Recent progress on random topology

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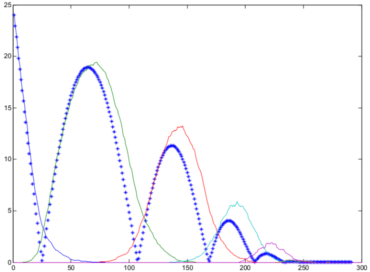

“I predict a new subject of statistical topology. Rather than count the number of holes, Betti numbers, etc., one will be more interested in the distribution of such objects on noncompact manifolds as one goes out to infinity.” — Isadore Singer, 2004.

Motivation

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- Certain situations in physics seem to be well modeled by random topology: cosmology, black holes, quantum gravity, etc.
- Can we explain why so many groups / manifolds / simplicial complexes / etc. seem to have a certain topological property?

  E.g. many simplicial complexes and posets arising in combinatorics are homotopy equivalent to bouquets of spheres. But why does this happen so often?

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- Geometric group theory — e.g. Gromov, Żuk
- Expander graphs — e.g. Pinsker, Barzdin & Kolmogorov

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Random graphs

Define $G(n, p)$ to be the probability space of graphs on vertex set $[n] = \{1, 2, \ldots, n\}$, where each edge has probability p, independently.

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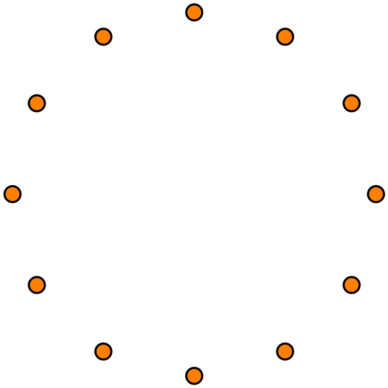
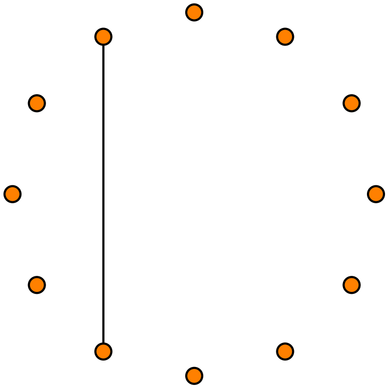
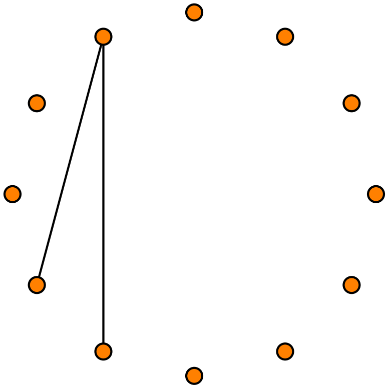
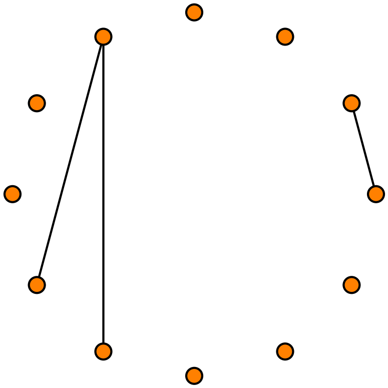
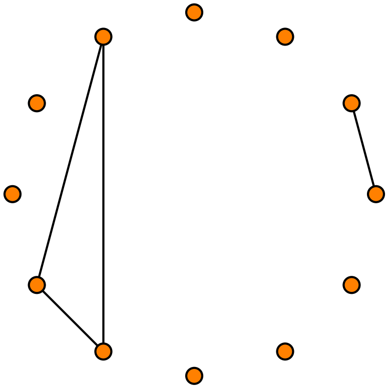
It is often useful to think of a growth process associated with $G(n, p)$ where edges are added one at a time.

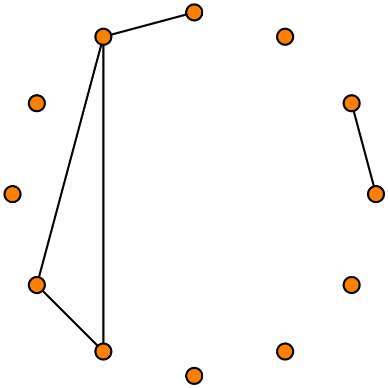


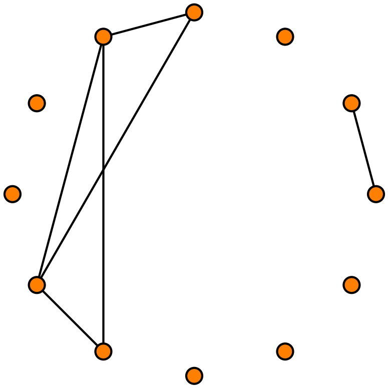


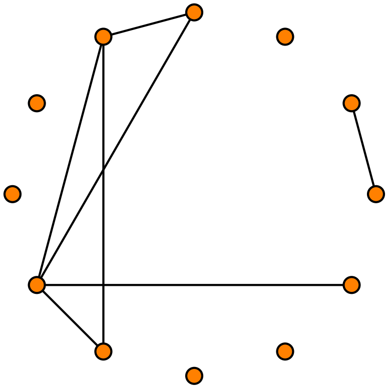


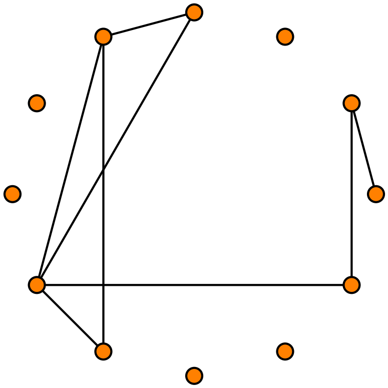


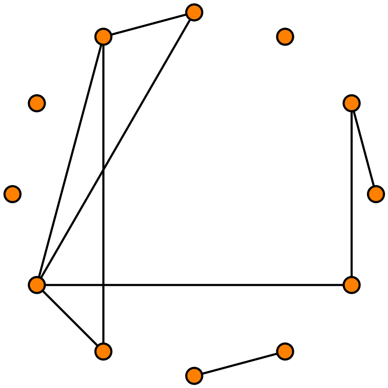


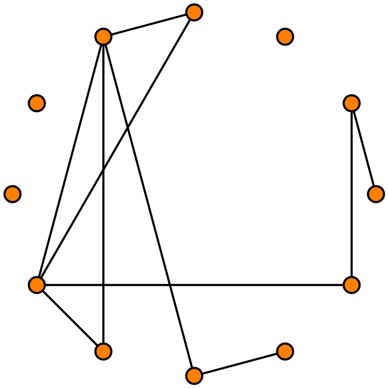


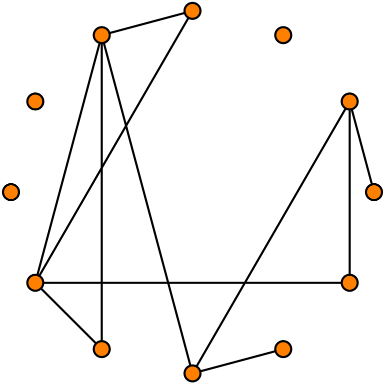


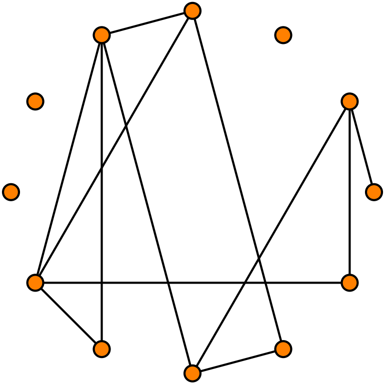


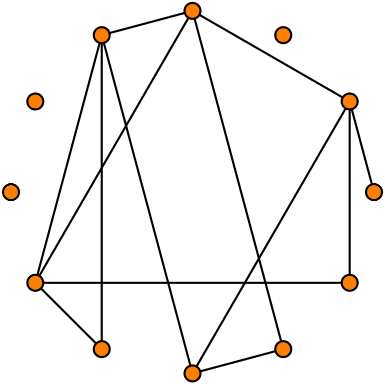


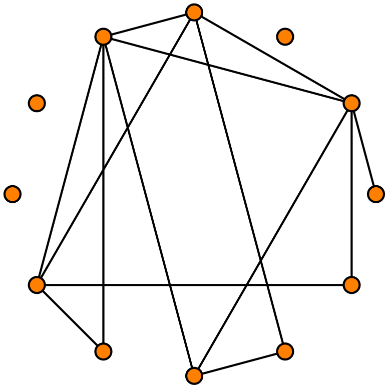


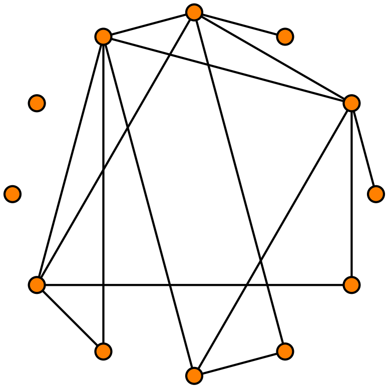


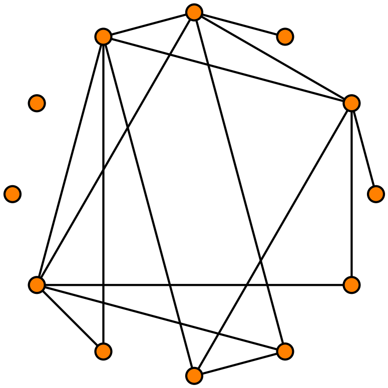


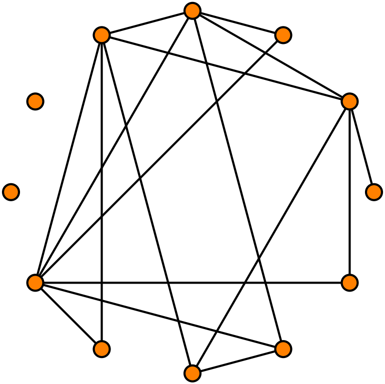


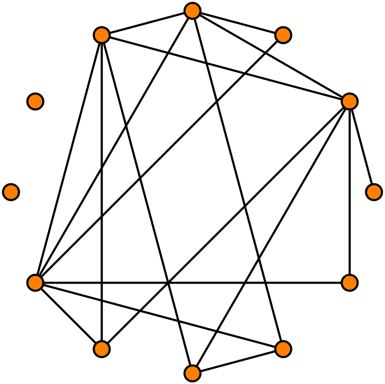


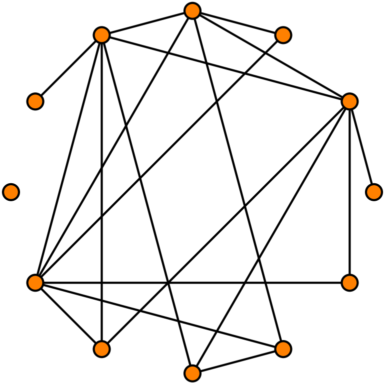


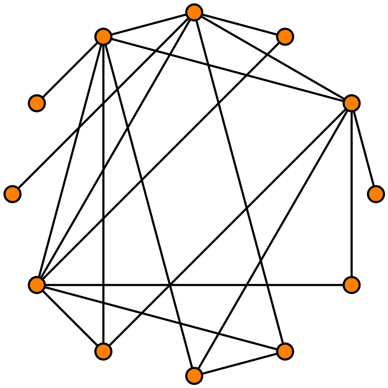












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We are often interested in *thresholds* for graph properties. An archetypal example is the Erdős–Rényi theorem.

Theorem (Erdős–Rényi, 1959)

Let $\epsilon > 0$ *be fixed and* $G \sim G(n, p)$. *Then*

$$\mathbb{P}[G \text{ is connected}] \to \begin{cases} 1 & : p \geq (1+\epsilon)\log n/n \\ 0 & : p \leq (1-\epsilon)\log n/n \end{cases}$$

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They actually proved a slightly sharper result.

Theorem (Erdős–Rényi, 1959)

Let $c \in \mathbb{R}$ be fixed and $G \sim G(n, p)$. If

$$p = \frac{\log n + c}{n},$$

then $\widetilde{\beta}_0(G)$ is asymptotically Poisson distributed with mean e^{-c} and in particular

$$\mathbb{P}[G \text{ is connected}] \to e^{-e^{-c}}$$

as $n \to \infty$.

The first step is to show that if $p \approx \log n/n$, then the probability that there are any components of order i, with $2 \leq i \leq n/2$ tends to 0 as $n \to \infty$.

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A “union bound” argument shows that it is sufficient to show that

$$\sum_{k=2}^{\lfloor n/2 \rfloor} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)} \to 0,$$

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So if $p \approx \log n/n$, then w.h.p. $G(n,p)$ consists of a giant component and isolated vertices.

Now set $p = (\log n + c)/n$, where $c \in \mathbb{R}$ is fixed. By linearity of expectation, the expected number of isolated vertices V is

$$\mathbb{E}[V] = n(1-p)^{n-1},$$

and since $1 - p \approx e^{-p}$ for $p \approx 0$, we have

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By computing the higher moments, or by Stein's method, one can show that V approaches a Poisson distribution with mean e^{-c}.

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- The ultimate obstruction to connectivity is isolated vertices. A theorem of Bollobás and Thomason takes this idea to its natural conclusion.
- At the connectivity threshold, $G \sim G(n, p)$ is already an *expander*.

The normalized graph Laplacian $\mathcal{L}$ of a connected graph H is the matrix

$$\mathcal{L} = I - D^{-1/2} A D^{-1/2},$$

where I is the identity matrix, D is the diagonal matrix with degrees down the diagonal, and A is the adjacency matrix.

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If $\{H_i\}$ is a sequence of graphs such that $\# V(H_i) \to \infty$ and such that

$$\liminf \lambda_2[H_i] > 0,$$

as $i \to \infty$, we say that $\{H_i\}$ is an *expander family*.

Theorem (Hoffman, K., Paquette, 2012)

Fix $k \geq 0$ and let $\omega \to \infty$ arbitrarily slowly. Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$ be the eigenvalues of the normalized graph Laplacian of $G \sim G(n,p)$. If

$$p \geq \frac{(k+1)\log n + \omega}{n},$$

then

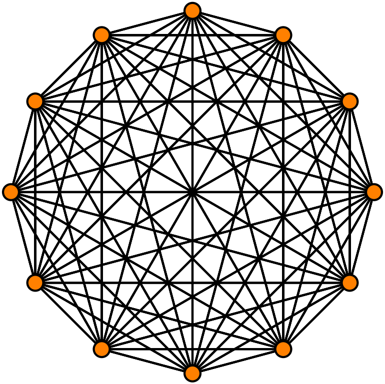
$$1 - \sqrt{\frac{C}{np}} \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 1 + \sqrt{\frac{C}{np}},$$

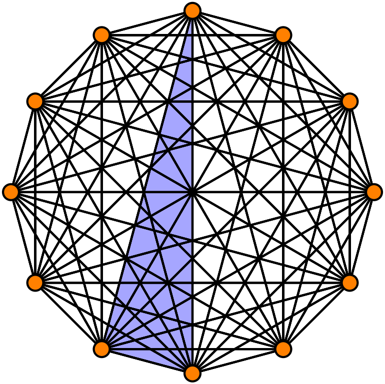
with probability at least $1 - o\left(n^{-k}\right)$. Here C is a constant which only depends on k.

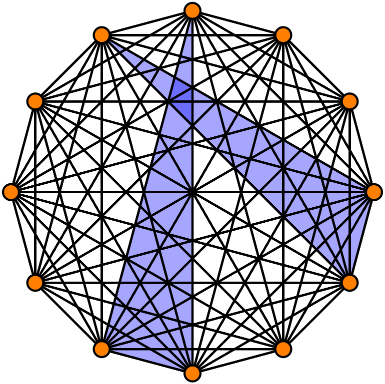
Random simplicial complexes

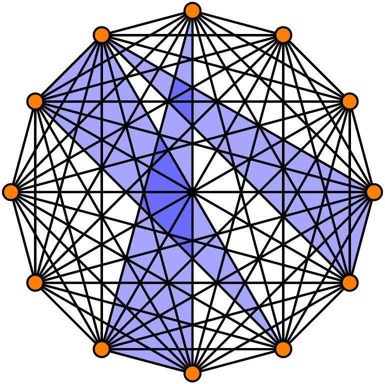
Linial and Meshulam defined $Y_2(n, p)$ to be the probability space of 2-dimensional simplicial complexes with vertex set $[n]$, edge set $\binom{[n]}{2}$, and such that each 2-face has probability p.

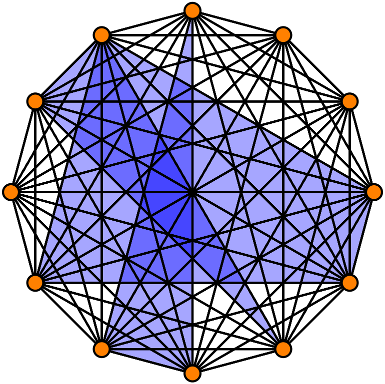
More generally, Meshulam and Wallach defined the random d-dimensional simplicial complex $Y_d(n, p)$ to be the probability distribution over all simplicial complexes on n vertices with complete $d - 1$-skeleton, and such that every d-dimensional face is included with probability p independently.

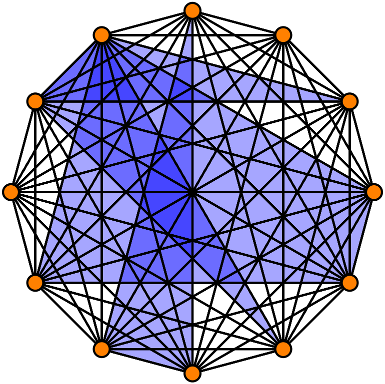


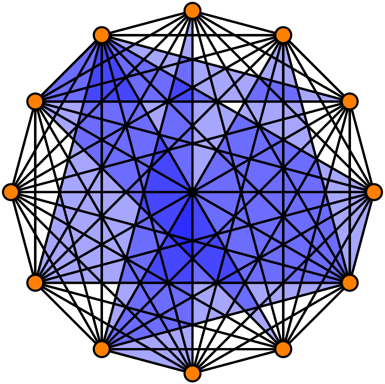


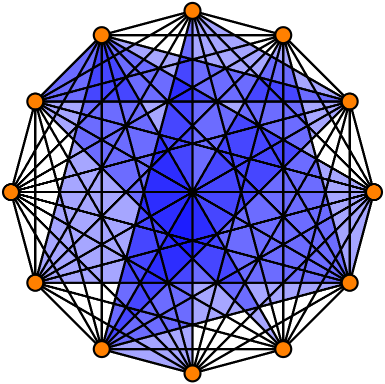


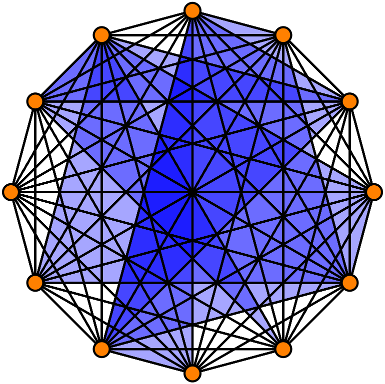


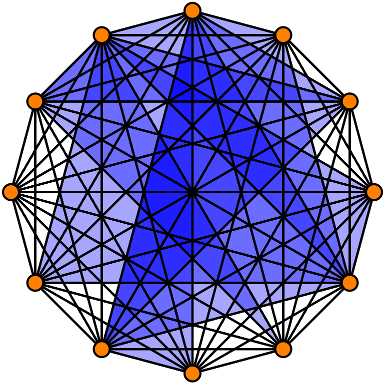












The following theorems generalized the Erdős–Rényi theorem to higher dimensions, and inspired a lot of future work by various teams of researchers.

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Theorem

(Linial–Meshulam, 2006) Let $\epsilon > 0$ *be fixed and* $Y \sim Y_2(n, p)$. *Then*

$$\mathbb{P}[H^1(Y, \mathbb{Z}/2) = 0] \to \begin{cases} 1 & : p \geq \frac{(2+\epsilon)\log n}{n} \\ 0 & : p \leq \frac{(2-\epsilon)\log n}{n} \end{cases}$$

Theorem

(Meshulam–Wallach, 2008) Let $d \geq 2$, $\ell \geq 2$, *and* $\epsilon > 0$ *be fixed and* $Y \sim Y_d(n, p)$. *Then*

$$\mathbb{P}[H^{d-1}(Y, \mathbb{Z}/\ell) = 0] \to \begin{cases} 1 & : p \geq \frac{(d+\epsilon)\log n}{n} \\ 0 & : p \leq \frac{(d-\epsilon)\log n}{n} \end{cases}$$

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Theorem

(Babson–Hoffman–K., 2011) Let $\epsilon > 0$ *be fixed and* $Y \sim Y_2(n, p)$. *Then*

$$\mathbb{P}[\pi_1(Y) = 0] \to \begin{cases} 1 & : p \geq \frac{n^{\epsilon}}{\sqrt{n}} \\ 0 & : p \leq \frac{n^{-\epsilon}}{\sqrt{n}} \end{cases}$$

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Theorem (Costa–Farber)

Let $Y \sim Y_2(n, p)$, and set $p = n^{-\alpha}$. If $\alpha > 1$ then w.h.p. $\operatorname{cd} \pi_1(Y) = 1$, if $3/5 < \alpha < 1$ then w.h.p. $\operatorname{cd} \pi_1(Y) = 2$, and if $1/2 < \alpha < 3/5$ then w.h.p. $\operatorname{cd} \pi_1(Y) = \infty$.

We recently revisited homology vanishing theorems for random d-dimensional complexes.

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Theorem (Garland, Ballman–Swiatkowski)

If Δ is a finite, pure d-dimensional, simplicial complex, such that

$$\lambda_2[lk(\sigma)] > 1 - \frac{1}{d}$$

for every $(d-2)$-dimensional face σ, then $H^{d-1}(\Delta, \mathbb{Q}) = 0$.

Using Garland's method, together with the new spectral gap theorem, we immediately recover the following theorem of Meshulam and Wallach.

Theorem

Let $d \geq 2$ and $\epsilon > 0$ be fixed and $Y \sim Y_d(n, p)$. Then

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We also get stronger "stopping time" results, analogous to the Bollobás–Thomason theorem, and these results are new.

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- Sharp vanishing thresholds for cohomology of random flag complexes.

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- A sharp threshold for $\pi_1(Y)$ to have Kazhdan's property (T) (with Hoffman and Paquette).
- Sharp vanishing thresholds for cohomology of random flag complexes.
- Random right angled Coxeter groups are rational duality groups (with Davis).

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The Meshulam–Wallach theorem tells that for every fixed prime ℓ, the threshold for $H_1(Y, \mathbb{Z})$ but what about ℓ growing with n?

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For example, there is a $\mathbb{Q}$-acyclic simplicial complex Δ on 31 vertices with

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In fact, Gil Kalai showed in 1983 that a uniform random $\mathbb{Q}$-acyclic complex on n vertices has enormous torsion. Indeed,

$$\mathbb{E}\left[|H_1(\Delta, \mathbb{Z})|\right] \geq \exp\left(cn^2\right),$$

for some constant $c > 0$.

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Theorem (Hoffman, K., Paquette)

Let $d \geq 2$ be fixed and $Y \sim Y_d(n,p)$. If

$$p \geq \frac{80d \log n}{n}$$

then $H_{d-1}(Y; \mathbb{Z}) = 0$ w.h.p.

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Let $d \geq 2$ be fixed and $Y \sim Y_d(n, p)$. If

$$p \geq \frac{80 d \log n}{n}$$

then $H_{d-1}(Y; \mathbb{Z}) = 0$ w.h.p.

(This is tight, up to the factor of 80. We have not tried to optimize this constant.)

An important intermediate result is the following.

Lemma (Hoffman, K., Paquette)

Let ℓ be any prime, $d \geq 2$ be fixed, and $Y \sim Y_d(n,p)$. If

$$p \geq \frac{40 d \log n}{n},$$

then

$$\mathbb{P}\left[H_{d-1}(Y;\ell) \neq 0\right] \leq \frac{1}{n^{d+1}}.$$

Given the lemma, the proof of the theorem follows quickly, as follows.

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Suppose that

$$p \geq \frac{40 d \log n}{n}.$$

Then w.h.p. we have that $H_{d-1}(Y;\mathbb{Q}) = 0$, so $H_{d-1}(Y;\mathbb{Z})$ is some finite abelian group. A key estimate is that

$$H_{d-1}(Y;\mathbb{Z}) = \exp\left(O(n^d)\right),$$

so there are at most $O(n^d)$ distinct primes ℓ such that $H_{d-1}(Y;\mathbb{Z}/\ell) \neq 0$.

Now take two independent random complexes $Y_1, Y_2 \sim Y_2(n,p)$ and let $Y = Y_1 \cup Y_2$ be their union. Standard random graph theory techniques give that Y is stochastically dominated by a complex drawn from $Y(n, 2p)$.

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There at most $O(n^d)$ primes ℓ such that $H_{d-1}(Y_1, \mathbb{Z}/\ell) \neq 0)$. But by the lemma, for every such prime,

$$\mathbb{P}\left[H_{d-1}(Y_2, \mathbb{Z}/\ell) \neq 0\right] \leq \frac{1}{n^{d+1}}.$$

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Applying a union bound, the probability that there is any prime ℓ such that $H_{d-1}(Y_2, \mathbb{Z}/\ell) \neq 0$ is less than $O(1/n) \to 0$.

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Consider the random d-complex process on n vertices, adding one face at a time.

$$Y_d(n, 0) \subset Y_d(n, 1) \subset Y_d(n, 2) \subset \ldots.$$

So let us give a heuristic for a proof of the key lemma.
Consider the random d-complex process on n vertices, adding one face at a time.

$$Y_d(n,0) \subset Y_d(n,1) \subset Y_d(n,2) \subset \dots.$$

Let E_m be the event that

$$\dim H_{d-1}(Y_d(n,m),\ell) > \dim H_{d-1}(Y_d(n,m+1),\ell),$$

and set $\tilde{m}$ be the smallest m such that $\mathbb{P}(E_m) < 1/2$.

It seems reasonable to guess that $\tilde{m} < 2\binom{n}{d}$, since the starting dimension

$$\dim H_{d-1}(Y_d(n,0), \ell) < \binom{n}{d}.$$

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If so, then we are in good shape. Indeed, considering $k = C \log n$ independent complexes $Y^1, \ldots, Y^k$, each with distribution $Y_d(n, \tilde{m})$ makes the probability that the d-dimensional face f is in the ℓ-homology reducing set for $Y_d(n, k\tilde{m})$ less than $n^{-\alpha}$ where α can be made arbitrarily large by increasing C.

Then by making C large enough that $\alpha > d$ and applying a union bound, we have that the probability that there is *any* face that reduces homology is $o(1)$. But the only way that there is no face that reduces homology $H_{d-1}(Y, \ell)$ is if $H_{d-1}(Y, \ell) = 0$.

Open problems

There is still so much to do in this area, but I'll finish by mentioning two of my favorite open problems.

Let $X(n, p)$ be a random flag complex, i.e. the maximal simplicial complex compatible with $G(n, p)$ as its 1-skeleton.

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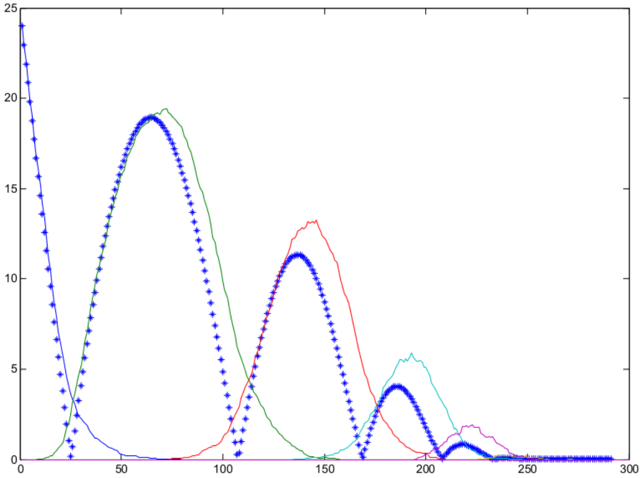
Conjecture

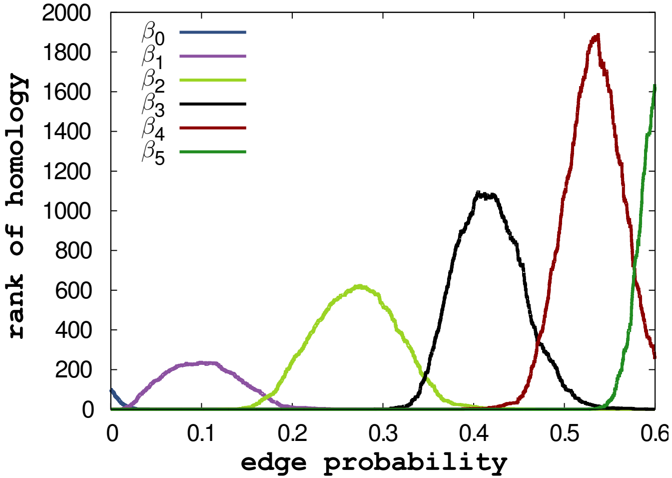
If $k \geq 3$ is fixed, and

$$\frac{1}{n^{1/k}} \ll p \ll \frac{1}{n^{1/(k+1)}},$$

then $X \sim X(n,p)$ homotopy equivalent to a wedge of k-dimensional spheres w.h.p.



A random flag complex on $n = 25$ vertices, together with expected Euler characteristic. Computation and image courtesy of Vidit Nanda.



A random flag complex on $n = 100$ vertices. Computation and image courtesy of Afra Zomorodian.

A question from geometric group theory: is every hyperbolic group residually finite?

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Conjecture

There exist hyperbolic groups which are not residually finite.

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There exist hyperbolic groups which are not residually finite.

For all we know, random fundamental group $\pi_1(Y)$ for $Y \sim Y_2(n, p)$ and a certain range of p may be a candidate for a group which does not admit *any* maps to finite groups.

Thanks for your time and attention!

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