Quantization for finite frames

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In this lecture, we will mainly follow the survey article [30]. In addition, we will discuss several recent results on finite frame quantization with a focus on applications in compressive sampling. Below, we outline our lecture and provide some key references.

1 Generalities on finite frames and quantization

- Finite frame $\{f_n\}_{n=1}^N \subset \mathbb{R}^d$ and dual frame $\{h_n\}_{n=1}^N \subset \mathbb{R}^d$
- Frame expansion:

$$\forall x \in \mathbb{R}^d, \quad x = \sum_{n=1}^{N} \langle x, f_n \rangle h_n$$

- Frame expansion discretely encodes $x \in \mathbb{R}^d$ by the frame coefficients $\{\langle x, f_n \rangle\}_{n=1}^N$.
- Frame coefficients $\langle x, f_n \rangle$ can take on a continuum of values.
- To be amenable to digital processing and storage, the coefficients $\langle x, f_n \rangle$ need to be discretized in amplitude. This discretization step is known as *quantization*.

Suggested readings:

- Primary suggested reading: The survey article [30] will serve as a main template for the lecture. This will be supplemented by a discussion of recent and ongoing work on quantization for finite frames.
- Other suggested readings: The references [16, 2, 12, 18] have influenced the general perspective of the lecture. Other references from the large literature on quantization for finite frames are summarized below.

We shall focus on the following methods for quantization of finite frames:

- **Memoryless scalar quantization (MSQ):** This is a simple classical method but is not particularly adept at exploiting redundancy present in frames.
- **First order Sigma-Delta ($\Sigma\Delta$) quantization:** This is a more sophisticated low complexity approach which effectively exploits redundancy but still leaves much room for theoretical improvements.

- **Higher order Sigma-Delta ($\Sigma\Delta$) quantization:** This method yields strong error bounds at the cost of increased complexity by exploiting a class of noncanonical dual frames known as Sobolev duals.

2 Memoryless scalar quantization (MSQ)

- Let $\mathcal{A} \subset \mathbb{R}$ be a finite set known as a *quantization alphabet*.
- Let $Q : \mathbb{R} \to \mathcal{A}$ be a function satisfying $|u - Q(u)| \leq |u - a|$ for all $u \in \mathbb{R}$ and $a \in \mathcal{A}$. In other words, $Q(u)$ rounds $u \in \mathbb{R}$ to a nearest element of $\mathcal{A}$. We shall refer to Q as a *scalar quantizer* associated to the quantization alphabet $\mathcal{A}$.
- MSQ quantizes frame coefficients $\langle x, f_n \rangle$ by $q_n = Q(\langle x, f_n \rangle)$.
- Reconstruction algorithms? Error bounds? Frame theoretic issues?
- Suggested references:
  - General discussion about MSQ for frames: [16, 15, 10]
  - Error bounds for MSQ: [37]
  - Uniform noise models and dithering in quantization: [4, 17, 24, 7]
  - Consistent reconstruction: [34, 35, 33, 29, 32]
  - β expansions: [13]

3 First order Sigma-Delta ($\Sigma\Delta$) quantization

- MSQ does not utilize the redundancy of frames.
- First order $\Sigma\Delta$ quantization exploits correlations among frame coefficients.
- First order $\Sigma\Delta$ quantization:
  Set $u_0 = 0$ and produce quantized coefficients $\{q_n\}_{n=1}^N$ by iterating for $n = 1, \cdots, N$:

$$q_n = Q(u_{n-1} + \langle x, f_n \rangle)$$
$$u_n = u_{n-1} + \langle x, f_n \rangle - q_n$$

- Intuition? Reconstruction algorithms? Error bounds? Stability of the algorithm? Frame theoretic issues? Robustness?
- Suggested references:
  - Basic and refined error bounds for $\Sigma\Delta$ quantization of finite frames: [2]
  - $\Sigma\Delta$ quantization for Shannon sampling expansions: [12, 19, 31]
  - Comparison of MSQ and $\Sigma\Delta$: [1]
  - Frame paths and error bounds: [8]
  - Robustness: [20]
  - Connections with travelling salesman problem: [36]
  - Origins and engineering issues: [23, 28]

4 Higher order Sigma-Delta $(\Sigma\Delta)$ quantization

- rth order $\Sigma\Delta$ quantization:

$$q_n = Q(F(\{u_j\}_{j=n-T}^{n}, \{\langle x, f_j\rangle\}_{j=n-S}^{n})$$
$$(\Delta^r u)_n = \langle x, f_n\rangle - q_n$$

  Here Δ is defined by $(\Delta u)_n = u_n - u_{n-1}$ and Δ^r is defined by $\Delta^r = \Delta\Delta^{r-1}$. The function F is a specially designed quantization rule.
- Why higher order $\Sigma\Delta$? Reconstruction algorithms? Canonical dual frames versus non-canonical dual frames? Error rates as a function of frame redundancy? Frame theoretic issues? Applications to compressed sensing?
- Suggested references:
  - Dual frames for $\Sigma\Delta$ quantization: [27, 5, 25, 9]
  - Compressed sensing, Gaussian and sub-Gaussian random frames: [21, 26]
  - Stability of higher order $\Sigma\Delta$ quantization: [12, 38]
  - Ergodic dyamics and tilings for $\Sigma\Delta$ quantization: [22]
  - Nonlinear reconstruction: [34]

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