AMS Short Course on Sums of Squares

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1. For a polynomial $p(x_1, \dots, x_n)$ in n variables of degree d, define *homogenization* $\bar{p}$ of p with respect to a new variable x_0 to be

$$\bar{p}(x_0, \dots, x_n) = x_0^d\, p\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right).$$

For example, homogenization of $x_1 + x_2^2$ is $x_0x_1 + x_2^2$ and homogenization of $x_1^3 - x_1x_2 + 3$ is $x_1^3 - x_0x_1x_2 + 3x_0^3$. Show that if a polynomial p is nonnegative then its homogenization is nonnegative as well. Does the same hold for strictly positive polynomials?

2. The *Newton Polytope* N_p of a polynomial p is the convex hull of the vectors of monomial exponents that occur in p. For example, the Newton Polytope of $x^2 + xy + z^2$ is the convex hull of vectors $(2, 0, 0), (1, 1, 0)$ and $(0, 0, 2)$.

Show that if a polynomial $p = \sum_i q_i^2$ is a sum of squares then the Newton Polytope of each q_i is contained in $\frac{1}{2}N_p$.

3. Use Exercise 3 to show that the Motzkin polynomial $M = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ is not a sum of squares.

4. (Choi-Lam) Show that the form $S(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$ is nonnegative but not a sum of squares.

5. Formulate a sum of squares relaxation for computing the minimum of $1 - x + x^2$ on the interval $x \geq 0$. Write down the corresponding semidefinite program and solve it.

6. Show that a univariate nonnegative polynomial is a sum of squares. What can be said about polynomials nonnegative on an interval (not necessarily bounded)?

7. The Max-Cut problem asks given a graph G to find the partition of vertices of G into two classes V_1 and V_2 such that the number of edges between V_1 and V_2 is maximal. We can model the choice of two classes as assigning $+1$ to vertices in V_1 and -1 to vertices in V_2.

(a) Write down the size of a cut as a quadratic function (that depends on the graph G) on the $\{+1, -1\}$ hypercube $\{+1, -1\}^n$.

(b) Formulate a sum of squares relaxation for the Max-Cut problem as a semidefinite program.

8. (Independent Set Polytope). Let G be a graph on n vertices. A subset S of vertices of G is called an independent set, if there are no edges between vertices of S. Any subset S of vertices of G can be encoded as a 0/1 subset S' of $\mathbb{R}^n$, where 1's in S' correspond to elements of S. Let $\mathcal{I}(G)$ be the set of 0/1 points in $\mathbb{R}^n$ which encode all the independent subsets of G. The convex hull of $\mathcal{I}(G)$ is called the independent (stable) set polytope of G.

(a) Find the vanishing ideal of $\mathcal{I}(G)$.

(b) Formulate a hierarchy of sum of squares relaxations for computing linear functionals non-negative on $\mathcal{I}(G)$.

(c)* When can we compute all linear functionals nonnegative on $\mathcal{I}(G)$ using only degree 2 sums of squares relaxation (sums of squares of linear polynomials)?

(d) Can you generalize the above procedure to other polytopes?