ALGEBRO-GEOMETRIC TECHNIQUES AND GEOMETRIC INSIGHTS FOR FINITE FRAMES

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ABSTRACT. Through a series of examples, we explore varieties of finite unit norm tight frames (FUNTFs) using techniques from Geometry and Algebraic Geometry. First, we identify the FUNTF varieties as almost-everywhere transversal intersections of tori and Stiefel manifolds. This allows us to characterize the singular points on these varieties, compute the tangent spaces at regular points, and determine when the FUNTF variety is a manifold. Next, using elimination theory, we explicitly solve the system of equations defining the FUNTF varieties. We show an alternative parameterization using eigensteps, and we use eigensteps to examine the path connectivity of FUNTF varieties and their non-singular points. We conclude by using these facts to show that the FUNTF varieties are irreducible.

CONTENTS

1. BACKGROUND

Some recent breakthroughs in Finite Frame Theory have invoked Algebro-Geometric and Geometric techniques. The line of reasoning traversed in these notes first began with the work of Dykema and Strawn [3]. In this paper, spaces of FUNTFs were first examined as algebraic varieties and their geometric structure was examined closely. In that same paper, path-connectivity was demonstrated for the spaces of FUNTFs in two dimensions with at least four vectors, and path-connectivity was conjectured to hold for spaces of FUNTFs in arbitrary dimension as long as the

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number of frame vectors exceeded the dimension by at least two. Some partial results were obtained on this homotopy problem by Giol et al. [7], and the problem was conjecture was completely verified in [2].

The pivotal development that enabled the verification of this conjecture has been explicit parameterizations of spaces of FUNTFs. In [9], explicit parameterizations of FUNTF spaces were constructed for the first time. In particular, that work used techniques from elimination theory to solve the system of polynomials representing the frame constraints. While solving this system of equations could be carried out, the solution involved the quartic equation and writing out the equations required dozens of pages.

In [1], a much more powerful parameterization of finite frames was uncovered using the theory of eigensteps. If one considers a rank-one update of a symmetric positive-definite matrix

$$A + xx^*,$$

there is no known general closed-form solution for the updated eigenvalues and eigenvectors. On the other hand, if one specifies the eigenvalues for the rank-one update, then there are convenient equations for x and the eigenvectors of the updated system. Thus, eigensteps circumvent all the potentially intractable calculations, and parameterizations using eigensteps allow for the construction of paths connecting arbitrary frames to frames with additional internal structure. This latter property is what fuels the solution to the homotopy problem.

FUNTFs are generally used in applications because they satisfy certain optimality properties (see [6, 8, 11]). Since there are potentially many degrees of freedom on a FUNTF variety, it is desirable to perform optimization on these spaces for various applications. A general procedure for such optimization utilizes the geometry of FUNTF varieties to perform geometric gradient descent directly on spaces of FUNTFs [10]. The fact that these spaces are connected means that such a procedure is not doomed to fail because it is initialized on a component that does not contain the optimizer. While [10] develops a convergence theory for this procedure, it is only valid in the case when the spaces of FUNTFs are manifolds. In order to develop a proper theory for optimization over these spaces, it will be necessary to understand the singular points on these varieties in a deeper manner.

2. Notation

In these notes, we consider only real-valued frames, and we identify finite frames of N members in $\mathbb{R}^d$ with their realization as d by N matrices (the synthesis operator). We denote the set of all d by N matrices with real entries by $M_{d,N}$, we let $\mathcal{F}_{d,N} \subset M_{d,N}$ denote the space of N-member FUNTFs in $\mathbb{R}^d$.

We let I_d denote the d by d identity matrix, $\mathbf{0}_d \in \mathbb{R}^d$ denotes the zero vector, and $\mathbf{1}_d \in \mathbb{R}^d$ denotes the vector with all entries equal to 1. For a d by d matrix A, we let $\operatorname{tr}(A)$ denote the trace of the matrix A and $\operatorname{diag}(A) \in \mathbb{R}^d$ denotes the vector consisting of all diagonal entries of A.

For $F = [f_1\ f_2\ \cdots\ f_N] \in \mathcal{F}_{d,N}$ we have that FF^* is the frame operator of F and F^*F is the Grammian, where F^* is the transpose of F (the analysis operator). Also, note that

$$FF^* = \sum_{i=1}^{N} f_i f_i^*.$$

Throughout these notes, we shall use the frames

$$\mathcal{E} = [\phi_1\ \phi_2\ \phi_3\ \phi_4\ \phi_5\ \phi_6] = \begin{pmatrix} 1 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & 1 & 0 & 0 & -\frac{\sqrt{6}}{3} \\ 0 & \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{6}}{6} \end{pmatrix} \text{ and } \Xi = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

as our central examples. It is not difficult to check that $\mathcal{E}, \Xi \in \mathcal{F}_{3,6}$. The Gram matrix of $\mathcal{E}$ is

$$\begin{pmatrix} 1 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 1 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} & 1 & 0 & 0 & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{2} & 0 & 0 & 1 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} & 0 & -\frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{6}}{6} & 1 \end{pmatrix}.$$

Setting $\rho_{d,N} = \frac{N}{d}$, we let

$$\mathrm{St}_{d,N} = \{F \in M_{d,N} : FF^* = \rho_{d,N} I_d\}$$

denote the scaled Stiefel manifold, which is in correspondence with the tight frames of N members in $\mathbb{R}^d$ with frame bound $\rho_{d,N}$. We let

$$\mathcal{S}_d = \{u \in \mathbb{R}^d : \|u\| = 1\}$$

and

$$\mathcal{S}_{d,N} = \{V \in M_{d,N} : \mathrm{tr}(VV^*) = N\}$$

respectively denote the unit spheres in $\mathbb{R}^d$ under the Euclidean norm, and the radius $\sqrt{N}$ sphere in $M_{d,N}$ under the Hilbert-Schmidt norm. Finally, define the torus

$$\mathbb{T}_{d,N} = \{V \in M_{d,N} : \mathrm{diag}(V^*V) = \mathbf{1}_N\}.$$

A real algebraic variety is any subset of $\mathbb{R}^d$ which is the zero set of a system of polynomials, and an irreducible algebraic variety is a variety that is not a non-trivial union of two varieties. For a reference on algebraic geometry, refer to [5], and for a reference on manifolds, diffeomorphisms, and transversal intersections of manifolds, see [4].

3. The intersection of tori and Stiefel manifolds in the Hilbert-Schmidt sphere

By definition, $F \in \mathbb{T}_{d,N}$ if and only if the diagonal of F^*F consists entirely of 1's. Thus,

$$\mathrm{tr}(FF^*) = \mathrm{tr}(F^*F) = N.$$

Similarly, for $F \in \mathrm{St}_{d,N}$, we have that

$$\mathrm{tr}(FF^*) = \mathrm{tr}(\rho_{d,N} I_d) = d\rho_{d,N} = N.$$

These facts are summarized in Proposition 3.1.

Proposition 3.1. *For $N \geq d$, we have that $\mathbb{T}_{d,N} \subset \mathcal{S}_{d,N}$, $St_{d,N} \subset \mathcal{S}_{d,N}$, and $\mathcal{F}_{d,N} = \mathbb{T}_{d,N} \cap St_{d,N}$. This also verifies that $\mathcal{F}_{d,N}$ is a quadratic variety.*

Because both $\mathbb{T}_{d,N}$ and $\mathrm{St}_{d,N}$ lie in the $\mathcal{S}_{d,N}$, we may examine their intersection and the transversality of that intersection relative to $\mathcal{S}_{d,N}$. It is now well-known that $\mathbb{T}_{d,N} \cap \mathrm{St}_{d,N}$ is not empty for all $N \geq d \geq 1$.

3.1. **Local transversality of the intersection of $\mathbb{T}_{d,N}$ and $\mathbf{St}_{d,N}$ in $\mathcal{S}_{d,N}$.** We now show that the intersection of $\mathbb{T}_{3,6}$ and $\mathrm{St}_{3,6}$ is transversal at $\mathcal{E}$, but not at Ξ. We see that the tangent space of $\mathbb{T}_{3,6}$ at $\mathcal{E}$ is equal to

$$T_{\mathcal{E}}\mathbb{T}_{3,6} = \{Y \in M_{3,6} : \mathrm{diag}(Y^*\mathcal{E}) = \mathbf{0}_N\}$$
$$T_{\mathcal{E}}\mathrm{St}_{3,6} = \{\mathcal{E}Z \in M_{3,6} : Z \in M_{6,6},\ Z = -Z^*\}.$$

To show local transversality of the intersection of $\mathbb{T}_{3,6}$ and $\mathrm{St}_{3,6}$ at $\mathcal{E}$, we must show that, for each $X \in T_{\mathcal{E}}\mathcal{S}_{3,6} = \{X \in M_{3,6} : \mathrm{tr}(X^*\mathcal{E}) = 0$, there is a skew-symmetric $Z \in M_{6,6}$ and a $Y \in T_{\mathcal{E}}\mathbb{T}_{3,6}$ such that $X = \mathcal{E}Z + Y$. In columns, this becomes the system of equations

$$\begin{array}{ccccccccccccccc}
x_1 & = & 0 & + & z_{21}\phi_2 & + & z_{31}\phi_3 & + & z_{41}\phi_4 & + & z_{51}\phi_5 & + & z_{61}\phi_6 & + & y_1 \\
x_2 & = & -z_{21}\phi_1 & + & 0 & + & z_{32}\phi_3 & + & z_{42}\phi_4 & + & z_{52}\phi_5 & + & z_{62}\phi_6 & + & y_2 \\
x_3 & = & -z_{31}\phi_1 & - & z_{32}\phi_2 & + & 0 & + & z_{43}\phi_4 & + & z_{53}\phi_5 & + & z_{63}\phi_6 & + & y_3 \\
x_4 & = & -z_{41}\phi_1 & - & z_{42}\phi_2 & - & z_{43}\phi_3 & + & 0 & + & z_{54}\phi_5 & + & z_{64}\phi_6 & + & y_4 \\
x_5 & = & -z_{51}\phi_1 & - & z_{52}\phi_2 & - & z_{53}\phi_3 & - & z_{54}\phi_4 & + & 0 & - & z_{65}\phi_6 & + & y_5 \\
x_6 & = & -z_{61}\phi_1 & - & z_{62}\phi_2 & - & z_{63}\phi_3 & - & z_{64}\phi_4 & - & z_{65}\phi_5 & + & 0 & + & y_6
\end{array}.$$

Noting that $\langle y_i, \phi_i\rangle = 0$, these equations imply the system of equations

$$\begin{array}{ccccccccccccc}
\langle x_1,\phi_1\rangle & = & 0 & + & \frac{\sqrt{3}}{3}z_{21} & + & 0 & + & \frac{\sqrt{2}}{2}z_{41} & + & 0 & + & \frac{\sqrt{6}}{6}z_{61} \\
\langle x_2,\phi_2\rangle & = & -\frac{\sqrt{3}}{3}z_{21} & + & 0 & + & \frac{\sqrt{3}}{3}z_{32} & + & 0 & + & \frac{\sqrt{3}}{3}z_{52} & + & 0 \\
\langle x_3,\phi_3\rangle & = & 0 & - & \frac{\sqrt{3}}{3}z_{32} & + & 0 & + & 0 & + & 0 & - & \frac{\sqrt{6}}{3}z_{63} \\
\langle x_4,\phi_4\rangle & = & -\frac{\sqrt{2}}{2}z_{41} & + & 0 & + & 0 & + & 0 & - & \frac{\sqrt{2}}{2}z_{54} & + & 0 \\
\langle x_5,\phi_5\rangle & = & 0 & - & \frac{\sqrt{3}}{3}z_{52} & + & 0 & + & \frac{\sqrt{2}}{2}z_{54} & + & 0 & + & \frac{\sqrt{6}}{6}z_{65} \\
\langle x_6,\phi_6\rangle & = & -\frac{\sqrt{6}}{6}z_{61} & + & 0 & + & \frac{\sqrt{6}}{3}z_{63} & + & 0 & - & \frac{\sqrt{6}}{6}z_{65} & + & 0
\end{array}. \tag{3.2}$$

We can express this system more compactly in the form

$$\begin{pmatrix}\langle x_1,\phi_1\rangle\\ \langle x_2,\phi_2\rangle\\ \langle x_3,\phi_3\rangle\\ \langle x_4,\phi_4\rangle\\ \langle x_5,\phi_5\rangle\\ \langle x_6,\phi_6\rangle\end{pmatrix} = (\Gamma_{\mathcal{E}} \odot Z)\mathbf{1}_6 = \begin{pmatrix} 0 & \frac{\sqrt{3}}{3}z_{21} & 0 & \frac{\sqrt{2}}{2}z_{41} & 0 & \frac{\sqrt{6}}{6}z_{61} \\ -\frac{\sqrt{3}}{3}z_{21} & 0 & \frac{\sqrt{3}}{3}z_{32} & 0 & \frac{\sqrt{3}}{3}z_{52} & 0 \\ 0 & -\frac{\sqrt{3}}{3}z_{32} & 0 & 0 & 0 & -\frac{\sqrt{6}}{3}z_{63} \\ -\frac{\sqrt{2}}{2}z_{41} & 0 & 0 & 0 & -\frac{\sqrt{2}}{2}z_{54} & 0 \\ 0 & -\frac{\sqrt{3}}{3}z_{32} & 0 & \frac{\sqrt{2}}{2}z_{54} & 0 & \frac{\sqrt{6}}{6}z_{65} \\ -\frac{\sqrt{6}}{6}z_{61} & 0 & \frac{\sqrt{6}}{3}z_{63} & 0 & -\frac{\sqrt{6}}{6}z_{65} & 0 \end{pmatrix}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix},$$

where $\odot$ denotes the Hadamard, or entry-wise product. Note that $\Gamma_{\mathcal{E}} \odot Z$ is skew-symmetric. Because of the skew-symmetry of the terms arising in these equations, there is a very intuitive analogy for the system. Consider the banks B_1, B_2, B_3, B_4, B_5, and B_6. Suppose that we want to ship money in between the banks so that by the end of the month the net changes in the total cash deposits (excluding all other transactions) at the banks are $\langle x_1,\phi_1\rangle$, $\langle x_2,\phi_2\rangle$, $\langle x_3,\phi_3\rangle$, $\langle x_4,\phi_4\rangle$, $\langle x_5,\phi_5\rangle$, and $\langle x_6,\phi_6\rangle$ respectively. Skew-symmetry in the above equations enforces the condition that shipping money from bank B_i to bank B_j decreases the deposits at B_i by the specified amount, and increases the deposits at B_j by the specified amount. Every 0 in the above equations represents a shipping route that is prohibited (it takes too long to ship between those routes, or maybe it is not safe to ship money along those routes). Finally, the net transactions over the entire network must be zero.

Now, if we can find z_{ij} values that solve this system, then we immediately obtain $X - \mathcal{E}Z \in T_{\mathcal{E}}\mathbb{T}_{3,6}$. Noting that $X \in T_{\mathcal{E}}\mathcal{S}_{3,6}$ if and only if $\sum_{i=1}^{6}\langle x_i,\phi_i\rangle = 0$, we naively begin constructing Z column by column to satisfy all the equations. For the first column, we set $z_{61} = \sqrt{6}\langle x_1,\phi_1\rangle$ and all the remaining entries of the column are set to zero. Similarly, we can set $z_{52} = \sqrt{3}\langle x_2, \phi_2\rangle$, $z_{63} = -\frac{\sqrt{6}}{2}\langle x_3, \phi_3\rangle$, $z_{54} = -\sqrt{2}\langle x_4, \phi_4\rangle$, and all the other entries of the first four columns equal to zero. This ensures that the first four equations are all satisfied, and by imposing skew-symmetry we have the following partial definition of Z:

$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\sqrt{6}\langle x_1, \phi_1\rangle \\ 0 & 0 & 0 & 0 & -\sqrt{3}\langle x_2, \phi_2\rangle & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{2}\langle x_3, \phi_3\rangle \\ 0 & 0 & 0 & 0 & \sqrt{2}\langle x_4, \phi_4\rangle & 0 \\ 0 & \sqrt{3}\langle x_2, \phi_2\rangle & 0 & -\sqrt{2}\langle x_4, \phi_4\rangle & 0 & -z_{65} \\ \sqrt{6}\langle x_1, \phi_1\rangle & 0 & -\frac{\sqrt{6}}{2}\langle x_3, \phi_3\rangle & 0 & z_{65} & 0 \end{pmatrix}$$

Substituting in these values for z_{ij}, the fifth and sixth equations become

$$\begin{aligned} \langle x_5, \phi_5\rangle &= -\frac{\sqrt{3}}{3}\left(\sqrt{3}\langle x_2, \phi_2\rangle\right) + \frac{\sqrt{2}}{2}\left(-\sqrt{2}\langle x_4, \phi_4\rangle\right) + \frac{\sqrt{6}}{6} z_{65} \\ \langle x_6, \phi_6\rangle &= -\frac{\sqrt{6}}{6}\left(\sqrt{6}\langle x_1, \phi_1\rangle\right) + \frac{\sqrt{6}}{3}\left(-\frac{\sqrt{6}}{2}\langle x_3, \phi_3\rangle\right) - \frac{\sqrt{6}}{6} z_{65}. \end{aligned}$$

Solving both equations for z_{65} yields

$$\begin{aligned} z_{65} &= \sqrt{6}\left(\langle x_2, \phi_2\rangle + \langle x_4, \phi_4\rangle + \langle x_5, \phi_5\rangle\right) \\ z_{65} &= -\sqrt{6}\left(\langle x_1, \phi_1\rangle + \langle x_3, \phi_3\rangle + \langle x_6, \phi_6\rangle\right). \end{aligned}$$

By virtue of the fact that $\sum_{i=1}^{6}\langle x_i, \phi_i\rangle = 0$, we have that

$$\langle x_2, \phi_2\rangle + \langle x_4, \phi_4\rangle + \langle x_5, \phi_5\rangle = -\left(\langle x_1, \phi_1\rangle + \langle x_3, \phi_3\rangle + \langle x_6, \phi_6\rangle\right)$$

and we see that these expressions for z_{65} are consistent. Consequently, we have a general formula for Z solving the desired equations, and we conclude that $\mathbb{T}_{3,6}$ and $\mathrm{St}_{3,6}$ intersect transversally at $\mathcal{E}$ in $\mathcal{S}_{3,6}$.

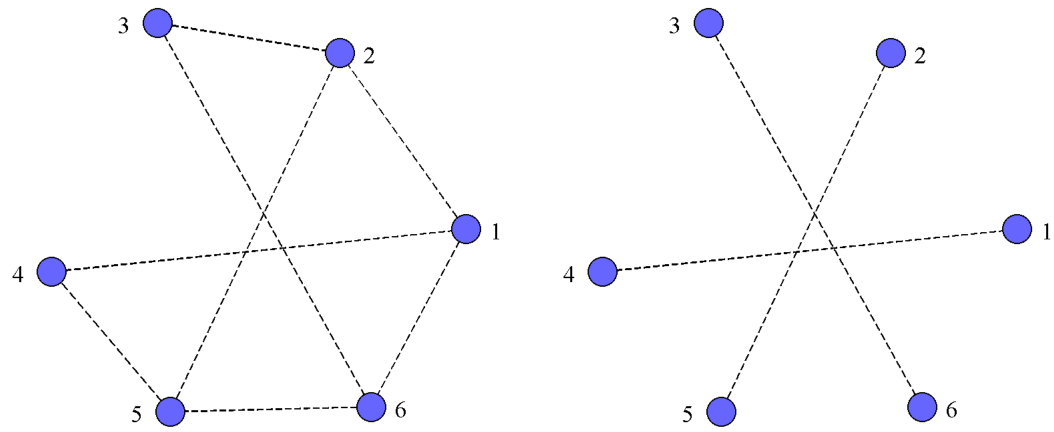
Now, let us consider the same scenario for Ξ. By considering the Gram matrix of Ξ, the equations are

$$\begin{aligned} \langle x_1, e_1\rangle &= z_{41} \\ \langle x_2, e_2\rangle &= z_{52} \\ \langle x_3, e_3\rangle &= z_{63} \\ \langle x_4, e_1\rangle &= -z_{41} \\ \langle x_5, e_2\rangle &= -z_{52} \\ \langle x_6, e_3\rangle &= -z_{63}. \end{aligned}$$

By simply choosing $x_1 = e_1$, $x_2 = -2e_2$, $x_3 = e_1$, $x_4 = 0$, $x_5 = 0$, and $x_6 = 0$, we clearly have that

$$\langle x_1, e_1\rangle + \langle x_2, e_2\rangle + \langle x_3, e_3\rangle + \langle x_4, e_1\rangle + \langle x_5, e_2\rangle + \langle x_6, e_3\rangle = 0$$

but the above system of equations is inconsistent because $1 = z_{41}$ and $1 = -z_{41}$. Thus, we see that the transversality of the intersection fails for Ξ. Returning to the bank analogy from before, it becomes intuitively clear why the system of equations does not have a general solution. The banks B_1 and B_4 are only connected to each other, so it is not possible to ship money to either of them from bank B_2. Since the zeros in the system of equations are governed precisely by the zeros of the Gram matrix, we are naturally led to the following definition.



(A) $\mathcal{G}_{\mathcal{E}}$ (B) $\mathcal{G}_{\Xi}$

FIGURE 1. Correlation networks of our examples.

Definition 3.3. For a frame $F = [f_1 f_2 \cdots f_N] \in M_{d,N}$, the **correlation network** is the symmetric graph $\mathcal{G}_F = (V, E)$ with vertices $V = \{1, 2, \ldots, N\}$ and edge set

$$E = \{(i,j) : \langle f_i, f_j \rangle \neq 0\}.$$

Figure 1 depicts the correlation networks of $\mathcal{E}$ and Ξ, and Figure 2 depicts the reduction of $\mathcal{G}_{\mathcal{E}}$ used to solve the linear system for our demonstration of local transversality. It is also clear from that this system cannot be solved for Ξ because $\mathcal{G}_{\Xi}$ is not connected. With the definition of the correlation network in hand, we may prove the following theorem that connects the correlation network and transversal intersection at F, and as a corollary we obtain the local dimension of the smooth structure.

Theorem 3.4. *Suppose $N \geq d \geq 2$. The manifolds $\mathbb{T}_{d,N}$ and $St_{d,N}$ intersect transversally in $\mathcal{S}_{d,N}$ at $F \in \mathcal{F}_{d,N}$ if and only if $\mathcal{G}_F$ is connected. Moreover, the local dimension of $\mathcal{F}_{d,N}$ around such an F is given by*

$$(d-1)N + \left(dN - \binom{d+1}{2}\right) - (dN - 1) = (d-1)N - \binom{d+1}{2} + 1.$$

Connectivity of $\mathcal{G}_F$ thus provides us with a combinatorial condition for local transversal intersection. On the other hand, connectivity of $\mathcal{G}_F$ depends on the structure of the orthogonal pairs inside of F. We may therefore obtain an equivalent geometric condition for local transversal intersection.

Proposition 3.5. *The correlation network $\mathcal{G}_F$ is connected if and only if F cannot be partitioned into two non-trivial subsets of matrices with orthogonal column spaces. If F does admit such a partition, we say that F is* **orthodecomposable**.

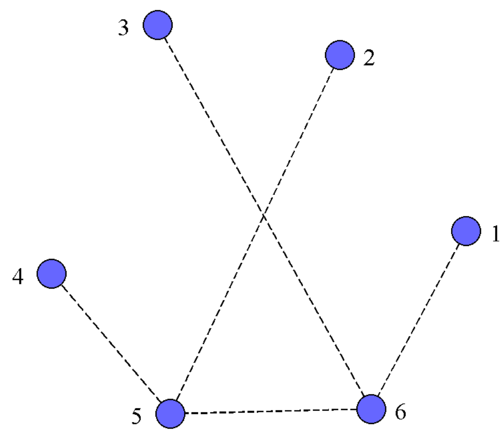


FIGURE 2. To solve the system (3.2), we ignore all the edges from $\mathcal{G}_{\mathcal{E}}$ that do not connect to the vertices labeled 5 and 6. We then fix the transfers to/from $1, 2, 3, 4$ from/to 5 and 6 that satisfy the first four equations, and then the required transfer between 5 and 6 is shown to be consistent by invoking the fact that the net change over the entire graph is zero.

Very little is known about the local geometry of $\mathcal{F}_{d,N}$ around orthodecomposable frames, and the existence of these points introduces an additional level of complexity when one would like to carry out optimization programs on $\mathcal{F}_{d,N}$ (such as in CITE).

Problem 3.6. *If $\mathcal{F}_{d,N}$ is not a manifold, describe the local geometry around the orthodecomposable frames in $\mathcal{F}_{d,N}$.*

3.2. **When is $\mathcal{F}_{d,N}$ a manifold?** Now that we have a characterization of the singular points on $\mathcal{F}_{d,N}$, we can determine exactly when $\mathcal{F}_{d,N}$ is a manifold. Suppose $F \in \mathcal{F}_{d,N}$ is orthodecomposable, so there is a partition of F into $F_1 \in M_{d,N_1}$ and $F_2 \in M_{d,N_2}$, where $N_1, N_2 > 0$. Moreover, $\mathrm{col}(F_1) \perp \mathrm{col}(F_2)$ and hence

$$FF^* x_1 = F_1 F_1^* x_1 = \frac{N}{d} x_1 \text{ and } FF^* x_2 = F_2 F_2^* x_2 = \frac{N}{d} x_2$$

for $x_1 \in \mathrm{col}(F_1)$ and $x_2 \in \mathrm{col}(F_2)$. Consequently, we see that F_1 and F_2 are FUNTFs for subspaces of dimension $d_1 > 0$ and $d_2 > 0$, respectively. Moreover, the above can occur if and only if the frame bounds satisfy

$$\frac{N_1}{d_1} = \frac{N_2}{d_2} = \frac{N}{d} = \frac{N_1 + N_2}{d_1 + d_2}.$$

Since $N_1 = N - N_2 < N$, these equations imply the existence of an integer c so that $cN_1 = N$ and $cd_1 = d$, and we conclude that N and d are not relatively prime.

On the other hand, if N and d are not relatively prime with $cN_1 = cd_1$ for some $c > 1$, we set $N_2 = (c-1)N_1$ and $d_2 = (c-1)d_1$, choose $F_1 \in \mathcal{F}_{d_1,N_1}$ and $F_2 \in \mathcal{F}_{d_2,N_2}$, and observe that

$$\begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \in \mathcal{F}_{d,N}$$

is orthodecomposable. Gathering together all of our observations, we obtain the following proposition.

Proposition 3.7. *The variety $\mathcal{F}_{d,N}$ is a manifold if and only if N and d are relatively prime.*

4. Explicit, locally-defined, analytic coordinate functions on $\mathcal{F}_{d,N}$

In this section we show how to construct explicit local coordinate systems around all non-orthodecomposable $F \in \mathcal{F}_{d,N}$.

4.1. **Local coordinates via elimination theory.** First, let us build up our intuition by exploring perturbations of $\mathcal{E}$. Since $\mathcal{E}$ is a non-singular point of $\mathcal{F}_{d,N}$, Theorem 3.4 gives us that the local dimension of the manifold structure is

$$(3-1)6 - \binom{4}{2} + 1 = 7.$$

Now, perturbations of $\mathcal{E}$ that remain in $\mathbb{T}_{3,6}$ consist of angular perturbations of each column in $\mathcal{E}$. Because of the dimension constraint, we see that we can perform simultaneous angular perturbations of at most three vectors in $\mathcal{E}$. Let

$$\Theta = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \end{pmatrix}$$

denote the parameters associated to the angular perturbation of the first three columns of $\mathcal{E}$, and set

$$\phi_i(\Theta) = \phi_i(\theta_{i1}, \theta_{i2}) \text{ and } \phi_i(0,0) = \phi_i$$

for $i = 1, 2, 3$. This only accounts for six of the seven parameters, so we let τ denote the additional parameter and assume that the first three columns are independent of τ. Our goal is to find functions $\phi_i(\Theta, \tau)$ for $i = 4, 5, 6$ such that

$$(\Theta, \tau) \longrightarrow \Phi(\Theta, \tau) = \begin{pmatrix} \phi_1(\Theta) & \phi_2(\Theta) & \phi_3(\Theta) & \phi_4(\Theta,\tau) & \phi_5(\Theta,\tau) & \phi_6(\Theta,\tau) \end{pmatrix} \in \mathcal{F}_{3,6}$$

is an embedding at $(\Theta_0, \tau_0) = (0, 0)$ and $\Phi(\Theta_0, \tau_0) = \mathcal{E}$. Now, a necessary condition is that

$$\Phi(\Theta,\tau)\Phi(\Theta,\tau)^* = \sum_{i=1}^{3} \phi_i(\Theta)\phi_i(\Theta)^* + \sum_{i=4}^{6} \phi_i(\Theta,\tau)\phi_i(\Theta,\tau)^* = 2I_3$$

which is equivalent to

$$\begin{aligned} \widetilde{\Phi}(\Theta,\tau)\widetilde{\Phi}(\Theta,\tau)^* &= \begin{pmatrix} \phi_4(\Theta,\tau) & \phi_5(\Theta,\tau) & \phi_6(\Theta,\tau) \end{pmatrix} \begin{pmatrix} \phi_4(\Theta,\tau) & \phi_5(\Theta,\tau) & \phi_6(\Theta,\tau) \end{pmatrix}^* \\ &= \sum_{i=4}^{6} \phi_i(\Theta,\tau)\phi_i(\Theta,\tau)^* \\ &= 2I_3 - \sum_{i=1}^{3} \phi_i(\Theta)\phi_i(\Theta)^* \\ &= \widetilde{S}(\Theta) \end{aligned}$$

Since $\widetilde{S}(0) = 2I_3 - \sum_{i=1}^{3} \phi_i \phi_i^*$ is positive definite and invertible, in a small enough neighborhood around Θ_0 we shall also have that $\widetilde{S}(\Theta)$ is positive definite and invertible. It then follows that any $\widetilde{\Phi}(\Theta, \tau)$ satisfying this condition is also invertible. Thus, for Θ in a small neighborhood around Θ_0, we may perform a few algebraic manipulations to obtain the equivalent condition

$$\widetilde{\Phi}(\Theta, \tau)^* \widetilde{S}(\Theta)^{-1} \widetilde{\Phi}(\Theta, \tau) = I_3$$

or

$$\langle \widetilde{S}(\Theta)^{-1} \phi_i(\Theta, \tau), \phi_j(\Theta, \tau) \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

for all $i, j = 4, 5, 6$. Consider the conditions that only involve $\phi_4(\Theta, \tau)$:

$$\begin{aligned} \phi_4(\Theta, \tau)^* \widetilde{S}(\Theta)^{-1} \phi_4(\Theta, \tau) &= 1 \\ \phi_4(\Theta, \tau)^* \phi_4(\Theta, \tau) &= 1. \end{aligned}$$

Thus, the possible choices for $\phi_3(\Theta, \tau)$ come from the intersection of an ellipsoid and a sphere, but where the ellipsoid is a function of Θ. We see that the parameter τ comes from the single degree of freedom obtained in this intersection, and this motivates the set of conditions

$$\begin{aligned} \phi_4(\Theta, \tau)^* \widetilde{S}(\Theta)^{-1} \phi_4(\Theta, \tau) &= 1 \\ \phi_4(\Theta, \tau)^* \phi_4(\Theta, \tau) &= 1 \\ \phi_4(\Theta, \tau)^* \eta &= \tau \end{aligned}$$

where η is a unit vector satisfying $\phi_4^* \widetilde{S}(0)^{-1} \eta = 0$ and $\phi_4^* \eta = 0$, or equivalently η is tangent to the intersection of the ellipsoid and the sphere at (Θ_0, τ_0). For each fixed τ, we can use the third equation to eliminate one of the variables in the first two equations. This reduces the first two equations to a system of the form

$$\begin{aligned} \alpha_2(x) y^2 + \alpha_1(x) y + \alpha_0(x) &= 0 \\ \beta_2(x) y^2 + \beta_1(x) y + \beta_0(x) &= 0 \end{aligned}$$

where the polynomials α_i, β_i have degree at most $2 - i$ for $i = 0, 1, 2$ and the coefficients of the polynomials are functions of (Θ, τ).

Proposition 4.1. *The above system admits a solution y if and only if there is an x such that the Bézout determinant vanishes:*

$$\mathrm{Bz}(x) = (\alpha_2(x)\beta_1(x) - \alpha_1(x)\beta_2(x))(\alpha_1(x)\beta_0(x) - \alpha_0(x)\beta_1(x)) - (\alpha_2(x)\beta_0(x) - \alpha_0(x)\beta_2(x))^2 = 0.$$

By considering the degrees of α_i and β_i, we see that $\mathrm{Bz}(x)$ is a polynomial of degree at most four. Consequently, we may use the quartic formula to analytically solve for $x = x(\Theta, \tau)$, and then we can back solve for y and the variable eliminated by the linear constraint. In this manner, we see that it is formally possible to construct an analytic solution to $\phi_4(\Theta, \tau)$. It then remains to solve the systems

$$\begin{aligned} \phi_i(\Theta, \tau)^* \widetilde{S}(\Theta)^{-1} \phi_i(\Theta, \tau) &= 1 \\ \phi_i(\Theta, \tau)^* \phi_i(\Theta, \tau) &= 1 \\ \phi_i(\Theta, \tau)^* \widetilde{S}(\Theta)^{-1} \phi_4(\Theta, \tau) &= 0 \end{aligned}$$

for $i = 5, 6$. Since the final equation is a linear constraint, an analytic solution is obtained using the same approach we used to construct $\phi_4(\Theta, \tau)$.

Thus, we can formally construct analytic coordinates on $\mathcal{F}_{3,6}$ in a neighborhood around $\mathcal{E}$. In [9], it was shown that these formal coordinates provide a local parameterization of $\mathcal{F}_{3,6}$ around $\mathcal{E}$. While the coordinate functions themselves can be computed explicitly, the equations use the quartic formula, which is a very involved expression and the explicit coordinate functions can fill many pages.

4.2. **Local coordinates from eigensteps.** In contrast to the above coordinate systems, liftings of eigensteps provides explicit parameterizations of $\mathcal{F}_{d,N}$ that are far more concise. On the other hand, it is not clear that there is such a lifting exists for every non-singular point in $\mathcal{F}_{d,N}$, which leads to an interesting open problem.

Let $\lambda' : M_{d,d} \to \mathbb{R}^d$ denote the map taking a matrix to its eigenvalues listed in non-increasing order (note that the list is not strictly decreasing because the map includes multiplicities). Now, we define the eigensteps map $\lambda : \mathcal{F}_{d,N} \to \Delta_{d,N}$ by setting

$$\lambda(F) = \begin{pmatrix} \lambda'(F_1F_1^*) & \lambda'(F_2F_2^*) & \cdots & \lambda'(FF^*) \end{pmatrix}$$

where $F_k = (f_1 f_2 \cdots f_k)$ are the first k columns of F. For example, a somewhat laborious calculation give us

$$\lambda(\mathcal{E}) = \begin{pmatrix} 1 & (3+\sqrt{3})/3 & (3+\sqrt{6})/3 & 2 & 2 & 2 \\ 0 & (3-\sqrt{3})/3 & 1 & (6+\sqrt{6})/6 & 2 & 2 \\ 0 & 0 & (3-\sqrt{6})/3 & (6-\sqrt{6})/6 & 1 & 2 \end{pmatrix}.$$

Letting $\mathcal{F}_{d,N}^B$ denote the set of all frames whose first d columns form a basis of $\mathbb{R}^d$, we note that the Q matrix in the QR decomposition of the first d columns of $F \in \mathcal{F}_{d,N}^B$ is unique under the assumption that the diagonal of R is strictly positive. Moreover, this identification is a smooth function of F and we obtain the continuous map $\mathcal{Q} : \mathcal{F}_{d,N}^B \to \mathcal{O}(d)$ by setting $\mathcal{Q}(F)$ equal to the Q in this QR decomposition.

Now, define $\Psi : \mathcal{F}_{d,N}^B \to \mathcal{O}(d) \times \Delta_{d,N}$ by setting $\Psi(F) = (\mathcal{Q}(F), \lambda(F))$. Note that if $F \in \mathcal{F}_{d,N}^B$ is such that $\lambda(F) \notin \partial\Delta_{d,N}$, then Ψ is locally analytic at F.

We shall now construct $\Phi : \mathcal{O}(3) \times \Delta_{3,6}$ that is the local inverse of Ψ at $\mathcal{E}$. First, we fix the matrices V_k, P_k, and Q_k according to the table in Figure 3.

Next, we define the functions $v_k(\lambda)$, $w_k(\lambda)$, $W_k(\lambda)$ according to the table in Figure 5. The columns of Φ are then defined by first setting $U_1(U,\lambda) = U$, $\phi_1(U,\lambda) = U_1(U,\lambda)e_1$, and then iterating

$$\begin{aligned} \phi_{k+1}(U,\lambda) &= U_k(U,\lambda)V_kP_k^*v_k(\lambda) \\ U_{k+1}(U,\lambda) &= U_k(U,\lambda)V_kP_k^*W_k(\lambda)Q_k \end{aligned}$$

for $k = 1,2,3,4,5$. Observing that

$$\mathcal{Q}(\mathcal{E}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix},$$

one may verify that $\Phi(\mathcal{Q}(\mathcal{E}), \lambda(\mathcal{E})) = \mathcal{E}$.

Noting that $\mathcal{E} \in \mathcal{F}_{3,6}^B$ and $\mathcal{E} \in \Delta_{3,6}^\circ$, continuity of Φ around $(U_0, \lambda_0) = (\mathcal{Q}(\mathcal{E}), \lambda(\mathcal{E}))$ implies that there is an open neighborhood $(U_0, \lambda_0) \in \mathcal{U} \subset \mathcal{O}(3) \times \Delta_{3,6}^\circ$ such that $\Phi(U,\mu) \in \mathcal{F}_{3,6}^B$ for all $(U,\mu) \in \mathcal{U}$. Consequently, Ψ is defined and analytic on $\Phi(\mathcal{U})$. Clearly, $\mathcal{Q}(\Phi(U,\mu)) = U$ and

k	V_k	P_k	Q_k
1	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
2	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
3	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
4	$-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$
5	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

FIGURE 3. Auxiliary matrices used in the eigensteps parameterization around $\mathcal{E}$.

$\lambda(\Phi(U, \mu)) = \mu$ by construction, and hence we have that $\Psi \circ \Phi$ is the identity function on $\mathcal{U}$. By analyticity of Φ and Ψ we conclude that the differentials satisfy

$$I_{T_{(U_0,\lambda_0)}\mathcal{O}(3)\times\Delta_{3,6}} = (D(\Psi \circ \Phi))(U_0, \lambda_0) = D\Psi(\Phi(U_0, \lambda_0))D\Phi(U_0, \lambda_0)$$

where $I_{T_{(U_0,\lambda_0)}}$ is the identity operator on the tangent space of $\mathcal{O}(3) \times \Delta_{3,6}$ at (U_0, λ_0). This implies that $D\Phi(U_0, \lambda_0)$ has maximal rank, and therefore we obtain the following proposition.

Proposition 4.2. *The map $\Phi : \mathcal{U} \to \Phi(\mathcal{U})$ is a diffeomorphism.*

Comparing this parameterization with the one obtained through the elimination theoretic arguments, this coordinate system is relatively simple to write down, but the paths are not quite as intuitive. In Figure 4, we simply vary the second column of the eigensteps around $\lambda(\mathcal{E})$. The resulting motion is a synchronized rotation of the two orthonormal bases, a motion that would be harder to describe in the intuitive parameterization. Additionally, the proof of Proposition 4.2 invoked the fact that $\lambda(\mathcal{E}) \in \Delta^\circ_{3,6}$ to obtain local analyticity of Ψ. For frames with eigensteps on the boundary of the eigensteps polytope, one might still be able to permute the frame vectors to obtain eigensteps in $\Delta^\circ_{3,6}$ to obtain a valid parameterization. However, there is no guarantee that such a permutation exists, which leads to the following open problem.

Problem 4.3. *Prove that for any non-orthodecomposable $F \in \mathcal{F}_{d,N}$ there is a permutation matrix $\Pi \in M_{N,N}$ such that $\lambda(F\Pi) \in \Delta^\circ_{d,N}$, or else find a counterexample.*

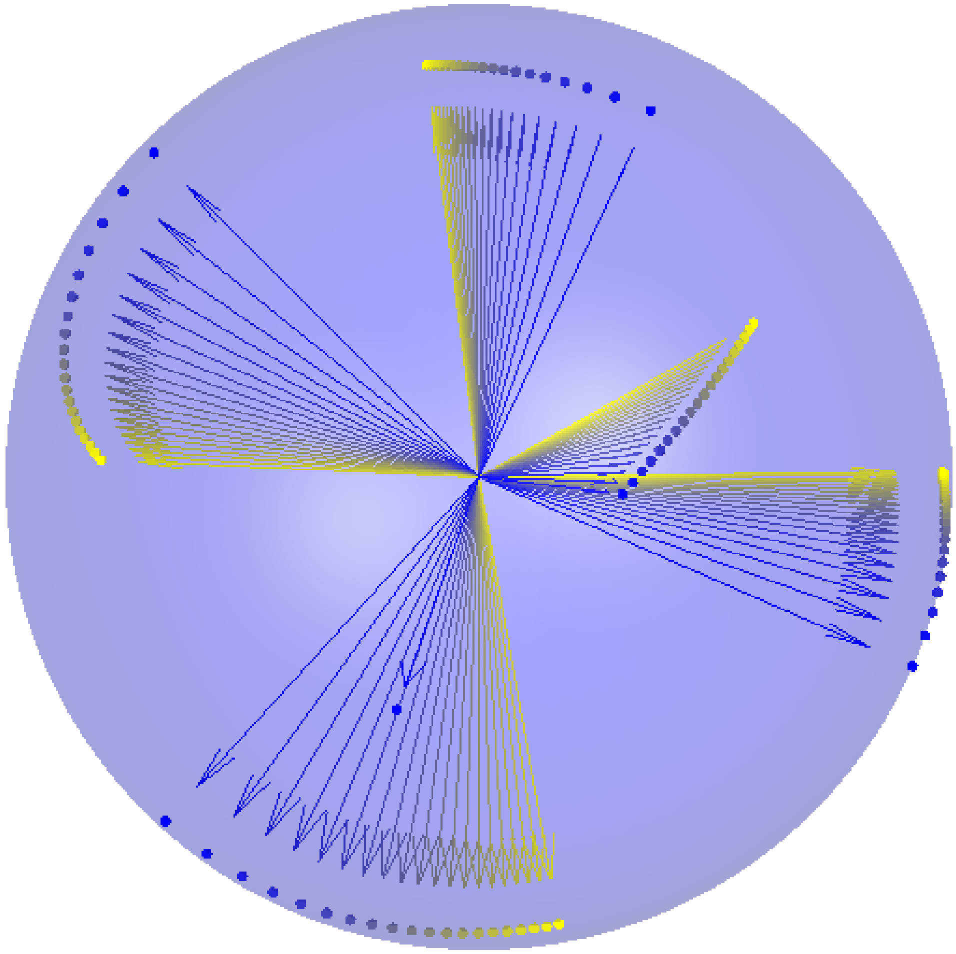


FIGURE 4. The trajectories carved out by the frame vectors while varying the second column of $\lambda(\mathcal{E})$. The resulting trajectory is a synchronized rotation of the two orthonormal bases.

k	$v_k(\lambda)$	$w_k(\lambda)$	$W_k(\lambda)$
1	$\begin{pmatrix}\sqrt{-\frac{(\lambda_{11}-\lambda_{12})(\lambda_{11}-\lambda_{22})(\lambda_{11}-\lambda_{32})}{(\lambda_{11}-\lambda_{21})(\lambda_{11}-\lambda_{31})}}\\ \sqrt{-\frac{(\lambda_{21}-\lambda_{12})(\lambda_{21}-\lambda_{22})}{(\lambda_{21}-\lambda_{11})}}\\ 0\end{pmatrix}$	$\begin{pmatrix}\sqrt{\frac{(\lambda_{12}-\lambda_{11})(\lambda_{12}-\lambda_{21})(\lambda_{12}-\lambda_{31})}{(\lambda_{12}-\lambda_{22})(\lambda_{12}-\lambda_{32})}}\\ \sqrt{\frac{(\lambda_{22}-\lambda_{11})(\lambda_{22}-\lambda_{21})}{(\lambda_{21}-\lambda_{11})}}\\ 0\end{pmatrix}$	$\begin{pmatrix}\frac{v_{11}(\lambda)w_{11}(\lambda)}{\lambda_{12}-\lambda_{11}} & \frac{v_{11}(\lambda)w_{21}(\lambda)}{\lambda_{22}-\lambda_{11}} & 0\\ \frac{v_{21}(\lambda)w_{11}(\lambda)}{\lambda_{12}-\lambda_{21}} & \frac{v_{21}(\lambda)w_{21}(\lambda)}{\lambda_{22}-\lambda_{21}} & 0\\ 0 & 0 & 1\end{pmatrix}$
2	$\begin{pmatrix}\sqrt{-\frac{(\lambda_{12}-\lambda_{13})(\lambda_{12}-\lambda_{23})(\lambda_{12}-\lambda_{33})}{(\lambda_{12}-\lambda_{22})(\lambda_{12}-\lambda_{32})}}\\ \sqrt{-\frac{(\lambda_{22}-\lambda_{13})(\lambda_{22}-\lambda_{23})(\lambda_{22}-\lambda_{33})}{(\lambda_{22}-\lambda_{12})(\lambda_{22}-\lambda_{32})}}\\ \sqrt{-\frac{(\lambda_{32}-\lambda_{13})(\lambda_{32}-\lambda_{23})(\lambda_{32}-\lambda_{33})}{(\lambda_{32}-\lambda_{12})(\lambda_{32}-\lambda_{22})}}\end{pmatrix}$	$\begin{pmatrix}\sqrt{\frac{(\lambda_{13}-\lambda_{12})(\lambda_{13}-\lambda_{22})(\lambda_{13}-\lambda_{32})}{(\lambda_{13}-\lambda_{23})(\lambda_{13}-\lambda_{33})}}\\ \sqrt{\frac{(\lambda_{23}-\lambda_{12})(\lambda_{23}-\lambda_{22})(\lambda_{23}-\lambda_{32})}{(\lambda_{23}-\lambda_{13})(\lambda_{23}-\lambda_{33})}}\\ \sqrt{\frac{(\lambda_{33}-\lambda_{12})(\lambda_{33}-\lambda_{22})(\lambda_{33}-\lambda_{32})}{(\lambda_{33}-\lambda_{13})(\lambda_{33}-\lambda_{23})}}\end{pmatrix}$	$\begin{pmatrix}\frac{v_{12}(\lambda)w_{12}(\lambda)}{\lambda_{13}-\lambda_{12}} & \frac{v_{12}(\lambda)w_{22}(\lambda)}{\lambda_{23}-\lambda_{12}} & \frac{v_{12}(\lambda)w_{32}(\lambda)}{\lambda_{33}-\lambda_{12}}\\ \frac{v_{22}(\lambda)w_{12}(\lambda)}{\lambda_{13}-\lambda_{22}} & \frac{v_{22}(\lambda)w_{22}(\lambda)}{\lambda_{23}-\lambda_{22}} & \frac{v_{22}(\lambda)w_{32}(\lambda)}{\lambda_{33}-\lambda_{22}}\\ \frac{v_{32}(\lambda)w_{12}(\lambda)}{\lambda_{13}-\lambda_{32}} & \frac{v_{32}(\lambda)w_{22}(\lambda)}{\lambda_{23}-\lambda_{32}} & \frac{v_{32}(\lambda)w_{32}(\lambda)}{\lambda_{33}-\lambda_{32}}\end{pmatrix}$
3	$\begin{pmatrix}\sqrt{-\frac{(\lambda_{13}-\lambda_{14})(\lambda_{13}-\lambda_{24})(\lambda_{13}-\lambda_{34})}{(\lambda_{13}-\lambda_{23})(\lambda_{13}-\lambda_{33})}}\\ \sqrt{-\frac{(\lambda_{23}-\lambda_{14})(\lambda_{23}-\lambda_{24})(\lambda_{23}-\lambda_{34})}{(\lambda_{23}-\lambda_{13})(\lambda_{23}-\lambda_{33})}}\\ \sqrt{-\frac{(\lambda_{33}-\lambda_{14})(\lambda_{33}-\lambda_{24})(\lambda_{33}-\lambda_{34})}{(\lambda_{33}-\lambda_{13})(\lambda_{33}-\lambda_{23})}}\end{pmatrix}$	$\begin{pmatrix}\sqrt{\frac{(\lambda_{14}-\lambda_{13})(\lambda_{14}-\lambda_{23})(\lambda_{14}-\lambda_{33})}{(\lambda_{14}-\lambda_{24})(\lambda_{14}-\lambda_{34})}}\\ \sqrt{\frac{(\lambda_{24}-\lambda_{13})(\lambda_{24}-\lambda_{23})(\lambda_{24}-\lambda_{33})}{(\lambda_{24}-\lambda_{14})(\lambda_{24}-\lambda_{34})}}\\ \sqrt{\frac{(\lambda_{34}-\lambda_{13})(\lambda_{34}-\lambda_{23})(\lambda_{34}-\lambda_{33})}{(\lambda_{34}-\lambda_{14})(\lambda_{34}-\lambda_{24})}}\end{pmatrix}$	$\begin{pmatrix}\frac{v_{13}(\lambda)w_{13}(\lambda)}{\lambda_{14}-\lambda_{13}} & \frac{v_{13}(\lambda)w_{23}(\lambda)}{\lambda_{24}-\lambda_{13}} & \frac{v_{13}(\lambda)w_{33}(\lambda)}{\lambda_{34}-\lambda_{13}}\\ \frac{v_{23}(\lambda)w_{13}(\lambda)}{\lambda_{14}-\lambda_{23}} & \frac{v_{23}(\lambda)w_{23}(\lambda)}{\lambda_{24}-\lambda_{23}} & \frac{v_{23}(\lambda)w_{33}(\lambda)}{\lambda_{34}-\lambda_{23}}\\ \frac{v_{33}(\lambda)w_{13}(\lambda)}{\lambda_{14}-\lambda_{33}} & \frac{v_{33}(\lambda)w_{23}(\lambda)}{\lambda_{24}-\lambda_{33}} & \frac{v_{33}(\lambda)w_{33}(\lambda)}{\lambda_{34}-\lambda_{33}}\end{pmatrix}$
4	$\begin{pmatrix}\sqrt{-\frac{(\lambda_{24}-\lambda_{15})(\lambda_{24}-\lambda_{25})(\lambda_{24}-\lambda_{35})}{(\lambda_{24}-\lambda_{14})(\lambda_{24}-\lambda_{34})}}\\ \sqrt{-\frac{(\lambda_{34}-\lambda_{15})(\lambda_{34}-\lambda_{25})(\lambda_{34}-\lambda_{35})}{(\lambda_{34}-\lambda_{14})(\lambda_{34}-\lambda_{24})}}\\ 0\end{pmatrix}$	$\begin{pmatrix}\sqrt{\frac{(\lambda_{25}-\lambda_{24})(\lambda_{25}-\lambda_{34})}{(\lambda_{25}-\lambda_{35})}}\\ \sqrt{\frac{(\lambda_{35}-\lambda_{14})(\lambda_{35}-\lambda_{24})(\lambda_{35}-\lambda_{34})}{(\lambda_{35}-\lambda_{15})(\lambda_{35}-\lambda_{25})}}\\ 0\end{pmatrix}$	$\begin{pmatrix}\frac{v_{14}(\lambda)w_{14}(\lambda)}{\lambda_{25}-\lambda_{24}} & \frac{v_{14}(\lambda)w_{24}(\lambda)}{\lambda_{35}-\lambda_{24}} & 0\\ \frac{v_{24}(\lambda)w_{14}(\lambda)}{\lambda_{25}-\lambda_{34}} & \frac{v_{24}(\lambda)w_{24}(\lambda)}{\lambda_{35}-\lambda_{34}} & 0\\ 0 & 0 & 1\end{pmatrix}$
5	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

Figure 5. Auxilary functions for the eigensteps parameterization around $\mathcal{E}$.

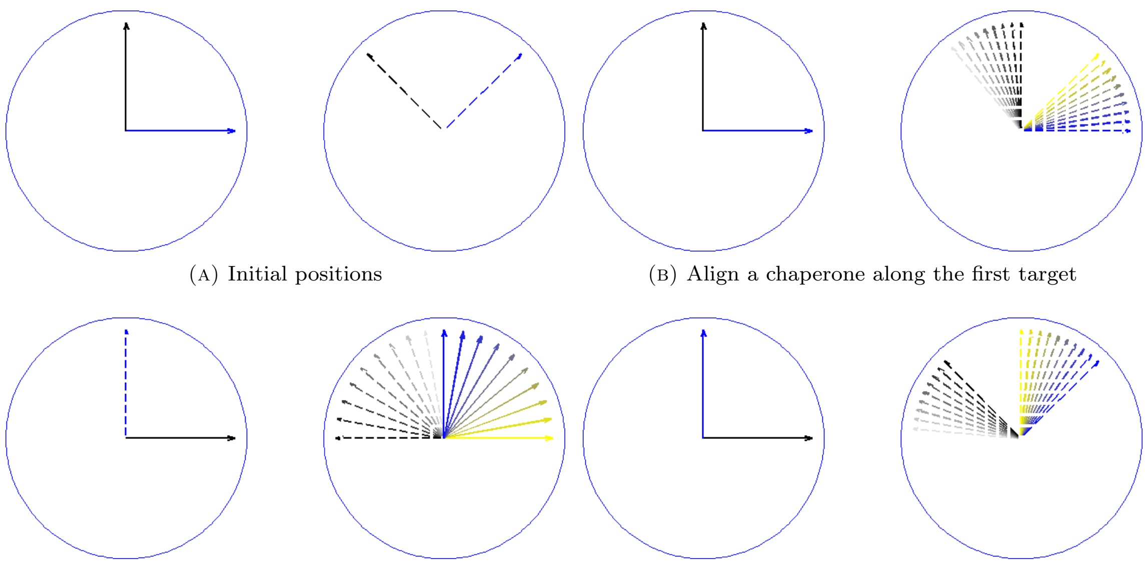
5. Connectivity and irreducibility of $\mathcal{F}_{d,N}$

5.1. **Connectivity.** Using the eigensteps parameterizations, we can also show that $\mathcal{F}_{d,N}$ is path-connected whenever $N \geq d+2 \geq 4$. For example, we can lift the path

$$\lambda(t) = (1-t)\lambda(\mathcal{E}) + t\lambda(\Xi)$$

to a continuous path $\gamma(t)$ in $\mathcal{F}_{3,6}$ such that $\gamma(0) = \mathcal{E}$ and $\gamma(1)$ has the same eigensteps as Ξ. It is then easy to see that $\gamma(1)$ must be the union of two orthonormal bases. It then remains for us to show that the set of frames consisting of a union of two orthonormal bases is path connected. Since the orthogonal group has two connected components, this is easy to show if there is a motion which swaps any pair of vectors. While swapping vectors between the two orthonormal bases is as simple as aligning the two vectors by a continuous rotation, and then reversing the rotation while the roles of the vectors are reversed.

If the pair is within the same orthonormal basis, by aligning a pair from the other orthonormal basis in the plane of the targets, this process can be reduced to the two dimensional case. Figure 6 illustrates a continuous motion that swaps the targets and returns the other vectors to their original position.



(A) Initial positions

(B) Align a chaperone along the first target

(C) Use the other chaperone to align the first target along the second target

(D) Return the chaperones to their original position

Figure 6. Swapping vectors within an orthonormal basis. The vectors targeted for swapping are the solid blue and solid black pair. The dashed pair play the role of chaperones during this transposition.

The main trick here is that we use liftings of the eigensteps map to connect any frame to a particular set of frames or a "hub," and then we show that this "hub" is connected. Generalizing this argument leads to the following theorem.

Theorem 5.1 (Cahill, Mixon, Strawn, 2014). *If $N \geq d+2 \geq 4$, then $\mathcal{F}_{d,N}$ is path-connected.*

5.2. **Irreducibility.** By the following proposition, if we can also show that the non-singular points of $\mathcal{F}_{3,6}$ are path-connected and dense in $\mathcal{F}_{3,6}$, then we know that $\mathcal{F}_{3,6}$ is an irreducible algebraic variety.

Proposition 5.2 (Cahill, Mixon, Strawn, 2014)**.** *Suppose V is an algebraic variety such that*

(i) *the set of non-singular points of V is path-connected, and*
(ii) *the set of non-singular points is dense in V.*

Then V is an irreducible algebraic variety.

First, we argue that the non-orthodecomposable frames of $\mathcal{F}_{3,6}$ are dense in $\mathcal{F}_{3,6}$. The following characterization of the orthodecomposable frames makes this argument go through very easily.

Proposition 5.3. *A frame $F \in \mathcal{F}_{3,6}$ is orthodecomposable if and only if there are two distinct frame vectors f_i and f_j that are parallel.*

Proof. If F is orthodecomposable, then there is a partition of F into F_1 and F_2 so that the linear spans of the vectors in F_1 and F_2 (denoted V_1 and V_2) are non-trivial orthogonal subspaces of $\mathbb{R}^3$. Consequently, either V_1 or V_2 has dimension equal to 1, and hence by the tight frame bound condition either F_1 or F_2 consists of two parallel vectors.

On the other hand, assuming that there are vectors f_i and f_j which are parallel, the tight frame bound condition requires that all other vectors in F are orthogonal to f_i and f_j. Consequently, F is orthodecomposable. $\square$

Coupling this with the fact that all frames in $\mathcal{F}_{2,4}$ consist of a union of two orthonormal bases, we deduce the following proposition.

Proposition 5.4. *If $F \in \mathcal{F}_{3,6}$ is orthodecomposable, then it is a union of two orthonormal bases.*

Observing that there is always a small rotation of one orthonormal basis that can bring all of its vectors out of alignment with another orthonormal basis (and thus the union is arbitrarily close to a non-orthodecomposable union of two orthonormal bases), and since all unions of two orthonormal bases are in $\mathcal{F}_{3,6}$, we conclude that any orthodecomposable frame in $\mathcal{F}_{3,6}$ is a limit of non-orthodecomposable frames in $\mathcal{F}_{3,6}$ and we obtain the following proposition.

Proposition 5.5. *The non-singular points of $\mathcal{F}_{3,6}$ are dense in $\mathcal{F}_{3,6}$.*

Now, we need to show that the non-singular points of $\mathcal{F}_{3,6}$ form a path-connected set. This is somewhat similar to the connectivity proof, but we need to take great care to avoid the orthodecomposable frames along our path. While we shall still attempt to connect to the same "hub" of unions of two orthonormal bases, we do not directly move to this hub using the eigensteps liftings since it is technically possible that the path ends on an orthodecomposable frame. Instead, we lift the eigensteps map to connect to any frame that has the eigensteps

$$\begin{pmatrix} 1 & 3/2 & 3/2 & 2 & 2 & 2 \\ 0 & 1/2 & 3/2 & 3/2 & 2 & 2 \\ 0 & 0 & 0 & 1/2 & 1 & 2 \end{pmatrix}$$

Any frame with these eigensteps can be rotated, permuted, and negated to get it to the block form

$$F = \begin{pmatrix} F_1 & \sqrt{1/3}F_2 \\ \mathbf{0}_3^* & \sqrt{2/3}\mathbf{1}_3^* \end{pmatrix}$$

where

$$F_1 = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix} \in \mathcal{F}_{2,3}$$

and $F_2 \in \mathcal{F}_{2,3}$. Note that any frame of this form is non-orthodecomposable. Moreover, since $F_2 \in \mathcal{F}_{2,3}$,

$$\begin{pmatrix} F_1 & \sqrt{1/3}UF_2 \\ \mathbf{0}_3^* & \sqrt{2/3}\mathbf{1}_3^* \end{pmatrix} \in \mathcal{F}_{3,6}$$

for any orthogonal matrix U. This means that we can rotate so that

$$F_2 = \begin{pmatrix} 1 & * & * \\ 0 & * & * \end{pmatrix}.$$

Orthogonality of the rows of F_2 with $\mathbf{1}_3$ and the fact that $F_2 \in \mathcal{F}_{2,3}$ therefore fixes the first row to be $(1\ -1/2\ -1/2)$. By possibly performing one final permutation, we may therefore assume that $F_1 = F_2$. Now, consider the path

$$\gamma(t) = \begin{pmatrix} \sqrt{1-t^2}F_1 & \sqrt{\frac{1}{3}+t^2}F_1 \\ -t\mathbf{1}_3^* & \sqrt{\frac{2}{3}-t^2}\mathbf{1}_3^* \end{pmatrix}$$

for $t \in [0, 1/\sqrt{3}]$. Observe that

$$\begin{aligned}
\gamma(t)\gamma(t)^* &= \begin{pmatrix} \sqrt{1-t^2}F_1 & \sqrt{\frac{1}{3}+t^2}F_1 \\ -t\mathbf{1}_3^* & \sqrt{\frac{2}{3}-t^2}\mathbf{1}_3^* \end{pmatrix} \begin{pmatrix} \sqrt{1-t^2}F_1^* & -t\mathbf{1}_3 \\ \sqrt{1/3+t^2}F_1^* & \sqrt{\frac{2}{3}-t^2}\mathbf{1}_3 \end{pmatrix} \\
&= \begin{pmatrix} (1-t^2)F_1F_1^* + (1/3+t^2)F_1F_1^* & -t\sqrt{1-t^2}F_1\mathbf{1}_3 + \sqrt{\frac{2}{3}-t^2}\sqrt{\frac{1}{3}+t^2}F_1\mathbf{1}_3 \\ \left(-t\sqrt{(1-t^2})F_1\mathbf{1}_3 + \sqrt{\frac{2}{3}-t^2}\sqrt{\frac{1}{3}+t^2}F_1\mathbf{1}_3\right)^* & 3t^2 + 3(2/3-t^2) \end{pmatrix} \\
&= \begin{pmatrix} (4/3)(3/2)I_2 & \mathbf{0}_2 \\ \mathbf{0}_2^* & 2 \end{pmatrix} \\
&= 2I_3,
\end{aligned}$$

and noting that the columns of $\gamma(t)$ are always on the unit sphere, we conclude that γ is continuous path in $\mathcal{F}_{3,6}$. In particular, this path does not pass through any orthodecomposable frames since none of the rows can be parallel. Now,

$$\gamma(1/\sqrt{3}) = \begin{pmatrix} \sqrt{\frac{2}{3}}F_1 & \sqrt{\frac{2}{3}}F_1 \\ -\sqrt{\frac{1}{3}}\mathbf{1}_3^* & \sqrt{\frac{1}{3}}\mathbf{1}_3^* \end{pmatrix}$$

and it is easy to verify that

$$\begin{pmatrix} \sqrt{\frac{2}{3}}F_1 \\ -\sqrt{\frac{1}{3}}\mathbf{1}_3^* \end{pmatrix} \text{ and } \begin{pmatrix} \sqrt{\frac{2}{3}}F_1 \\ \sqrt{\frac{1}{3}}\mathbf{1}_3^* \end{pmatrix}$$

are both orthonormal bases, and $\gamma(1/\sqrt{3})$ is a union of orthonormal bases which is non-orthodecomposable. Figure 7 depicts how the frame vectors move along this path.

Now that we have arrived at the "hub", we observe that all non-orthodecomposable unions of two orthonormal bases arising from a particular column ordering are path-connected. Thus,

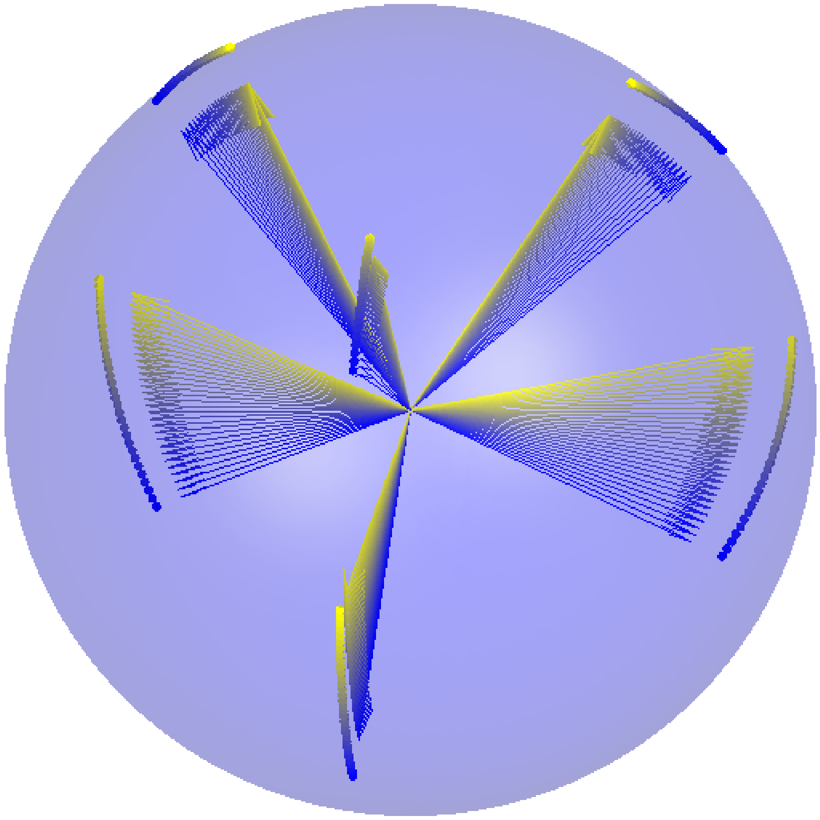


FIGURE 7. The motion arriving at a non-orthodecomposable union of two orthonormal bases.

the last step of our proof involves showing that there are continuous paths in $\mathcal{F}_{3,6}$ connecting a non-orthodecomposable union of two orthonormal bases to a reordering of its columns.

In the proof of path-connectivity of $\mathcal{F}_{3,6}$, we showed how to construct paths that acted as transpositions, and composing transpositions gives us the full permutation group. However, the paths in this case require the alignment of vectors, and hence pass through the orthodecomposable frames. Now that such a path is not an option, we must provide an alternative path that avoids the orthodecomposable frames.

Assume that we begin at the following non-orthodecomposable system, which is a union of two orthonormal bases:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Our goal is to swap the second and fifth column without putting any of the vectors in a parallel position. Figure 8(A) depicts this frame, and the second and fifth columns are indicated by the blue and cyan markers, respectively.

During this motion, we shall make extensive use of the fact that continuously rotating any orthonormal pair in their span induces a continuous path through $\mathcal{F}_{3,6}$. As a first application of this idea, we spin the first and fourth columns, effectively moving away from the "hub" of unions of two orthonormal bases. This motion is depicted in Figure 8(B). Next, Figure 9(A) shows how we spin the four vectors sticking out (essentially inducing a 4-cycle). The remaining motions will essentially perform the action of a 3-cycle on the vectors marked blue, green, and black.

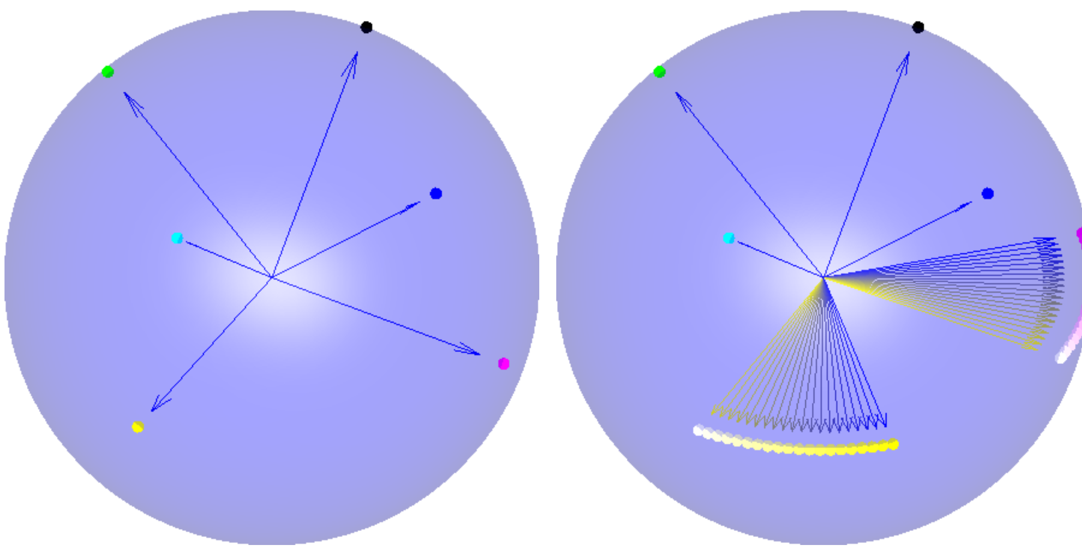
First, we spin the blue and green vectors by $\pi/4$ radians in their span as depicted in Figure 9(B). After this motion, the vectors marked green and black are now orthogonal and Figure 9(C) illustrates how we rotate these two $\pi/2$ radians in their span. Lastly, we spin the vectors marked with black and blue by $\pi/4$ radians. If we return the first and third columns to their original position, we see that we have swapped the vectors marked with blue and cyan.

This completes the proof of connectivity of the non-singular points of $\mathcal{F}_{3,6}$, and Proposition 5.2 now implies the following theorem.

Theorem 5.6. *The algebraic variety $\mathcal{F}_{3,6}$ is irreducible.*

The case when $N = 2d$ is the most complicated part of the general proof of this result for arbitrary N and d.

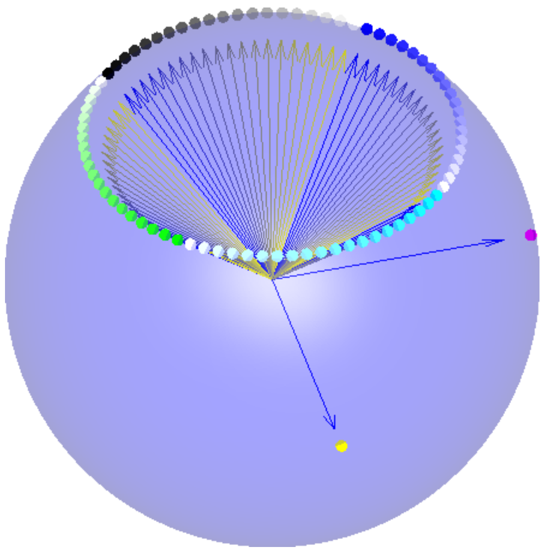
Theorem 5.7 (Cahill, Mixon, Strawn, 2014)**.** *If $N \geq d + 2$ and $N > 4$, then $\mathcal{F}_{d,N}$ is an irreducible algebraic variety.*



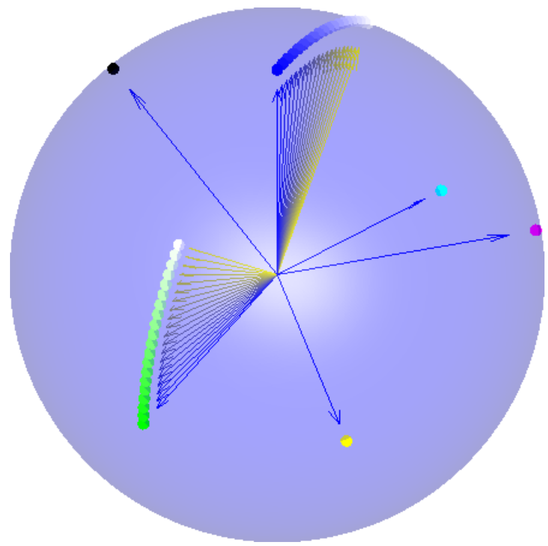
(A) The starting point for the transposition.

(B) This first motion ensures that the ensuing path does not pass through an orthodecompsable frame.

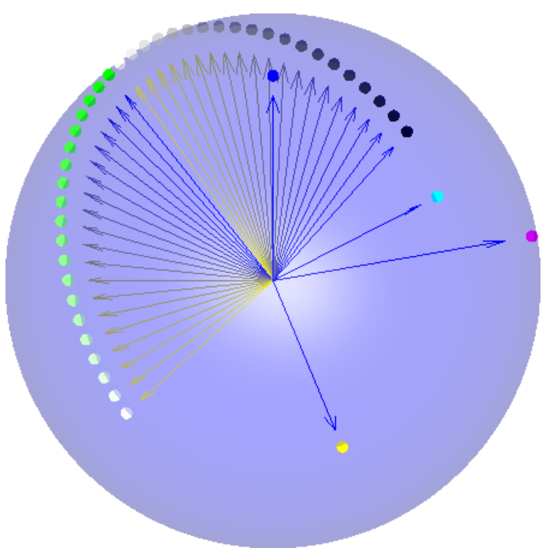
Figure 8. Initial positioning for the transposition



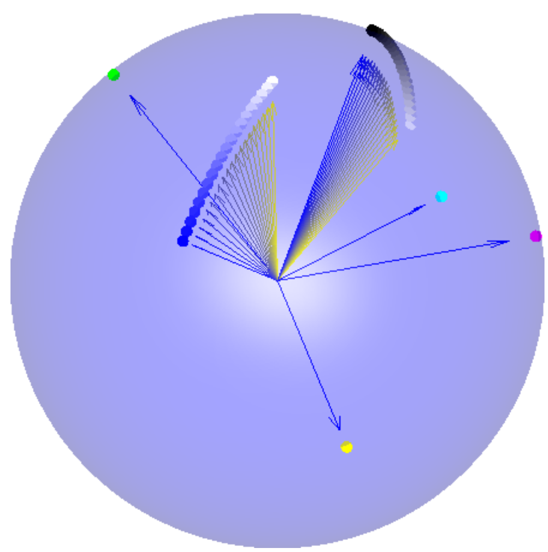
(A) Rotation of the four vector that are "sticking out" induces a 4-cycle.



(B) Continuous rotation of the green and blue vectors.



(C) Continuous rotation of the green and black vectors.



(D) Continuous rotation of the blue and black vectors. This completes the 3-cycle on the black, blue, and green vectors.

FIGURE 9. The motions for inducing the transposition.

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