STATIONARY VERSION AND FLUCTUATION EXPONENTS FOR THE EXACTLY SOLVABLE CORNER GROWTH MODEL

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1. Introduction

This lecture discusses the fluctuation exponents of the exactly solvable corner growth model. We construct the stationary version of the last-passage process, use it to derive the explicit limit shape, and then go over some of the ideas involved in the proof of the fluctuation exponents.

The results in this lecture should be contrasted with the following classical results. Let $\{X_k\}_{k\in\mathbb{N}}$ be a sequence of independent, identically distributed random variables with partial sums $S_n = X_1 + \cdots + X_n$. Let $\mu = E(X_i)$ be their common mean and $\sigma^2 = \mathrm{Var}(X_i) = E[(X_i-\mu)^2\,]$ their common variance, whenever these exist. Under the finite mean assumption, the *strong law of large numbers* says that $n^{-1}S_n$ converges to μ with probability 1. Under the finite variance assumption, the *central limit theorem* says that the random fluctuations of S_n around its mean $E(S_n) = n\mu$ have order of magnitude $n^{1/2}$ and in the limit they are Gaussian:

$$\lim_{n\to\infty} P\Big\{\frac{S_n - n\mu}{n^{1/2}\sigma} \leqslant x\Big\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-s^2/2}\,ds\,, \qquad x\in\mathbb{R}. \tag{1.1}$$

The CLT is an example of a limit in distribution: the probability distribution of the standardized sum $\frac{S_n-n\mu}{n^{1/2}\sigma}$ converges to the standard normal distribution whose probability density function is $\varphi(s) = e^{-s^2/2}/\sqrt{2\pi}$.

That the fluctuation exponent of S_n is 1/2 can be discovered with a variance calculation, without needing the full force of the CLT:

$$\begin{aligned}\mathrm{Var}(S_n) &= E\Big[\Big(\sum_{i=1}^{n}(X_i-\mu)\Big)^2\,\Big] = \sum_{i=1}^{n} E[(X_i-\mu)^2\,] + \sum_{i\neq j} E[(X_i-\mu)(X_j-\mu)]\\ &= n\sigma^2.\end{aligned} \tag{1.2}$$

The crucial use of independence happened in the last equality above where the cross terms (covariances) vanish for $i\neq j$: the expectation of a product of independent random variables is the product of expectations, and so

$$E[(X_i-\mu)(X_j-\mu)] = E[(X_i-\mu)]\cdot E[(X_j-\mu)] = 0\cdot 0 = 0.$$

Calculation (1.2) says that the standard deviation of S_n equals $n^{1/2}\sigma$. This we interpret as saying that the "typical" fluctuations of S_n have order of magnitude $n^{1/2}$.

In this lecture we go over results for the corner growth model that are analogues of the law of large numbers and the identification of the fluctuation exponent 1/2. In the KPZ class the corresponding fluctuation exponent is 1/3. The results in this lecture were originally published in article [1].

Notation, definitions and terminology. $\mathbb{Z}_+ = \{0,1,2,3,\dots\}$ and $\mathbb{N} = \{1,2,3,\dots\}$. The standard basis vectors of $\mathbb{R}^2$ are $e_1 = (1,0)$ and $e_2 = (0,1)$. For a point $x = (x_1,x_2)\in\mathbb{R}^2$ the ℓ^1-norm is $|x|_1 = |x_1| + |x_2|$ and integer parts are taken coordinatewise: $\lfloor x\rfloor = (\lfloor x_1\rfloor, \lfloor x_2\rfloor)$. We call the x-axis occasionally the e_1-axis, and similarly the y-axis and the e_2-axis are the same thing. C is a constant whose value can change from line to line.

$X\sim\mathrm{Exp}(\lambda)$ for $0<\lambda<\infty$ means that random variable X has exponential distribution with rate λ. This X is a positive random variable whose probability distribution satisfies $P(X>t) = e^{-\lambda t}$ for $t\geqslant 0$. It has mean $E(X) = \lambda^{-1}$ and variance $\mathrm{Var}(X) = \lambda^{-2}$.

We write ω_x and $\omega(x)$ interchangeably for the weight attached to lattice point x. $\overline{X} = X - EX$ is the centered random variable X.

2. The exactly solvable corner growth model

We repeat the definition of the planar corner growth model with independent and identically distributed (i.i.d.) weights. The model is defined on a probability space $(\Omega, \mathfrak{S}, \mathbb{P})$ whose ingredients are these: $\Omega = \mathbb{R}^{\mathbb{Z}^2}$ is the Cartesian product space of environments or weight configurations $\omega = (\omega_x)_{x \in \mathbb{Z}^2}$ where ω_x is a real-valued random weight attached to lattice vertex x, $\mathfrak{S}$ is the Borel σ-algebra of Ω, and $\mathbb{P}$ is a Borel product probability measure on Ω under which the weights $\{\omega_x\}$ are independent, identically distributed (i.i.d.) nonconstant random variables. That $\mathbb{P}$ is an i.i.d. product measure means that

$$\mathbb{P}\{\omega : \omega_{x_i} \in B_i \text{ for } i = 1, \dots, n\} = \prod_{i=1}^{n} \mathbb{P}\{\omega : \omega_0 \in B_i\}$$

for any distinct points $x_1, \dots, x_n \in \mathbb{Z}^2$ and Borel subsets $B_1, \dots, B_n \subseteq \mathbb{R}$.

Given an environment ω and two points $x, y \in \mathbb{Z}^2$ with $x \leqslant y$ coordinatewise, define the *point-to-point last-passage time* by

$$G_{x,y} = \max_{x_\bullet \in \Pi_{x,y}} \sum_{k=0}^{|y-x|_1} \omega_{x_k}. \tag{2.1}$$

$\Pi_{x,y}$ is the set of paths $x_\bullet = (x_k)_{k=0}^n$ that start at $x_0 = x$, end at $x_n = y$ with $n = |y - x|_1$, and have increments $x_{k+1} - x_k \in \{e_1, e_2\}$. Call such paths *admissible* or *up-right*. The extreme case is $G_{x,x} = \omega_x$.

There are two choices of weight distribution that make this model solvable, the exponential and geometric distributions. Solvability means that the model has special features that enable precise computations that are not accessible to the corner growth model with general weights. We work with the exponentially distributed weights, and so make the following assumption:

$$\text{the weights } \omega_x \text{ are rate 1 exponentially distributed random variables.} \tag{2.2}$$

This means that $\mathbb{P}\{\omega_x > t\} = e^{-t}$ for $t \geqslant 0$. This is abbreviated as $\omega_x \sim \text{Exp}(1)$.

The first consequence of choosing this particular weight distribution is that we can find the shape function g explicitly. For general weight distributions the finite limit in (2.3) below exists under a moment assumption, such as $\mathbb{E}|\omega_0|^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$, but very little is known about g, except for the soft properties of concavity and continuity.

Theorem 2.1. *Assume (2.2). Then we have the following law of large numbers. For every $\xi \in \mathbb{R}_+^2$ the limit below holds with probability 1, with the shape function g as given.*

$$\lim_{N \to \infty} N^{-1} G_{0, \lfloor N\xi \rfloor} = g(\xi) \equiv \big(\sqrt{\xi_1} + \sqrt{\xi_2}\,\big)^2. \tag{2.3}$$

The next result states that the fluctuation exponent of $G_{0,\lfloor N\xi \rfloor}$ is 1/3, as predicted by Kardar-Parisi-Zhang (KPZ) universality.

Theorem 2.2. *Fix $\xi \in (0, \infty)^2$. Then we have the following statements, where the constants can depend on ξ.*

There exist constants $0 < c_0, N_0 < \infty$ such that, for $b > 0$ and $N \geqslant N_0$

$$\mathbb{P}\big\{\, |G_{0,\lfloor N\xi \rfloor} - Ng(\xi)| \geqslant bN^{1/3} \big\} \leqslant c_0 b^{-3/2}. \tag{2.4}$$

For each $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for $N \geqslant 1$

$$\mathbb{P}\big\{\, |G_{0,\lfloor N\xi \rfloor} - Ng(\xi)| \geqslant \delta N^{1/3} \big\} \geqslant \varepsilon_0. \tag{2.5}$$

There exists a constant $0 < C = C(\xi, p) < \infty$ such that for $N \geqslant 1$ and $1 \leqslant p < 3/2$,

$$C^{-1} N^{p/3} \leqslant \mathbb{E}\big[\, |G_{0,\lfloor N\xi \rfloor} - Ng(\xi)|^p \,\big] \leqslant C N^{p/3}. \tag{2.6}$$

In order to prove Theorem 2.1 we construct and study a stationary version of the growth process. The current version of these notes does not prove Theorem 2.2, but does prove the corresponding fluctation theorem for the stationary corner growth process.

3. The corner growth process with stationary increments

Stationarity in probability theory means in general that the probability distribution of a model is suitably invariant under a group, which in this case is the group of lattice translations $\{\theta_x\}_{x\in\mathbb{Z}^2}$ that act on Ω by $(\theta_x\omega)_y = \omega_{x+y}$ for $x, y \in \mathbb{Z}^2$. To create a stationary version of the corner growth model, we restrict the i.i.d. Exp(1) weights to the first positive quadrant $\mathbb{N}^2$ by considering $(\omega_x)_{x\in\mathbb{N}^2}$, and then add to the space of weights boundary weights $(\omega_{ie_1})_{i\in\mathbb{N}}$ on the x-axis and $(\omega_{je_2})_{j\in\mathbb{N}}$ on the y-axis. We can also put an inconsequential weight $\omega_0 = 0$ at the origin. We fix a parameter value $0 < \rho < 1$ and set the distributions of the weights as follows: the entire collection $(\omega_x)_{x\in\mathbb{Z}^2_+}$ is a collection of mutually independent random variables with these marginal distributions:

$$\begin{aligned} \omega_{ie_1} &\sim \text{Exp}(1-\rho) \quad \text{for } i \in \mathbb{N} \\ \omega_{je_2} &\sim \text{Exp}(\rho) \quad \text{for } j \in \mathbb{N} \\ \omega_x &\sim \text{Exp}(1) \quad \text{for } x \in \mathbb{N}^2. \end{aligned} \tag{3.1}$$

In this setting define the last-passage time

$$G^\rho_x = \max_{x_\bullet\in\Pi_{0,x}} \sum_{k=1}^{|x|_1} \omega_{x_k}, \qquad x \in \mathbb{Z}^2_+. \tag{3.2}$$

(It is immaterial now whether the sum starts at $k = 0$ or $k = 1$ because $\omega_0 = 0$.) The superscript ρ is there to distinguish this last-passage problem from the generic one (2.1) with i.i.d. weights. Note that $G^\rho_0 = 0$. The initial point in (3.2) is always 0, and so we simplified the notation from $G^\rho_{0,x}$ to G^ρ_x.

Define *increment variables*

$$I_{x,y} = G^\rho_y - G^\rho_x, \qquad x, y \in \mathbb{Z}^2_+. \tag{3.3}$$

Note that the distributions of the I-variables on the axes are determined by the boundary conditions in (3.1), because, for example, on the x-axis for $0 \leqslant k < \ell$,

$$I_{ke_1,\ell e_1} = G^\rho_{\ell e_1} - G^\rho_{ke_1} = \sum_{i=1}^{\ell} \omega_{ie_1} - \sum_{i=1}^{k} \omega_{ie_1} = \sum_{i=k+1}^{\ell} \omega_{ie_1}.$$

By additivity the nearest-neighbor increments $I_{x,x+e_r}$, $x \in \mathbb{Z}^2_+$ and $r \in \{1, 2\}$, determine all $I_{x,y}$. A useful way to think of the nearest-neighbor increment variables is that their boundary conditions $I_{(k-1)e_r,\, ke_r} = \omega_{ke_r}$ for $k \in \mathbb{N}$ and $r \in \{1, 2\}$ are given in (3.1), and for $x \in \mathbb{N}^2$ the increment variables are determined by the following *north-east induction*:

$$\begin{aligned} I_{x-e_1,\,x} &= \omega_x + (I_{x-e_1-e_2,\,x-e_2} - I_{x-e_1-e_2,\,x-e_1})^+ \\ I_{x-e_2,\,x} &= \omega_x + (I_{x-e_1-e_2,\,x-e_1} - I_{x-e_1-e_2,\,x-e_2})^+. \end{aligned} \tag{3.4}$$

These equations come from the induction satisfied by the last-passage process:

$$G^\rho_x = \omega_x + G^\rho_{x-e_1} \vee G^\rho_{x-e_2}, \qquad x \in \mathbb{N}^2,$$

from which

$$\begin{aligned} I_{x-e_1,\,x} = G^\rho_x - G^\rho_{x-e_1} &= \omega_x + 0 \vee (G^\rho_{x-e_2} - G^\rho_{x-e_1}) = \omega_x + (G^\rho_{x-e_2} - G^\rho_{x-e_1-e_2} - G^\rho_{x-e_1} + G^\rho_{x-e_1-e_2})^+ \\ &= \omega_x + (I_{x-e_1-e_2,\,x-e_2} - I_{x-e_1-e_2,\,x-e_1})^+. \end{aligned}$$

Define further

$$\breve{\omega}_x = I_{x,x+e_1} \wedge I_{x,x+e_2} \qquad \text{for } x \in \mathbb{Z}^2_+. \tag{3.5}$$

Then the induction can be carried out by a sequence of involutions that takes place over each unit cell $\{x - e_1 - e_2, x - e_1, x - e_2, x\}$: the mapping

$$(I_{x-e_1-e_2,\,x-e_2},\, I_{x-e_1-e_2,\,x-e_1},\, \omega_x) \mapsto (I_{x-e_1,\,x},\, I_{x-e_2,x},\, \breve{\omega}_{x-e_1-e_2}) \tag{3.6}$$

defined by (3.4)–(3.5) is an involution. The induction begins at the left bottom corner $x = e_1 + e_2$ of $\mathbb{Z}^2_+$ and then proceeds up and right one corner at a time.

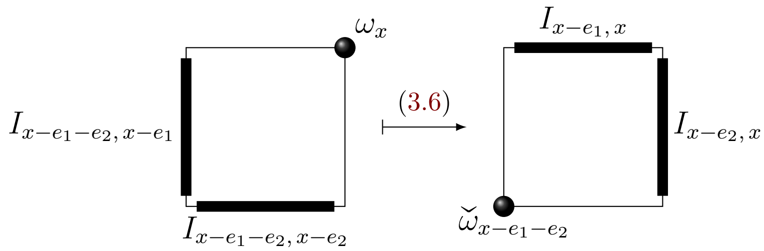

FIGURE 1. Mapping (3.6) on a single lattice square. The figure illustrates how southwest corners are flipped into northeast corners in the inductive construction of the increment variables.

The translation invariance of this process is a property of the increment variables, as stated in the next theorem.

Theorem 3.1. *For each $u \in \mathbb{Z}_+^2$, the processes $\{I_{x,y} : x, y \in \mathbb{Z}_+^2\}$ and $\{I_{u+x,u+y} : x, y \in \mathbb{Z}_+^2\}$ have the same probability distribution.*

Theorem 3.1 follows from a more general statement that can be proved conveniently by induction. Given a bi-infinite down-right path $\mathcal{Y} = (y_k)_{k\in\mathbb{Z}}$ in $\mathbb{Z}_+^2$, that is, a path whose steps satisfy $y_k - y_{k-1} \in \{e_1, -e_2\}$ for $k \in \mathbb{Z}$, define the lattice regions $\mathcal{G}_- = \{x \in \mathbb{Z}_+^2 : \exists j \in \mathbb{N} \text{ s.t. } x + (j, j) \in \mathcal{Y}\}$ strictly to the south and west of $\mathcal{Y}$ and $\mathcal{G}_+ = \{x \in \mathbb{Z}_+^2 : \exists j \in \mathbb{N} \text{ s.t. } x - (j, j) \in \mathcal{Y}\}$ strictly to the north and east of $\mathcal{Y}$. Note that $\mathcal{G}_+$ is necessarily unbounded but $\mathcal{G}_-$ is finite iff all but finitely many points y_k lie on the e_1 and e_2 axes. In the extreme case $\mathcal{Y} = \{ie_1, je_2 : 0 \leqslant i, j < \infty\}$ consists of the axes, $\mathcal{G}_- = \varnothing$ and $\mathcal{G}_+ = \mathbb{N}^2$.

Proposition 3.2. *Given a bi-infinite down-right path $\mathcal{Y} = (y_k)_{k\in\mathbb{Z}}$ in $\mathbb{Z}_+^2$, the random variables*

$$\{\breve\omega_z,\, I_{y_{k-1},y_k},\, \omega_x : z \in \mathcal{G}_-,\, k \in \mathbb{Z},\, x \in \mathcal{G}_+\}$$

are mutually independent with marginal distributions

$$\breve\omega_z,\, \omega_x \sim \text{Exp}(1) \quad \text{and} \quad I_{y_{k-1},y_k} \sim \begin{cases} \text{Exp}(1-\rho) & y_k = y_{k-1} + e_1 \\ -\text{Exp}(\rho) & y_k = y_{k-1} - e_2. \end{cases}$$

Theorem 3.1 is a corollary of Proposition 3.2 applied to the case where $\mathcal{Y}$ is obtained by translating the origin to u: $y_k = u + ke_1$ and $y_{-k} = u + ke_2$ for $k \geqslant 0$.

We see an immediate enormous benefit of the invariance: the last-passage values G_x^ρ grow on average linearly. First rewrite in terms of nearest-neighbor increments:

$$G^\rho_{(m,n)} = I_{0,\,(m,0)} + I_{(m,0),\,(m,n)} = \sum_{i=1}^m I_{(i-1,0),\,(i,0)} + \sum_{j=1}^n I_{(m,j-1),\,(m,j)}. \tag{3.7}$$

Then use the translation invariance of the distributions which says that each nearest-neighbor increment has the exponential distribution imposed on the boundary variables in (3.1):

$$\mathbb{E}[G^\rho_{(m,n)}] = \sum_{i=1}^m \mathbb{E}I_{(i-1)e_1,\,ie_1} + \sum_{j=1}^n \mathbb{E}I_{(m,j-1),\,(m,j)} = \frac{m}{1-\rho} + \frac{n}{\rho}. \tag{3.8}$$

The limit of the stationary last-passage process is an application of the classical law of large numbers and some large deviation estimates, applied separately to the two sums: the limit below holds almost surely for any given $(s,t) \in \mathbb{R}_+^2$.

$$\begin{aligned} \lim_{N\to\infty} N^{-1} G^\rho_{(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} &= \lim_{N\to\infty} \Bigl\{ N^{-1} \sum_{i=1}^{\lfloor Ns\rfloor} I_{(i-1,0),\,(i,0)} + N^{-1} \sum_{j=1}^{\lfloor Nt\rfloor} I_{(\lfloor Ns\rfloor,\,j-1),\,(\lfloor Nt\rfloor,\,j)} \Bigr\} \\ &= s\mathbb{E}(I_{0,\,e_1}) + t\mathbb{E}(I_{0,\,e_2}) = \frac{s}{1-\rho} + \frac{t}{\rho}. \end{aligned} \tag{3.9}$$

To further illustrate the usefulness of this invariance together with the known boundary distributions in (3.1), we show how it yields the explicit form of the shape function g in (2.3).

Proof of Theorem 2.1, assuming that the limit in (2.3) exists and that g is finite, concave and continuous. The proof begins by coupling together the last-passage processes $G_{x,y}$ and G^ρ_x. Fix $t > 0$. Re-express (3.2) for $x = (\lfloor Nt \rfloor, \lfloor Nt \rfloor)$ by separating the boundary weights:

$$G^\rho_{(\lfloor Nt\rfloor,\lfloor Nt\rfloor)} = \sup_{0\leqslant a\leqslant t}\Big\{\sum_{i=1}^{\lfloor Na\rfloor} I_{(i-1,0),\,(i,0)} + G_{(\lfloor Na\rfloor,1),(\lfloor Nt\rfloor,\lfloor Nt\rfloor)}\Big\} \bigvee \sup_{0\leqslant b\leqslant t}\Big\{\sum_{j=1}^{\lfloor Nb\rfloor} I_{(0,j-1),\,(0,j)} + G_{(1,\lfloor Nb\rfloor),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}\Big\}.$$

After letting $N \to \infty$ we have

$$\frac{t}{1-\rho} + \frac{t}{\rho} = \sup_{0\leqslant a\leqslant t}\Big\{\frac{a}{1-\rho} + g(t-a,t)\Big\} \vee \sup_{0\leqslant b\leqslant t}\Big\{\frac{b}{\rho} + g(t,t-b)\Big\}.$$

Use the symmetry of g and assume that $0 < \rho \leqslant 1/2$:

$$\begin{aligned}\frac{t}{1-\rho} + \frac{t}{\rho} &= \sup_{0\leqslant a\leqslant t}\Big\{\frac{a}{1-\rho} + g(t-a,t)\Big\} \vee \sup_{0\leqslant b\leqslant t}\Big\{\frac{b}{\rho} + g(t-b,t)\Big\}\\ &= \sup_{0\leqslant b\leqslant t}\Big\{\frac{b}{\rho} + g(t-b,t)\Big\}.\end{aligned}$$

Let

$$f(b) = \begin{cases} -g(t-b,t), & 0 \leqslant b \leqslant t \\ \infty, & b < 0 \text{ or } b > t. \end{cases}$$

Then f is convex and lower semicontinuous. After a change of variable $x = 1/\rho \in [2,\infty)$, the equation above becomes

$$t\Big(x + 1 + \frac{1}{x-1}\Big) = \sup_{b\in\mathbb{R}}\{bx - f(b)\}, \qquad x \geqslant 2.$$

This is an instance of convex duality, so the convex conjugate f^* of f satisfies

$$f^*(x) = t\Big(x + 1 + \frac{1}{x-1}\Big) \qquad \text{for } x \geqslant 2.$$

The derivatives $(f^*)'(2+) = 0$ and $(f^*)'(\infty-) = t$ tell us that we can restrict the supremum in the double convex duality as below, for $0 \leqslant b \leqslant t$. Then find the supremum by calculus:

$$f(b) = f^{**}(b) = \sup_{x\geqslant 2}\{xb - f^*(x)\} = -\frac{b\sqrt{t(t-b)} + bt - t(\sqrt{t} + \sqrt{t-b}\,)^2}{\sqrt{t(t-b)}}.$$

Taking $b = t - s$ for $s \in [0,t]$ in the definition of f in terms of g gives

$$g(s,t) = (\sqrt{s} + \sqrt{t}\,)^2 \qquad \text{for } 0 \leqslant s \leqslant t.$$

Symmetry of g completes the proof. ∎

Next we turn to the fluctuations of the last-passage value G^ρ in the increment-stationary process. This result would also serve as an intermediate step towards proving Theorem 2.2. We do not have a closed form expression for $\mathbb{V}\mathrm{ar}[G^\rho_{(m,n)}]$ but we can access it well enough to show that it obeys the fluctuation exponent $1/3$ characteristic of the KPZ class. However, there in an extra twist. Notice in (3.1) that the boundary weights ω_{ie_1} and ω_{je_2} are larger on average than the bulk weights $\{\omega_x\}_{x\in\mathbb{N}^2}$. This implies that the boundaries are attractive to the maximizing path. It turns out that only when we take the point x to infinity in a particular *characteristic direction* $((1-\rho)^2, \rho^2)$, the pull of the boundaries balance out and G^ρ_x obeys KPZ fluctuations. Otherwise the boundaries swamp the effects of the percolation and G^ρ_x obeys the classical central limit theorem.

Let N be a scaling parameter that increases to ∞. We consider the point-to-point last-passage percolation from 0 to a point $(m,n) = (m(N), n(N))$ that is taken to infinity as $N \to \infty$. Let κ_N denote the deviation of (m,n) from the characteristic direction:

$$\kappa_N = |\,m - N(1-\rho)^2\,| + |\,n - N\rho^2\,|. \tag{3.10}$$

Theorem 3.3. *Assume weight distributions* (3.1) *and* $\kappa_N \leqslant a_0 N^{2/3}$ *for some constant* a_0. *Then* $\exists$ *constant* $0 < C = C(\rho, a_0) < \infty$ *such that*

$$C^{-1} N^{2/3} \leqslant \mathbb{V}\mathrm{ar}[G^{\rho}_{(m,n)}] \leqslant C N^{2/3} \qquad \text{for } N \geqslant 1. \tag{3.11}$$

We prove the upper bound in the theorem above completely and the lower bound for the case where κ_N is bounded by a constant.

As a fairly immediate corollary we obtain the behavior in off-characteristic directions. For concreteness, we state the result for the case where the horizontal direction is abnormally large.

Corollary 3.4. *Assume weight distributions* (3.1). *Suppose* $m, n \to \infty$. *Define parameter* N *by* $n = N\rho^2$, *and assume that*

$$N^{-\alpha}\bigl(m - N(1-\rho)^2\bigr) \to c_1 > 0 \quad \text{as } m, n \to \infty$$

for some $\alpha > 2/3$. *Then as* $m, n \to \infty$,

$$N^{-\alpha/2}\bigl\{G^{\rho}_{(m,n)} - \mathbb{E}(G^{\rho}_{(m,n)})\bigr\}$$

converges in distribution to a centered normal distribution with variance $c_1(1-\rho)^{-2}$.

Proof. Recall that overline means centering of a random variable.

$$N^{-\alpha/2}\,\overline{G}^{\rho}_{(m,n)} = N^{-\alpha/2}\,\overline{G}^{\rho}_{(\lfloor N(1-\rho)^2,\, N\rho^2)\rfloor} \;+\; N^{-\alpha/2} \sum_{i=\lfloor N(1-\rho)^2\rfloor+1}^{m} \bar{I}_{(i-1,n),(i,n)}$$

The mean square of the first term on the right is of order $N^{-\alpha} \cdot N^{2/3}$ and hence in the limit vanishes in L^2 and in probability. The second term is a sum of approximately $c_1 N^{\alpha}$ mean zero terms with variance $\mathbb{E}[\,\bar{I}^2_{(i-1,n),(i,n)}] = (1-\rho)^{-2}$. This sum gives the normal limit, by the CLT. □

The first step towards the proof of Theorem 3.3 is an explicit formula that ties together $\mathbb{V}\mathrm{ar}[G^{\rho}_{(m,n)}]$ and the amount of weight the maximizing path collects on the boundary.

For a given x, the last-passage problem (3.2) has an almost surely unique maximizing path $\bar{x}_{\bullet} = (\bar{x}_k)_{k=0}^{n}$ from $\bar{x}_0 = 0$ to $\bar{x}_n = x$ that satisfies $G^{\rho}_x = \sum_{k=1}^{n} \omega_{\bar{x}_k}$. For $r = 1, 2$, define the exit time (or exit point) of this path from the e_r axis by

$$\tau_r = \max\{k \geqslant 0 : \bar{x}_k \cdot e_{3-r} = 0\}, \qquad r = 1, 2.$$

If the first step of the path $\bar{x}_{\bullet}$ from the origin is e_r, then $1 \leqslant \tau_r \leqslant x \cdot e_r$ and $\tau_{3-r} = 0$. In other words, almost surely exactly one of τ_1 and τ_2 is positive (but which one is positive varies with the realization of the weights ω).

Further, introduce the sums of weights along the axes:

$$S_{r,k} = \sum_{i=1}^{k} \omega_{ie_r}, \qquad r = 1, 2.$$

Then S_{r,τ_r} is the amount of weight that the maximizing path collects on the e_r-axis. Again, for each weight configuration ω, exactly one of S_{1,τ_1} and S_{2,τ_2} is positive and the other one zero. When necessary for distinguishing processes with different boundary weights (3.1), these variables will be adorned with superscripts, as in τ^{ρ}_r and $S^{\rho}_{r,k}$.

Next we state the variance formula for the last-passage value in the increment-stationary CGM.

Theorem 3.5. *Assume weight distributions* (3.1).

$$\begin{aligned} \mathbb{V}\mathrm{ar}[G^{\rho}_{(m,n)}] &= -\frac{m}{(1-\rho)^2} + \frac{n}{\rho^2} + \frac{2}{1-\rho}\mathbb{E}[S_{1,\tau_1}] \\ &= \frac{m}{(1-\rho)^2} - \frac{n}{\rho^2} + \frac{2}{\rho}\mathbb{E}[S_{2,\tau_2}]. \end{aligned} \tag{3.12}$$

We skip the proof of this lemma for now. It involves explicit computations with exponential distributions and covariances.

4. Proof of the fluctuation exponent for the stationary process

This section proves Theorem 3.3. The section is divided into an upper bound proof and a lower bound proof.

4.1. **Upper bound.** We couple the boundary variables for two different parameters $0 < \rho < \lambda < 1$ as follows:

$$\omega^{\lambda}_{ie_1} = \frac{1-\rho}{1-\lambda}\omega^{\rho}_{ie_1} > \omega^{\rho}_{ie_1} \qquad \text{and} \qquad \omega^{\lambda}_{je_2} = \frac{\rho}{\lambda}\omega^{\rho}_{je_1 2} < \omega^{\rho}_{je_2}. \tag{4.1}$$

From this follows for example that $\tau_1^{\lambda} \geqslant \tau_1^{\rho}$ and $\tau_2^{\lambda} \leqslant \tau_2^{\rho}$, and also

$$S^{\lambda}_{1,\ell} - S^{\rho}_{1,\ell} \leqslant S^{\lambda}_{1,k} - S^{\rho}_{1,k} \qquad \text{for } 0 \leqslant \ell \leqslant k. \tag{4.2}$$

We begin with auxiliary lemmas.

Lemma 4.1. *Let $0 < \varepsilon < 1$. Then there exists a constant $C = C(\varepsilon)$ such that, for $\varepsilon \leqslant \rho < \lambda \leqslant 1-\varepsilon$,*

$$\mathbb{V}\mathrm{ar}[G^{\lambda}_{(m,n)}] \leqslant \mathbb{V}\mathrm{ar}[G^{\rho}_{(m,n)}] + Cm(\lambda - \rho).$$

Proof. From (4.1)

$$S^{\lambda}_{2,\tau_2} = \sum_{j=1}^{\tau_2^{\lambda}} \omega^{\lambda}_{(0,j)} \leqslant \sum_{j=1}^{\tau_2^{\rho}} \omega^{\lambda}_{(0,j)} = \frac{\rho}{\lambda}\sum_{j=1}^{\tau_2^{\rho}} \omega^{\rho}_{(0,j)} = \frac{\rho}{\lambda} S^{\rho}_{2,\tau_2}.$$

Using the second line of (3.12),

$$\begin{aligned}
\mathbb{V}\mathrm{ar}[G^{\lambda}_{(m,n)}] &= \frac{m}{(1-\lambda)^2} - \frac{n}{\lambda^2} + \frac{2}{\lambda}\mathbb{E}[S^{\lambda}_{2,\tau_2}] \\
&\leqslant \frac{(1-\rho)^2}{(1-\lambda)^2}\cdot\frac{m}{(1-\rho)^2} - \frac{\rho^2}{\lambda^2}\cdot\frac{n}{\rho^2} + \frac{\rho^2}{\lambda^2}\cdot\frac{2}{\rho}\mathbb{E}[S^{\rho}_{2,\tau_2}] \\
&= \frac{\rho^2}{\lambda^2}\cdot\mathbb{V}\mathrm{ar}[G^{\rho}_{(m,n)}] + \frac{m}{(1-\rho)^2}\left(\frac{(1-\rho)^2}{(1-\lambda)^2} - \frac{\rho^2}{\lambda^2}\right) \\
&\leqslant \mathbb{V}\mathrm{ar}[G^{\rho}_{(m,n)}] + Cm(\lambda-\rho).
\end{aligned}$$

□

Lemma 4.2. *Let $0 < \varepsilon < 1$. Then there exists a constant $C = C(\varepsilon)$ such that, for $\varepsilon \leqslant \rho \leqslant 1-\varepsilon$,*

$$\mathbb{E}[S^{\rho}_{1,\tau_1}] \leqslant C\big(\mathbb{E}[\tau_1^{\rho}] + 1\big). \tag{4.3}$$

We skip the proof of the above lemma.

The main estimate for the upper bound in (3.11) is contained in the next proposition.

Proposition 4.3. *Consider the increment-stationary CGM $G^{\rho}_{(m,n)}$ with weight distributions (3.1) for a given $0 < \rho < 1$. Let κ_N be defined by (3.10). Three positive constants a_0, a_1 and N_0 are given and the assumption is that*

$$\kappa_N \leqslant a_0 N^{2/3} \quad \textit{and} \quad m \leqslant a_1 N \quad \textit{for} \quad N \geqslant N_0. \tag{4.4}$$

Then there exist constants $c_2, c_3 < \infty$ such that the following two bounds hold:

$$\mathbb{P}\{\tau_1^{\rho} \geqslant \ell\} \leqslant c_3\Big(\frac{N^2}{\ell^3} + (1+a_0)\frac{N^{8/3}}{\ell^4}\Big) \qquad \textit{for } N \geqslant N_0 \textit{ and } 1 \vee c_2\kappa_N \leqslant \ell \leqslant m \tag{4.5}$$

and

$$\mathbb{E}[(\tau_1^{\rho})^q] \leqslant \Big(c_2 a_0 + \frac{c_3}{3-q}\Big)N^{2q/3} \qquad \textit{for } N \geqslant N_0 \textit{ and } 1 \leqslant q < 3. \tag{4.6}$$

The functional dependencies of the constants c_2, c_3 on the parameters is as follows:

$$c_2 = c_2(\rho) \quad \textit{and} \quad c_3 = c_3(a_1, \rho). \tag{4.7}$$

Furthermore, c_2 and c_3 are locally bounded functions of their arguments.

The upper variance bound in (3.11) follows from a combination of (3.12), assumption (3.10), (4.3), and (4.6) for $q = 1$.

Proof of Proposition 4.3. Consider $N \geqslant N_0$ so that the assumptions are in force. Assume that, for some $0 < c_2 < \infty$, the integer ℓ satisfies

$$1 \vee c_2\kappa_N \leqslant \ell \leqslant m \leqslant a_1 N.$$

The proof will choose $c_2 = c_2(\rho)$ large enough. Let $0 < r < 1$ be a constant that will be set small enough in the proof. Let

$$\lambda = \rho + \frac{r\ell}{N}. \tag{4.8}$$

We take $r = r(a_1, \rho)$ at least small enough so that $ra_1 < \frac{1}{2}(1-\rho)$. This guarantees that for $N \geqslant 1$, $\lambda \in (\rho, \frac{1+\rho}{2})$ is also a legitimate parameter for an increment-stationary CGM.

In the first inequality below use $S^\lambda_{1,k} + G_{(k,1),(m,n)} \leqslant G^\lambda_{(m,n)}$. In the last inequality below use (4.2).

$$\begin{aligned}
\mathbb{P}\{\tau_1^\rho \geqslant \ell\} &= \mathbb{P}\{\,\exists k \geqslant \ell:\ S^\rho_{1,k} + G_{(k,1),(m,n)} = G^\rho_{(m,n)}\,\} \\
&\leqslant \mathbb{P}\{\,\exists k \geqslant \ell:\ S^\lambda_{1,k} - S^\rho_{1,k} \leqslant G^\lambda_{(m,n)} - G^\rho_{(m,n)}\,\} \\
&\leqslant \mathbb{P}\{\,S^\lambda_{1,\ell} - S^\rho_{1,\ell} \leqslant G^\lambda_{(m,n)} - G^\rho_{(m,n)}\,\} \qquad (4.9)
\end{aligned}$$

Next we compute and bound the means of the random variables in the probability above. First

$$\mathbb{E}[S^\lambda_{1,\ell} - S^\rho_{1,\ell}] = \ell\Big(\frac{1}{1-\lambda} - \frac{1}{1-\rho}\Big) = \frac{\ell}{(1-\lambda)(1-\rho)}(\lambda-\rho) = \frac{1}{(1-\lambda)(1-\rho)}\cdot\frac{r\ell^2}{N}$$

Introduce the quantities

$$\kappa^1_N = m - N(1-\rho)^2 \quad\text{and}\quad \kappa^2_N = n - N\rho^2 \tag{4.10}$$

that satisfy (with κ_N as in (3.10))

$$|\kappa^1_N| + |\kappa^2_N| \leqslant \kappa_N.$$

Then the LPP values.

$$\begin{aligned}
\mathbb{E}[G^\lambda_{(m,n)} - G^\rho_{(m,n)}] &= m\Big(\frac{1}{1-\lambda} - \frac{1}{1-\rho}\Big) + n\Big(\frac{1}{\lambda} - \frac{1}{\rho}\Big) \\
&= \Big(\frac{m}{(1-\lambda)(1-\rho)} - \frac{n}{\lambda\rho}\Big)(\lambda-\rho) \\
&= N\Big(\frac{1-\rho}{1-\lambda} - \frac{\rho}{\lambda}\Big)(\lambda-\rho) + \Big(\frac{\kappa^1_N}{(1-\lambda)(1-\rho)} - \frac{\kappa^2_N}{\lambda\rho}\Big)(\lambda-\rho) \\
&= \frac{N}{\lambda(1-\lambda)}(\lambda-\rho)^2 + \Big(\frac{\kappa^1_N}{(1-\lambda)(1-\rho)} - \frac{\kappa^2_N}{\lambda\rho}\Big)(\lambda-\rho) \qquad (4.11) \\
&= \frac{r^2\ell^2}{\lambda(1-\lambda)N} + \Big(\frac{\kappa^1_N}{(1-\lambda)(1-\rho)} - \frac{\kappa^2_N}{\lambda\rho}\Big)\frac{r\ell}{N} \\
&\leqslant \frac{r^2\ell^2}{\lambda(1-\lambda)N} + \frac{1}{(1-\lambda)(1-\rho)\wedge\lambda\rho}\cdot\frac{r\ell^2}{c_2N}
\end{aligned}$$

The last inequality came from $\kappa_N \leqslant \ell/c_2$.

Comparison of the means shows that if we choose c_2 large enough and then r small enough, both as functions of (λ, ρ), then for a large enough constant $c_3 = c_3(\lambda, \rho)$,

$$\mathbb{E}[S^\lambda_{1,\ell} - S^\rho_{1,\ell}] > \mathbb{E}[G^\lambda_{(m,n)} - G^\rho_{(m,n)}] + \frac{r\ell^2}{c_3N}.$$

Since the range $\lambda \in (\rho, \frac{1+\rho}{2})$ is determined by ρ, the dependence on λ can be dropped and we have $c_2 = c_2(\rho)$, $r = r(a_1, \rho)$ and $c_3 = c_3(\rho)$.

We continue from line (4.9). Below we subsume r, a_1, ρ, λ dependent factors into a constant $C = C(a_1, \rho)$. Along the way we use Lemma 4.1, Theorem 3.5, (4.10), $\kappa_N \leqslant c_2^{-1}\ell$, $m \leqslant a_1 N$, and Lemma 4.2.

$$\begin{aligned}
\mathbb{P}\{\tau_1^\rho \geqslant \ell\} &\leqslant \mathbb{P}\Big\{\, \overline{S^\lambda_{1,\ell} - S^\rho_{1,\ell}} \leqslant \overline{G^\lambda_{(m,n)} - G^\rho_{(m,n)}} - \frac{r\ell^2}{c_3 N}\,\Big\} \\
&\leqslant \mathbb{P}\Big\{\, \overline{S^\lambda_{1,\ell} - S^\rho_{1,\ell}} \leqslant -\frac{r\ell^2}{2c_3 N}\,\Big\} + \mathbb{P}\Big\{\, \overline{G^\lambda_{(m,n)} - G^\rho_{(m,n)}} \geqslant \frac{r\ell^2}{2c_3 N}\,\Big\} \\
&\leqslant \frac{CN^2}{\ell^4}\mathbb{V}\mathrm{ar}[S^\lambda_{1,\ell} - S^\rho_{1,\ell}] + \frac{CN^2}{\ell^4}\mathbb{V}\mathrm{ar}[G^\lambda_{(m,n)} - G^\rho_{(m,n)}] \\
&\leqslant \frac{CN^2}{\ell^3} + \frac{CN^2}{\ell^4}\big(\mathbb{V}\mathrm{ar}[G^\lambda_{(m,n)}] + \mathbb{V}\mathrm{ar}[G^\rho_{(m,n)}]\big) \\
&\leqslant \frac{CN^2}{\ell^3} + \frac{CN^2}{\ell^4}\big(\mathbb{V}\mathrm{ar}[G^\rho_{(m,n)}] + m(\lambda-\rho)\big) \\
&= \frac{CN^2}{\ell^3} + \frac{CN^2}{\ell^4}\Big(-\frac{m}{(1-\rho)^2} + \frac{n}{\rho^2} + \frac{2}{1-\rho}\mathbb{E}[S^\rho_{1,\tau_1}] + a_1 N \cdot \frac{r\ell}{N}\Big) \\
&\leqslant \frac{CN^2}{\ell^3} + \frac{CN^2}{\ell^4}\big(\mathbb{E}[\tau_1^\rho] + \ell\big) \leqslant \frac{CN^2}{\ell^3} + \frac{CN^2}{\ell^4}\mathbb{E}[\tau_1^\rho]. \qquad (4.12)
\end{aligned}$$

Now use the assumption $\kappa_N \leqslant a_0 N^{2/3}$. Let $b = c_2 a_0 + 1 + C$, with C as above. This ensures $bN^{2/3} \geqslant c_2\kappa_N$ which lets us use the bound above for integers $\ell \geqslant bN^{2/3}$. By adjusting the constant C in the front we can apply the bound to all real $\ell \geqslant bN^{2/3}$.

$$\begin{aligned}
\mathbb{E}[\tau_1^\rho] = \int_0^\infty \mathbb{P}(\tau_1^\rho \geqslant s)\, ds \;&\leqslant\; bN^{2/3} + C\int_{bN^{2/3}}^\infty \Big(\frac{N^2}{s^3} + \frac{N^2}{s^4}\mathbb{E}[\tau_1^\rho]\Big)\, ds \\
&= bN^{2/3} + \frac{CN^{2/3}}{2b^2} + \frac{C}{3b^3}\mathbb{E}[\tau_1^\rho] \;\leqslant\; bN^{2/3} + \tfrac{1}{2}N^{2/3} + \tfrac{1}{3}\mathbb{E}[\tau_1^\rho].
\end{aligned}$$

From this we obtain the bound

$$\mathbb{E}[\tau_1^\rho] \leqslant \big(c_2(\rho)a_0 + C_1(a_1,\rho)\big)N^{2/3}$$

and thereby (4.6) has been proved for $q = 1$. Substituting this bound back into line (4.12) gives

$$\mathbb{P}\{\tau_1^\rho \geqslant \ell\} \leqslant C_2\Big(\frac{N^2}{\ell^3} + (1+a_0)\frac{N^{8/3}}{\ell^4}\Big)$$

for a constant $C_2 = C_2(a_1, \rho)$, verifying (4.5). Another integration with $b = c_2(\rho)a_0 + 1 + C_2$ proves (4.6) for $1 < q < 3$:

$$\begin{aligned}
\mathbb{E}[(\tau_1^\rho)^q] = \int_0^\infty \mathbb{P}(\tau_1^\rho \geqslant s)\, qs^{q-1}\, ds \;&\leqslant\; bN^{2/3} + C_2\int_{bN^{2/3}}^\infty \big(N^2 s^{q-4} + (1+a_0)N^{8/3}s^{q-5}\big)\, ds \\
&= bN^{2/3} + \frac{C_2 b^{q-3}}{3-q}N^{\frac{2}{3}q} + \frac{C_2}{b^{4-q}}(1+a_0)N^{\frac{2}{3}q} \leqslant \Big(c_2(\rho)a_0 + \frac{C_3}{3-q}\Big)N^{2q/3}
\end{aligned}$$

where we summarized the (a_1,ρ)-dependent constants into $C_3 = C_3(a_1,\rho)$. This completes the proof of Proposition 4.3, with c_3 defined as the constant that appears in front of the right-hand sides of (4.5)–(4.6). □

4.2. **Lower bound.** The parameter $0 < \rho < 1$ of the increment-stationary LPP process is fixed. Let N be the scaling parameter that is sent to infinity, and define the endpoint of the point-to-point LPP process by

$$(m,n) = \big(\lfloor N(1-\rho)^2\rfloor,\, \lfloor N\rho^2\rfloor\big)$$

going in the characteristic direction for ρ. Introduce another parameter $r > 0$ that will be fixed in the course of the proof. Define another parameter for the increment-stationary LPP by

$$\lambda = \rho + \frac{r}{N^{1/3}}.$$

To guarantee that $\lambda \in (\rho, \frac{1+\rho}{2})$ we assume that

$$N \geqslant N_0 = N_0(\rho, r) = 8\big(\tfrac{r}{1-\rho}\big)^3.$$

Notational comment: in this section we shall find it convenient to attach the parameters ρ and λ to the measure $\mathbb{P}$ to indicate which boundary variables are used together with the Exp(1) bulk weights $(\omega_x)_{x\in\mathbb{N}^2}$.

For $N \in \mathbb{N}$ and $r > 0$ define the event

$$A_{N,r} = \big\{(1-\rho)rN^{2/3} \leqslant \tau_1 \leqslant 4\rho^{-1}rN^{2/3}\big\}. \tag{4.13}$$

We develop a lower bound for the probability of $A_{N,r}$ under $\mathbb{P}^{\lambda}_{(m,n)}$, that is, for the increment-stationary LPP process with parameter λ, restricted to the rectangle $[0, (m, n)]$. Note that this rectangle is *not* of the characteristic shape for λ, and we take advantage of this in the proof.

Lemma 4.4. *There exists a constant $C_1 = C_1(\rho)$ such that the bound below holds for $r \geqslant 1$ and $N \geqslant N_0$:*

$$\mathbb{P}^{\lambda}_{(m,n)}(A_{N,r}) \geqslant 1 - C_1 r^{-3} \tag{4.14}$$

Proof. We derive first an upper bound for $\mathbb{P}^{\lambda}_{(m,n)}\{\tau_1 > 4\rho^{-1}rN^{2/3}\}$. Define

$$\widetilde{m} = \big\lfloor N\rho^2\lambda^{-2}(1-\lambda)^2\big\rfloor$$

so that $(\widetilde{m}, n)$ points in the characteristic direction for λ, up to an $O(1)$ error κ_N coming from integer parts. Furthermore,

$$\begin{aligned} m - \widetilde{m} &\leqslant N\big((1-\rho)^2 - \rho^2\lambda^{-2}(1-\lambda)^2\big) + 1 = N\frac{\lambda+\rho-2\lambda\rho}{\lambda^2}(\lambda-\rho) + 1 \\ &\leqslant 2\rho^{-1}rN^{2/3} \end{aligned}$$

for $N \geqslant N_0(\rho, r)$, for a suitably chosen $N_0(\rho, r)$. By Lemma 5.2 in the appendix, and then by the upper bound (4.5),

$$\begin{aligned} \mathbb{P}^{\lambda}_{(m,n)}\{\tau_1 > 4\rho^{-1}rN^{2/3}\} &= \mathbb{P}^{\lambda}_{(\widetilde{m},n)}\{\tau_1 > 4\rho^{-1}rN^{2/3} - (m-\widetilde{m})\} \\ &\leqslant \mathbb{P}^{\lambda}_{(\widetilde{m},n)}\{\tau_1 > 2\rho^{-1}rN^{2/3}\} \leqslant \frac{c_4}{r^3} \end{aligned}$$

where $c_4 = c_4(\rho)$ contains c_3 from (4.5).

Next we derive an upper bound for $\mathbb{P}^{\lambda}_{(m,n)}\{\tau_1 < (1-\rho)rN^{2/3}\}$. Let

$$(\bar{m}, \bar{n}) = \big(\lfloor N(1-\lambda)^2\rfloor, \lfloor N\lambda^2\rfloor\big) \tag{4.15}$$

point in the characteristic direction λ. Bound these differences:

$$\begin{aligned} m - \bar{m} &\geqslant N((1-\rho)^2 - (1-\lambda)^2) - 1 = N(\lambda-\rho)(2-\rho-\lambda) - 1 = (2-\rho-\lambda)rN^{2/3} - 1 \\ &\geqslant (1-\rho)rN^{2/3} \end{aligned}$$

and

$$\bar{n} - n \geqslant N(\lambda^2 - \rho^2) - 1 = N(\lambda-\rho)(\rho+\lambda) - 1 \geqslant \rho rN^{2/3},$$

again for large enough N relative to (ρ, r). By Lemma 5.3 in the appendix, and then by the upper bound (4.5),

$$\begin{aligned} \mathbb{P}^{\lambda}_{(m,n)}\big\{\tau_1 < (1-\rho)rN^{2/3}\big\} &\leqslant \mathbb{P}^{\lambda}_{(m,n)}\big\{\tau_1 < m - \bar{m}\big\} = \mathbb{P}^{\lambda}_{(\bar{m},\bar{n})}\big\{\tau_2 > \bar{n} - n\big\} \\ &\leqslant \mathbb{P}^{\lambda}_{(\bar{m},\bar{n})}\big\{\tau_2 > \rho rN^{2/3}\big\} \leqslant \frac{c_5}{r^3} \end{aligned}$$

where $c_5 = c_5(\rho)$ contains c_3 from (4.5).

Combine the bounds:

$$\begin{aligned} \mathbb{P}^{\lambda}_{(m,n)}\big\{(1-\rho)rN^{2/3} \leqslant \tau_1 \leqslant 4\rho^{-1}rN^{2/3}\big\} &= 1 - \mathbb{P}^{\lambda}_{(m,n)}\{\tau_1 > 4\rho^{-1}rN^{2/3}\} - \mathbb{P}^{\lambda}_{(m,n)}\{\tau_1 < (1-\rho)rN^{2/3}\} \\ &\geqslant 1 - C_1 r^{-3}. \end{aligned}$$

□

Computing as in (4.11),

$$\begin{aligned}\mathbb{E}[G^{\lambda}_{(m,n)} - G^{\rho}_{(m,n)}] &= m\Big(\frac{1}{1-\lambda} - \frac{1}{1-\rho}\Big) + n\Big(\frac{1}{\lambda} - \frac{1}{\rho}\Big) \\ &= \frac{N}{\lambda(1-\lambda)}(\lambda-\rho)^2 + \bigg\{ \frac{\lfloor N(1-\rho)^2\rfloor - N(1-\rho)^2}{(1-\lambda)(1-\rho)} - \frac{\lfloor N\rho^2\rfloor - N\rho^2}{\lambda\rho} \bigg\}(\lambda-\rho) \\ &= \frac{r^2N^{1/3}}{\lambda(1-\lambda)} + O(1)\cdot\frac{r}{N^{1/3}} \geqslant c_6 r^2 N^{1/3}\end{aligned} \tag{4.16}$$

where $c_6 = c_6(\rho) > 0$ is a constant chosen small enough to satisfy the inequality above for all $N \geqslant N_0$ and $r \geqslant 1$.

Let Λ_N denote the set of paths $x_\bullet \in \Pi_{0,(m,n)}$ that satisfy $x_1 = e_1$ and $x_k \cdot e_2 > 0$ for $k > \lfloor 4\rho^{-1} r N^{2/3}\rfloor$. In other words, the path stays on the x-axis for a while after leaving the origin, but does not stay on the x-axis beyond the point $\lfloor 4\rho^{-1} r N^{2/3}\rfloor e_1$. For any given weights $\{\omega_x\}$ on the rectangle $\{0,\dots,m\}\times\{0,\dots,n\}$, let

$$G_{(m,n)}(\Lambda_N) = \max_{x_\bullet\in\Lambda_N} \sum_{k=1}^{m+n} \omega_{x_k} \tag{4.17}$$

denote the LPP value for paths from 0 to (m,n) whose maximum is restricted to the paths in Λ_N. Observe that $G_{(m,n)}(\Lambda_N) = G_{(m,n)}$ if event $A_{N,r}$ of (4.13) occurs for weights $\{\omega_x\}$. (This would be true even if the lower bound in $A_{N,r}$ would be relaxed to $\tau_1 \geqslant 1$ instead of $\tau_1 \geqslant (1-\rho)rN^{2/3}$.)

We derive our second probability bound. Define the event

$$B_{N,r} = \big\{\omega : G_{(m,n)}(\Lambda_N) \geqslant \mathbb{E}[G^{\rho}_{(m,n)}] + \tfrac{1}{2}c_6 r^2 N^{1/3}\big\}. \tag{4.18}$$

Lemma 4.5. *There exists a constant $C_2 = C_2(\rho)$ such that the bound below holds for $r \geqslant 1$ and $N \geqslant N_0$:*

$$\mathbb{P}^{\lambda}_{(m,n)}(B_{N,r}) \geqslant 1 - C_2 r^{-3}. \tag{4.19}$$

Proof. Since by (4.16)

$$\mathbb{E}[G^{\rho}_{(m,n)}] + \tfrac{1}{2}c_6 r^2 N^{1/3} \leqslant \mathbb{E}[G^{\lambda}_{(m,n)}] - \tfrac{1}{2}c_6 r^2 N^{1/3},$$

we can bound the complementary probability as follows. Constant C changes from line to line. Below we use Lemma 4.1.

$$\begin{aligned}\mathbb{P}^{\lambda}_{(m,n)}(B^c_{N,r}) &= \mathbb{P}^{\lambda}_{(m,n)}\big\{G^{\lambda}_{(m,n)}(\Lambda_N) < \mathbb{E}[G^{\rho}_{(m,n)}] + \tfrac{1}{2}c_6 r^2 N^{1/3}\big\} \\ &\leqslant \mathbb{P}^{\lambda}_{(m,n)}\big\{G^{\lambda}_{(m,n)}(\Lambda_N) < \mathbb{E}[G^{\lambda}_{(m,n)}] - \tfrac{1}{2}c_6 r^2 N^{1/3}\big\} \\ &\leqslant \mathbb{P}^{\lambda}_{(m,n)}\big\{G^{\lambda}_{(m,n)} < \mathbb{E}[G^{\lambda}_{(m,n)}] - \tfrac{1}{2}c_6 r^2 N^{1/3}\big\} + \mathbb{P}^{\lambda}_{(m,n)}(A^c_{N,r}) \\ &\leqslant \frac{C}{r^4N^{2/3}}\mathbb{V}\mathrm{ar}[G^{\lambda}_{(m,n)}] + \frac{C_1}{r^3} \\ &\leqslant \frac{C}{r^4N^{2/3}}\big(\mathbb{V}\mathrm{ar}[G^{\rho}_{(m,n)}] + m(\lambda-\rho)\big) + \frac{C_1}{r^3} \\ &\leqslant \frac{C}{r^4} + \frac{C_1}{r^3} \leqslant \frac{C_2}{r^3}.\end{aligned}$$

□

We construct a coupling of three environments. Let ω^ρ and ω^λ denote environments as described in (3.1) with parameters ρ and λ. We assume that these environments are coupled so that in the bulk, for $x\in\mathbb{N}^2$, $\omega^\rho_x = \omega^\lambda_x = \omega_x$, while the boundary variables $\{\omega^\rho_{ie_1}, \omega^\rho_{je_2}, \omega^\lambda_{ie_1}, \omega^\lambda_{je_2} : i,j\in\mathbb{N}\}$ are mutually independent.

Construct a mixed environment $\widehat{\omega}$ as follows:

$$\begin{aligned}\widehat{\omega}_{ie_1} &= \omega^\lambda_{ie_1} \quad \text{for} \quad 1\leqslant i \leqslant \lfloor 4\rho^{-1}rN^{2/3}\rfloor \\ \text{and} \qquad \widehat{\omega}_x &= \omega^\rho_x \quad \text{for} \quad x \notin \big\{ie_1 : 1\leqslant i\leqslant \lfloor 4\rho^{-1}rN^{2/3}\rfloor\big\}.\end{aligned}$$

Thus in the bulk all weights agree and are i.i.d. Exp(1): for $x\in\mathbb{N}^2$, $\widehat{\omega}_x = \omega^\rho_x = \omega^\lambda_x = \omega_x$. On the boundary $\widehat{\omega}$ follows ω^λ on the segment that is relevant for the event $B_{N,r}$ and elsewhere $\widehat{\omega}$ follows ω^ρ. Note that $\omega^\lambda \in B_{N,r}$ iff $\widehat{\omega}\in B_{N,r}$.

Let the distributions of the three environments ω^ρ, ω^λ and $\widehat{\omega}$, restricted to the rectangle $\{0,\dots,m\}\times\{0,\dots,n\}$, be denoted by $\mathbb{P}^\rho_{(m,n)}$, $\mathbb{P}^\lambda_{(m,n)}$ and $\widehat{\mathbb{P}}_{(m,n)}$, respectively. These are all probability measures on the product space $\mathbb{R}_+^{\{0,\dots,m\}\times\{0,\dots,n\}}$. The Radon-Nikodym derivative

$$f_N(\omega) = \frac{d\widehat{\mathbb{P}}_{(m,n)}}{d\mathbb{P}^\rho_{(m,n)}}(\omega) = \prod_{i=1}^{\lfloor 4\rho^{-1}rN^{2/3}\rfloor} \frac{\lambda}{\rho}\, e^{-(\lambda-\rho)\omega_{ie_1}}$$

is a product of the Radon-Nikodym derivatives of the exponential single weight marginal distributions on that segment of the boundary where ω^ρ and $\widehat{\omega}$ differ. Computation of the mean square gives

$$\begin{aligned}
\mathbb{E}^\rho_{(m,n)}[f_N^2] &= \biggl(\frac{\lambda^2}{\rho^2}\int_0^\infty e^{-2(\lambda-\rho)s}\rho e^{-\rho s}\,ds\biggr)^{\lfloor 4\rho^{-1}rN^{2/3}\rfloor} = \biggl(\frac{\lambda^2}{\rho(2\lambda-\rho)}\biggr)^{\lfloor 4\rho^{-1}rN^{2/3}\rfloor} \\
&= \exp\biggl\{\lfloor 4\rho^{-1}rN^{2/3}\rfloor\biggl[2\log\Bigl(1+\frac{r}{\rho N^{1/3}}\Bigr) - \log\Bigl(1+\frac{2r}{\rho N^{1/3}}\Bigr)\biggr]\biggr\} \\
&\leqslant e^{4r^3\rho^{-3}}.
\end{aligned}$$

Now fix $r\geqslant 1$ large enough relative to C_2 from (4.19) so that $C_2 r^{-3}<1$.

$$\begin{aligned}
1 - C_2r^{-3} &\leqslant \mathbb{P}^\lambda_{(m,n)}(B_{N,r}) = \widehat{\mathbb{P}}_{(m,n)}(B_{N,r}) = \mathbb{E}^\rho_{(m,n)}[\mathbb{1}_{B_{N,r}}f_N] \leqslant \bigl\{\mathbb{P}^\rho_{(m,n)}(B_{N,r})\bigr\}^{1/2}\bigl\{\mathbb{E}^\rho_{(m,n)}[f_N^2]\bigr\}^{1/2} \\
&\leqslant \bigl\{\mathbb{P}^\rho_{(m,n)}(B_{N,r})\bigr\}^{1/2} e^{2r^3\rho^{-3}}.
\end{aligned}$$

Since $G_{(m,n)}\geqslant G_{(m,n)}(\Lambda_N)$, from this comes the lower bound

$$\begin{aligned}
&\mathbb{P}^\rho_{(m,n)}\bigl\{\omega : G_{(m,n)} \geqslant \mathbb{E}[G^\rho_{(m,n)}] + \tfrac12 c_6 r^2 N^{1/3}\bigr\} \\
&\geqslant \mathbb{P}^\rho_{(m,n)}\bigl\{\omega : G_{(m,n)}(\Lambda_N) \geqslant \mathbb{E}[G^\rho_{(m,n)}] + \tfrac12 c_6 r^2 N^{1/3}\bigr\} \\
&\geqslant e^{-4r^3\rho^{-3}}\bigl(1-C_2r^{-3}\bigr)^2 \equiv \delta_1(\rho,r).
\end{aligned} \tag{4.20}$$

This gives the lower variance bound in (3.11) for the case when κ_N is bounded:

$$\mathbb{V}\mathrm{ar}[G^\rho_{(m,n)}] = \mathbb{E}\bigl[\bigl(G_{(m,n)} - \mathbb{E}[G^\rho_{(m,n)}]\bigr)^2\bigr] \geqslant \tfrac14 c_6^2 r^4 N^{2/3}\delta_1(\rho,r) = \delta_2(\rho,r)N^{2/3}$$

for a constant $\delta_2(\rho,r)>0$.

5. Technical appendix

5.1. **Coupling different rectangles.** We first prove a lemma for deterministic weights. Fix a point $a\in\mathbb{Z}^2$. Suppose boundary weights $\{\omega_{a+ke_r} : k\in\mathbb{N}, r\in\{1,2\}\}$ on the south and west boundaries of $a+\mathbb{Z}^2_+$ and bulk weights $\{\omega_x\}_{x\in a+\mathbb{N}^2}$ are given. Put an irrelevant weight $\omega_a=0$ in the corner a. Let $G_{a,x}$ denote the LPP value for points $x\in a+\mathbb{Z}^2_+$ and let $\pi^{a,x}_{\scriptscriptstyle\blacksquare}$ be a maximizing path from a to x. (If it is not unique, make an arbitrary choice.)

Let $b\geqslant a$ on $\mathbb{Z}^2$. On the lattice $b+\mathbb{Z}^2_+$, put a corner weight $\eta_b=0$ and define boundary weights

$$\eta_{b+ke_r} = G_{a,\,b+ke_r} - G_{a,\,b+(k-1)e_r} \qquad \text{for } k\in\mathbb{N} \text{ and } r\in\{1,2\}. \tag{5.1}$$

In the bulk use $\eta_x=\omega_x$ for $x\in b+\mathbb{N}^2$. Denote the LPP process in $b+\mathbb{Z}^2_+$ that uses weights $\{\eta_x\}_{x\in b+\mathbb{Z}^2_+}$ by

$$\widetilde{G}_{b,x} = \max_{x_{\scriptscriptstyle\blacksquare}\in\Pi_{b,x}} \sum_{i=0}^{|x-b|_1} \eta_{x_i}, \qquad x\in b+\mathbb{Z}^2_+. \tag{5.2}$$

Lemma 5.1. *Let $a\leqslant b\leqslant v$ in $\mathbb{Z}^2$. Then $G_{a,v}=G_{a,b}+\widetilde{G}_{b,v}$. The restriction of any maximizing path for $G_{a,v}$ to $b+\mathbb{Z}^2_+$ is part of a maximizing path for $\widetilde{G}_{b,v}$. The edges in the interior of $b+\mathbb{Z}^2_+$ of any maximizing path for $\widetilde{G}_{b,v}$ extend to a maximizing path for $G_{a,b}$.*

Proof. If $v = b + ke_r$ (that is, v is on the boundary of $b + \mathbb{Z}_+^2$) the situation is straightforward. Suppose $v > b$ coordinatewise. Suppose a maximal path from a to v enters $b + \mathbb{N}^2$ by taking the step from $x = b + ke_r$ to $y = b + ke_r + e_{3-r}$. Suppose a maximal path for $\widetilde{G}_{b,v}$ enters $b + \mathbb{N}^2$ by taking the step from $\widetilde{x} = b + \ell e_s$ to $\widetilde{y} = b + \ell e_s + e_{3-s}$. Then

$$\begin{aligned} G_{a,v} &= G_{a,x} + G_{y,v} = G_{a,b} + \sum_{i=1}^{k} \eta_{b+ie_r} + G_{y,v} \\ &\leqslant G_{a,b} + \widetilde{G}_{b,v} = G_{a,b} + \sum_{i=1}^{\ell} \eta_{b+ie_s} + G_{\widetilde{y},v} \\ &= G_{a,\widetilde{x}} + G_{\widetilde{y},v} \leqslant G_{a,v}. \end{aligned}$$

Thus the inequalities above are in fact equalities. □

Write $\mathbb{P}_v$ for the probability measure of the LPP process in the rectangle $[0, v]$ with boundary and bulk weights (3.1).

Lemma 5.2. *Let $1 \leqslant k < k + \ell \leqslant m$. Then $\mathbb{P}_{(m,n)}(\tau_1 \geqslant k + \ell) = \mathbb{P}_{(m-k,n)}(\tau_1 \geqslant \ell)$.*

Proof. Take $a = 0$, $b = (k, 0)$ and $v = (m, n)$ in Lemma 5.1. Then, under $\mathbb{P}_{(m,n)}$, the LPP process $\widetilde{G}_{b,x}$ in $[b, v]$ has the same distribution, modulo the translation of the origin to b, as an LPP process under $\mathbb{P}_{(m-k,n)}$. By Lemma 5.1 the maximizing paths from a and b to v agree in their portions inside $[k + 1, m] \times [0, n]$. □

Lemma 5.3. *Let $1 \leqslant \bar{m} < m$ and $1 \leqslant n < \bar{n}$. Then $\mathbb{P}_{(m,n)}(\tau_1 < m - \bar{m}) = P_{(\bar{m},\bar{n})}(\tau_2 > \bar{n} - n)$.*

Proof. We couple these LPP processes as follows. Let

$$a = (\bar{m} - m, 0), \quad a' = (0, n - \bar{n}) \quad \text{and} \quad v = (\bar{m}, n).$$

The origin 0 takes the role of b in Lemma 5.1.

Let i.i.d. Exp(1) weights $\{\omega_x\}_{x\in\mathbb{Z}^2}$ be given. Then place independent boundary edge weights with distributions dictated by (3.1) on the south and west boundaries of the lattice region $(a + \mathbb{Z}_+^2) \cup (a' + \mathbb{Z}_+^2)$:

(a) On horizontal boundary edges put Exp$(1 - \rho)$ weights $\sigma_{(i-1)e_1, ie_1}$ for $\bar{m} - m + 1 \leqslant i \leqslant 0$ and $\sigma_{a'+(i-1)e_1, a'+ie_1}$ for $i \in \mathbb{N}$.

(b) On vertical boundary edges put Exp(ρ) weights $\sigma_{(j-1)e_2, je_2}$ for $n - \bar{n} + 1 \leqslant j \leqslant 0$ and $\sigma_{a+(j-1)e_2, a+je_2}$ for $j \in \mathbb{N}$.

Next consider two LPP processes that emanate from a and a' and use the boundary weights described above in (a) and (b): $G_{a,y}$ for points y on the y-axis, and $G_{a',x}$ for points x on the x-axis. (The restriction put on y implies that $G_{a,y}$ does not need boundary weights on the x-axis beyond the interval $[a, 0]$, and similarly $G_{a',x}$ does not need boundary weights on the y-axis beyond the interval $[a', 0]$.) Let these processes define boundary weights on $\mathbb{Z}_+^2$: $\eta_{(i-1)e_1, ie_1} = G_{a', ie_1} - G_{a',(i-1)e_1}$ and $\eta_{(j-1)e_2, je_2} = G_{a, je_2} - G_{a,(j-1)e_2}$ for $i, j \in \mathbb{N}$.

Now consider three LPP processes with lower left corners a, 0 and a':

(i) $\widetilde{G}_{a,x}$ uses boundary weights $\sigma_{(i-1)e_1, ie_1}$ for $\bar{m} - m + 1 \leqslant i \leqslant 0$ and $\eta_{(i-1)e_1, ie_1}$ for $i \in \mathbb{N}$ on the horizontal axis emanating from a and $\sigma_{a+(j-1)e_2, a+je_2}$ for $j \in \mathbb{N}$ on the vertical axis emanating from a.

(ii) $\widetilde{G}_{0,x}$ uses boundary weights $\eta_{(i-1)e_1, ie_1}$ and $\eta_{(j-1)e_2, je_2}$ on the standard axes emanating from 0.

(iii) $\widetilde{G}_{a',x}$ uses boundary weights $\sigma_{a'+(i-1)e_1, a'+ie_1}$ for $i \in \mathbb{N}$ on the horizontal axis emanating from a' and weights $\sigma_{(j-1)e_2, je_2}$ for $n - \bar{n} + 1 \leqslant j \leqslant 0$ and $\eta_{(j-1)e_2, je_2}$ for $j \in \mathbb{N}$ on the vertical axis emanating from a'.

Let $\widetilde{P}$ denote the probability measure under which this coupling has been constructed, that is, the probability measure of the independent weights ω_x and $\sigma_{x, x+e_k}$.

Let A be the event that the (a.s. unique) maximal path for $\widetilde{G}_{a,v}$ does not go through the origin. Let B the event that the (a.s. unique) maximal path for $\widetilde{G}_{a',v}$ goes through the point e_2. Lemma 5.1 applies to the pair $\widetilde{G}_{a,v}$ and $\widetilde{G}_{0,v}$, and also to the pair $\widetilde{G}_{a',v}$ and $\widetilde{G}_{0,v}$. Thus the maximizing paths for $\widetilde{G}_{a,v}$ and $\widetilde{G}_{a',v}$ agree from that point onwards at which they exit the y-axis. Both A and B are equivalent to the statement that this point is strictly above the origin on the y-axis. Hence $A = B$.

On the other hand, LPP processes $\{\widetilde{G}_{a,\,a+x}\}_{x\in\mathbb{Z}_+^2}$ and $\{\widetilde{G}_{a',\,a'+x}\}_{x\in\mathbb{Z}_+^2}$ both have the same distribution as the LPP process $\{G^\rho_x\}_{x\in\mathbb{Z}_+^2}$ with stationary increments. Event A is equivalent to the condition that the maximizing path for $\widetilde{G}_{a,v}$ takes at most $m - \bar{m} - 1$ consecutive e_1-steps from a, which is the same as $\tau_1 < m - \bar{m}$ for $G^\rho_{(m,n)}$. Similarly, event B says that the maximizing path for $\widetilde{G}_{a',v}$ takes at least $\bar{n} - n + 1$ consecutive e_2-steps from a', which for $G^\rho_{(\bar{m},\bar{n})}$ is the same as $\tau_2 > \bar{n} - n$. Thus

$$\mathbb{P}_{(m,n)}(\tau_1 < m - \bar{m}) = \widetilde{P}(A) = \widetilde{P}(B) = P_{(\bar{m},\bar{n})}(\tau_2 > \bar{n} - n). \qquad \square$$

References

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