INFINITE GEODESICS, ASYMPTOTIC DIRECTIONS, AND BUSEMANN FUNCTIONS

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1. Geodesics and infinite geodesics

1.1. **Setting and assumptions.** We consider the setting of FPP on $(\mathbb{Z}^d, \mathcal{E}^d)$ — we recall some of the notation here for readability. For each $e \in \mathcal{E}^d$, there is an associated edge weight $t_e \geq 0$. We assume for definiteness that the collection (t_e) is independent and identically distributed with common distribution F. However, it is important to note that in many of the results below, the i.i.d. restriction can be considerably relaxed to more general translation-invariant cases. We generally write $\mathbf{P}$ for the joint distribution of the edge weights.

In all cases that follow, we consider F having at least the following properties.

Assumption 1 (Standing assumptions). Unless otherwise specified, from now on we assume:

(1) $\mathbf{E}\min\{t_1, \ldots, t_{2d}\}^d < \infty$.
(2) F is continuous — i.e., $\mathbf{P}(t_e = a) = 0$ for each real a.

Note that (1) from Assumption 1 is the moment assumption made in the statement of the shape theorem in earlier lectures. Item (2) is stronger than the assumption $\mathbf{P}(t_e = 0) < p_c$ from that theorem, so these assumptions give us the existence of a limit shape $\mathcal{B}$ and a norm g such that $\mathcal{B} = \{x : g(x) \leq 1\}$.

For some results (the results of Newman below — see Theorem 8), we will need the following strong assumptions in addition to the above.

Assumption 2 (Strong assumptions).

(ExpM) There is some $\alpha > 0$ such that $\mathbf{E}\exp(\alpha t_e) < \infty$.
(Curve) The limit shape $\mathcal{B}$ is uniformly curved, in the following sense. There exists a uniform constant $c_{\mathcal{B}}$ such that, for any z_1 and z_2 having $g(z_1) = g(z_2) = 1$ and any $a \in [0,1]$,

$$g(z) \leq 1 - c\left[\min\{g(z - z_1),\, g(z - z_2)\}\right]^2 \ .$$

Some comments on these assumptions are in order. Assumption (ExpM) is straightforward but strong, and it is (as we will see) unnecessary for many of the results on infinite geodesics. In fact, it is mainly applied to guarantee concentration results for passage times hold, and any result which relies too heavily on these will not adapt well to the non-i.i.d. settings mentioned above.

Assumption (Curve) is widely believed to hold for most "reasonable" edge weight distributions, and this claim is bolstered by comparison to some exactly solvable models closely related to FPP. But as it stands, (Curve) is far from proof! Indeed, as discussed in the first lectures, it is not even currently known that there exists some F such that $\mathcal{B}$ is strictly convex for i.i.d. edge weights, though this is also highly plausible.

1.2. **Types of geodesics.** We recall the definition of a geodesic between two terminal points x and y.

Definition 1. A nearest-neighbor path $\gamma = (x = x_0, x_1, \ldots, x_n = y)$ is called a **(finite) geodesic** between x and y if $T(\gamma) = T(x, y)$.

Lemma 2. Under Assumption 1, there a.s. exists a unique finite geodesic between each x and y in $\mathbb{Z}^d$.

We call the event on which the global existence and uniqueness of Lemma 2 holds by the name Ω_u.

Proof. The existence claim follows easily from the shape theorem; we sketch the argument. Namely, we have under our assumptions that (a.s.) $T(x, z) \to \infty$ as $|z| \to \infty$. On the other hand, taking fixing a nonrandom shortest lattice path (in the $\mathbb{Z}^d$ metric, not the FPP metric) γ from x to y, we have that $T(x, y) \leq T(\gamma)$. In particular, for a.e. realization of the edge weights, there is some Euclidean ball such that $T(x, z) > T(x, y)$ for any z outside this ball. This implies that the infimum over paths in the definition of $T(x,y)$ is a.s. effectively over a finite set and ensures the existence of the geodesic.

We show uniqueness. Suppose that $\gamma_1 \neq \gamma_2$ are two finite nearest-neighbor paths from x to y — in particular, that $e \in \gamma_1 \setminus \gamma_2$. For both γ_1 and γ_2 to be geodesics from x to y, we must have $T(\gamma_1) = T(\gamma_2)$; we show this has probability zero.

Condition on the values of the weights of all edges other than e, and view $h = T(\gamma_1) - T(\gamma_2)$ conditionally as a function of t_e. We have $h'(t_e) = 1$ for all values of t_e, so there exists at most one value a such that $h(a) = 0$. But $\mathbf{P}(t_e = a) = 0$, so there is conditional probability 0 that $T(\gamma_1) = T(\gamma_2)$. □

Definition 3. The unique geodesic from x to y whose existence is guaranteed by Lemma 2 is denoted $G(x,y)$.

Of course, a subpath of a geodesic is also a geodesic. In particular, if x_i and x_j are vertices of $G(x,y)$, then the subpath of $G(x,y)$ between them is identical to $G(x_i,x_j)$.

One can ask many interesting questions about the behavior of $G(x,y)$ — for instance, how far does it typically deviate or "wander" from the straight line segment between x and y? One such question of particular importance, which is closely related to the preceding wandering question, is:

- For what sequences $(y_n)_n \subseteq \mathbb{Z}^d$ with $|y_n| \to \infty$ does the sequence of geodesics $(G(x,y_n))_n$ converge?

Here we say a sequence of (nearest-neighbor) paths $(\gamma_n)_n$ converges to a path γ if for each fixed k and all large n, the first k vertices of γ_n are identical with the first k vertices of γ. It is not hard to see (and we make precise in the proof of Proposition 5 below) that the following holds for almost every edge weight configuration: if there are a vertex x, a sequence $(y_n)_n$, and some infinite path γ such that $\lim_n G(x,y_n) = \gamma$, then γ must be an infinite geodesic, in the sense of the next definition.

Definition 4. An infinite path $\gamma = (x_0,\, x_1,\, \ldots)$ is an **infinite geodesic** or **unigeodesic** if its finite subsegments are all finite geodesics – that is, for every $i < j$,

$$(x_i,\, x_{i+1},\, ,\ldots,\, x_j) = G(x_i,x_j)\ .$$

Of course, simpler than convergence statements would just be existence statements, and this is mainly what we will focus on. Do there exist unigeodesics? How many? And what can we say about their wandering?

Proposition 5. With probability one, there exists an infinite geodesic $\gamma = (0 = x_0,\, x_1, \ldots)$.

Proof. This is a standard "diagonal argument". Consider the sequence $(G(0,ne_1))_n$. The second vertex of each $G(0,ne_1)$ is an element of $\{\pm e_i\}_{i=1}^d$, a finite set; there thus exists an $e \in \{\pm e_i\}$ and a sequence $n_k \to \infty$ such that the first two vertices of $G(0,n_k)$ are identical for all k — i.e., $G(0,n_k e_1) = (0, e, \ldots)$ for all k. Repeating this argument on the subsequence $(G(0,n_k e_1))$ and inducting produces a subsequential limit γ.

Since any finite subsegment of γ is also a finite subsegment of $G(0,ne_1)$ for some n, and since finite subsegments of geodesics are also geodesics, we see that γ is a unigeodesic. □

The proof of Proposition 5 uses practically nothing from the model, but for this reason it is difficult to improve. One obvious consequence of translation-invariance is that we can replace 0 by any x and get an infinite geodesic from x as well. But perhaps every unigeodesic is essentially they same.

To clarify, let us call unigeodesics γ_1 and γ_2 **distinct** if $|\gamma_1 \triangle \gamma_2| = \infty$. It is easy to see that uniqueness implies that unigeodesics cannot touch, subsequently separate, then touch again, as this would imply that the separated segments are two different geodesics between vertices where they touch. This can be formalized as:

Proposition 6. On the probability one event Ω_u of Lemma 2, geodesics γ_1, γ_2 are distinct if and only if they touch at most finitely often: $|\gamma_1 \cap \gamma_2| < \infty$. More specifically, if $\gamma_1 = (z_1, z_2, \ldots)$ and $\gamma_2 = (z'_1, z'_2, \ldots)$ are distinct, the intersection is a single subpath: $\gamma_1 \cap \gamma_2 = (z_K, \ldots, z_{K+\ell}) = (z'_M, \ldots, z'_{M+\ell})$.

In particular, geodesics γ_1 and γ_2 (almost surely) cannot touch infinitely often without **merging** or **coalescing**: becoming asymptotically identical. We leave the formal proof of Proposition 6 as an exercise, since it gives some intuition about the "treelike" behavior of geodesics when they are known to be unique.

It is not clear at all a priori how to show that there exist even two distinct unigeodesics. It is intuitively plausible, however that subsequential limits of (for instance) $(G(0,ne_1))_{n>0}$ and $(G(0,-ne_1))_{n>0}$ should look very different. If some γ arose as a subsequential limit of both sequences, it would have to in some sense represent a "fast path" in both the e_1 and $-e_1$ directions, which is difficult to imagine.

Let $\mathcal{N}$ denote the random number of distinct unigeodesics (the supremal cardinality among collections of distinct unigeodesics). The above reasoning suggests that $\mathbf{P}(\mathcal{N} \geq 2) > 0$. It also suggests that limits of $G(0, ne_1)$ and $G(0, -ne_1)$ could be differently directed, in the following sense:

Definition 7. Let $\gamma = (x_0, x_1, \ldots)$ be a unigeodesic. We say that θ is a limiting direction of γ if θ is a limit point of the sequence $x_i/|x_i|$. The set of all limiting directions for γ shall be denoted $\Theta(\gamma)$.

We will spend the rest of our time on the following questions:

Question 1. Can we make the above reasoning precise to show $\mathbf{P}(\mathcal{N} \geq 2) > 0$? What is the largest value $\mathcal{N}$ can take — for instance, is $\mathbf{P}(\mathcal{N} = \infty) > 0$?

Question 2. What $\Theta(\gamma)$ are realizable for unigeodesics γ? Can we find γ such that $\Theta(\gamma)$ is strictly smaller than $\mathbb{S}^{d-1}$? Or indeed, is there positive probability that there exists γ with $\Theta(\gamma) = \mathbb{S}^{d-1}$?

2. Newman's results under curvature

The first results on these questions are due to Newman[7] (and related results with collaborators C. Licea [6] and M. Piza [8]). We say that a path γ has **direction** $\theta \in \mathbb{S}^{d-1}$ if $\Theta(\gamma) = \{\theta\}$. That is, if $\gamma = (x_0, x_1, \ldots)$, then γ has direction θ if and only if $x_i/|x_i| \to \theta$.

Theorem 8 ([7]). Assume (ExpM) and (Curve) in addition to the usual assumptions. Then:

(1) With probability one, for each $\theta \in \mathbb{S}^{d-1}$ there is a unigeodesic beginning at 0 having direction θ.
(2) With probability one, each unigeodesic has direction; i.e., for each geodesic γ, there exists a $\theta \in \mathbb{S}^{d-1}$ such that $\Theta(\gamma) = \{\theta\}$.

Since geodesics in distinct directions must be distinct, Newman's theorem suggests that "reasonable" edge weight distributions have $\mathbf{P}(\mathcal{N} = \infty) = 1$. The directedness statements of the theorem also arise from a version of the limiting procedure described after Proposition 5, by taking limits of finite geodesics $G(0, v_n)$ with $v_n/|v_n| \to \theta$. Implementing the strategy amounts to rigorously establishing a strong version of our heuristic that geodesics to points far off in direction θ should not be geodesics for points in direction θ ("no backtracking"). Here one must show a version of this statement not just for $-\theta$, but also for directions even very close to θ.

We note here briefly that a subsequent work of Licea and Newman [6] shows a limited form of a "uniqueness" statement for the geodesics described above. Namely, when $d = 2$, we have, fixing θ in some full-measure subset of $\mathbb{S}^1$, that there is a.s. at most one distinct geodesic γ such that $\Theta(\gamma) = \{\theta\}$. This will be a partial motivation for our assumption (LimG) introduced below, which allows the construction of a well-behaved "Busemann function".

We will not attempt to prove the theorem in the time allotted here, but will give a very basic idea of the connections between $\mathcal{B}$ and statements of a "no backtracking" flavor. Suppose we wish to show a much weaker statement; namely, that the geodesic from 0 to $ne_1 + ne_2$ is not likely to be identical to the concatenation of the geodesic $G(0, ne_1)$ with $G(ne_1, ne_1 + ne_2)$. One way to do this would be to show that

$$T(0, ne_1 + ne_2) \ll T(0, ne_1) + T(ne_1, ne_1 + ne_2). \tag{2.1}$$

On can approximate the left- and right-hand sides of (2.1), for n large, by $ng(e_1 + e_2)$ and $n[g(e_1) + g(e_2)]$ (up to $o(n)$ terms). Then (2.1) will certainly hold if $g(e_1 + e_2) < g(e_1) + g(e_2)$ or equivalently if $g(e_1/2 + e_2/2) < (1/2)[g(e_1) + g(e_2)]$. This follows, for instance, if $\mathcal{B}$ is strictly convex.

3. Showing $\mathcal{N} \geq 2$

The first unconditional result showing that $\mathbf{P}(\mathcal{N} \geq 2) > 0$ is due to Häggström and Pemantle [4] in the case that $d = 2$ and F is an exponential distribution. We will not discuss the details of their proof, since it is very specialized to that setting. We shall instead show a more general result due to both Garet-Marchand [3] and Hoffman [5]:

Theorem 9 (Garet-Marchand [3], Hoffman [5]). Under our standing assumptions, $\mathbf{P}(\mathcal{N} \geq 2) > 0$.

We will give an adaptation of the proof of Garet-Marchand in this section, with an eye towards subsequently introducing the Busemann function methods of Hoffman.

Our main goal is to show that with positive probability there exist sequences m_k, $n_k \to \infty$ such that $\Gamma_0 := \lim_k G(0, -n_k e_1)$ and $\Gamma_1 := \lim_k G(e_1, m_k e_1)$ are distinct. We will actually show something more general, which illustrates a general principle for finding multiple geodesics; see Lemma 10 below. Let us define, for each $z \in \mathbb{Z}^d$, $B_z(x, y) := T(x, z) - T(y, z)$. Note that $B_z(x, y) = -B_z(y, x)$.

We shall see we can find distinct unigeodesics Γ_0 and Γ_1 if we can show $B_{ne_1}(0, e_1)$ is "positively biased" for large positive n. In fact, it is easier to show this with e_1 replaced by ke_1, because the control we get over B will be asymptotic.

Lemma 10. Fix a finite set of vertices $\{x_1, \dots, x_k\} \subseteq \mathbb{Z}^d$. Let us define G_i, $1 \le i \le k$ by

$$G_i := \{z \in \mathbb{Z}^d : B_z(x_i, x_j) < 0 \quad \text{for all } j \neq i\}$$

— i.e., G_i is the set of vertices "closer to x_i". On the event that $|G_i| = \infty$ for all $1 \le i \le k$, we have $\mathcal{N} \ge k$.

Corollary 11. Assume that for some $\ell > 0$,

$$\mathbf{P}\left(\left\{\limsup_{n\to\infty} B_{ne_1}(0, \ell e_1) > 0\right\} \cap \left\{\limsup_{n\to\infty} B_{-ne_1}(\ell e_1, 0) > 0\right\}\right) > 0 \ . \tag{3.1}$$

Then $\mathbf{P}(\mathcal{N} \ge 2) > 0$.

Proof. Letting $x_1 = 0$, $x_2 = \ell e_1$, on the event in (3.1) we have $|G_i| = \infty$, $i = 1, 2$. The result now follows from Lemma 10 .

□

Proof of Lemma 10. As a preliminary, we observe that G_i and G_j are disjoint by construction, and that with probability one $x_i \in G_i$. We will show that for each i there is an infinite path Γ_i of vertices of G_i, having x_i as its initial vertex, which is actually a unigeodesic.

For each i, choose a sequence $(z_k(i))_k$ of distinct elements of G_i such that $\Gamma_i := \lim_k G(x_i, z_k(i))$ exists; that this is possible follows via the diagonal argument of the proof of Proposition 5. We claim that each vertex of Γ_i is an element of G_i. If there were some $y \in \Gamma_i$ with $y \notin G_i$, then for some j, $T(x_i, y) \ge T(x_j, y)$. But since $z \in G(x_i, z_k(i))$ for some k, we have $T(x_i, z_k(i)) = T(x_i, z) + T(z, z_k(i))$.

On the other hand, subadditivity gives

$$T(x_j, z_k(i)) \le T(x_j, z) + T(z, z_k(i)) \le T(x_i, z) + T(z, z_k(i)) = T(x_i, z_k(i)) \ ,$$

so $z_k(i) \notin G_i$. This is a contradiction. □

The structure of Lemma 10 is based on viewing the FPP model as defining a "competition model" (G_i is the set of sites "infected" by a disease with initial patient located at site x_i; sites can be infected only by a single type of disease). This type of perspective is due to the original paper of Häggström-Pemantle. The proof of Lemma 10 amounts to showing that each infection spreads along FPP geodesics.

Proof of Theorem 9. We show that hypothesis (3.1) of Corollary 11 holds by contradiction. So choose ℓ so large that $\mathbf{E}T(0, \ell e_1) < 3\ell g(e_1)/2$ (this choice will become clear in a moment), and defining

$$A_0 := \left\{\limsup_{n\to\infty} B_{ne_1}(0, \ell e_1) > 0\right\} \quad \text{and} \quad A_\ell := \left\{\limsup_{n\to\infty} B_{-ne_1}(\ell e_1, 0) > 0\right\} \ ,$$

assume that $\mathbf{P}(A_0 \cap A_\ell) = 0$. We begin with a simple bound: note that the triangle inequality implies that $T(0, ne_1) \le T(0, \ell e_1) + T(\ell e_1, ne_1)$, so $B_{ne_1}(0, \ell e_1) \le T(0, \ell e_1)$ for all n. (A similar argument gives $B_{ne_1}(0, \ell e_1) \ge -T(0, \ell e_1)$ and in particular that $B_{ne_1}(0, \ell e_1)$ is integrable).

We can of course bound $B_{ne_1}(0, \ell e_1)$ by 0 on the event A_0^c, and give a similar bound for $B_{-ne_1}(\ell e_1, 0)$. In particular,

$$\limsup_{n\to\infty} B_{ne_1}(0, \ell e_1) + \limsup_{m\to\infty} B_{-me_1}(\ell e_1, 0) \le T(0, \ell e_1)[\mathbf{1}_{A_0} + \mathbf{1}_{A_\ell}] \ . \tag{3.2}$$

Since $\mathbf{1}_{A_0} + \mathbf{1}_{A_\ell} \leq 1$ almost surely, we can take expectations in (3.2) and use our choice of ℓ to see

$$\mathbf{E}\left[\limsup_{n\to\infty} B_{ne_1}(0,\ell e_1) + \limsup_{m\to\infty} B_{-me_1}(\ell e_1, 0)\right] \leq \mathbf{E}T(0,\ell e_1) < 3\ell g(e_1)/2 \ . \tag{3.3}$$

So far we have not done anything to show that there is anything wrong with what has been derived. The key step is to show that $B_{ne_1}(0,\ell e_1)$ is typically of order $(\ell-\varepsilon)g(e_1)$ for a large density of n. Combined with a corresponding bound for the other term, we show that the left-hand side of (3.3) is of order $(2\ell-\varepsilon)g(e_1)$, in contradiction to the bound on the right-hand side. This contradicts the assumption that $\mathbf{P}(A_0 \cap A_\ell) = 0$.

The estimate described above is shown following an averaging trick; a more sophisticated version of this will appear in the proof of Theorem 24 below. Considering multiples of ℓe_1, we consider terms of the form

$$B_{K\ell e_1}(0,\ell e_1) = T(0, K\ell e_1) - T(\ell e_1, K\ell e_1). \tag{3.4}$$

Note that the model is invariant under lattice shifts, so

$$\mathbf{E}B_{K\ell e_1}(0,\ell e_1) = \mathbf{E}T(0,K\ell e_1) - \mathbf{E}T(\ell e_1, K\ell e_1) = \mathbf{E}T(0,K\ell e_1) - \mathbf{E}T(0,(K-1)\ell e_1) \ . \tag{3.5}$$

This shifts the perspective of the infection process and allows us to apply shape theorem results.

Applying (3.5) to terms of the type (3.4) gives

$$\mathbf{E}\sum_{K=1}^{n} B_{K\ell e_1}(0,\ell e_1) = \mathbf{E}T(0,n\ell e_1) \ .$$

In particular, dividing by n and using the shape theorem gives that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{K=1}^{n} \mathbf{E}B_{K\ell e_1}(0,\ell e_1) = g(\ell e_1) = \ell g(e_1). \tag{3.6}$$

In particular, (3.6) gives $\limsup_{n\to\infty} \mathbf{E}B_{ne_1}(0,\ell e_1) \geq \ell g(e_1)$, with a similar bound for $\limsup_{m\to\infty} B_{-me_1}(\ell e_1, 0)$. Since $|B_z(0,\ell e_1)| \leq T(0,\ell e_1)$, it follows (e.g. by Fatou's lemma) that

$$\mathbf{E}\limsup_{n\to\infty} B_{ne_1}(0,\ell e_1) \geq \limsup_{n\to\infty} \mathbf{E}B_{ne_1}(0,\ell e_1) \geq \ell g(e_1) \ .$$

Applying this in (3.3) (with the corresponding bound for the other term) completes the proof. □

4. Hoffman's method and Busemann functions

One can think of the function $B_{ne_1}(x,y)$ from the preceding section as a sort of "relative distance to infinity in direction e_1" between points x and y, for n very large. If one could show that the limit $B := \lim_n B_{ne_1}$ existed, one could work directly with the object B in Garet-Marchand's proof, which would make certain results more natural. For instance, a part of their proof amounts to showing that $\mathbf{E}B_{ne_1}(0,e_1) \approx g(e_1)$ for a large density of but potentially not all n — this type of "density" caveat could be removed under the existence of B.

We will come back to this perspective later, and focus instead on Hoffman's approach to Theorem 9. The key idea is to notice that there is a natural object similar to B_{ne_1} for which limits are easily shown to exist and which has similar implications for geodesics. This object is called a **Busemann function** and was originally introduced by Busemann in the study of (nonrandom!) metric geometry.

Definition 12. Consider a fixed realization of edge weights and suppose $\gamma = (z_1, z_2, \ldots)$ is a unigeodesic in this configuration. The Busemann function $B_\gamma(\cdot,\cdot)$ is defined by

$$B_\gamma(x,y) := \lim_{n\to\infty}\left[T(x,z_n) - T(y,z_n)\right] \ . \tag{4.1}$$

Claim 13. The limit (4.1) exists for any fixed configuration of edge weights, x, y unigeodesic γ.

Proof. Note that the triangle inequality gives that the terms on the right-hand side of (4.1) are bounded in magnitude by $T(x,y)$. For instance, $T(x,z_n) \leq T(x,y) + T(y,z_n)$ gives $T(x,z_n) - T(y,z_n) \leq T(x,y)$. This implies that if we can show the monotonicity (in n) of these terms for a fixed pair x, y, we will have established existence of the limit.

We first consider the case $y = z_1$. By geodesicity of γ, we have for each $1 < i < n$ that $T(z_1, z_n) = T(z_1, z_i) + T(z_i, z_n)$. On the other hand, subadditivity of course yields $T(x, z_n) \leq T(x, z_i) + T(z_i, z_n)$. Thus,

$$T(x, z_n) - T(z_1, z_n) \leq T(x, z_i) + T(z_i, z_n) - [T(z_1, z_i) - T(z_i, z_n)] = T(x, z_i) - T(z_1, z_i) \ .$$

Therefore, the terms on the right-hand side of (4.1) are decreasing in n, which (combined with the boundedness already established) shows existence of the limit.

Now we consider the case $y \neq z_1$. We can rewrite the typical term on the right-hand side of (4.1) as

$$T(x, z_n) - T(y, z_n) = [T(x, z_n) - T(z_1, z_n)] - [T(y, z_n) - T(z_1, z_n)] \ . \tag{4.2}$$

Since the limits of each of the bracketed terms on the right-hand side of (4.2) exist, so does the limit of the left-hand side.

□

Being a difference of passage times, B_γ inherits several properties which are key to its analysis.

Lemma 14. On the event Ω_u, the following hold for each unigeodesic $\gamma = (z_1, z_2, \ldots)$ and all vertices x, y, $z \in \mathbb{Z}^d$:

(1) $|B_\gamma(x, y)| \leq T(x, y)$;
(2) $B_\gamma(x, y) + B_\gamma(y, z) = B_\gamma(x, z)$;
(3) $B_\gamma(y, x) = -B_\gamma(x, y)$;
(4) If $x = z_i$ and $y = z_j$ with $i < j$ (so x appears before y in the unigeodesic), then $B_\gamma(x, y) = T(x, y)$.
(5) If $\widetilde{\gamma}$ is another unigeodesic which is not distinct from γ, then $B_\gamma = B_{\widetilde{\gamma}}$.

Proof. Property (1) already was shown during the proof of Claim 13 to establish boundedness of the sequence appearing in (4.1). Properties (2)–(3) follow easily from the fact that B_γ is defined as a difference: for instance, to see (3), write

$$T(y, z_n) - T(x, z_n) = -[T(x, z_n) - T(y, z_n)] \tag{4.3}$$

and take the limit of (4.3) as $n \to \infty$.

To see (4), note that since γ is a geodesic, we have for $n > j$:

$$T(x, z_n) = T(z_i, z_n) = T(z_i, z_j) + T(z_j, z_n) = T(x, y) + T(y, z_n) \ .$$

Subtracting $T(y, z_n)$ from both sides and taking limits establishes (4).

Property (5) is where we use Ω_u. Let $\widetilde{\gamma} = (z'_1, z'_2, \ldots)$. By Proposition 6, we see that the unigeodesics coalesce, so there are some i and j such that $z_i = z'_j$, $z_{i+1} = z'_{j+1}, \ldots$. In particular, the limit appearing in (4.1) is unchanged if we replace z_n by z'_n. □

We now are almost equipped to give Hoffman's version of a proof of Theorem 9. Since the argument involves applying an ergodic theorem, we take a moment here to define the operators which shift configurations by integer vectors.

Definition 15. Let $z \in \mathbb{Z}^d$, and let ω be a realization of the edge weights (t_e). We define the shift θ_z to be the operator which acts on ω, producing a new configuration $\theta_z \omega$, as follows: $\theta_z \omega$ is the realization of edge weights (t'_e), where $t'_{\{x,y\}} = t_{\{x-z,\, y-z\}}$.

Proof of Theorem 9. Assume that $\mathbf{P}(\mathcal{N} = 1) = 1$. Using Proposition 5, we see that we are guaranteed the existence of a unigeodesic γ beginning from 0. This γ must be a.s. unique, in the strong sense that there cannot exist an infinite path from 0 which is a unigeodesic, other than γ. Indeed, if there were another unigeodesic $\widetilde{\gamma}$ starting from 0, then by the non-distinctness assumption there must be some $0 \neq z \in \gamma \cap \widetilde{\gamma}$ beyond which the two geodesics are the same. But by a.s. uniqueness of finite geodesics, the subpaths of both γ and $\widetilde{\gamma}$ from 0 to z must be identical, so these two unigeodesics must be exactly the same path.

The translation-invariance of the model gives that we can apply Proposition 5 to find a unigeodesic $\Gamma(x)$ from each x in $\mathbb{Z}^d$. Since $\mathcal{N} = 1$, Proposition 6 gives us that each $\Gamma(x)$ must coalesce with γ. Let us define $B = B_\gamma$, where γ is as in the preceding paragraph. The proof of Proposition 5 in fact shows that every subsequential limit of finite geodesics from any initial vertex x must produce $\Gamma(x)$, which coalesces with γ — so B is in some sense a quite explicit function of the edge weights, and it is not hard to see that this implies the measurability of B. The property (1) of Lemma 14 and the integrability of T gives that $\mathbf{E}B(x,y)$ is defined and finite for all x, y.

In fact, B is translation-covariant, in the following sense. Writing explicitly the dependence of each object on the configuration ω, our assumption that $\mathcal{N} = 1$ gives that, almost surely, $\gamma[\theta_z\omega] = \Gamma(z)[\omega] - z$, which coalesces with $\gamma[\omega]$. This implies the translation-covariance of B: $B(x,y)[\theta_z\omega] = B(z+x, z+y)[\omega]$. We can also notice that reflections and rotations which fix 0 and leave $\mathbb{Z}^d$ invariant rotate / reflect γ, but that B must be invariant in distribution under these operations (by the invariance of the model and the construction of B). So $\mathbf{E}B(x,y) = 0$ for all x and y.

Fix $0 \neq x \in \mathbb{Z}^d$. Using additivity (property (2) of Lemma 14), shift-covariance, and the ergodic theorem, we have for almost every ω:

$$\frac{B(0,nx)[\omega]}{n} = \frac{1}{n}\sum_{i=1}^{n} B((i-1)x, ix)[\omega] = \frac{1}{n}\sum_{i=1}^{n} B(0,x)[\theta_x^{i-1}\omega] \overset{n\to\infty}{\longrightarrow} \mathbf{E}B(0,x) = 0 \ . \tag{4.4}$$

Thus, B grows sublinearly in each direction with probability one. We claim a stronger statement: analogous to the usual shape theorem, we can make this sublinearity hold simultaneously in all directions with probability one:

$$\mathbf{P}\left(\lim_{|x|\to\infty}\frac{|B(0,x)|}{|x|} = \limsup_{|x|\to\infty}\frac{|B(0,x)|}{|x|} = 0\right) = 1 \ . \tag{4.5}$$

The proof of (4.5) from (4.4) is (*mutatis mutandis*) identical to the proof of the shape theorem, so we only recall the main idea here. Fix some configuration ω such that (4.4) holds for each $x \in \mathbb{Z}^d$, and assume for the sake of contradiction that there were some sequence z_k along which $|B(0,z_k)| > \varepsilon|z_k|$ uniformly. Without loss of generality, we may assume that $z_k/|z_k| \to z$ for some z. We can find some $x \in \mathbb{Z}^d$ with $x/|x|$ close to z. For each k large, let n_k minimize $|n_k x - z|$; by careful choice of x, we have $|n_k x - z| \ll \min\{n_k|x|, |z_k|\}$. In particular, since $|B(0,n_k x) - B(0,z_k)| \le T(n_k x, z_k)$ (and since this passage time is order $|n_k x - z|$), we see that $|B(0,n_k x)|$ is large relative to $n_k|x|$. This is in contradiction to the fact that $B(0,nx)/n|x| \to 0$ on ω.

On the other hand, we can see that sublinearity of B is absurd by considering property (4) of Lemma 14. Letting $\gamma = (y_1, y_2, \ldots)$, we have $B(y_i, y_j) = T(y_i, y_j)$ for $i < j$. For j large, we have by the shape theorem that $T(0,y_j) \approx g(y_j)$. To be precise, fix $\delta > 0$ such that $g(z) > \delta$ for all z with $|z| = 1$; this is possible by the boundedness of the limit shape. The shape theorem implies that with probability one,

$$\lim_{j\to\infty}\frac{B(0,y_j)}{|y_j|} = \lim_{j\to\infty}\frac{T(0,y_j)}{|y_j|} \ge \delta > 0 \ .$$

This is in contradiction to (4.5), showing that our assumption that $\mathbf{P}(\mathcal{N} = 1) = 1$ is false.

□

5. Directedness and Busemann functions

There is a clear similarity between the Busemann function B_γ and the object B_{ne_1} considered in the proof of Garet-Marchand. We will take some time to develop this idea here under a strong assumption (different from (Curve) and (Expm)). We will not push these ideas as far as we could, since we will in the next section give a framework for getting around these sorts of strong assumptions.

Suppose we wish to avoid assuming (Curve), but still believe the results of Theorem 8 should hold. With this as our guidepost, it seems perhaps reasonable to replace assumption (Curve) with the following:

Assumption 3.(LimG) For each $x \in \mathbb{Z}^d$, the geodesics $G(x, ne_1)$ have a limit $\Gamma(x)$. Moreover, these geodesics all coalesce.

One immediate consequence of Assumption (LimG) is that the Busemann function $B(x,y) := \lim_n[T(x,ne_1) - T(y,ne_1)]$ exists, and is covariant with respect to translations by e_1, similarly to the Busemann function in Hoffman's argument. If we continue taking Theorem 8 as a goal, one could be led to believe that $\Gamma(x)$ should be directed: $\Theta(\Gamma(x)) = \{e_1\}$, or at least that $\Theta(\Gamma(x)) \neq \mathbb{S}^{d-1}$.

We will not try to prove anything as strong as directedness, but instead just the following much weaker claim. In what follows, let S_δ denote the sector of aperture δ around $-e_1$:

$$S_\delta = \{z \in \mathbb{S}^{d-1} : |z + e_1| < \delta\} \ .$$

Theorem 16. Assume (LimG) (along with the standard assumptions of Assumption 1). Then there is some δ such that $\Theta(\Gamma(x)) \cap S_\delta = \varnothing$ almost surely.

The strategy of the theorem's proof is easy to outline given what we have seen. As in the argument of Garet-Marchand, we can use an "averaging trick" to show that $\mathbf{E}B(0, -me_1)$ grows linearly as $mg(e-1)$ for large m. In fact, the translation-covariance of B gives that the growth of B occurs almost surely. On the other hand, property 4 of Lemma 14 holds for B, giving that $B(0, \cdot)$ grows like the passage time along $\Gamma(0)$. These asymptotics would conflict if $-me_1$ were on $\Gamma(0)$, and in fact exclude the geodesic coming within some conical region of the axis.

Proof of Theorem 16. Let us strengthen the assumptions even further to include boundedness: $\mathbf{P}(t_e \le M) = 1$ for some finite M; we prove the theorem in this setting. Assume for the contradiction that the statement of the theorem fails when $\delta = (g(e_1)/16M)$. Consider an outcome ω on which $\Theta(\Gamma(x)) \cap S_\delta \neq \varnothing$ (we will also need to assume that ω lies in various probability one events on which, e.g., $t_e \le M$ for all e; this will become clear in the course of the proof).

The translation-covariance of B gives, as in (4.4),

$$\lim_{m\to\infty} B(0, -me_1)/m = \mathbf{E}B(0, -e_1) \quad \text{a.s. and in } L^1 \ . \tag{5.1}$$

To compute $\mathbf{E}B$, let us as before define $B_n(x, y) := T(x, ne_1) - T(y, ne_1)$. Note that the shape theorem (in fact, the work used to establish the shape theorem) shows that $\mathbf{E}T(0, ne_1)/n \to g(e_1)$. We use the Garet-Marchand averaging trick:

$$\begin{aligned}\frac{\mathbf{E}T(0, ne_1)}{n} &= \frac{1}{n}\Big([\mathbf{E}T(0, ne_1) - \mathbf{E}T(0, (n-1)e_1)] + [\mathbf{E}T(0, (n-1)e_1) - \mathbf{E}T(0, (n-2)e_1)] + \ldots + \mathbf{E}T(0, e_1)\Big) \\ &= \frac{1}{n}\left(\mathbf{E}B_n(0, e_1) + \mathbf{E}B_{n-1}(0, e_1) + \ldots + \mathbf{E}T(0, e_1)\right) \ .\end{aligned}$$

Since $B_n(0, e_1) \to B(0, e_1)$ almost surely and since $|B_n(0, e_1)| \le M$ almost surely, the typical term above converges to $\mathbf{E}B(0, e_1)$. On the other hand, the left-hand side converges to $g(e_1)$, so we see

$$\mathbf{E}B(0, -e_1) = -\mathbf{E}B(0, e_1) = -g(e_1) \ . \tag{5.2}$$

Combining (5.2) with (5.1), we see that $B(0, -me_1)/m \to -g(e_1)$ almost surely. We now move toward a contradiction similarly to Hoffman's argument. Write $\Gamma(0) = (z_1, z_2, \ldots)$. As in Lemma 14, we have $B(0, z_i) = T(0, z_i)$ for all i. In particular, for all i, $B(0, z_i) \ge 0$. Let $m_i = \lfloor |z_i| \rfloor$.

$$|B(0, z_i) - B(0, -m_ie_1)| = |B(-m_ie_1, z_i)| \le T(-m_ie_1, z_i) \le M|m_ie_1 + z_i| \le 2M(|z_i|)|(z_i/|z_i|) + e_1| \ , \tag{5.3}$$

where the last inequality only holds for i large (so $m_i \approx |z_i|$).

On the other hand, as $i \to \infty$, we have

$$\lim_i B(0, -m_ie_1)/m_i = \lim_i B(0, -m_ie_1)/|z_i| = -g(e_1) \ .$$

If i is sufficiently large, then $|z_i/|z_i| + e_1| \le 2\delta = (g(e_1)/8M)$ and $B(0, -m_ie_1) \le -|z_i|g(e_1)/2$. Then the above implies along with (5.3) that

$$B(0, z_i)/|z_i| \le -g(e_1)/2 + (2M)(g(e_1)/8M) \le -g(e_1)/4.$$

This contradicts the fact that $B(0, z_i) \ge 0$. ∎

6. Busemann subsequential limits and general directedness statements

We have shown that averaging properties of Busemann functions can be used to control directedness properties of infinite geodesics, and have some idea of how to implement this strategy in practice. Unfortunately, without assumption (LimG), we are lacking a Busemann function and corresponding geodesics on which to run this program. It is obviously reasonable to want to try to construct geodesics without making any unproven assumptions! Our main goal in the remainder of the notes is to run a more sophisticated version of last section's argument which allows us to circumvent (LimG).

Fix some $z \in \partial\mathcal{B}$. In all our work in this section, we replace (LimG) with the following assumption:

Assumption 4. (Dif) $\partial\mathcal{B}$ is differentiable at z.

Recall the meaning of this statement is that there is a unique supporting hyperplane H for $\mathcal{B}$ at z (recall Michael's lectures). There is a unique ρ such that $H = \{x : x \cdot \rho = 1\}$. Recall that $\mathcal{B}$ is convex by the shape theorem, so (Dif) is a much weaker assumption than it seems at first glance.

Given a z as in (Dif), we can of course define the "sector of contact" of H with $\partial\mathcal{B}$:

$$S = H \cap \partial\mathcal{B} \ .$$

The corresponding set of angles is

$$\Theta_S := \{z \in \mathbb{S}^{d-1} : z/g(z) \in S\} \ . \tag{6.1}$$

Our main theorem is that, under (Dif), we can produce a geodesic which is directed in a sector no wider than Θ_S.

Theorem 17. [2] Assume the standard assumptions and (Dif). Then with probability one, there is an infinite geodesic γ from 0 which is directed in Θ_S, in the sense that $\Theta(\gamma) \subseteq \Theta_S$.

In particular, if z is an exposed point (i.e., if $S = \{z\}$), then we can produce a directed geodesic.

In our construction, we build a limiting Busemann function B and corresponding limiting geodesics via a particularly chosen limiting procedure. In our analysis of the asymptotics of this Busemman function, we also make a technical improvement on the methods of the last section. Note that our contradiction there came from the observation B_γ grows (at least remaining positive) along its base geodesic γ and becomes negative at a linear rate along the $-e_1$ axis. In fact, we can sharpen these observations by considering the growth of B in a global sense. The main goal here will be to provide a version of (4.5) which gives a "shape theorem" for a particular Busemman function which characterizes completely the linear-order growth.

We note here that the presentation of Theorem 17 and its proof are influenced by the versions appearing in [1]; in particular, unlike the original paper [2], we make clear the generalization to $d > 2$.

6.1. Construction of geodesics, Busemann functions.

Our construction benefits greatly from considering point-to-set geodesics. For a subset $S \subseteq \mathbb{R}^d$ and $x \in \mathbb{Z}^d$, let $T(x, S) = \inf_{z \in S} T(x, z)$ (recall we extended T to points of $\mathbb{R}^d$). A point-to-set geodesic from x to S is a path γ from x to some $[z]$ with $z \in S$, such that $T(\gamma) = T(x, S)$.

For each $\alpha \in \mathbb{R}$, let $H_\alpha = \{z : z \cdot \rho = \alpha\}$. It is not hard to see following the proof of Lemma 2 that geodesics to H_α exist and are unique, and we let $\Gamma_\alpha(x)$ denote the geodesic from $x \in \mathbb{Z}^d$ to H_α, and $B_\alpha(x, y) = T(x, H_\alpha) - T(y, H_\alpha)$. We would be very happy if each $\Gamma_\alpha(x)$ and $B_\alpha(x, y)$ had limits as $\alpha \to \infty$, as we could then work with the limiting objects! Because it is not clear how to show this, we will take a particular sort of subsequential limit instead. Particular desiderata which guide the choice of limiting procedure are that the limits have appropriate translation-invariance properties (so we can do versions of averaging tricks and apply ergodic theorems) and that they preserve the relationship between the limiting analogues of Γ_α and B_α (so we can say something about geodesics from Busemann asymptotics!).

We consider the edge weight configuration $\omega = (t_e)_e$ to live on the canonical probability space $\Omega_1 := [0, \infty)^{\mathcal{E}^d}$. We will need to consider an enlarged version of this space. Let $\vec{\mathcal{E}}^d$ denote the set of **directed** edges (i.e., an ordered pair (x, y) is in $\vec{\mathcal{E}}^d$ if $\{x, y\} \in \mathcal{E}^d$). We let $\widetilde{\Omega} := \Omega_1 \times \Omega_2 \times \Omega_3$, where $\Omega_2 = \mathbb{R}^{\mathbb{Z}^d}$ and $\Omega_3 = \{0, 1\}^{\vec{\mathcal{E}}^d}$. For each α, we will push forward our original measure $\mathbf{P}$ on Ω_1 to a measure μ_α on $\widetilde{\Omega}$.

For $(x,y)=\vec{e}\in\vec{\mathcal{E}}^d$, let $e=\{x,\, y\}$ be the undirected version of $\vec{e}$ and define the random variable

$$\eta_\alpha(\vec{e}) := \begin{cases} 1 & \text{if } T(x,H_\alpha)=T(y,H_\alpha)+t_e \\ 0 & \text{otherwise.}\end{cases}$$

For each α , define the map $\Phi_\alpha : \Omega_1 \to \widetilde{\Omega}$ as follows:

$$\Phi_\alpha(\omega) = (\omega, B_\alpha, \eta_\alpha)\ .$$

For each α, we define the probability measure μ_α (on $\widetilde{\Omega}$ with the Borel sigma-algebra) to be the push-forward of $\mathbf{P}$ by the map Φ_α.

μ_α keeps track of the joint distribution of the edge weights, Busemann functions, and edges in geodesics. The geodesics are kept track of by the edge variables η_α; for instance, we have almost surely that $\eta_\alpha((x,y))=1$ if and only if (x,y) is in $\Gamma_\alpha(x)$. Indeed, if $(x,y)\in\Gamma_\alpha(x)$, then $\eta_\alpha((x,y))=1$ by definition of a geodesic; conversely, if $\eta_\alpha((x,y))=1$, then by concatenating (x,y) with $\Gamma_\alpha(y)$ we produce a path γ satisfying $T(\gamma)=T(x,H_\alpha)$ (which must be the unique geodesic).

Recall the translation operators from Definition 15. We extend them to $\widetilde{\Omega}$ as follows. For $z\in\mathbb{Z}^d$, consider a typical point $((t_e),(B(x,y)),(\eta_\alpha(\vec{e})))$ of $\widetilde{\Omega}$; then

$$\theta_z((t_e),(B(x,y)),(\eta(\vec{e}))) = ((t_{e-z}),(B(x-z,y-z)),(\eta(\vec{e}-z)))\ .$$

Lemma 18. If $z\in\mathbb{Z}^d$ and $A\subseteq\widetilde{\Omega}$ is an event, then $\mu_\alpha\circ\theta_z^{-1}=\mu_{\alpha+\rho\cdot z}$.

Proof. Let us just demonstrate via the event $\{\eta((x,y))=1\}$. We have

$$\begin{aligned}\mu_\alpha\circ\theta_z^{-1}(\eta((x,y))=1) &= \mu_\alpha(\eta((x-z,y-z))=1) = \mathbf{P}((x-z,y-z)\in\Gamma(x-z,H_\alpha)) \\ &= \mathbf{P}((x,y)\in\Gamma(x,H_{\alpha+z\cdot\rho})) = \mu_{\alpha+\rho\cdot z}(\eta((x,y))=1)\ .\end{aligned}$$

Here we have used the invariance of $\mathbf{P}$ under shifts, and the fact that $H_\alpha+z=H_{\alpha+\rho\cdot z}$. ☐

The following properties of B_α follow via very similar arguments of those of Lemma 14, so we omit their proofs.

Lemma 19. The following hold $\mathbf{P}$-a.s. for each $x,y,z\in\mathbb{Z}^d$:

(1) $|B_\alpha(x,y)|\le T(x,y)$;
(2) $B_\alpha(x,z)=B_\alpha(x,y)+B_\alpha(y,z)$;
(3) $B_\alpha(y,x)=-B_\alpha(x,y)$;
(4) If $y\in\Gamma_\alpha(x)$, then $B_\alpha(x,y)=T(x,y)$.

In particular, if we replace B_α and η_α by typical points B and η of Ω_2 and Ω_3, the analogues of the above hold μ_α-a.s.

When we consider a limit of μ_α, since we do not know that Γ_α's converge, there is no clear way to reconstruct an infinite geodesic corresponding to "the $\alpha=\infty$ version of Γ_α". We wish to use the η variables above to read off an infinite geodesic from a configuration on $\widetilde{\Omega}$. We will be greatly helped here by the following graphical construction.

Definition 20. For each $\eta\in\Omega_3$, define a directed graph $\mathbb{G}$ with vertex set $\mathbb{Z}^d$ and directed edge set as follows: (x,y) is an edge of $\mathbb{G}$ if and only if $\eta((x,y))=1$. When the configuration η is understood, we write $x\to y$ if there is a directed path in $\mathbb{G}(\eta)$ from x to y.

Lemma 21. The following hold for μ_α-a.e. configuration of $\widetilde{\Omega}$.

(1) For each x, there is a directed path (possibly equal to (x)) from x to H_α.
(2) If γ is a directed path in $\mathbb{G}$, then γ is a geodesic.
(3) If $x\to y$, then $B(x,y)=T(x,y)$.

Proof. Item (1) is clear by considering the fact that the edges $\vec{e}$ of $\Gamma_\alpha(x)$ have $\eta_\alpha(\vec{e})=1$. Item (3) follows from the definition of B_α and property (4) of Lemma 19.

Consider an outcome such that geodesics exist and are unique. To prove property (2), let γ start at x and tranverse the directed edges $\vec{e}_1\ \ldots,\vec{e}_n$ in order. Write $K\le n$ be the maximal index such that the path γ_K which traverses (in order) $\vec{e}_1\ \ \ldots,\ \vec{e}_K$ is a subpath of $\Gamma_\alpha(x)$. We will show $K=n$.

By the observation that $\eta_\alpha((y,z)) = 1$ if and only if $(y,z) \in \Gamma_\alpha(y)$, we see that $\vec{e}_1$ is the first vertex of $\Gamma_\alpha(x)$ and so $K \geq 1$. We now show that if $K < n$, then γ_{K+1} is also a subpath of $\Gamma_\alpha(x)$. Note that there is some path γ' which extends γ_K to a geodesic from x to H_α. Letting $\vec{e}_K = (y,z)$, the subpath of γ' from z to H_α must be the unique geodesic from z to H_α. On the other hand, since $\vec{e}_{K+1}$ is in $\mathbb{G}$ (and is thus in $\Gamma_\alpha(Z)$, we have that the edge of γ' following $\vec{e}_K$ must be $\vec{e}_{K+1}$. □

We average the μ_α's to produce a new measure μ_n^* on $\widetilde{\Omega}$: for $n = 1, 2, \ldots$, set

$$\mu_n^* = \frac{1}{n} \int_0^n \mu_\alpha \, d\alpha \ .$$

There is a technical argument required for this definition: namely, we need to show that for each measurable A, the map $\alpha \mapsto \mu_\alpha(A)$ is measurable (so the integral above makes sense). We refer the interested reader to Appendix A of [2].

Lemma 22. There is a subsequence (n_k) and a measure μ on $\widetilde{\Omega}$ such that

$$\lim_{k\to\infty} \mu_{n_k}^* = \mu \quad \text{(weakly)}.$$

Proof. The distribution $\mathbf{P}$ on Ω_1 is easily seen to be tight, in the usual sense that for each $\varepsilon > 0$, we can find a compact measurable $K_\varepsilon \subseteq \Omega_1$ such that $\mathbf{P}(\Omega_1 \setminus K_\varepsilon) < \varepsilon$. By property (1) of Lemma 19, we see that the sequence (μ_n^*) is also tight. Prokhorov's theorem now gives the existence of a subsequential weak limit. □

Choose some μ as in the statement of Lemma 22. This will be the object we use to construct the geodesic of Theorem 17. The bulk of this construction will be done in the next subsection, by analyzing the graphs $\mathbb{G}$ and Busemann functions B sampled from μ. For now, we just give the main reason for using the averaging procedure which constructed μ (and not, say, choosing μ just as a limit of μ_α).

Lemma 23. μ is translation-invariant: for any $z \in \mathbb{Z}^d$ and any event $A \subseteq \widetilde{\Omega}$,

$$\mu \circ \theta_z^{-1}(A) = \mu^*(A) \ .$$

Proof sketch. Let n be a positive integer. We can write (using Lemma 18)

$$\mu_n^* \circ \theta_z^{-1}(A) = \frac{1}{n} \int_{z\cdot\rho}^{n+z\cdot\rho} \mu_\alpha(A) d\alpha \ .$$

In particular,

$$\left|\mu_n^* \circ \theta_z^{-1}(A) - \mu_n^*(A)\right| \leq \frac{1}{n} \left| \int_n^{n+z\cdot\rho} \mu_\alpha(A)\, d\alpha + \int_0^{z\cdot\rho} \mu_\alpha(A)\, d\alpha \right| \ ,$$

which tends to zero in n for any fixed A. The result now follows by approximating μ by μ_n^* for n large. □

6.2. **Asymptotics for samples from μ^*.** We will now study the asymptotics of a sample $B \in \Omega_2$ from the marginal of μ. Our goal is to replicate the previous "averaging" arguments but in a strong sense. Our first step, as before, is to control the expectation.

Theorem 24. For any $x, y \in \mathbb{Z}^d$, we have

$$\mathbf{E}_\mu B(x,y) = \rho \cdot (y - x) \ .$$

We need the following lemma:

Lemma 25. Almost surely and in L^1,

$$\lim_{\alpha\to\infty} T(0, H_\alpha)/\alpha = 1 \ .$$

This lemma follows from the shape theorem in a fairly straightforward manner, so we omit the proof.

Proof of Theorem 24. Let n be a positive integer. Write

$$\begin{aligned}\mathbf{E}_{\mu_n^*} B(-x,0) &= \frac{1}{n}\left[\int_0^n \mathbf{E}T(-x,H_\alpha)\,d\alpha - \int_0^n \mathbf{E}T(0,H_\alpha)\,d\alpha\right]\\ \text{(by translation invariance)} &= \frac{1}{n}\left[\int_0^n \mathbf{E}T(0,H_{\alpha+x\cdot\rho})\,d\alpha - \int_0^n \mathbf{E}T(0,H_\alpha)\,d\alpha\right]\\ &= \frac{1}{n}\left[\int_n^{n+x\cdot\rho} \mathbf{E}T(0,H_\alpha)\,d\alpha - \int_0^{x\cdot\rho} \mathbf{E}T(0,H_\alpha)\,d\alpha\right]\ . \end{aligned} \tag{6.2}$$

Taking limits in (6.2), the second term goes to zero with n. To deal with the first term, note that Lemma 25 implies that for each $\alpha \in [0, x\cdot\rho]$,

$$\lim_{n\to\infty} \frac{\mathbf{E}T(0,H_{\alpha+n})}{n} = \lim_{n\to\infty} \frac{\mathbf{E}T(0,H_{\alpha+n})}{n+\alpha}\frac{n+\alpha}{n} = 1\ .$$

Applying this in (6.2) (with a dominated convergence theorem argument to take the limit under the integral) gives $\mathbf{E}_{\mu_n^*} B(-x,0) \to \rho\cdot x$.

It remains only to show that $\mathbf{E}_{\mu_{n_k}^*} B(-x,0) \to \mathbf{E}_\mu B(-x,0)$ (since the expectation at arguments x, y now follows by translation-invariance). For $R > 0$, defining the continuous truncation

$$B_R(-x,0) = B(-x,0)\mathbf{1}_{|B(-x,0)|\le R} + R\,\mathrm{sign}(B(-x,0))\mathbf{1}_{|B(-x,0)>R|}\ ,$$

we have by continuity and boundedness that

$$\mathbf{E}_{\mu_{n_k}^*} B_R(-x,0) \to \mathbf{E}_\mu B_R(-x,0)\ .$$

The claim now follows by taking $R\to\infty$. Indeed, we have that $B(-x,0)^2 \le T(0,-x)^2$, which has a finite second moment (bounded uniformly in n), so

$$\lim_{R\to\infty}\limsup_{n\to\infty} \mathbf{E}_{\mu_n^*} |B(-x,0)|\mathbf{1}_{|B(-x,0)|>R} = 0\ .$$

□

As we said earlier, we want to prove a shape theorem for B, establishing the a.s. leading-order behavior of $B(0,x)$ under μ. The preceding lemma suggests a target for this shape theorem: for $|x|$ large, one expects $|B(0,x) - \rho\cdot x| \ll |x|$. Unfortunately, a result like this does not immediately follow in general without information about $\mathcal{B}$, and it is here we use assumption (Dif).

A brief explanation of the problem is as follows. We want to use the ergodic theorem, as in the proof of Garet-Marchand's result, to establish that $B(0,nx)/n \to \rho\cdot x$ for each fixed x; if we could do this, we could patch together a global result by continuity. Unfortunately, the ergodic theorem does not give convergence of $B(0,nx)/n$ to a deterministic limit, because the measure μ is not guaranteed to be ergodic. So the best we can hope for *a priori* is convergence to some functional whose mean is ρ.

The above obstacle is the main reason we assumed (Dif) in the first place. We will see that the differentiability gives us the ability to say the random limiting functional is in fact identical to ρ, almost surely.

Theorem 26. There exists a random vector ϖ on $\widetilde{\Omega}$ such that

$$\mu\left(\limsup_{|z|\to\infty} \frac{|B(0,z) - \varpi\cdot z|}{|z|} = 0\right) = 1\ .$$

Moreover, $\mathbf{E}_\mu \varpi = \rho$.

We give a sketch of the main ideas of the proof, highlighting the issues with ergodicity.

Proof sketch. We first show that for $e = e_1, e_2, \ldots, e_d$, we have

$$\lim_{n\to\infty} B(0,ne)/n =: \varpi(e) \quad \text{exists } \mu\text{-a.s.} \tag{6.3}$$

To establish (6.3), we write $n^{-1}B(0,ne) = n^{-1}\sum_{j=1}^n B((j-1)e, je)$, which follows from (2) in Lemma 19 after some work to pass this property through the limit which produces μ. We then rewrite this sum as a sum of copies of $B(0,e)$ evaluated in shifted environments as usual, and apply the ergodic theorem. Since μ is not ergodic but rather merely translation-invariant, we are only guaranteed that the limit exists and defines some random variable: it need not be constant.

We now define $\varpi = (\varpi(e_1), \ldots, \varpi(e_d))$; ϖ is invariant under translations of the realization $((t_e), (B), (\eta))$. Our next step is to see that for each fixed $x = (x(1), x(2), \ldots, x(d)) \in \mathbb{Z}^d$, we have $B(0, nx)/n \to x \cdot \varpi$. This follows by writing

$$B(0, nx) = B(0, nx(1)e_1) + B(nx(1)e_1, nx(1)e_1 + nx(2)e_2) + \ldots$$

and using the previous convergence result (and invariance of ϖ) to approximate each of these terms by $nx(1)\varpi(1)$, $nx(2)\varpi(2)$, etc.

This gives us convergence in fixed directions. To give the global convergence as in the statement of the theorem, we follow the proof of the shape theorem (just as we did in the proof of (4.5)). The form of the mean of ϖ is a consequence of Theorem 24. □

As promised, we claimed that we can in fact show that ϖ is ρ under our assumptions. We conclude this subsection by giving the argument.

Lemma 27. μ-a.s., the hyperplane $H_\varpi := \{x : x \cdot \varpi = 1\}$ is a supporting hyperplane for $\partial\mathcal{B}$ at z. In particular, under (Dif), we have $\varpi = \rho$ almost surely.

Proof. Note that almost surely, for any fixed $x \in \mathcal{B}$

$$\varpi \cdot x = \lim_{n\to\infty} \frac{B(0, nx)}{n} \leq \lim_{n\to\infty} \frac{T(0, nx)}{n} = g(x) \leq 1 \ .$$

In particular, $\mathcal{B}$ a.s. lies on one side of H_ϖ. On the other hand,

$$\mathbf{E}_\mu \varpi \cdot z = \rho \cdot z = 1 \ .$$

Thus $\varpi \cdot z = 1$ almost surely, and H_ϖ is a supporting hyperplane. □

6.3. **Directedness.** We now prove Theorem 17. We show that for μ-almost every element of $\widetilde{\Omega}$, there is an infinite path from 0 in $\mathbb{G}(\eta)$ (which must be a geodesic, by the limiting version of Lemma 21 (2)) and which has the required directedness.

Proof of Theorem 17. It is not hard to show a limiting version of Property (1) of Lemma 21 which says that with μ-probabiility one, there is an infinite path in $\mathbb{G}$ from 0. As mentioned just prior, this path is also easily seen to be a geodesic. So the main argument is to show

$$\text{a.s., for each infinite path } \gamma \text{ in } \mathbb{G},\ \Theta(\gamma) \subseteq \Theta_S \ . \tag{6.4}$$

Let $\gamma = (0 = x_0, x_1, \ldots)$, and let (x_{n_k}) be a subsequence such that $x_{n_k}/|x_{n_k}| \to \theta$. We show $\theta \in \Theta_S$. Applying Theorem 26 (with $\varpi = \rho$) gives

$$\lim_k B(0, x_k)/|x_k| = \rho \cdot \theta \ .$$

On the other hand, this limit also equals $\lim_k T(0, x_k)/|x_k| = g(\theta)$. In particular, $\theta/g(\theta)$ is on $\partial\mathcal{B}$ and in the set H, so it is in S.

□

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