, a partial isometry) if and only if $T^*T = Id$ if and only if T is an isometry.

Proof. (1) \Leftrightarrow (2) is immediate by Proposition 4.3, and (2) \Leftrightarrow (3) is immediate by Lemma 3.6. Proposition 3.11 gives the moreover part. \square

The **frame operator** for the frame is $S = T^*T : \mathbb{H}^N \rightarrow \mathbb{H}^N$ given by

$$Sx = T^*Tx = T^* \left(\sum_{i=1}^M \langle x, \varphi_i \rangle e_i \right) = \sum_{i=1}^M \langle x, \varphi_i \rangle T^* e_i = \sum_{i=1}^M \langle x, \varphi_i \rangle \varphi_i.$$

A direct calculation now yields

$$(4.2) \quad \langle Sx, x \rangle = \sum_{i=1}^M |\langle x, \varphi_i \rangle|^2.$$

Proposition 4.5. *The frame operator is a positive, self-adjoint, and invertible operator on \mathbb{H}^N . Moreover,*

$$A \cdot Id \leq S \leq B \cdot Id.$$

Proof. This is all clear except perhaps the **operator inequality**:

$$\langle Ax, x \rangle = A\|x\|^2 \leq \sum_{i=1}^M |\langle x, \varphi_i \rangle|^2 = \langle Sx, x \rangle \leq B\|x\|^2 = \langle Bx, x \rangle. \quad \square$$

The frame operator can be used to **reconstruct** vectors in the space using the computation

$$\begin{aligned} x &= SS^{-1}x \\ &= S^{-1}Sx \\ &= \sum_{i=1}^M \langle S^{-1}x, \varphi_i \rangle \varphi_i \\ &= \sum_{i=1}^M \langle x, S^{-1}\varphi_i \rangle \varphi_i \\ &= \sum_{i=1}^M \langle x, S^{-1/2}\varphi_i \rangle S^{-1/2}\varphi_i. \end{aligned}$$

Since S is invertible, the family $\{S^{-1}\varphi_i\}_{i=1}^M$ is also a frame for \mathbb{H}^N called the **canonical dual frame**. See Subsection 4.2 for the definition and properties of dual frames.

The following result is basically just a restatement of the definitions and known facts.

Proposition 4.6. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a frame for \mathbb{H}^N with analysis operator T and frame operator S . The following are equivalent:*

- (1) $\{\varphi_i\}_{i=1}^M$ is an A -tight frame for \mathbb{H}^N .
- (2) $S = A \cdot Id$.

(3) For every $x \in \mathbb{H}^N$,

$$x = A^{-1} \sum_{i=1}^M \langle x, \varphi_i \rangle \varphi_i.$$

(4) For every $x \in \mathbb{H}^N$,

$$A\|x\|^2 = \sum_{i=1}^M |\langle x, \varphi_i \rangle|^2.$$

(5) T/\sqrt{A} is an isometry.

Moreover, if Φ is a Parseval frame then $A = 1$ above.

Tight frames have the property that all of the eigenvalues of the frame operator coincide and are equal the frame bound. For an arbitrary frame, it turns out the smallest and largest eigenvalues of the frame operator are the optimal lower and upper frame bounds, respectively.

Theorem 4.7. *Suppose $\{\varphi_i\}_{i=1}^M$ is a frame for \mathbb{H}^N with frame operator S with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$. Then λ_1 is the optimal upper frame bound and λ_N is the optimal lower frame bound.*

Proof. Let $\{e_j\}_{j=1}^N$ be an orthonormal eigenbasis of the frame operator S with associated eigenvalues $\{\lambda_j\}_{j=1}^N$ given in decreasing order. Now, given an $x \in \mathbb{H}^N$, write

$$x = \sum_{j=1}^N \langle x, e_j \rangle e_j$$

to obtain

$$Sx = \sum_{j=1}^N \lambda_j \langle x, e_j \rangle e_j.$$

By Equation (4.2), this gives that

$$\begin{aligned} \sum_{i=1}^M |\langle x, \varphi_i \rangle|^2 &= \langle Sx, x \rangle = \sum_{i,j=1}^M \langle \lambda_i \langle x, e_i \rangle e_i, \langle x, e_j \rangle e_j \rangle \\ &= \sum_{j=1}^M \lambda_j |\langle x, e_j \rangle|^2 \leq \lambda_1 \|x\|^2 \end{aligned}$$

proving that λ_1 is an upper frame bound. To see that it is optimal, note

$$\sum_{i=1}^M |\langle e_1, \varphi_i \rangle|^2 = \langle Se_1, e_1 \rangle = \lambda_1.$$

The lower bound is proven similarly. □

Another type of sequence that we often deal with are *Riesz bases*, which rids of the orthonormality assumption, but retains a unique composition.

Definition 4.8. A family of vectors $\{\varphi_i\}_{i=1}^N$ in a Hilbert space \mathbb{H}^N is a **Riesz basis** if there are constants $A, B > 0$ so that for all families of scalars $\{a_i\}_{i=1}^N$ we have

$$A \sum_{i=1}^N |a_i|^2 \leq \left\| \sum_{i=1}^N a_i \varphi_i \right\|^2 \leq B \sum_{i=1}^N |a_i|^2.$$

The following gives an equivalent formulation. It follows directly from the definitions.

Proposition 4.9. Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a family vectors in \mathbb{H}^N . Then the following are equivalent.

- (1) Φ is a Riesz basis for \mathbb{H}^N with Riesz bounds A and B .
- (2) For any orthonormal basis $\{e_i\}_{i=1}^N$ for \mathbb{H}^N , the operator F on \mathbb{H}^N given by $F e_i = \varphi_i$ for all $i = 1, 2, \dots, N$ is an invertible operator with $\|F\|^2 \leq B$ and $\|F^{-1}\|^{-2} \geq A$.

It follows that if Φ is a Riesz basis with bounds A and B , then it is a frame with these same bounds.

Next we see that applying an invertible operator to a frame still gives a frame.

Proposition 4.10. Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a sequence of vectors in \mathbb{H}^N with analysis operator T and let F be a linear operator on \mathbb{H}^N . Then the analysis operator of the sequence $F\Phi = \{F\varphi_i\}_{i=1}^M$ is given by

$$T_{F\Phi} = TF^*.$$

Moreover, if Φ is a frame for \mathbb{H}^N and F is invertible, then $F\Phi$ is also a frame for \mathbb{H}^N .

Proof. For $x \in \mathbb{H}^N$,

$$T_{F\Phi} x = \left\{ \langle x, F\varphi_i \rangle \right\}_{i=1}^M = \left\{ \langle F^* x, \varphi_i \rangle \right\}_{i=1}^M = TF^* x.$$

The moreover part follows from Theorem 4.4. \square

Furthermore, if we apply an invertible operator to a frame, there is a specific form that the frame operator of the new frame must take.

Proposition 4.11. Suppose $\Phi = \{\varphi_i\}_{i=1}^M$ is a frame for \mathbb{H}^N with frame operator S and F is an invertible operator on \mathbb{H}^N . Then the frame operator of the frame $F\Phi = \{F\varphi_i\}_{i=1}^M$ is the operator FSF^* .

Proof. This follows immediately from Proposition 4.10 and the definition. \square

Corollary 3.26 concerning the trace formula for diagonalizable operators, has a corresponding result for Parseval frames.

Proposition 4.12 (Trace formula for Parseval Frames). *Let $\{\varphi_i\}_{i=1}^M$ be a parseval frame on \mathbb{H}^N and let F be a bounded operator on \mathbb{H}^N . Then*

$$\text{Tr}(F) = \sum_{i=1}^M \langle F\varphi_i, \varphi_i \rangle.$$

Proof. If $\{e_i\}_{i=1}^N$ is an orthonormal basis for \mathbb{H}^N then by definition

$$\text{Tr}(F) = \sum_{j=1}^N \langle Fe_j, e_j \rangle.$$

This implies:

$$\begin{aligned} \text{Tr}(F) &= \sum_{j=1}^N \left\langle \sum_{i=1}^M \langle Fe_j, \varphi_i \rangle \varphi_i, e_j \right\rangle \\ &= \sum_{j=1}^N \sum_{i=1}^M \langle e_j, F^* \varphi_i \rangle \langle \varphi_i, e_j \rangle \\ &= \sum_{i=1}^M \left\langle \sum_{j=1}^N \langle \varphi_i, e_j \rangle e_j, F^* \varphi_i \right\rangle \\ &= \sum_{i=1}^M \langle \varphi_i, F^* \varphi_i \rangle \\ &= \sum_{i=1}^M \langle F\varphi_i, \varphi_i \rangle. \quad \square \end{aligned}$$

Definition 4.13. *Two sequences of vectors $\{\varphi_i\}_{i=1}^M$ and $\{\psi_i\}_{i=1}^M$ in a Hilbert space \mathbb{H}^N are **isomorphic** if the operator $L(\varphi_i) = \psi_i$ is a well defined, bounded invertible operator. We say they are **unitarily isomorphic** if L is a unitary operator.*

Proposition 4.14. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ and $\Psi = \{\psi_i\}_{i=1}^M$ be frames for \mathbb{H}^N with analysis operators T_1 and T_2 respectively. The following are equivalent:*

- (1) Φ and Ψ are isomorphic.
- (2) $\text{ran } T_1 = \text{ran } T_2$.
- (3) $\ker T_1^* = \ker T_2^*$.

Moreover, in this case $F = T_2^*(T_1^*|_{\text{ran } T_1})^{-1}$ satisfies $F\varphi_i = \psi_i$ for all $i = 1, 2, \dots, M$.

Proof. The equivalence of (2) and (3) follows by Lemma 3.6.

(1) \Rightarrow (3): Let $F\varphi_i = \psi_i$ be a well-defined invertible operator on \mathbb{H}^N . Then by Proposition 4.10 we have that $T_2 = T_1F^*$ and hence $FT_1^* = T_2^*$. Since F is invertible, (3) follows.

(2) \Rightarrow (1): Let P be the orthogonal projection onto $W = \text{ran } T_1 = \text{ran } T_2$. Then

$$\varphi_i = T_1^*e_i = T_1^*Pe_i + T_1^*(Id - P)e_i = T_1^*Pe_i$$

and similarly $\psi_i = T_2^*Pe_i$. The operators T_1^* and T_2^* both map W bijectively onto \mathbb{H}^N . Therefore, the operator $F =: T_2^*(T_1^*|_W)^{-1}$ maps \mathbb{H}^N bijectively onto itself. Consequently, for all $i = 1, 2, \dots, M$ we have

$$F\varphi_i = T_2^*(T_1^*|_W)^{-1}T_1^*Pe_i = T_2^*Pe_i = \psi_i. \quad \square$$

As a consequence of Proposition 4.11 we have:

Theorem 4.15. *Every frame $\{\varphi_i\}_{i=1}^M$ (with frame operator S) is isomorphic to the Parseval frame $\{S^{-1/2}\varphi_i\}_{i \in I}$.*

Proof. The frame operator of $\{S^{-1/2}\varphi_i\}_{i=1}^M$ is $S^{-1/2}S(S^{-1/2})^* = Id$. \square

As a consequence, only unitary operators can map Parseval frames to Parseval frames.

Corollary 4.16. *If two Parseval frames $\Phi = \{\varphi_i\}_{i=1}^M$ and $\Psi = \{\psi_i\}_{i=1}^M$ are isomorphic, then they are unitarily isomorphic.*

Proof. Since both frames have the identity as their frame operator, if F maps one Parseval frame to another and is invertible, then by Proposition 4.11,

$$Id = F(Id)F^* = FF^*.$$

Since F^*F is injective, F is a unitary operator by Proposition 3.11. \square

We can always “move” one frame operator to another.

Proposition 4.17. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ and $\Psi = \{\psi_i\}_{i=1}^M$ be frames for \mathbb{H}^N with frame operators S_1 and S_2 respectively. There there exists an invertible operator F on \mathbb{H}^N so that S_1 is the frame operator of the frame $F\Psi = \{F\psi_i\}_{i=1}^M$.*

Proof. If S is the frame operator of $F\Psi$, letting $F = S_1^{1/2}S_2^{-1/2}$ we have

$$S = FS_2F^* = (S_1^{1/2}S_2^{-1/2})S_2(S_1^{1/2}S_2^{-1/2})^* = S_1. \quad \square$$

4.2. Dual Frames. We begin with the definition.

Definition 4.18. *If $\Phi = \{\varphi_i\}_{i=1}^M$ is a frame for \mathbb{H}^N , a frame $\{\psi_i\}_{i=1}^M$ for \mathbb{H}^N is called a **dual frame** for Φ if*

$$\sum_{i=1}^M \langle x, \psi_i \rangle \varphi_i = x, \text{ for all } x \in \mathbb{H}^N.$$

It follows that the canonical dual frame $\{S^{-1}\varphi_i\}_{i=1}^M$, where S is the frame operator of Φ , is a dual frame. But there are many other dual frames in general.

Proposition 4.19. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ and $\Psi = \{\psi_i\}_{i=1}^M$ be frames for \mathbb{H}^N with analysis operators T_1, T_2 respectively. The following are equivalent:*

- (1) Ψ is a dual frame of Φ .
- (2) $T_1^*T_2 = Id$.

Proof. Note that for any $x \in \mathbb{H}^N$,

$$T_1^*T_2x = T_1^* \left[(\langle x, \psi_i \rangle)_{i=1}^M \right] = \sum_{i=1}^M \langle x, \psi_i \rangle \varphi_i.$$

The result is now immediate. \square

Theorem 4.20. *Suppose $\Phi = \{\varphi_i\}_{i=1}^M$ is a frame with analysis operator T_1 and frame operator S . The class of all dual frames Φ are all frames of the form: $\{\eta_i\}_{i=1}^M := \{S^{-1}\varphi_i + \psi_i\}_{i=1}^M$, where if T_2 is the analysis operator of $\Psi = \{\psi_i\}_{i=1}^M$, then $T_1^*T_2 = 0$. That is, $\text{ran } T_1 \perp \text{ran } T_2$.*

Proof. Note that the analysis operator of $\{\eta_i\}_{i=1}^M$ is $(T_1^*)^{-1} + T_2$. Now,

$$T_1^*((T_1^*)^{-1} + T_2) = T_1^*(T_1^*)^{-1} + T_1^*T_2 = Id + 0 = Id.$$

By Proposition 4.19, $\{\eta_i\}_{i=1}^M$ is a dual frame of Φ .

Conversely, if $\{\eta_i\}_{i=1}^M$ is a dual frame for Φ , let $\psi_i = \eta_i - S^{-1}\varphi_i$, for all $i = 1, 2, \dots, M$. Then for all $x \in \mathbb{H}^N$,

$$\begin{aligned} T_1^*T_2x &= \sum_{i=1}^M \langle x, \psi_i \rangle \varphi_i \\ &= \sum_{i=1}^M \langle x, \eta_i - S^{-1}\varphi_i \rangle \varphi_i \\ &= \sum_{i=1}^M \langle x, \eta_i \rangle \varphi_i + \sum_{i=1}^M \langle x, S^{-1}\varphi_i \rangle \varphi_i \\ &= x - x = 0. \end{aligned}$$

This implies that for all $x, y \in \mathbb{H}^N$,

$$\langle T_1x, T_2y \rangle = \langle x, T_1^*T_2y \rangle = 0,$$

which is precisely $\text{ran } T_1 \perp \text{ran } T_2$. \square

Proposition 4.21. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a frame for \mathbb{H}^N with frame operator S . Then the only dual frame of Φ which is isomorphic to Φ is $\{S^{-1}\varphi_i\}_{i=1}^M$.*

Proof. Let $\{\psi_i\}_{i=1}^M$ be a dual frame for Φ and assume there is an invertible operator F so that $\psi_i = FS^{-1}\varphi_i$ for all $i = 1, 2, \dots, M$. Then, for every $x \in \mathbb{H}^N$ we have

$$\begin{aligned} F^*x &= \sum_{i=1}^M \langle F^*x, S^{-1}\varphi_i \rangle \varphi_i \\ &= \sum_{i=1}^M \langle x, FS^{-1}\varphi_i \rangle \varphi_i \\ &= \sum_{i=1}^M \langle x, \psi_i \rangle \varphi_i \\ &= x. \end{aligned}$$

It follows that $F^* = Id$ and so $F = Id$. \square

4.3. Redundancy. The main property of frames which makes them so useful in applied problems is their **redundancy**. That is, each vector in the space has infinitely many representations with respect to the frame but it also has one natural representation given by the frame coefficients. The role played by redundancy varies with specific applications. One important role is its **robustness**. That is, by spreading our information over a wider range of vectors, we are better able to sustain **losses** (called **erasures** in this setting) and still have accurate reconstruction. This shows up in internet coding (for transmission losses), distributed processing (where “sensors” are constantly fading out), modeling the brain (where memory cells are constantly dying out) and a host of other applications. Another advantage of spreading our information over a wider range is to mitigate the effects of noise in our signal or to make it prominent enough so it can be removed as in signal/image processing. Another advantage of redundancy is in areas such as quantum tomography where we need classes of orthonormal bases which have “constant” interactions with one another or we need vectors to form a Parseval frame but have the absolute values of their inner products with all other vectors the same. In speech recognition, we need a vector to be determined by the absolute value of its frame coefficients. This is a very natural frame theory problem since this is impossible for a linearly independent set to achieve. Redundancy is a fundamental issue in this setting.

Our next proposition shows the relationship between the frame elements and the frame bounds.

Proposition 4.22. *Let $\{\varphi_i\}_{i=1}^M$ be a frame for H with frame bounds A, B . Then for any $1 \leq i \leq M$ so that $\|\varphi_i\|^2 \leq B$ and $\|\varphi_i\|^2 = B$ holds implies that $\varphi_i \perp \text{span}\{\varphi_j\}_{j \neq i}$. If $\|\varphi_i\|^2 < A$, then $\varphi_i \in \overline{\text{span}}\{\varphi_j\}_{j \neq i}$.*

Proof. If we replace x in the frame definition by φ_i we see that

$$A\|\varphi_i\|^2 \leq \|\varphi_i\|^4 + \sum_{j \neq i} |\langle \varphi_i, \varphi_j \rangle|^2 \leq B\|\varphi_i\|^2.$$

The first part of the result is now immediate. For the second part, assume to the contrary that $E = \overline{\text{span}} \{\varphi_j\}_{j \neq i}$ is a proper subspace of \mathbb{H}^N . Replacing φ_i in the above inequality by $P_{E^\perp} \varphi_i$ and using the left hand side of the inequality yields an immediate contradiction. \square

As a particular case of Proposition 4.22 we have for a Parseval frame $\{\varphi_i\}_{i=1}^M$ that $\|\varphi_i\|^2 \leq 1$ and $\|\varphi_i\| = 1$ if and only if $\varphi_i \perp \overline{\text{span}} \{\varphi_j\}_{j \neq i}$. We call $\{\varphi_i\}_{i=1}^M$ an **exact frame** if it ceases to be a frame when any one of its vectors is removed. If $\{\varphi_i\}_{i=1}^M$ is an exact frame then $\langle S^{-1} \varphi_i, \varphi_j \rangle = \langle S^{-1/2} \varphi_i, S^{-1/2} \varphi_j \rangle = \delta_{ij}$ (where δ_{ij} is the Kronecker delta) since $\{S^{-1/2} \varphi_i\}_{i=1}^M$ is now an orthonormal basis for \mathbb{H}^N . That is, $\{S^{-1} \varphi_i\}_{i=1}^M$ and $\{\varphi_i\}_{i=1}^M$ form a **biorthogonal system**. Also, it follows that $\{e_i\}_{i=1}^N$ is an orthonormal basis for \mathbb{H}^N if and only if it is an exact, Parseval frame. Another consequence of Proposition 4.22 is the following.

Proposition 4.23. *The removal of a vector from a frame leaves either a frame or an incomplete set.*

Proof. By Theorem 4.15, we may assume that $\{\varphi_i\}_{i=1}^M$ is a Parseval frame. Now by Proposition 4.22, for any i , either $\|\varphi_i\| = 1$ and $\varphi_i \perp \overline{\text{span}} \{\varphi_j\}_{j \neq i}$, or $\|\varphi_i\| < 1$ and $\varphi_i \in \overline{\text{span}} \{\varphi_j\}_{j \neq i}$. \square

4.4. Minimal Moments. Since a frame is not independent (unless it is a Riesz basis) a vector in the space will have many representations relative to the frame besides the natural one given by the frame coefficients. However, the natural representation of a vector is the unique representation of minimal ℓ_2 -norm as the following result of Duffin and Schaeffer [48] shows.

Theorem 4.24. *Let $\{\varphi_i\}_{i=1}^M$ be a frame for a Hilbert space \mathbb{H}^N with frame operator S and $x \in \mathbb{H}^N$. If $\{b_i\}_{i=1}^M$ is any sequence of scalars such that*

$$x = \sum_{i=1}^M b_i \varphi_i,$$

then

$$(4.3) \quad \sum_{i=1}^M |b_i|^2 = \sum_{i=1}^M |\langle S^{-1} x, \varphi_i \rangle|^2 + \sum_{i=1}^M |\langle S^{-1} x, \varphi_i \rangle - b_i|^2.$$

Proof. We have by assumption

$$\sum_{i=1}^M \langle S^{-1} x, \varphi_i \rangle \varphi_i = \sum_{i=1}^M b_i \varphi_i.$$

Now, taking the inner product of both sides with $S^{-1}x$ we get

$$\sum_{i=1}^M |\langle S^{-1}x, \varphi_i \rangle|^2 = \sum_{i=1}^M \overline{\langle S^{-1}x, \varphi_i \rangle} b_i,$$

and (4.3) follows easily. \square

4.5. Orthogonal Projections and Naimark's Theorem. A major advantage of frames over wavelets is that orthogonal projections take frames to frames but do not map wavelets to wavelets.

Proposition 4.25. *Let $\{\varphi_i\}_{i=1}^M$ be a frame for \mathbb{H}^N with frame bounds A, B , and let P be an orthogonal projection on H . Then $\{P\varphi_i\}_{i=1}^M$ is a frame for $P(H)$ with frame bounds A, B . In Particular, an orthogonal projection of a orthonormal basis for a Parseval frame is a Parseval frame.*

Proof. For any $x \in P(H)$ we have

$$\sum_{i=1}^M |\langle x, P\varphi_i \rangle|^2 = \sum_{i=1}^M |\langle Px, \varphi_i \rangle|^2 = \sum_{i=1}^M |\langle x, \varphi_i \rangle|^2.$$

The result is now immediate. \square

Proposition 4.25 gives that an orthogonal projection P applied an orthonormal basis $\{e_i\}_{i=1}^M$ for \mathbb{H}^M leaves a Parseval frame $\{Pe_i\}_{i=1}^M$ for $P(\mathbb{H}^M)$. The converse of this is also true and is a result of Naimark (see [30] and Han and Larson [59]).

Theorem 4.26 (Naimark's Theorem). *A sequence $\{\varphi_i\}_{i=1}^M$ is a Parseval frame for a Hilbert space \mathbb{H}^N if and only if there is a larger Hilbert space $\mathbb{H}^M \supset \mathbb{H}^N$ and an orthonormal basis $\{e_i\}_{i=1}^M$ for \mathbb{H}^M so that the orthogonal projection P of \mathbb{H}^M onto \mathbb{H}^N satisfies $Pe_i = \varphi_i$ for all $i = 1, 2, \dots, M$.*

Proof. The “only if” part follows from Proposition 4.25. For the “if” part, note that if $\{\varphi_i\}_{i=1}^M$ is a Parseval for \mathbb{H}^N , then the synthesis operator $T^* : \ell_2^M \rightarrow \mathbb{H}^N$ is a partial isometry. Let $\{e_i\}_{i=1}^M$ be an orthonormal basis for ℓ_2^M for which $T^*e_i = \varphi_i$ for all $i = 1, 2, \dots, M$. Since the analysis operator T is an isometry we can identify \mathbb{H}^N with $T(\mathbb{H}^N)$. Now let $\mathbb{H}^M = \ell_2^M$ and let P be the orthogonal projection of \mathbb{H}^M onto $T(\mathbb{H}^N)$. Then for all $i = 1, 2, \dots, M$ and all $y = Tx \in T(\mathbb{H}^N)$ we have

$$\langle Tx, Pe_i \rangle = \langle Tx, e_i \rangle = \langle x, T^*e_i \rangle = \langle x, \varphi_i \rangle = \langle Tx, T\varphi_i \rangle.$$

It follows that $Pe_i = T\varphi_i$, and the result follows from the association of \mathbb{H}^N with $T(\mathbb{H}^N)$. \square

Definition 4.27. *The standard N -simplex (or regular N -gon) is the subset of \mathbb{R}^{N-1} given by unit norm $\{\varphi_i\}_{i=1}^N$ which are equiangular.*

It follows from Proposition 4.25:

Corollary 4.28. *A regular simplex is an equiangular tight frame.*

Proof. Given a regular N -simplex, it can be realized by letting $\{e_i\}_{i=1}^N$ be an orthonormal basis for \mathbb{R}^N and letting

$$x = \sum_{i=1}^N \varepsilon_i e_i, \text{ for any } \varepsilon_i = 1, \text{ for } i = 1, 2, \dots, N.$$

Now let P be the orthogonal projection onto $[\text{span } x]$, which is given by:

$$Py = \left[\frac{1}{N} \sum_{i=1}^N \varepsilon_i \langle y, e_i \rangle \right] x$$

Then the N -simplex is a scaling by $1/\|(Id - P)e_i\|$ of $\{(Id - P)e_i\}_{i=1}^N$, which is N -vectors in an $N - 1$ -dimensional Hilbert space and is a Parseval frame by Proposition 4.25. \square

4.6. Frame Representations. Another important property of frames is given in the next proposition.

Proposition 4.29. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a sequence of vectors in \mathbb{H}^N , let $\{e_j\}_{j=1}^N$ be an orthonormal basis for \mathbb{H}^N , and let $\{\lambda_j\}_{j=1}^N$ be positive real numbers. The following are equivalent:*

- (1) Φ is a frame for \mathbb{H}^N with frame operator S having eigenvectors $\{e_j\}_{j=1}^N$ and respective eigenvalues $\{\lambda_j\}_{j=1}^N$.
- (2) The following hold:
 - (a) If for all $j = 1, 2, \dots, N$,

$$\psi_j := (\langle \varphi_1, e_j \rangle, \langle \varphi_2, e_j \rangle, \dots, \langle \varphi_M, e_j \rangle),$$

then for all $1 \leq j \neq k \leq N$ we have

$$\langle \psi_j, \psi_k \rangle = 0.$$

- (b) For all $j = 1, 2, \dots, N$ we have $\|\psi_j\|_2^2 = \lambda_j$.

Proof. If $\{\varphi_i\}_{i=1}^M$ is a frame for \mathbb{H}^N with frame operator S having eigenvectors $\{e_j\}_{j=1}^N$ and respective eigenvalues $\{\lambda_j\}_{j=1}^N$, then for all $j = 1, 2, \dots, N$ we have

$$\sum_{i=1}^M \langle e_j, \varphi_i \rangle \varphi_i = \lambda_j e_j.$$

Hence, for $1 \leq j \neq k \leq N$ we have

$$\langle \psi_j, \psi_k \rangle = \sum_{i=1}^M \overline{\langle \varphi_i, e_j \rangle} \langle \varphi_i, e_k \rangle = \sum_{i=1}^M \langle e_j, \varphi_i \rangle \langle \varphi_i, e_k \rangle = \langle \lambda_j e_j, e_k \rangle = 0.$$

Similarly,

$$\|\psi_j\|_2^2 = \langle \lambda_j e_j, e_j \rangle = \lambda_j. \quad \square$$

This gives some important properties of the matrix representation of the synthesis operator. As stated before, these criterion are often used in constructing frames with a given synthesis operator.

Proposition 4.30. *If $\{\varphi\}_{i=1}^M$ is a frame for \mathbb{H}^N and $T^* = [a_{ij}]_{i=1,j=1}^{N,M}$ is the synthesis matrix with respect to the eigenvectors of the frame operator, then the following hold.*

- (1) *The rows of T^* are orthogonal.*
- (2) *The square sum of the columns are the norms of the frame vectors.*
- (3) *The square sum of the rows are the eigenvalues of the frame operator.*

5. CONSTANTS RELATED TO FRAMES

Let $\{\varphi_i\}_{i=1}^M$ be a frame for \mathbb{H}^N with frame operator S .

General Frame: The sum of the eigenvalues of S equals the sum of the squares of the lengths of the frame vectors:

$$\sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|\varphi_i\|^2.$$

Equal Norm Frame: For an equal norm frame in which $\|\varphi_i\| = c$ holds for all $i = 1, 2, \dots, M$ we have

$$\sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|\varphi_i\|^2 = M \cdot c^2.$$

Tight Frame: Since tightness means $A = B$, we have

$$\sum_{i=1}^M |\langle x, \varphi_i \rangle|^2 = A \|x\|^2, \quad \text{for all } x \in \mathbb{H}^N.$$

We have that $S = A \cdot I_N$ and thus the sum of the eigenvalues becomes:

$$N \cdot A = \sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|\varphi_i\|^2.$$

Parseval Frame: If the frame is Parseval, then $A = B = 1$ and so

$$\sum_{i=1}^M |\langle x, \varphi_i \rangle|^2 = \|x\|^2, \quad \text{for all } x \in \mathbb{H}_N.$$

We have that $S = Id$ and

$$N = \sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|\varphi_i\|^2.$$

Equal Norm Tight Frame: For an equal norm A -tight frame in which $\|\varphi_i\| = c$ holds for all $i = 1, 2, \dots, M$ we have

$$N \cdot A = \sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|\varphi_i\|^2 = M \cdot c^2.$$

Hence $A = M \cdot c^2/N$ and thus

$$\sum_{i=1}^M |\langle x, \varphi_i \rangle|^2 = \frac{M}{N} c^2 \|x\|^2, \quad \text{for all } x \in \mathbb{H}^N.$$

Equal Norm Parseval Frame: For an equal norm Parseval frame we have

$$N = \sum_{j=1}^M \lambda_j = \sum_{i=1}^M \|\varphi_i\|^2 = c^2 M.$$

6. CONSTRUCTING FINITE FRAMES

For applications, we need to construct finite frames with extra properties such as:

- (1) Prescribing in advance the norms of the frame vectors. (See for example [25, 35, 36]).
- (2) Constructing equiangular frames. That is, frames $\{\varphi_i\}_{i=1}^M$ for which there is a constant $c > 0$ and

$$|\langle \varphi_i, \varphi_j \rangle| = c, \quad \text{for all } i \neq j.$$

(See for example [37, 60, 75, 76]).

- (3) Frames for which the operator

$$\pm x \mapsto \{|\langle x, \varphi_i \rangle|\}_{i=1}^M, \quad \text{is one-to-one.}$$

(See for example [3, 11]).

For a good introduction to constructive methods for frames see [25]

6.1. Finding Parseval Frames. There is a unique way to get Parseval frames of M vectors in \mathbb{H}^N . Take a $M \times M$ unitary matrix $U = (a_{ij})_{i,j=1}^M$. Take the submatrix

$$V = (a_{ij})_{i=1,j=1}^{M,N}.$$

The rows of V form a parseval frame for \mathbb{H}^N , since these vectors are just the rows of the matrix U (which are an orthonormal basis for \mathbb{H}^M) projected onto \mathbb{H}^N and so form a Parseval frame. The converse to this is also true - and follows directly from **Naimark's Theorem** (Theorem 4.26).

6.2. Adding vectors to a frame to make it tight. Every finite frame for \mathbb{H}_N can be turned into a tight frame with the addition of at most $N - 1$ -vectors.

Proposition 6.1. *If $\{\varphi_i\}_{i=1}^M$ is a frame for \mathbb{H}^N , then there are vectors $\{\psi_j\}_{j=2}^N$ so that $\{\varphi_i\}_{i=1}^M \cup \{\psi_j\}_{j=2}^N$ is a tight frame.*

Proof. Let S be the frame operator for the frame with eigenvectors $\{e_j\}_{j=1}^N$ and respective eigenvalues $\{\lambda_j\}_{j=1}^N$ that satisfy $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$. We define ψ_j for $j = 2, 3, \dots, N$ by

$$\psi_j = \sqrt{\lambda_1 - \lambda_j} e_j.$$

This family $\{\varphi_i\}_{i=1}^M \cup \{\psi_j\}_{j=2}^N$ is a λ_1 -tight frame. \square

6.3. Majorization. One of the main constructive methods for frames is due to Casazza and Leon [35, 36]. They gave a construction for the important results of Benedetto and Fickus [8] and Casazza, Fickus, Kovačević, Leon, Tremain [29].

Theorem 6.2. [29] *Fix $N \leq M$ and $a_1 \geq a_2 \geq \dots \geq a_M > 0$. The following are equivalent:*

- (1) *There is a tight frame $\{\varphi_i\}_{i=1}^M$ for \mathbb{H}^N satisfying $\|\varphi_i\| = a_i$, for all $i = 1, 2, \dots, M$.*
- (2) *For all $1 \leq n < N$ we have*

$$a_n^2 \leq \frac{\sum_{i=n+1}^M a_i^2}{N - n}.$$

- (3) *We have*

$$\sum_{i=1}^M a_i^2 \geq N a_1^2.$$

- (4) *If*

$$\lambda = \sqrt{\frac{N}{\sum_{i=1}^M a_i^2}},$$

then

$$\lambda a_i \leq 1, \quad \text{for all } i = 1, 2, \dots, M.$$

This result was generalized by Casazza and Leon [35] to:

Theorem 6.3. *Let S be a positive self-adjoint operator on \mathbb{H}^N and let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N > 0$ be the eigenvalues of S . Fix $M \geq N$ and real numbers $a_1 \geq a_2 \geq \dots \geq a_M > 0$. The following are equivalent:*

- (1) *There is a frame $\{\varphi_i\}_{i=1}^M$ for \mathbb{H}^N with frame operator S satisfying $\|\varphi_i\| = a_i$ for all $i = 1, 2, \dots, M$.*
- (2) *For every $1 \leq k \leq N$ we have*

$$\sum_{j=1}^k a_j^2 \leq \sum_{j=1}^k \lambda_j,$$

and

$$\sum_{i=1}^M a_i^2 = \sum_{j=1}^N \lambda_j.$$

Theorem 6.3(2) is the so called *majorization* of one sequence over another. The next result follows readily from the above results.

Corollary 6.4. *Let S be a positive self-adjoint operator on \mathbb{H}^N . For any $M \geq N$ there is an equal norm sequence $\{\varphi_i\}_{i=1}^M$ in \mathbb{H}^N which has S as its frame operator.*

Proof. Let $\lambda_1 \geq \lambda_2 \geq \dots \lambda_N > 0$ be the eigenvalues of S and let

$$(6.1) \quad a^2 = \frac{1}{M} \sum_{j=1}^N \lambda_j.$$

Now we check condition (2) of Theorem 6.3 to see that there is a sequence $\{\varphi_i\}_{i=1}^M$ in \mathbb{H}^N with $\|\varphi_i\| = a$ for all $i = 1, 2, \dots, M$. That is, we check the condition with $a_1 = a_2 = \dots = a_M = a$. To check the equality in Theorem 6.3(2), note that by Equation (6.1) we have

$$(6.2) \quad \sum_{i=1}^M a_i^2 = Ma^2 = \sum_{i=1}^N \lambda_i.$$

For the first inequality with $k = 1$ in Theorem 6.3(2), we note that by Equation (6.1) we have that

$$a_1^2 = a^2 = \frac{1}{M} \sum_{i=1}^N \lambda_i \leq \frac{1}{N} \sum_{i=1}^N \lambda_i \leq \lambda_1.$$

So our inequality holds for $k = 1$. Suppose there is an $1 < k \leq N$ for which this inequality fails and k is the first time this fails. So,

$$\sum_{i=1}^{k-1} a_i^2 = (k-1)a^2 \leq \sum_{i=1}^{k-1} \lambda_i,$$

while

$$\sum_{i=1}^k a_i^2 = ka^2 > \sum_{i=1}^k \lambda_i.$$

It follows that

$$a_{k+1}^2 = \cdots = a_N = a^2 > \lambda_k \geq \lambda_{k+1} \geq \cdots \geq \lambda_N.$$

Hence,

$$\begin{aligned} Ma^2 &= \sum_{i=1}^M a_i^2 \geq \sum_{i=1}^k a_i^2 + \sum_{i=k+1}^N a_i^2 \\ &> \sum_{i=1}^k \lambda_i + \sum_{i=k+1}^N a_i^2 \\ &> \sum_{i=1}^k \lambda_i + \sum_{i=k+1}^N \lambda_i \\ &= \sum_{i=1}^N \lambda_i. \end{aligned}$$

But this contradicts Equation (6.2). \square

There was a recent **significant advance** on this subject due to due to Cahill/Fickus/Mixon/Poteet/Strawn [19, 18] where they give an algorithm for constructing all self-adjoint matrices with prescribed spectrum and diagonal and all finite frames with prescribed spectrum and diagonal. This work technically contains the solution to many of our frame questions. That is, if we could carry out their construction with an additional restriction on it (e.g. requiring “equiangular” - see Subsection 10.1) then we could construct equiangular tight frames.

6.4. Spectral Tetris. Another significant advance for frame theory came with the introduction of **spectral tetris** by Casazza/Fickus/Mixon/Wang/Zhou [27]. This is now a massive subject and we refer the reader to a comprehensive survey of Casazza/Woodland [39]. We will just give an illustrative example here.

Before we begin our example, let us go over a few necessary facts for construction. Recall, that in order to construct an M -element unit norm tight

frame (UNTF) in \mathbb{H}^N , we will construct an $N \times M$ synthesis matrix having the following properties:

- (1) The columns square sum to one, to obtain unit norm vectors.
- (2) The rows are orthogonal, which is equivalent to the frame operator, S , being a diagonal $N \times N$ matrix.
- (3) The rows have constant norm, to obtain tightness, meaning that $S = A \cdot Id$ for some constant A .

Remark 6.5. *Since we will be constructing M -element UNTFs in \mathbb{H}^N , recall that the frame bound will be $A = \frac{M}{N}$.*

Also, before the construction of a frame is possible, we must first ensure that such a frame exists by checking that the spectrum of the frame majorizes the square vector norms of the frame. However, this is not the only constraint. For Spectral Tetris to work, we also require that the frame has redundancy of at least 2, that is $M \geq 2N$, where M is the number of frame elements and N is the dimension of the Hilbert space. For a UNTF, since our unique eigenvalue is $\frac{M}{N}$, we see that this is equivalent to the requirement that the eigenvalue of the frame is greater than or equal to 2.

The main idea of Spectral Tetris is to iteratively construct a synthesis matrix, T^* , for a UNTF one to two vectors at a time, which satisfies properties (1) and (2) at each step and gets closer to and eventually satisfies property (3) when complete. When it is necessary to build two vectors at a time throughout the Spectral Tetris process, we will utilize the following key 2×2 matrix as a building block for our construction.

Spectral Tetris relies on the existence of 2×2 matrices $A(x)$, for given $0 \leq x \leq 2$, such that:

- (1) the columns of $A(x)$ square sum to 1,
- (2) $A(x)$ has orthogonal rows,
- (3) the square sum of the first row is x .

These properties combined are equivalent to

$$A(x) A^*(x) = \begin{bmatrix} x & 0 \\ 0 & 2 - x \end{bmatrix}.$$

A matrix which satisfies these properties and which is used as a building block in Spectral Tetris is:

$$A(x) = \begin{bmatrix} \sqrt{\frac{x}{2}} & \sqrt{\frac{x}{2}} \\ \sqrt{1 - \frac{x}{2}} & -\sqrt{1 - \frac{x}{2}} \end{bmatrix}.$$

We are now ready to give the example.

Example 6.6. We would like to use *Spectral Tetris* to construct a sparse, unit norm, tight frame with 11 elements in \mathbb{H}^4 , so our tight frame bound will be $\frac{11}{4}$.

To do this we will create a 4×11 matrix T^* , which satisfies the following conditions:

- (1) The columns square sum to 1.
- (2) T^* has orthogonal rows.
- (3) The rows square sum to $\frac{11}{4}$.
- (4) $S = T^*T = \frac{11}{4} \cdot Id$.

Note that (4) follows if (1), (2) and (3) are all satisfied. Also notice that the sequence of eigenvalues $\{\lambda_j\}_{j=1}^4 = \{\frac{11}{4}, \frac{11}{4}, \frac{11}{4}, \frac{11}{4}\}$ majorizes the sequence of square norms $\{a_i^2\}_{i=1}^{11} = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$, which, in general, is necessary for such a frame to exist.

Define $t_{i,j}$ to be the entry in the i^{th} row and j^{th} column of T^* . With an empty 4×11 matrix, we start at $t_{1,1}$ and work our way left to right to fill out the matrix. By requirement (1), we need the square sum of column one to be 1 and by requirement (2) we need the square sum of row one to be $\frac{11}{4} \geq 1$. Hence, we will start by being greedy and put the maximum weight of 1 in $t_{1,1}$. This forces the rest of the entries in column 1 to be zero, from requirement (1). We get:

$$T^* = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Next, since row one needs to square sum to $\frac{11}{4}$, by (3), and we only have a total weight of 1 in row one, then we need to add $\frac{11}{4} - 1 = \frac{7}{4} = 1 + \frac{3}{4} \geq 1$ more weight to row one. So we will again be greedy and add another 1 in $t_{1,2}$. This forces the rest of the entries in column 2 to be zero, by (1). Also note that we have a total square sum of 2 in row one. We get:

$$T^* = \begin{bmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

In order to have a total square sum of $\frac{11}{4}$ in the first row, we need to add a total of $\frac{11}{4} - 2 = \frac{3}{4} < 1$ more weight. If the remaining unknown entries are chosen so that T^* has orthogonal rows, then S will be a diagonal matrix. Currently, the diagonal entries of S are mostly unknowns, having the form $\{2+?, \cdot, \cdot, \cdot\}$. Therefore we need a way to add $\frac{3}{4}$ more weight in the first row without compromising the orthogonality of the rows of T^* nor the normality

of its columns. That is, if we get “greedy” and try to add $\sqrt{\frac{3}{4}}$ to position $t_{1,3}$ then the rest of row one must be zero, yielding:

$$T^* = \begin{bmatrix} 1 & 1 & \sqrt{\frac{3}{4}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

In order for column three to square sum to one, at least one of the entries $t_{2,3}$, $t_{3,3}$ or $t_{4,3}$ is non-zero. But then, it is impossible for the rows to be orthogonal and thus we cannot proceed. Hence, we need to instead add two columns of information in attempts to satisfy these conditions. The key idea is to utilize our 2×2 building block, $A(x)$, as defined at (a).

We define the third and fourth columns of T^* according to such a matrix $A(x)$, where $x = \frac{11}{4} - 2 = \frac{3}{4}$. Notice that by doing this, column three and column four now square sum to one within the first two rows, hence the rest of the unknown entries in these two columns will be zero. We get:

$$T^* = \begin{bmatrix} 1 & 1 & \sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

The diagonal entries of S are now $\{\frac{11}{4}, \frac{5}{4}+?, \cdot, \cdot\}$. The first row of T^* , and equivalently the first diagonal entry of S , now have sufficient weight and so its remaining entries are set to zero. The second row, however, is currently falling short by $\frac{11}{4} - \left(\left(\sqrt{\frac{5}{8}} \right)^2 + \left(-\sqrt{\frac{5}{8}} \right)^2 \right) = \frac{6}{4} = 1 + \frac{2}{4}$. Since $1 + \frac{2}{4} \geq 1$, we can be greedy and add a weight of 1 in $t_{2,5}$. Hence, column five becomes e_2 . Next, with a weight of $\frac{2}{4} < 1$ left to add to row two we utilize our 2×2 building block $A(x)$, with $x = \frac{2}{4}$. Adding this 2×2 block in columns six and seven yields sufficient weight in these columns and hence we finish these two columns with zeros. We get:

$$T^* = \begin{bmatrix} 1 & 1 & \sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & 1 & \sqrt{\frac{2}{8}} & \sqrt{\frac{2}{8}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{6}{8}} & -\sqrt{\frac{2}{8}} & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

The diagonal entries of T^* are now $\{\frac{11}{4}, \frac{11}{4}, \frac{6}{4}+?, \cdot\}$, where the third diagonal entry, and equivalently the third row, are falling short by $\frac{11}{4} - \frac{6}{4} = \frac{5}{4} = 1 + \frac{1}{4}$.

Since $1 + \frac{1}{4} \geq 1$, then we take the eighth column of T^* to be e_3 . We will complete our matrix following these same strategies, by letting the ninth and tenth columns arise from $A(\frac{1}{4})$, and making the final column e_4 , yielding the desired UNTF:

$$T^* = \begin{bmatrix} 1 & 1 & \sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & 1 & \sqrt{\frac{2}{8}} & \sqrt{\frac{2}{8}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{6}{8}} & -\sqrt{\frac{2}{8}} & 1 & \sqrt{\frac{7}{8}} & \sqrt{\frac{7}{8}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{7}{8}} & -\sqrt{\frac{7}{8}} & 1 \end{bmatrix}.$$

In this construction, column vectors are either introduced one at a time, such as columns 1, 2, 5, 8, and 11, or in pairs, such as columns $\{3, 4\}$, $\{6, 7\}$, and $\{9, 10\}$. Each singleton contributes a value of 1 to a particular diagonal entry of S , while each pair spreads two units of weight over two entries. Overall, we have formed a flat spectrum, $\{\frac{11}{4}, \frac{11}{4}, \frac{11}{4}, \frac{11}{4}\}$, from blocks of area one or two. This construction is reminiscent of the game Tetris, as we fill in blocks of mixed area to obtain a flat spectrum.

7. GRAMMIAN OPERATORS

If $\{\varphi_i\}_{i=1}^M$ is a frame for \mathbb{H}^N with analysis operator T , the **Grammian operator** is defined as

$$G := TT^*,$$

which has matrix representation

$$G = [\langle \varphi_j, \varphi_i \rangle]_{i,j=1}^M,$$

called the **Grammian matrix**. Since we know T^*T and TT^* have the same non-zero eigenvalues, we have:

Proposition 7.1. *Let $\{\varphi_i\}_{i=1}^M$ be a frame for \mathbb{H}^N with frame bounds A, B and frame operator S . The Grammian operator has the same non-zero eigenvalues as S . That is, the largest eigenvalue of G is less than or equal to B and the smallest non-zero eigenvalue is greater than or equal to A .*

Theorem 7.2. *If $\{\varphi_i\}_{i=1}^M$ is a Parseval frame with analysis operator T , then the Grammian operator is an orthogonal projection.*

Proof. It is clear that TT^* is self-adjoint and

$$(TT^*)(TT^*) = T(T^*T)T^* = T(I)T^* = TT^*. \quad \square$$

Corollary 7.3. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ be vectors in \mathbb{H}^N . The Grammian of Φ is invertible if and only if Φ is a Riesz basis (that is, when $M = N$).*

Proof. If $G = TT^*$ is invertible, by Proposition 7.1, T^*T is invertible. Hence, $\{\varphi_i\}_{i=1}^M$ is a frame for \mathbb{H}^N . Also, T^* is one-to-one, and further T^* is bounded, linear and onto. Hence, it is an isomorphism.

If $\{\varphi_i\}_{i=1}^M$ is a Riesz basis then T^* is an isomorphism and we have that T^* is invertible and so $G = TT^*$ is invertible. \square

Proposition 7.4. *Let $F = [a_{ij}]_{i,j=1}^M$ be a positive, self-adjoint matrix (operator) on \mathbb{H}^N with $\dim \ker F = M - N$. Then $\{F^{1/2}e_i\}_{i=1}^M$ spans an N -dimensional space and*

$$\langle F^{1/2}e_i, F^{1/2}e_j \rangle = \langle Fe_i, e_j \rangle = a_{ij}.$$

Hence, F is the Grammian matrix for the vectors $\{F^{1/2}e_i\}_{i=1}^M$. Moreover,

$$\|F^{1/2}e_i\|^2 = a_{ii},$$

and so if $a_{ii} = 1$ for all $i = 1, 2, \dots, M$ then $\{F^{1/2}e_i\}_{i=1}^M$ is a unit norm family.

8. FUSION FRAMES

A number of new applications have emerged which cannot be modeled naturally by one single frame system. Generally they share a common property that requires distributed processing. Furthermore, we are often overwhelmed by a deluge of data assigned to one single frame system, which becomes simply too large to be handled numerically. In these cases it would be highly beneficial to split a large frame system into a set of (overlapping) much smaller systems, and to process locally within each sub-system effectively.

A distributed frame theory for a set of local frame systems is therefore in demand. A variety of applications require distributed processing. Among them there are, for instance, wireless sensor networks [61], geophones in geophysics measurements and studies [43], and the physiological structure of visual and hearing systems [72]. To understand the nature, the constraints, and related problems of these applications, let us elaborate a bit further on the example of wireless sensor networks.

In wireless sensor networks, sensors of limited capacity and power are spread in an area sometimes as large as an entire forest to measure the temperature, sound, vibration, pressure, motion, and/or pollutants. In some applications, wireless sensors are placed in a geographical area to detect and characterize chemical, biological, radiological, and nuclear material. Such a sensor system is typically redundant, and there is no orthogonality among sensors, therefore each sensor functions as a frame element in the system. Due to practical and cost reasons, most sensors employed in such applications have severe constraints in their processing power and transmission bandwidth. They often have strictly metered power supply as well. Consequently, a typical large sensor network necessarily divides the network into redundant sub-networks –

forming a set of subspaces. The primary goal is to have local measurements transmitted to a local sub-station within a subspace for a subspace combining. An entire sensor system in such applications could have a number of such local processing centers. They function as relay stations, and have the gathered information further submitted to a central processing station for final assembly.

In such applications, distributed/local processing is built in the problem formulation. A staged processing structure is prescribed. We will have to be able to process the information stage by stage from local information and to eventually fuse them together at the central station. We see therefore that a mechanism of coherently collecting sub-station/subspace information is required.

Also, due to the often unpredictable nature of geographical factors, certain local sensor systems are less reliable than others. While facing the task of combining local subspace information coherently, one has also to consider weighting the more reliable sets of substation information more than suspected less reliable ones. Consequently, the coherent combination mechanism we just saw as necessary often requires a weighted structure as well. This all leads naturally to what is called a fusion frame.

Definition 8.1. *Let I be a countable index set, let $\{W_i\}_{i=1}^M$ be a family of closed subspaces in \mathbb{H}^N , and let $\{v_i\}_{i=1}^M$ be a family of weights, i.e. $v_i > 0$ for all $i \in I$. Then $\{(W_i, v_i)\}_{i=1}^M$ is a **fusion frame**, if there exist constants $0 < C \leq D < \infty$ such that*

$$(8.1) \quad C\|x\|^2 \leq \sum_{i=1}^M v_i^2 \|P_{W_i}(x)\|^2 \leq D\|x\|^2 \quad \text{for all } x \in \mathbb{H}^N,$$

where P_{W_i} is the orthogonal projection onto the subspace W_i . We call C and D the **fusion frame bounds**. The family $\{(W_i, v_i)\}_{i=1}^M$ is called a **C -tight fusion frame**, if in (8.1) the constants C and D can be chosen so that $C = D$, a **Parseval fusion frame** provided that $C = D = 1$, and an **orthonormal fusion basis** if $\mathbb{H}^N = \bigoplus_{i=1}^M W_i$. If $\{(W_i, v_i)\}_{i=1}^M$ possesses an upper fusion frame bound, but not necessarily a lower bound, we call it a **Bessel fusion sequence with Bessel fusion bound D** .

Often it will become essential to consider a fusion frame together with a set of local frames for its subspaces. In this case we speak of a fusion frame system.

Definition 8.2. *Let $\{(W_i, v_i)\}_{i=1}^M$ be a fusion frame for \mathbb{H}^N , and let $\{x_{ij}\}_{j=1}^{J_i}$ be a frame for W_i for each $1 \leq i \leq M$. Then we call $\{(W_i, v_i, \{\varphi_{ij}\}_{j=1}^{J_i})\}_{i=1}^M$ a **fusion frame system** for \mathbb{H}^N . The constants, C and D , are the associated **fusion frame bounds** if they are the fusion frame bounds for $\{(W_i, v_i)\}_{i=1}^M$,*

and A and B are the **local frame bounds** if these are the common frame bounds for the **local frames** $\{\varphi_{ij}\}_{j=1}^{J_i}$ for each $1 \leq i \leq M$. A collection of dual frames, $\{\psi_{ij}\}_{j=1}^{J_i}$ for each $1 \leq i \leq M$, associated with the local frames will be called **local dual frames**.

To provide a quick inside-look at some intriguing relations between properties of the associated fusion frame and the sequence consisting of all local frame vectors, we present the following theorem from [31] that provides a link between local and global properties.

Theorem 8.3. [31, Thm. 3.2] *For each $1 \leq i \leq M$, let $v_i > 0$, let W_i be a closed subspace of \mathbb{H}^N , and let $\{\varphi_{ij}\}_{j=1}^{J_i}$ be a frame for W_i with frame bounds A_i and B_i . Suppose that $0 < A = \inf_{1 \leq i \leq M} A_i \leq \sup_{1 \leq i \leq M} B_i = B < \infty$. Then the following conditions are equivalent.*

- (1) $\{(W_i, v_i)\}_{i=1}^M$ is a fusion frame for \mathbb{H}^N .
- (2) $\{v_i \varphi_{ij}\}_{j=1, i=1}^{J_i, M}$ is a frame for \mathbb{H}^N .

In particular, if $\{(W_i, v_i, \{\varphi_{ij}\}_{j=1}^{J_i})\}_{i=1}^M$ is a fusion frame system for \mathbb{H}^N with fusion frame bounds C and D , then $\{v_i \varphi_{ij}\}_{j=1, i=1}^{J_i, M}$ is a frame for \mathbb{H}^N with frame bounds AC and BD . Conversely, if $\{v_i \varphi_{ij}\}_{j=1, i=1}^{J_i, M}$ is a frame for \mathbb{H}^N with frame bounds C and D , then $\{(W_i, v_i, \{\varphi_{ij}\}_{j=1}^{J_i})\}_{i=1}^M$ is a fusion frame system for \mathbb{H}^N with fusion frame bounds $\frac{C}{B}$ and $\frac{D}{A}$.

Tight frames play a vital role in frame theory due to the fact that they provide easy reconstruction formulas. Tight fusion frames will turn out to be particularly useful for distributed reconstruction as well. Notice, that the previous theorem also implies that $\{(W_i, v_i)\}_{i=1}^M$ is a C -tight fusion frame for \mathbb{H}^N if and only if $\{v_i \varphi_{ij}\}_{j=1, i=1}^{J_i, M}$ is a C -tight frame for \mathbb{H}^N .

The following result from [33] proves that the fusion frame bound C of a C -tight fusion frame can be interpreted as the **redundancy** of this fusion frame.

Proposition 8.4. *If $\{(W_i, v_i)\}_{i=1}^M$ is a C -tight fusion frame for \mathbb{H}^N , then*

$$C = \frac{\sum_{i=1}^M v_i^2 \dim W_i}{N}.$$

Let $\mathcal{W} = \{(W_i, v_i)\}_{i=1}^M$ be a fusion frame for \mathbb{H}^N . In order to map a signal to the representation space, i.e., to analyze it, the **analysis operator** $T_{\mathcal{W}}$ is employed, which is defined by

$$T_{\mathcal{W}} : \mathbb{H}^N \rightarrow \left(\sum_{i=1}^M \oplus_{\ell_2} W_i \right) \quad \text{with} \quad T_{\mathcal{W}}(x) = \{v_i P_{W_i}(x)\}_{i=1}^M.$$

It can easily be shown that the **synthesis operator** $T_{\mathcal{W}}^*$, which is defined to be the adjoint operator of the analysis operator, is given by

$$T_{\mathcal{W}}^* : \left(\sum_{i=1}^M \oplus W_i \right)_{\ell_2} \rightarrow \mathbb{H}^N$$

with

$$T_{\mathcal{W}}^*(x) = \sum_{i=1}^M v_i x_i, \text{ where } x = \{x_i\}_{i=1}^M \in \left(\sum_{i=1}^M \oplus W_i \right)_{\ell_2}.$$

The **fusion frame operator** $S_{\mathcal{W}}$ for \mathcal{W} is defined by

$$S_{\mathcal{W}}(x) = T_{\mathcal{W}}^* T_{\mathcal{W}}(x) = \sum_{i \in I} v_i^2 P_{W_i}(x).$$

Interestingly, a fusion frame operator exhibits properties similar to a frame operator concerning invertibility. In fact, if $\{(W_i, v_i)\}_{i=1}^M$ is a fusion frame for \mathbb{H}^N with fusion frame bounds C and D , then the associated fusion frame operator $S_{\mathcal{W}}$ is positive and invertible on \mathbb{H}^N , and

$$(8.2) \quad C \cdot Id \leq S_{\mathcal{W}} \leq D \cdot Id.$$

We refer the reader to [31, Prop. 3.16] for details.

There has been a significant amount of recent work on fusion frames. This topic now has its own website and we recommend visiting it for the latest developments on Fusion frames and distributed processing/ sensor networks.

<http://www.fusionframe.org/>

But also visit the Frame Research Center Website:

<http://www.framerc.org/>

9. INFINITE DIMENSIONAL HILBERT SPACES

We work with two standard infinite dimensional Hilbert Spaces.

9.1. Hilbert Spaces of Sequences. We begin with the definition.

Definition 9.1. We define ℓ_2 by:

$$\{x = \{a_i\}_{i=1}^{\infty} : \|x\| := \sum_{i=1}^{\infty} |a_i|^2 < \infty.\}$$

The **inner product** of $x = \{a_i\}_{i=1}^{\infty}$ and $y = \{b_i\}_{i=1}^{\infty}$ is given by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} a_i \bar{b}_i.$$

The space ℓ_2 has a natural orthonormal basis $\{e_i\}_{i=1}^\infty$ where

$$e_i = (0, 0, \dots, 0, 1, 0, \dots)$$

where the 1 is in the i^{th} -coordinate. Most of the results on finite dimensional Hilbert spaces carry over here. The one major exception is that operators here may not have eigenvalues or be diagonalizable. We give an example in the next subsection. But all of the “identities” hold here. Another difference is the notion of **linear independence**.

Definition 9.2. A family of vectors $\{x_i\}_{i=1}^\infty$ in ℓ_2 is **linearly independent** if for every finite subset $I \subset \mathbb{N}$ and any scalars $\{a_i\}_{i \in I}$ we have

$$\sum_{i \in I} a_i x_i = 0 \Rightarrow a_i = 0, \text{ for all } i \in I.$$

The family is ω -independent if for any family of scalars $\{a_i\}_{i=1}^\infty$, with $\sum_{i=1}^\infty |a_i|^2 < \infty$, we have

$$\sum_{i=1}^\infty a_i x_i = 0 \Rightarrow a_i = 0 \text{ for all } i = 1, 2, \dots.$$

An orthonormal basis $\{e_i\}_{i=1}^\infty$ is clearly ω -independent. But, linearly independent vectors may not be ω -independent. For example, if we let

$$x_i = e_i + \frac{1}{2^i} e_{i+1}, \quad i = 1, 2, \dots,$$

it is easily checked that this family is finitely linearly independent. But,

$$\sum_{i=1}^\infty \frac{(-1)^{i-1}}{2^i} x_i = 0,$$

and so this family is not ω -independent.

9.2. Hilbert Spaces of Functions. We define a Hilbert space of functions:

Definition 9.3. If $A \subset \mathbb{R}$ (or \mathbb{C}) we define

$$L^2(A) = \left\{ f : A \rightarrow \mathbb{R} \text{ (or } \mathbb{C}) : \|f\|^2 := \int_A |f(t)|^2 dt < \infty \right\}.$$

The inner product of $f, g \in L^2(A)$ is

$$\langle f, g \rangle = \int_I f(t) \overline{g(t)} dt.$$

The two cases we work with the most are $L^2([0, 1])$ and $L^2(\mathbb{R})$.

The space $L^2([0, 1])$ has a natural orthonormal basis given by the complex exponentials:

$$\{e^{2\pi i n t}\}_{n \in \mathbb{Z}}.$$

If we choose $A \subset [0, 1]$, the orthogonal projection of $L^2([0, 1])$ onto $L^2(A)$ satisfies:

$$P(e^{2\pi int}) = \chi_A e^{2\pi int}.$$

This family of vectors is a Parseval frame (since it is the image of an orthonormal basis under an orthogonal projection) called the **Fourier frame on A** . Recently, Marcus/Spielman/Srivastava [68] solved the **Feichtinger Conjecture** for this class of frames (See subsection 10.6).

There are positive, self-adjoint invertible operators on infinite dimensional Hilbert spaces which have no eigenvectors. Consider the operator $S : L^2([0, 1]) \rightarrow L^2([0, 1])$ defined by $S(f)(x) := (1 + x)f(x)$. For any $f \in L^2[0, 1]$ we have:

$$\langle f, Sf \rangle = \int_0^1 (1 + x)f^2(x)dx \geq \int_0^1 f^2(x)dx = \|f\|^2.$$

So S is a positive self-adjoint invertible operator. Also, in order for $Sf = \lambda f$ we would need to have

$$(1 + x)f(x) = \lambda f(x), \text{ a.e. on } [0, 1].$$

This is clearly impossible unless $f = 0$.

10. MAJOR OPEN PROBLEMS IN FRAME THEORY

In this section we look at some of the major open problems in Frame Theory.

10.1. Equiangular Frames. One of the simplest stated yet deepest problems in mathematics is the equiangular line problem.

Problem 10.1. *How many equiangular lines can be drawn through the origin in \mathbb{R}^N or \mathbb{C}^N ?*

The easiest way to describe **equiangular lines** is to put a unit norm vector starting at the origin on each line, say $\{\varphi_i\}_{i=1}^M$, and the lines are **equiangular** if there is a constant $0 < c \leq 1$ so that

$$|\langle \varphi_i, \varphi_j \rangle| = c, \text{ for all } i \neq j.$$

These inner products represent the cosine of the acute angle between the lines. The problem of constructing any number (especially, the maximal number) of equiangular lines in \mathbb{R}^N is one of the most elementary and at the same time one of the most difficult problems in mathematics. After sixty years of research, we do not know the answer for all dimensions ≤ 20 in either the real or complex case. This line of research was started in 1948 by Hanntjes [57] in the setting of elliptic geometry where he identified the maximal number of equiangular lines in \mathbb{R}^N for $n = 2, 3$. Later, Van Lint and Seidel [67] classified the largest number of equiangular lines in \mathbb{R}^N for dimensions $N \leq 7$ and at the same time emphasized the relations to discrete mathematics. In 1973, Lemmens and Seidel [65] made a comprehensive study of real equiangular line sets which is

still today a fundamental piece of work. Gerzon [65] gave an upper bound for the maximal number of equiangular lines in \mathbb{R}^N :

Theorem 10.2 (Gerzon). *If we have M equiangular lines in \mathbb{R}^N then*

$$M \leq \frac{N(N+1)}{2}.$$

It is known that we cannot reach this maximum in most cases. It is also known that the maximal number of equiangular lines in \mathbb{C}^N is less than or equal to N^2 . It is believed that this number of lines exists in \mathbb{C}^N for all N but until now a positive answer does not exist for all $N \leq 20$.

Also, P. Neumann [65] produced a fundamental result in the area:

Theorem 10.3 (P. Neumann). *If \mathbb{R}^N has M equiangular lines at angle $1/\alpha$ and $M > 2N$, then α is an odd integer.*

Finally, there is a lower bound on the angle formed by equiangular line sets.

Theorem 10.4. *If $\{\varphi_m\}_{m=1}^M$ is a set of norm one vectors in \mathbb{R}^N , then*

$$(10.1) \quad \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle| \geq \sqrt{\frac{M-N}{N(M-1)}}.$$

Moreover, we have equality if and only if $\{\varphi_i\}_{i=1}^M$ is an equiangular tight frame and in this case the tight frame bound is $\frac{M}{N}$.

This inequality goes back to Welch [78]. Strohmer and Heath [75] and Holmes and Paulsen [60] give more direct arguments which also yields the “moreover” part. For some reason, in the literature there is a further assumption added to the “moreover” part of Theorem 10.4 that the vectors span \mathbb{R}^N . This assumption is not necessary. That is, equality in inequality 10.1 already implies that the vectors span the space [37].

Equiangular Tight Frames:

Good references for real equiangular frames are [37, 60, 75, 76]. A unit norm frame with the property that there is a constant c so that

$$|\langle \varphi_i, \varphi_j \rangle| = c, \quad \text{for all } i \neq j,$$

is called an **equiangular frame** at angle c . Equiangular tight frames first appeared in discrete geometry [67] but today (especially the complex case) have applications in signal processing, communications, coding theory and more [56, 75]. A detailed study of this class of frames was initiated by Strohmer and Heath [75] and Holmes and Paulsen [60]. Holmes and Paulsen [60] showed that equiangular tight frames give error correction codes that are robust against two erasures. Bodmann and Paulsen [13] analyzed arbitrary numbers of erasures

for equiangular tight frames. Recently, Bodmann, Casazza, Edidin and Balan [11] showed that equiangular tight frames are useful for signal reconstruction when all phase information is lost. Recently, Sustik, Tropp, Dhillon and Heath [76] made an important advance on this subject (and on the complex version). Other applications include the construction of capacity achieving signature sequences for multiuser communication systems in wireless communication theory [78]. The tightness condition allows equiangular tight frames to achieve the capacity of a Gaussian channel and their equiangularity allows them to satisfy an interference invariance property. Equiangular tight frames potentially have many more practical and theoretical applications. Unfortunately, we know very few of them and so their usefulness is largely untapped.

The main problem:

Problem 10.5. *Classify all equiangular tight frames, or find large classes of them.*

Fickus/Jasper/Mixon [50] gave a large class of **Kirkman equiangular tight frames** and used them in coding theory.

Theorem 10.6. *The following are equivalent:*

- (1) *The space \mathbb{R}^N has an equiangular tight frame with M elements at angle $1/\alpha$.*
- (2) *We have*

$$M = \frac{(\alpha^2 - 1)N}{\alpha^2 - N},$$

and there exist M equiangular lines in \mathbb{R}^N at angle $1/\alpha$.

Moreover, in this case we have:

- (a) $\alpha \leq N \leq \alpha^2 - 2$.
- (b) $N = \alpha$ if and only if $M = N + 1$.
- (c) $N = \alpha^2 - 2$ if and only if $M = \frac{N(N+1)}{2}$.
- (d) $M = 2N$ if and only if

$$\alpha^2 = 2N - 1 = a^2 + b^2, \quad a, b \text{ integers.}$$

If $M \neq N + 1, 2N$ then:

- (e) α is an odd integer.
- (f) M is even.
- (g) α divides $M-1$.
- (h) $\beta = \frac{M-1}{\alpha}$ is the angle for the complementary equiangular tight frame.

10.2. The Scaling Problem. The scaling problem is one of the deepest problems in frame theory.

Problem 10.7 (Scaling Problem). *Classify the frames $\{\varphi_i\}_{i=1}^M$ for \mathbb{H}^N so that there are scalars $\{a_i\}_{i=1}^M$ for which $\{a_i\varphi_i\}_{i=1}^M$ is a Parseval frame? Give an algorithm for finding $\{a_i\}_{i=1}^M$.*

This is really a special case of an even deeper problem.

Problem 10.8. *Given a frame $\{\varphi_i\}_{i=1}^M$ for \mathbb{H}^N , find the scalars $\{a_i\}_{i=1}^M$ so that $\{a_i\varphi_i\}_{i=1}^M$ has the minimal condition number with respect to all such scalings.*

For recent results on the scaling problem see Chen/Kutyniok/Okoudjou/Philipp/Wang [40], Cahill/Chen [15] and Kutyniok/Okoudjou/Philipp/Tuley [64].

10.3. Sparse Orthonormal Bases for Subspaces. We will look at two questions concerning the construction of sparse orthonormal bases.

Definition 10.9. *Given a vector $x = (a_1, a_2, \dots, a_N) \in \mathbb{H}^N$, we let*

$$\|x\|_0 = |\{1 \leq i \leq N : a_i \neq 0\}|.$$

A natural question in Hilbert space theory is:

Problem 10.10. *Given a Hilbert space \mathbb{H}^N with orthonormal basis $\{e_i\}_{i=1}^N$ and a K dimensional subspace W , find the sparsest orthonormal basis for W with respect to $\{e_i\}_{i=1}^N$. That is, find a orthonormal basis $\{g_i\}_{i=1}^K$ for W so that*

$$\sum_{i=1}^K \|g_i\|_0, \text{ is a minimum with respect to all orthonormal bases for } W.$$

Sparse Gram-Schmidt Orthogonalization:

There is a basic notion for turning a linearly independent set into an orthonormal set with the same partial spans: Gram-Schmidt Orthogonalization. Given a set $\{\varphi_i\}_{i=1}^M$ of linearly independent vectors in \mathbb{H}^N , first let

$$e_1 = \frac{\varphi_1}{\|\varphi_1\|}.$$

Assume we have constructed $\{e_i\}_{i=1}^K$ satisfying:

- (1) $\{e_i\}_{i=1}^K$ is orthonormal.
- (2) We have

$$\text{span}_{1 \leq i \leq j} e_i = \text{span}_{1 \leq i \leq j} \varphi_i, \text{ for all } j = 1, 2, \dots, K.$$

We then let

$$\psi_{K+1} = \varphi_{K+1} - \sum_{i=1}^K \langle \varphi_{K+1}, e_i \rangle e_i$$

and let

$$e_{K+1} = \frac{\psi_{K+1}}{\|\psi_{K+1}\|}.$$

If we have a fixed basis $\{g_i\}_{i=1}^N$ for \mathbb{H}^N , we can compute

$$\sum_{i=1}^K \|e_i\|_0 \text{ with respect to the basis } \{g_i\}_{i=1}^N.$$

But this sum is different for different orderings of the original vectors $\{\varphi_i\}_{i=1}^M$. Related to Problem 10.10 we have:

Problem 10.11. *What is the correct ordering of $\{\varphi_i\}_{i=1}^M$ so that Gram-Schmidt Orthogonalization produces the sparsest orthonormal sequence with respect to all possible orderings?*

10.4. The Paulsen Problem. To state the Paulsen Problem, we need some definitions.

Definition 10.12. *A frame $\{\varphi_i\}_{i=1}^M$ for \mathbb{H}^N with frame operator S is said to be ε -equal norm if*

$$(1 - \varepsilon) \frac{N}{M} \leq \|\varphi_i\|^2 \leq (1 + \varepsilon) \frac{N}{M}, \text{ for all } i = 1, 2, \dots, M,$$

and it is ε -nearly Parseval if

$$(1 - \varepsilon) \cdot Id \leq S \leq (1 + \varepsilon) \cdot Id.$$

Definition 10.13. *Given frames $\Phi = \{\varphi_i\}_{i=1}^M$ and $\Psi = \{\psi_i\}_{i=1}^M$, we define the distance between them by:*

$$d(\Phi, \Psi) = \sum_{i=1}^M \|\varphi_i - \psi_i\|^2.$$

Because we did not take the square-root on the right-hand-side of the above inequality, the function d is not really a distance function.

The **Paulsen Problem** now states:

Problem 10.14. *How close in terms of d is an ε -nearly equal norm and ε -nearly Parseval frame to an equal norm Parseval frame?*

The importance of this problem is that we have algorithms for finding frames which are equal norm and nearly Parseval. But we do not know that these are actually close to any equal norm Parseval frame.

The closest equal norm frame to a frame is known and the closest Parseval frame to a frame is known:

The closest equal norm frame to a frame $\{\varphi_i\}_{i=1}^M$ is

$$\left\{ C \frac{\varphi_i}{\|\varphi_i\|} \right\}_{i=1}^M \text{ where } C := \frac{\sum_{i=1}^M \|\varphi_i\|}{M}.$$

If $\Phi = \{\varphi_i\}_{i=1}^M$ is a frame with frame operator S , the closest Parseval frame to Φ is the canonical Parseval frame $\{S^{-1/2}\varphi_i\}_{i=1}^M$ [34] (see [10] for a better calculation).

Also, there is an algorithm for turning a frame into an equal norm frame without changing the frame operator [25].

Casazza/Cahill [16] showed that the Paulsen Problem is equivalent to an old deep problem in Operator Theory.

Problem 10.15. *Given a projection P on \mathbb{H}^N with ε -nearly equal diagonal elements of its matrix, what is the closest constant diagonal projection to P ?*

10.5. Concrete Construction of RIP Matrices. Compressive Sensing is one of the most active areas of research today. See the book [55] for an exhaustive coverage of this subject. A fundamental tool in this area matrices with the **Restricted Isometry Property**, denoted **RIP**. Compressive sensing is a method for solving **underdetermined systems** if we have some form of **sparsity** of the incoming signal.

Definition 10.16. *A vector $x = (a_1, a_2, \dots, a_N) \in \mathbb{H}^N$ is **K-sparse** if*

$$|\{1 \leq i \leq N : a_i \neq 0\}| \leq K.$$

The fundamental tool in compressive sensing is the class of **Restricted Isometry Property (RIP)** matrices.

Definition 10.17. *A matrix Φ has the (K, δ) - **Restricted Isometry Property, RIP** if*

$$(1 - \delta)\|x\|^2 \leq \|\Phi x\|^2 \leq (1 + \delta)\|x\|^2,$$

*for every K -sparse vector x . The smallest δ for which Φ is (K, δ) -RIP is the **restricted isometry constant (RIC)** δ_K .*

The main result here is (see [55]):

Theorem 10.18. *Given $\delta < 1$, there exist $N \times M$ matrices with restricted isometry constant $\delta_K \leq \delta$ for*

$$K \leq c \frac{N}{\ln(N/K)},$$

for a universal constant c .

This means, for example, in an N -dimensional Hilbert space, we can find a set of $100N$ norm one vectors $\{\varphi_i\}_{i=1}^{100N}$ for which every subset of size $N/100$ is a δ -Riesz basic sequence. This is a quite amazing result. We know that in an N -dimensional Hilbert space, any orthogonal set must have $\leq N$ elements. This result says that if we relax this requirement just a little, we can find huge sets of vectors for which every subset of size a proportion of the dimension of the space is nearly orthogonal.

In the language of frame theory, we are looking for a family of unit norm vectors $\Phi = \{\varphi_i\}_{i=1}^M$ in \mathbb{H}^N so that every subset of Φ of size a proportion of N is nearly orthogonal and M is much larger than N . The existence of such matrices has been carried out by random matrix theory. Which means we do not know concretely a single such matrix, despite the fact that these are essential for compressive sensing. For years, the closest thing to **concrete** here was a result of DeVore [45] which constructed $N \times M$ matrices for which subsets of size \sqrt{N} were δ -Riesz. But this is far from what we know is true which is subsets of size cN for $0 < c < 1$ independent of N . Bandira/Fickus/Mixon/Wong [5] investigated various methods for constructing RIP matrices. Bourgain [14] then **broke the square root barrier** by showing we can concretely construct RIP matrices with subsets of size $N^{1/2+\varepsilon}$ being δ -Riesz. There is also an important result of Rudelson/Vershynin [73] which says that if we take a random selection of rows from the Discrete Fourier Transform Matrix, then this submatrix will be a RIP matrix. Since these matrices are fundamental to compressive sensing, a longstanding, important and fundamental problem here is:

Problem 10.19. *Give a concrete construction of RIP matrices.*

10.6. An Algorithm for the Feichtinger Conjecture. For nearly 50 years the Kadison-Singer problem [62] has defied the best efforts of some of the most talented mathematicians of our time. It was just recently solved by Marcus/Spielman/Srivastava [68]. For a good summary of the history of this problem and consequences of this achievement see [26].

In his work on time-frequency analysis, Feichtinger [53, 38] noted that all of the Gabor frames he was using had the property that they could be divided into a finite number of subsets which were Riesz basic sequences. This led to a conjecture known as the **Feichtinger Conjecture** [38]. There is a significant body of work on this conjecture and we refer the reader to [26] for the best reference.

First we need:

Definition 10.20. *A family of vectors $\{\varphi_i\}_{i \in I}$ is an ε -Riesz basic sequence for $0 < \varepsilon < 1$ if for every family of scalars $\{a_i\}_{i \in I}$ we have*

$$(1 - \varepsilon) \sum_{i \in I} |a_i|^2 \leq \left\| \sum_{i \in I} a_i \varphi_i \right\|^2 \leq (1 + \varepsilon) \sum_{i \in I} |a_i|^2.$$

The following theorem gives the best quantitative solution to the **Feichtinger Conjecture** from the results of [68].

Theorem 10.21 (Marcus/Spielman/Srivastava). *Every unit norm B-Bessel sequence can be partitioned into r -subsets each of which is a ε -Riesz basic*

sequence, where

$$r = \left(\frac{6(B+1)}{\varepsilon} \right)^2 \text{ in the real case ,}$$

and

$$r = \left(\frac{6(B+1)}{\varepsilon} \right)^4 \text{ in the complex case .}$$

This theorem could be quite useful, except that it is an **existence proof**. Now what we really need is:

Problem 10.22. *Find an implementable algorithm for proving the Feichtinger Conjecture.*

10.7. Classifying Gabor Frames. Gabor frames form the basis for time-frequency analysis which is the mathematics behind signal processing. This is a huge subject which cannot be covered here except for a few remarks. We recommend the excellent book of Gröchenig [54] for a comprehensive coverage of this subject.

Definition 10.23. *Fix $a, b > 0$. For a function $f \in L^2(\mathbb{R})$ we define*

$$\text{Translation by } a : T_a f(x) = f(x - a),$$

$$\text{Modulation by } b : M_b f(x) = e^{2\pi i b x} f(x).$$

In 1946, Gabor [53] formulated a fundamental approach to signal decomposition in terms of elementary signals. Gabor's approach quickly became a paradigm for the spectral analysis associated with time-frequency methods, such as the short-time Fourier transform and the Wigner transform. For Gabor's method, we need to fix a **window function** $g \in L^\infty(\mathbb{R})$ and $a, b \in \mathbb{R}^+$. If the family

$$\mathcal{G}(g, a, b) = \{M_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$$

is a frame for $L^2(\mathbb{R})$ we call this a **Gabor frame**. Gabor frames are used in signal processing. It is a very deep question which values of a, b, g give Gabor frames. There are some necessary requirements however.

Theorem 10.24. *If the family given by (g, a, b) yields a Gabor frame then:*

- (1) $ab \leq 1$.
- (2) *If $ab = 1$ then this family is a frame if and only if it is a Riesz basis.*

Also, the Balian-Low Theorem puts some restrictions on the function $g \in L^2(\mathbb{R})$ for the case $ab = 1$.

Theorem 10.25 (Balian-Low Theorem). *If $g \in L^2(\mathbb{R})$, $ab = 1$ and (g, a, b) generates a Gabor frame, then either $xg(x) \notin L^2(\mathbb{R})$ or $g' \notin L^2(\mathbb{R})$.*

The Balian-Low Theorem implies that Gaussian functions e^{-ax^2} cannot yield Gabor frames for $ab = 1$.

The main problem here is:

Problem 10.26. *Find all functions g and positive constants a, b so that (g, a, b) forms a Gabor frame for $L^2(\mathbb{R})$.*

Recently, a significant advance was made on this problem by Dai/Sun [48] when they solved the old and famous **abc-problem**. We refer to [48] for the history of the problem. In particular, Dai/Sun classified all triples (a, b, c) so that

$$\mathcal{G}(\chi_I, a, b) \text{ is a Gabor frame when } |I| = c.$$

10.8. Phase Retrieval. Phase retrieval is one of the largest areas of engineering with applications to **x-ray crystallography, Electron Microscopy, Coherence Theory, Diffractive Imaging, Astronomical Imaging, x-ray tomography, Optics, Digital Holography, Speech Recognition** and more [12, 7, 51, 52, 69, 70, 71]. For an introduction to this subject see [17].

Phase retrieval is the problem of recovering a signal from the absolute values of linear measurement coefficients called *intensity measurements*. Note multiplying a signal by a global phase factor does not affect these coefficients, so we seek signal recovery mod a global phase factor.

There are two main approaches to this problem of phase retrieval. One is to restrict the problem to a subclass of signals on which the intensity measurements become injective. The other is to use a larger family of measurements so that the intensity measurements map any signal injectively. The latter approach in phase retrieval first appears in [3] where the authors examine injectivity of intensity measurements for finite Hilbert spaces. The authors completely characterize measurement vectors in the real case which yield such injectivity, and they provide a surprisingly small upper bound on the minimal number of measurements required for the complex case. This sparked an incredible volume of current phase retrieval research [1, 2, 4, 20, 35, 22, 46, 49] focused on algorithms and conditions guaranteeing injective and stable intensity measurements.

Definition 10.27. *A family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ does **phase retrieval** on \mathbb{H}^N if whenever $x, y \in \mathbb{H}^N$ satisfy*

$$|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle|, \text{ for all } i = 1, 2, \dots, M,$$

then $x = cy$ where $|c| = 1$.

A fundamental result in phase retrieval involves the **complement property**.

Definition 10.28. *A family of vectors $\{\varphi_i\}_{i=1}^M$ in \mathbb{H}^N has the **complement property** if whenever we choose $I \subset \{1, 2, \dots, M\}$, at least one of the sets $\{\varphi_i\}_{i \in I}$ or $\{\varphi_i\}_{i \in I^c}$ spans \mathbb{H}^N .*

Note that the complement property implies $M \geq 2N - 1$. For if $M \leq 2N - 2$ then we can choose $I = 1, 2, \dots, N - 1$ and since the two induced subsets of our vectors each has only $N - 1$ vectors, neither can span \mathbb{H}^N .

The fundamental result here is due to Balan/Casazza/Edidin [3]:

Theorem 10.29. *In \mathbb{R}^N , a family of vectors $\{\phi_i\}_{i=1}^M$ does phase retrieval if and only if it has the complement property. Moreover, there is a dense set of families of vectors $\{\varphi_i\}_{i=1}^{2N-1}$ which do phase retrieval.*

In the complex case, [3] showed that a dense set of families of $(4N - 2)$ -vectors does phase retrieval. Later, Bodmann [9] showed that phase retrieval can be done in the complex case with $(4N - 4)$ vectors. This was then improved by Conca/Edidin/Hering/Vinzant [42].

Theorem 10.30. *In \mathbb{C}^N , there are families (in fact a dense set of families) of vectors $\{\varphi_i\}_{i=1}^{4N-4}$ which do phase retrieval.*

Again, phase retrieval cannot be done with fewer vectors than $4N - 4$.

Given a signal x in a Hilbert space, intensity measurements may also be thought of as norms of x under rank one projections. Here the spans of measurement vectors serve as the one dimensional range of the projections. In some applications, however, a signal must be reconstructed from the norms of higher dimensional components. In X-ray crystallography for example, such a problem arises with crystal twinning [47]. In this scenario, there exists a similar phase retrieval problem: given subspaces $\{W_n\}_{n=1}^M$ of an N -dimensional Hilbert space \mathbb{H}^N and orthogonal projections $P_n : \mathbb{H}^N \rightarrow W_n$, can we recover any $x \in \mathbb{H}^N$ (up to a global phase factor) from the measurements $\{\|P_n x\|\}_{n=1}^M$? This problem was recently studied in [6] where the authors use semidefinite programming to develop a reconstruction algorithm for when the $\{W_n\}_{n=1}^M$ are equidimensional random subspaces. Most results using random intensity measurements require the cardinality of measurements to scale linearly with the dimension of the signal space along with an additional logarithmic factor [22], but this logarithmic factor was recently removed in [35]. Similarly, signal reconstruction from the norms of equidimensional random subspace components are possible with the cardinality of measurements scaling linearly with the dimension [6].

In [17] it was shown:

Theorem 10.31. *Phase retrieval can be done on \mathbb{R}^N with $2N - 1$ orthogonal projections of arbitrary rank.*

This theorem raises an important question:

Problem 10.32. *Can phase retrieval be done on \mathbb{R}^N (respectively, \mathbb{C}^N) with fewer than $2N - 1$ (respectively, $4N - 4$) projections? If so, what are the fewest number of projections needed in both the real and complex case?*

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