Some Topics in Computational Topology

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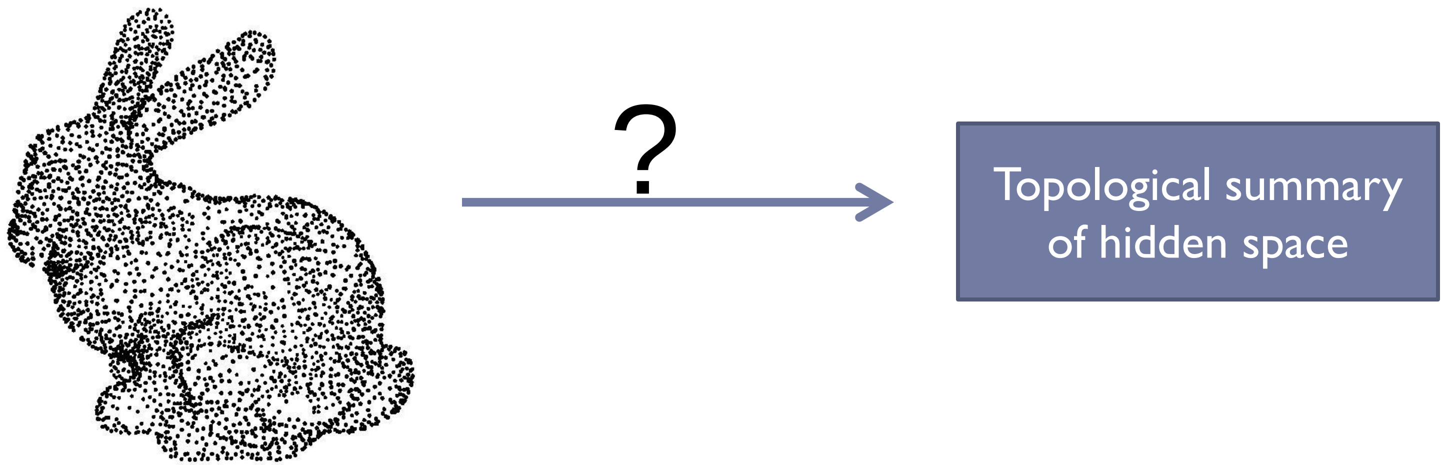
AMS Short Course 2014

Introduction

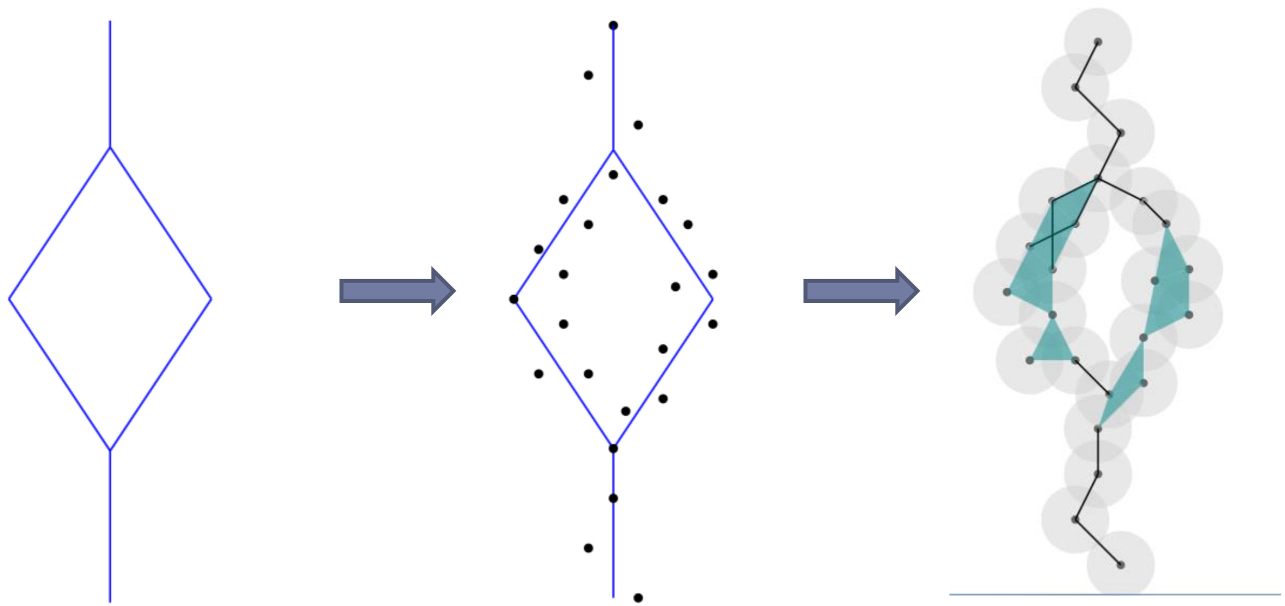
- Much recent developments in computational topology
  - Both in theory and in their applications
  - E.g, the theory of persistence homology
    - *[Edelsbrunner, Letscher, Zomorodian, DCG 2002], [Zomorodian and Carlsson, DCG 2005], [Carlsson and de Silva, FoCM 2010], …*

- This short course:
  - A computational perspective:
    - Estimation and inference of topological information / structure from point clouds data



- Develop discrete analog for the continuous case
- Approximation from discrete samples with theoretical guarantees
- Algorithmic issues

Main Topics



- From PCDs to simplicial complexes
- Sampling conditions
- Topological inferences

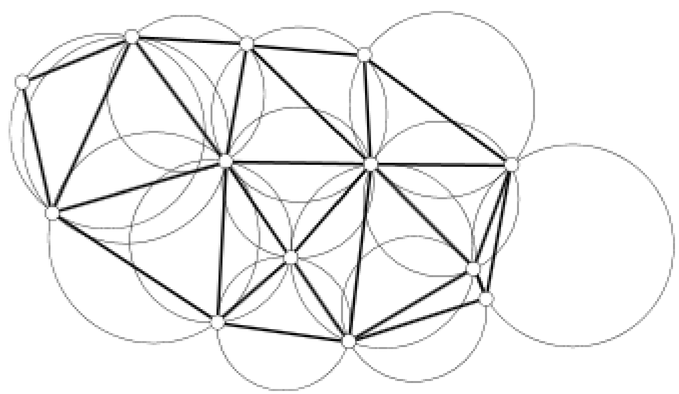
Outline

- From PCDs to simplicial complexes
  - Delaunay, Cech, Vietoris-Rips, witness complexes
  - Graph induced complex
- Sampling conditions
  - Local feature size, and homological feature size
- Topology inference
  - Homology inference
  - Handling noise
  - *Approximating cycles of shortest basis of the first homology group*
  - *Approximating Reeb graph*

Choice of Simplicial Complexes

Delaunay Complex

- Given a set of points $P = \{ p_1, p_2, \dots, p_n\} \subset R^d$
- Delaunay complex $Del(P)$
  - A simplex $\sigma = [p_{i_0}, p_{i_1}, \dots, p_{i_k}]$ is in $Del(P)$ if and only if
    - There exists a ball B whose boundary contains vertices of σ, and that the interior of B contains no other point from P.



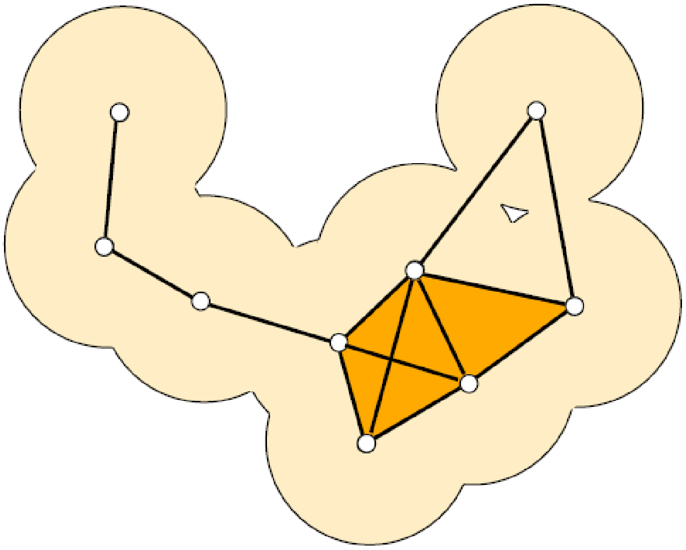
Delaunay Complex

- Many beautiful properties
  - Connection to Voronoi diagram
- Foundation for surface reconstruction and meshing in 3D
  - *[Dey, Curve and Surface Reconstruction, 2006],*
  - *[Cheng, Dey and Shewchuk, Delaunay Mesh Generation, 2012]*

- However,
  - Computationally very expensive in high dimensions

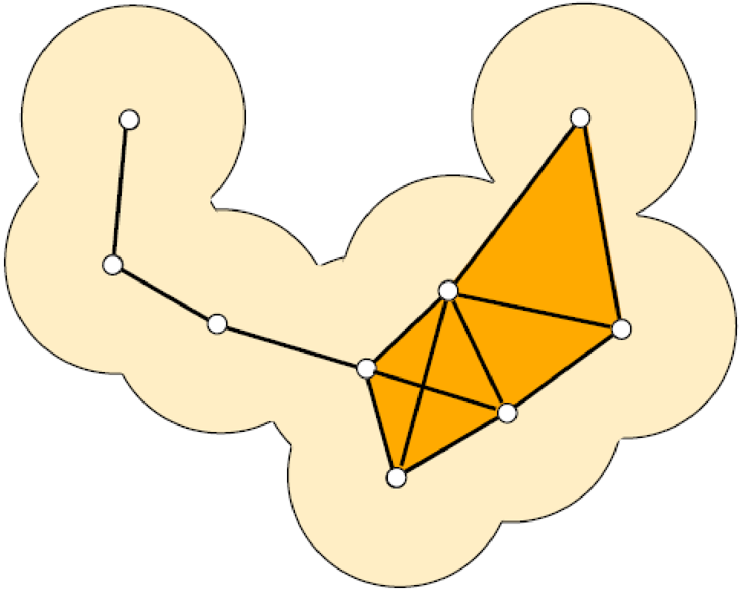
Čech Complex

- Given a set of points $P = \{ p_1, p_2, \ldots, p_n \} \subset R^d$
- Given a real value $r > 0$, the *Čech complex* $C^r(P)$ is the nerve of the set $\{ B(p_i, r) \}_{i \in [1,n]}$
  - where $B_r(p_i) = B(p_i, r) = \{ x \in R^d \mid d(p_i, x) < r \}$
- The definition can be extended to a finite sample P of a metric space.



Rips Complex

- Given a set of points $P = \{ p_1, p_2, \ldots, p_n\} \subset R^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $R^r(P)$ is:
  - $\{ \left(p_{i_0}, p_{i_1}, \ldots, p_{i_k}\right) \mid B_r\left(p_{i_l}\right) \cap B_r\left(p_{i_j}\right) \neq \emptyset, \forall\ l, j \in [0, k\ ]\}.$



Rips and Čech Complexes

- Relation in general metric spaces
  - $C^r(P) \subseteq R^r(P) \subseteq C^{2r}(P)$
  - Bounds better in Euclidean space
- Simple to compute
- Able to capture geometry and topology
  - We will make it precise shortly
  - One of the most popular choices for topology inference in recent years
- However:
  - Huge sizes
  - Computation also costly

Witness Complexes

- A simplex $\sigma = \{q_0, \ldots, q_k\}$ is *weakly witnessed* by a point x if $d(q_i, x) \leq d(q, x)$ for any $i \in [0,\ k]$ and $q \in Q \setminus \{q_0, \ldots, q_k\}$.

- Given a set of points $P = \{\, p_1, p_2, \ldots, p_n\} \subset R^d$ and a subset $Q \subseteq P$
- The *witness complex* $W(Q, P)$ is the collection of simplices with vertices from Q whose all subsimplices are weakly witnessed by a point in P.
  - *[de Silva and Carlsson, 2004] [de Silva 2003]*
  - Can be defined for a general metric space
  - P does not have to be a finite subset of points

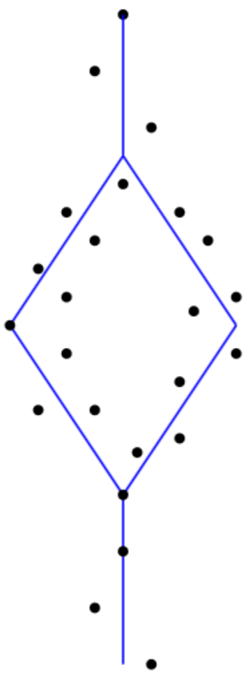
Witness Complexes

- Greatly reduce size of complex
  - Similar to Delaunay triangulation, remove redundancy
- Relation to Delaunay complex
  - $W(Q,P) \subseteq Del\ Q$
  - $W(Q,R^d) = Del\ Q$
  - $W(Q,M) = Del|_M\ Q$ if $M \subseteq R^d$ is a smooth 1- or 2-manifold
    - *[Attali et al, 2007]*
- However,
  - Does not capture full topology easily for high-dimensional manifolds

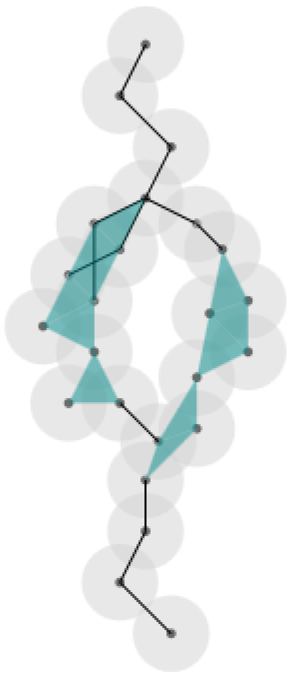
Remark

- Rips complex
  - Capture homology when input points are sampled dense enough
  - But too large in size
- Witness complex
  - Use a subsampling idea
  - Reduce size tremendously
  - May not be easy to capture topology in high-dimensions
- Combine the two ?
  - Graph induced complex

Subsampling

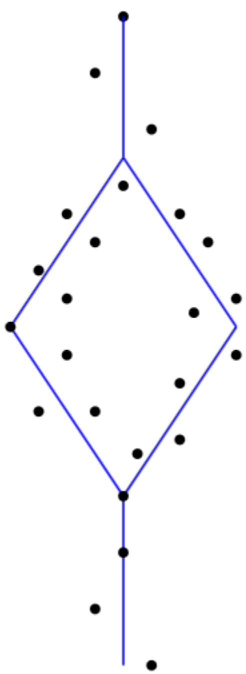


P



$R^{\epsilon}(P)$

Subsampling -cont



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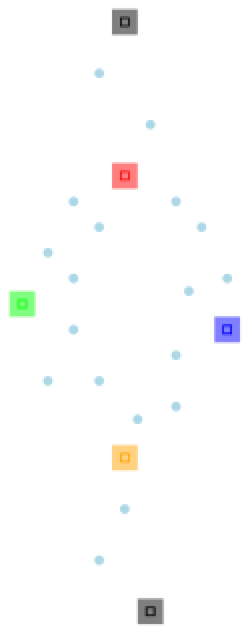
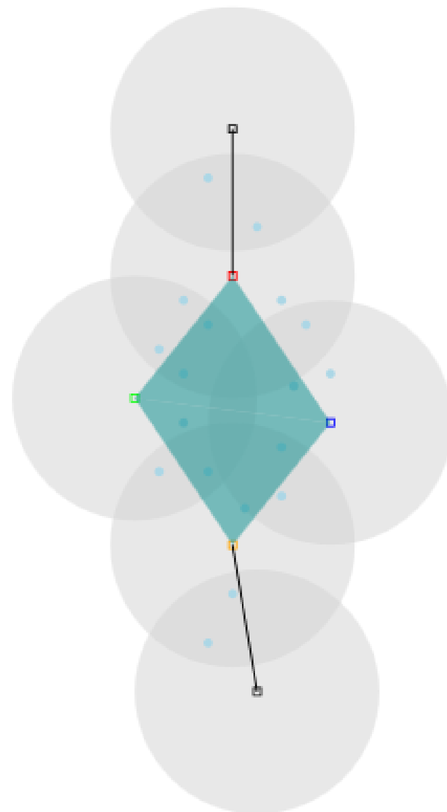
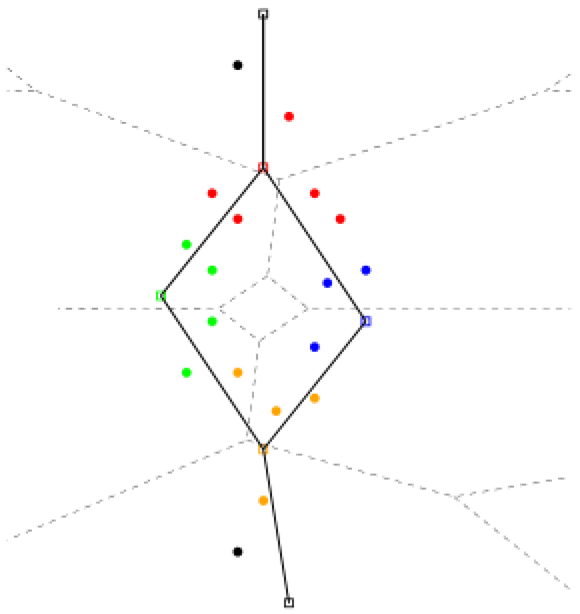


$Q \subseteq P$

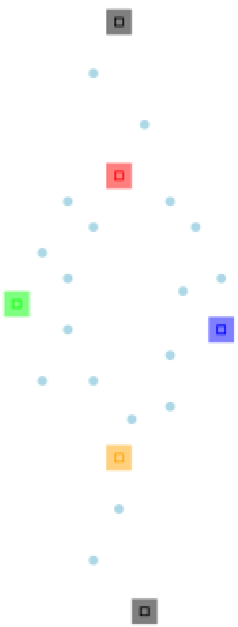


$R^r(Q)$



$W(Q, P)$

Subsampling -cont



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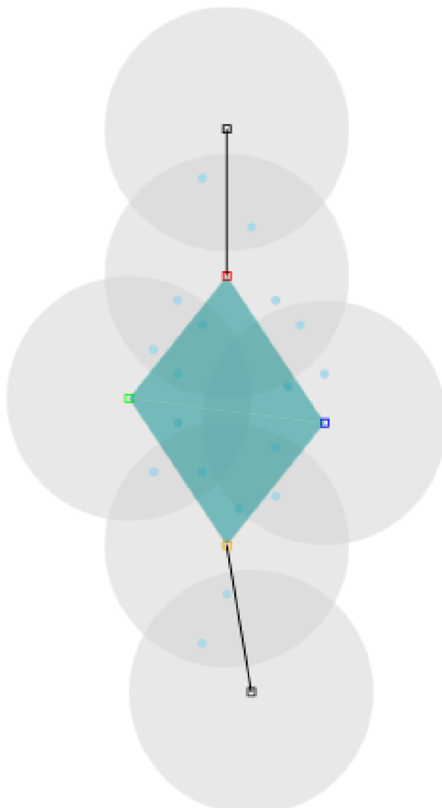
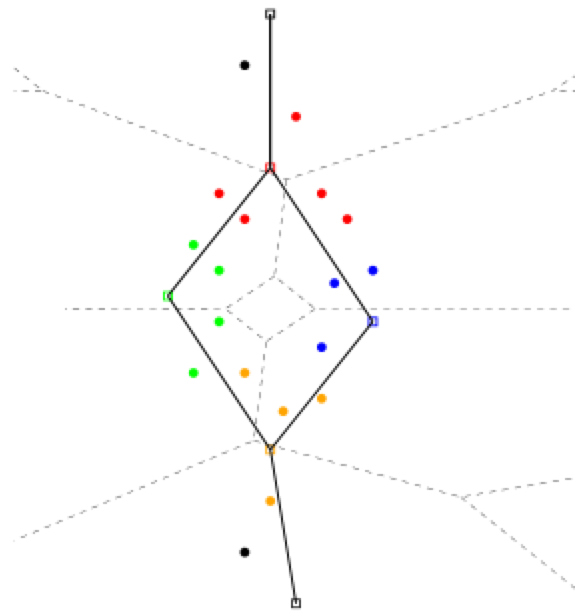


$R^r(Q)$



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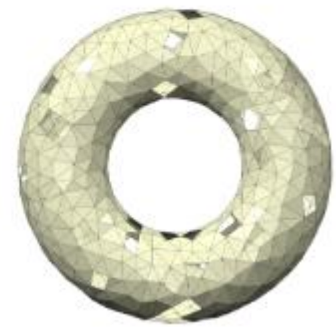
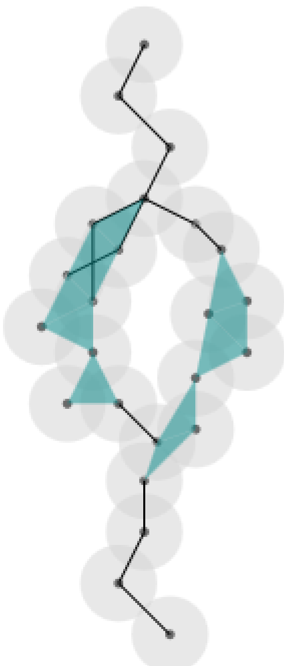
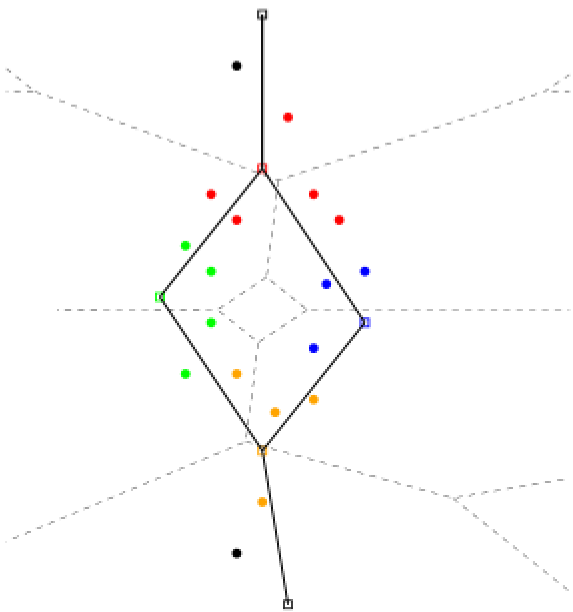


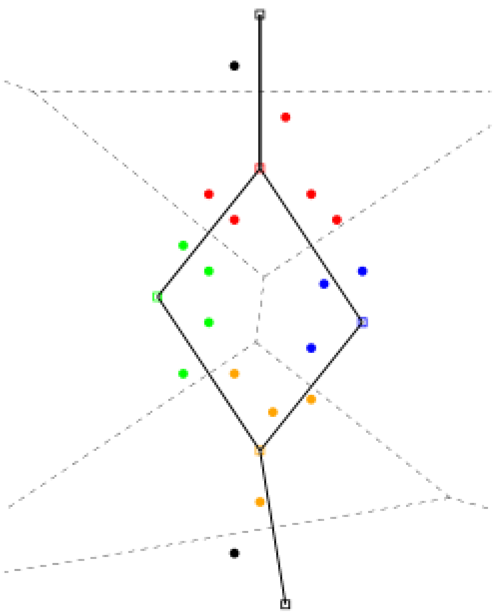
Subsampling - cont



$R^{\epsilon}(P)$



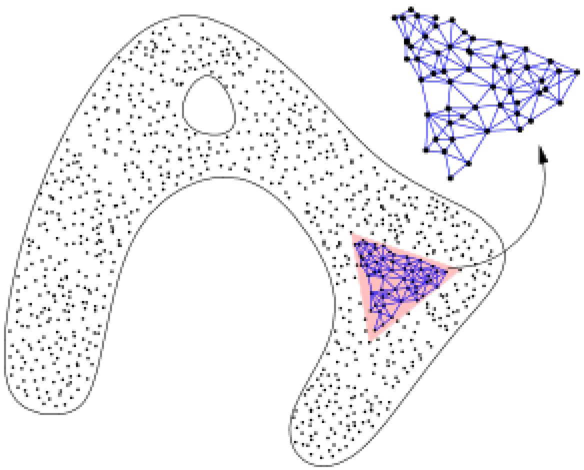
$W(Q, P)$



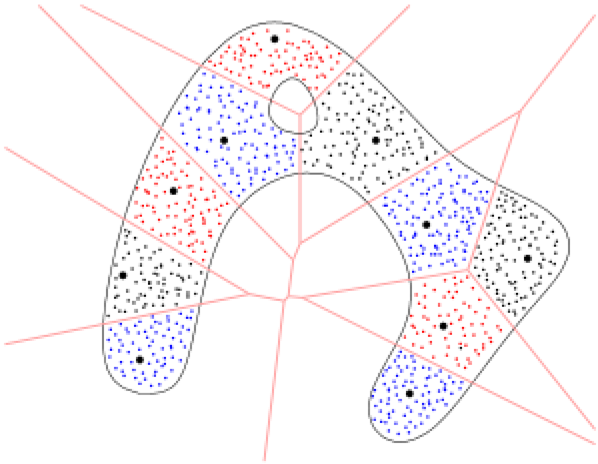
$\mathcal{G}^{r}(Q, P)$

Graph Induced Complex

- P: finite set of points
- (P, d): metric space
- $G(P)$: a graph

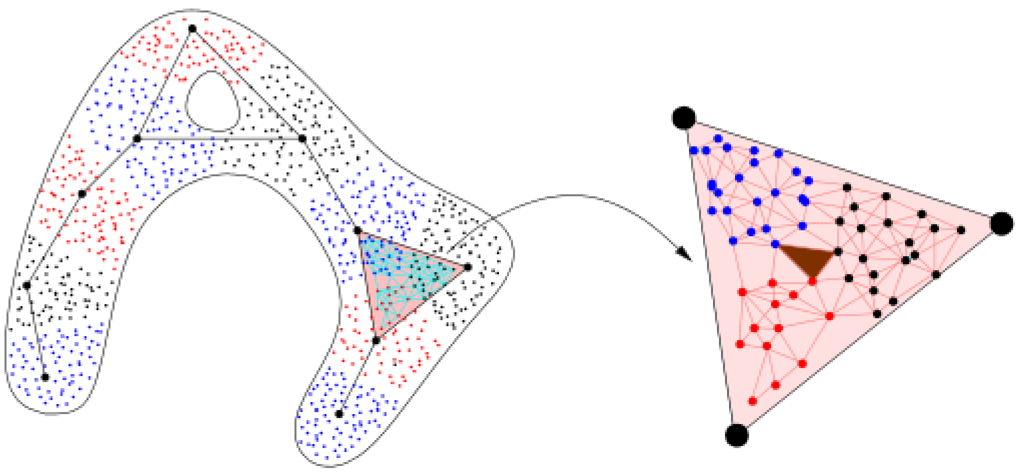


- $Q \subset P$: a subset
- $\pi(p)$: the closest point of $p \in P$ in Q



Graph Induced Complex

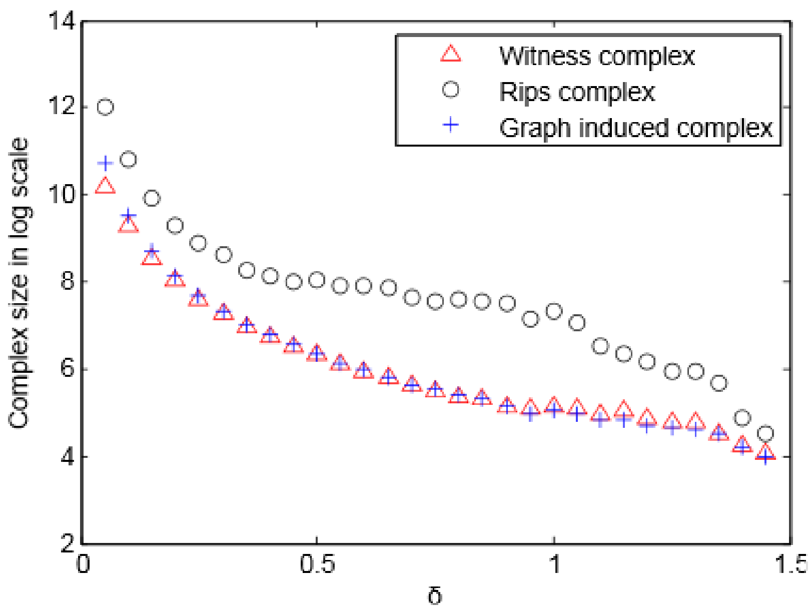
- *Graph induced complex* $\mathcal{G}(P, Q, d)$: $\{q_0, \ldots, q_k\} \subseteq Q$
  - if and only if there is a (k+1)-clique in $G(P)$ with vertices $p_0, \ldots, p_k$ such that $\pi(p_i) = q_i$, for any $i \in [0, k]$.

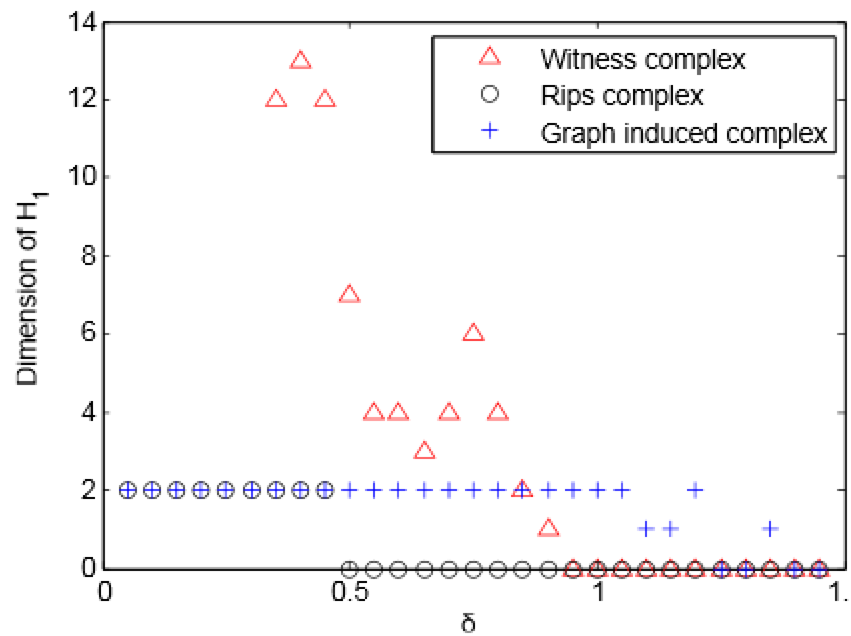


- Graph induced complex depends on the metric d:
  - Euclidean metric
  - Graph based distance d_G

Graph Induced Complex

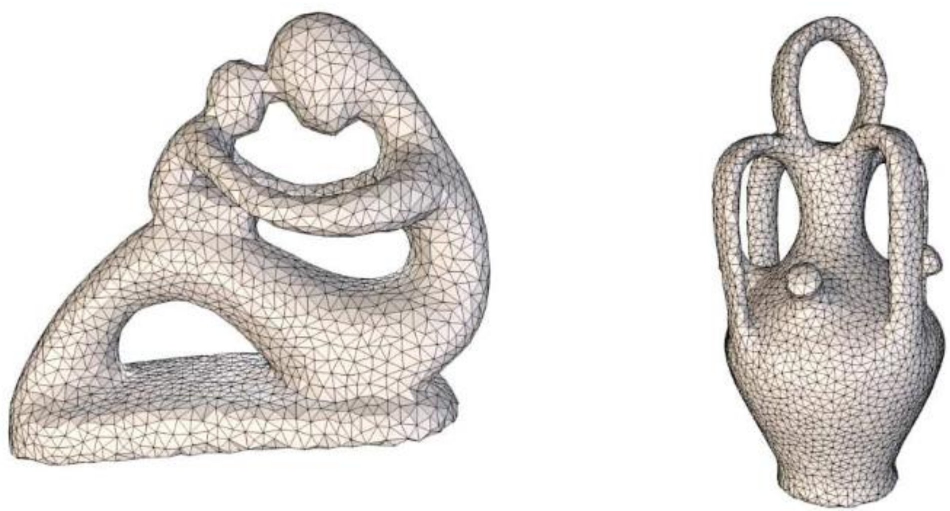
- Small size, but with homology inference guarantees
- In particular:
  - H_1 inference from a lean sample





Graph Induced Complex

- Small size, but with homology inference guarantees
- In particular:
  - H_1 inference from a lean sample
  - Surface reconstruction in R^3



  - Topological inference for compact sets in R^d using persistence

Outline

- From PCDs to simplicial complexes
  - Delaunay, Cech, Vietoris-Rips, witness complexes
  - Graph induced complex
- Sampling conditions
  - Local feature size, and homological feature size
- Topology inference
  - Homology inference
  - Handling noise
  - *Approximating cycles of shortest basis of the first homology group*
  - *Approximating Reeb graph*

Sampling Conditions

Motivation

- Theoretical guarantees are usually obtained when input points P sampling the hidden domain "well enough".

- Need to quantize the "wellness".
- Two common ones based on:
  - Local feature size
  - Weak feature size

Distance Function

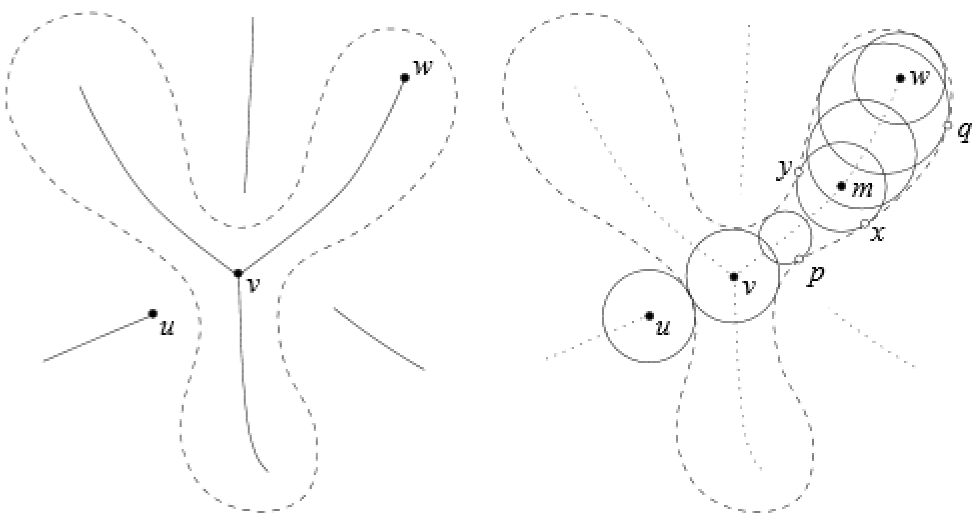
- $X \subset R^d$: a compact subset of R^d
- Distance function $d_X: R^d \to R^+ \cup \{0\}$
  - $d_X(x) = \min_{y \in X} d(x, y)$
  - d_X is a 1-Lipschitz function

- X^α: α-offset of X
  - $X^\alpha = \{y \in R^d \mid d(y, X) \leq \alpha\}$

- Given any point $x \in X$
  - $\Gamma(x) \coloneqq \{\, y \in X \mid d(x, y) = d_X(x) \,\}$

Medial Axis

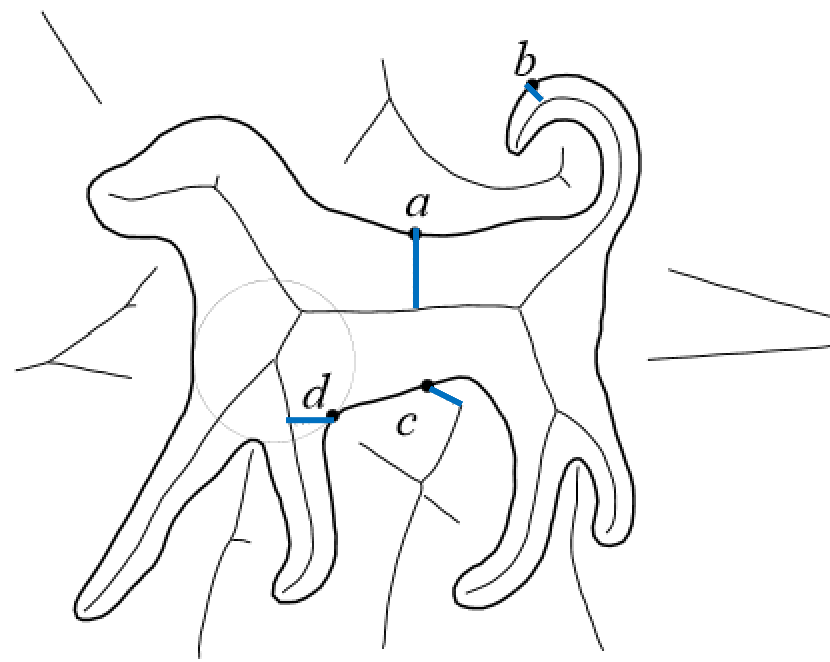
- The *medial axis* Σ of X is the closure of the set of points $x \in R^d$ such that $|\Gamma(x)| \geq 2$
  - $|\Gamma(x)| \geq 2$ means that there is a medial ball $B_r(x)$ touching X at more than 1 point and whose interior is empty of points from X.



Courtesy of [Dey, 2006]

Local Feature Size

- The local feature size $lfs(x)$ at a point $x \in X$ is the distance of x to the medial axis Σ of X
  - That is, $lfs(x) = d(x, \Sigma)$
- This concept is adaptive
  - Large in a place without "features"
- Intuitively:
  - We should sample more densely if local feature size is small.
- The *reach* $\rho(X) = \inf_{x \in X} lfs(x)$



Courtesy of [Dey, 2006]

Gradient of Distance Function

- Distance function not differentiable on the medial axis
- Still can define a generalized concept of gradient
  - *[Lieutier, 2004]*
- For $x \in R^d \setminus X$,
  - Let $c_X(x)$ and $r_X(x)$ be the center and radius of the smallest enclosing ball of point(s) in $\Gamma(x)$
  - The *generalized gradient* of distance function

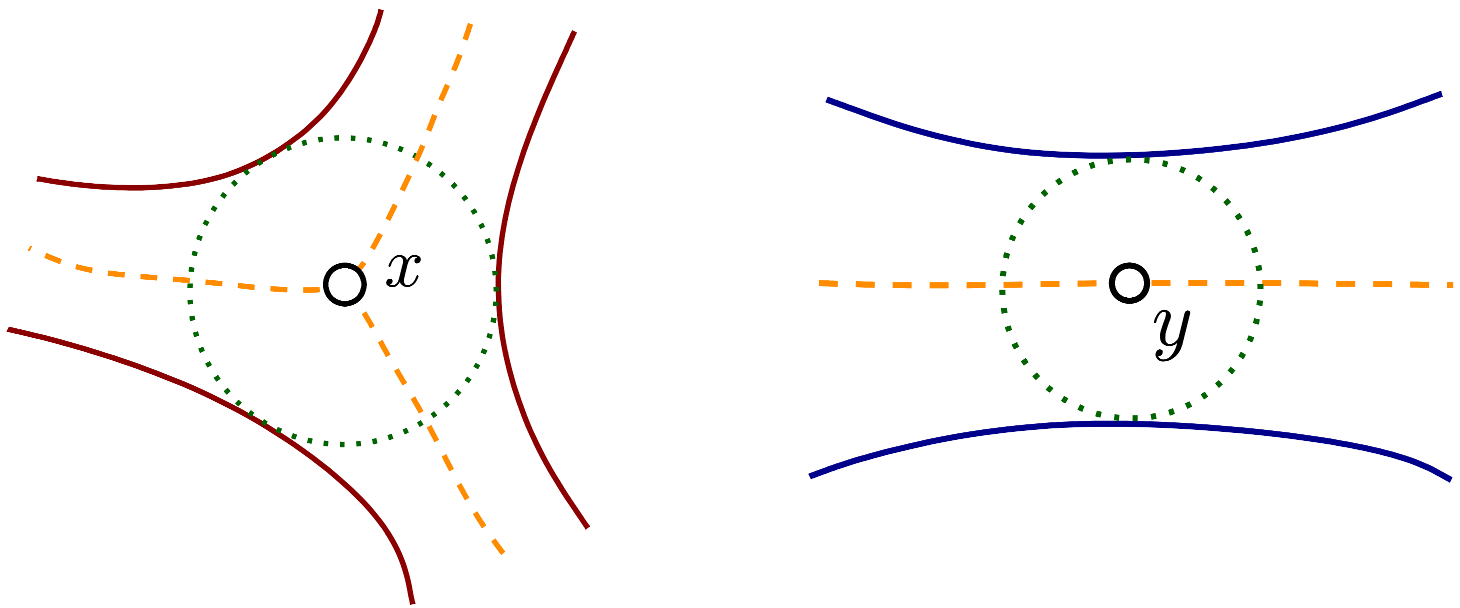
$$\nabla_X(x) = \frac{x - c_X(x)}{r_X(x)}$$

- Examples:

Critical Points

- A critical point of the distance function is a point whose generalized gradient $\nabla_X(x)$ vanishes

- A critical point is either in X or in its medial axis Σ



Weak Feature Size

- Given a compact $X \subset R^d$, let $C \subset R^d$ denote
  - the set of critical points of the distance function d_X that are not in X
- Given a compact $X \subset R^d$, the *weak feature size* is
  - $wfs(X) = \inf_{x \in X} d(x, C)$
- Equivalently,
  - $wfs(X)$ is the infimum of the positive critical value of d_X
- $\rho(X) \leq wfs(X)$

Typical Sampling Conditions

- Hausdorff distance $d_H(A, B)$ between two sets A and B
  - infimum value α such that $A \subseteq B^\alpha$ and $B \subseteq A^\alpha$

- No noise version:
  - A set of points P is an ϵ-sample of X if $P \subset X$ and $d_H(P, X) \leq \epsilon$

- With noise version:
  - A set of points P is an ϵ-sample of X if $d_H(P, X) \leq \epsilon$

Why Distance Field?

- Theorem [Offset Homotopy]

  If $0 < \alpha < \alpha'$ are such that there is no critical value of d_X in the closed interval $[\alpha, \alpha']$, then $X^{\alpha'}$ deformation retracts onto X^{α} . In particular, $H(X^{\alpha}) \cong H\left(X^{\alpha'}\right)$.

- Remarks:
  - For the case of compact set X, note that it is possible that X^{α}, for sufficiently small $\alpha > 0$, may not be homotopy equivalent to $X^{0} = X$.
  - Intuitively, by above theorem, we can approximate $H(X^{\alpha})$ for any small positive α from a thickened version (offset) of X^{α}.
  - The sampling condition makes sure that the discrete sample is sufficient to recover the offset homology.

Outline

- From PCDs to simplicial complexes
  - Delaunay, Cech, Vietoris-Rips, witness complexes
  - Graph induced complex
- Sampling conditions
  - Local feature size, and homological feature size
- Topology inference
  - Homology inference
  - Handling noise
  - *Approximating cycles of shortest basis of the first homology group*
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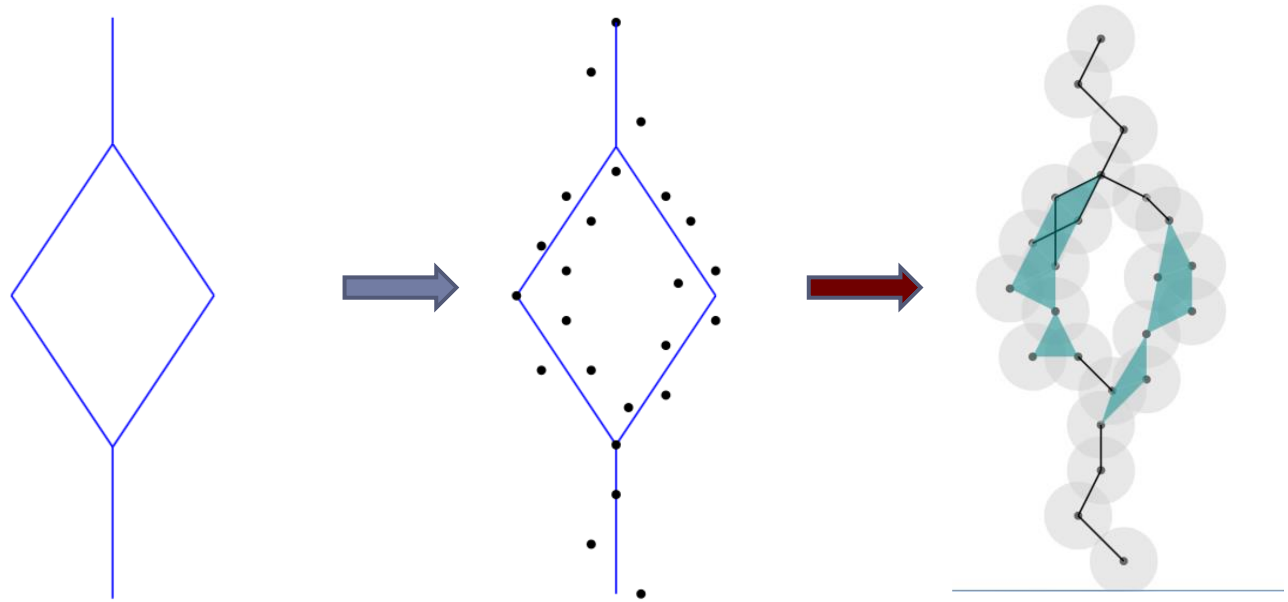
Homology Inference from PCD

Problem Setup



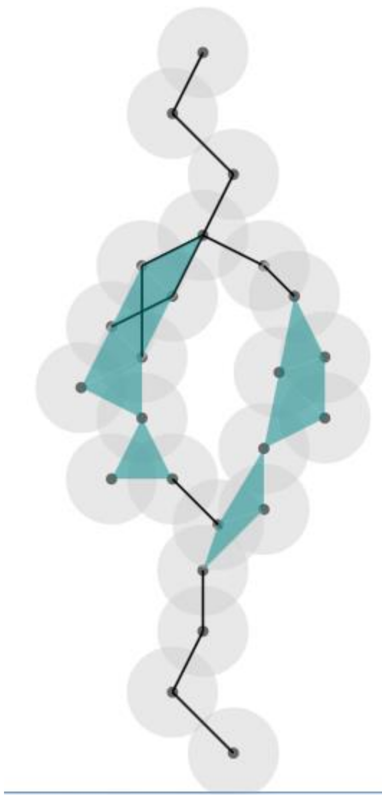
- A hidden compact X (or a manifold M)
- An ϵ-sample P of X
- Recover homology of X from some complex built on P
  - will focus on Čech complex and Rips complex

Union of Balls

- $X^\alpha = \bigcup_{x \in X} B(x, \alpha)$
- $P^\alpha = \bigcup_{p \in \mathrm{P}} B(p, \alpha)$

- Intuitively, P^α approximates offset X^α

- The Čech complex $C^\alpha(P)$ is the Nerve of P^α
- By Nerve Lemma, $C^\alpha(P)$ is homotopy equivalent to P^α



Smooth Manifold Case

- Let X be a smooth manifold embedded in R^d

Theorem [Niyogi, Smale, Weinberger]

Let $P \subset X$ be such that $d_H(X, P) \leq \epsilon$. If $2\epsilon \leq \alpha \leq \sqrt{\frac{3}{5}}\rho(X)$,

there is a deformation retraction from P^α to X.

Corollary A

Under the conditions above, we have

$$\mathrm{H}(\mathrm{X}) \cong H(P^\alpha) \cong H(C^\alpha(P)).$$

- How about using Rips complex instead of Čech complex?
- Recall that

$$C^r(P) \subseteq R^r(P) \subseteq C^{2r}(P)$$

inducing

$$H(C^r) \to H(R^r) \to H(C^{2r})$$

- Idea *[Chazal and Oudot 2008]*:
  - Forming interleaving sequence of homomorphism to connect them with the homology of the input manifold X and its offsets X^α

Convert to Cech Complexes

- Lemma A *[Chazal and Oudot, 2008]*:
- The following diagram commutes:

$$
\begin{array}{ccc}
H(P^{\alpha}) & \xrightarrow{\;i_*\;} & H(P^{\beta}) \\
\downarrow h_* & & \downarrow h_* \\
H(C^{\alpha}) & \xrightarrow{\;i_*\;} & H(C^{\beta})
\end{array}
$$

Corollary B

Let $P \subset X$ be s.t. $d_H(X, P) \leq \epsilon$. If $2\epsilon \leq \alpha \leq \alpha' \leq \sqrt{\frac{3}{5}}\rho(X)$, $\mathrm{H(X)} \cong H(C^{\alpha}) \cong H(C^{\beta})$ where the second isomorphism is induced by inclusion.

From Rips Complex

- Lemma B:

  Given a sequence $A \to B \to C \to D \to E$ of homomorphisms between finite dimensional vector spaces, if $rank(A \to E) = rank(C)$, then $rank(B \to D) = \dim C$.

- Rips and Čech complexes:

$$C^{\alpha}(P) \subseteq R^{\alpha}(P) \subseteq C^{2\alpha}(P) \subseteq R^{2\alpha}(P) \subseteq C^{4\alpha}(P)$$
$$\Rightarrow H(C^{\alpha}) \to H(R^{\alpha}) \to H(C^{2\alpha}) \to H(R^{2\alpha}) \to H(C^{4\alpha})$$

- Applying Lemma B

$$rank\big(H(R^{\alpha}) \to H(R^{2\alpha})\big) = rank\big(H(C^{2\alpha})\big) = rank\big(H(X)\big)$$

The Case of Compact

- In contrast to Corollary A, now we have the follow (using Lemma B).

- Lemma C *[Chazal and Oudot 2008]*:
  Let $P \subset R^d$ be a finite set such that $d_H(X, P) < \epsilon$ for some $\epsilon < \frac{1}{4} wfs(X)$. Then for all $\alpha, \beta \in [\epsilon, wfs(X) - \epsilon]$ such that $\beta - \alpha \geq 2\epsilon$, and for all $\lambda \in (0, wfs(X))$, we have $H(X^\lambda) \cong image(i_*)$, where $i_*: H(P^\alpha) \to H(P^\beta)$ is the homomorphism between homology groups induced by the canonical inclusion $i: P^\alpha \to P^\beta$.

The Case of Compacts

- One more level of interleaving.
- Use the following extension of Lemma B:

  Given a sequence $A \to B \to C \to D \to E \to F$ of homomorphisms between finite dimensional vector spaces, if $rank(A \to F) = rank(C \to D)$, then $rank(B \to E) = rank(C \to D)$.

- Theorem [Homology Inference] *[Chazal and Oudot 2008]*:

  Let $P \subset R^d$ be a finite set such that $d_H(X, P) < \epsilon$ for some $\epsilon < \frac{1}{9} wfs(X)$. Then for all $\alpha \in \left[2\epsilon, \frac{1}{4}(wfs(X) - \epsilon)\right]$ all $\lambda \in \left(0, wfs(X)\right)$, we have $H\left(X^\lambda\right) \cong image(j_*)$, where j_* is the homomorphism between homology groups induced by canonical inclusion $j: R^\alpha \to R^{4\alpha}$.

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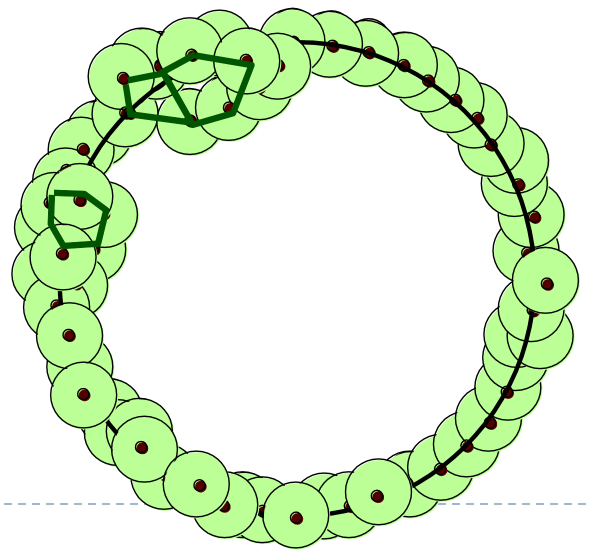

Summary of Homology Inference

- X^{α} homotopy equivalent to X^{β}
  - Critical points of distance field
- P^{α} approximates X^{α} (may be interleaving)
  - E.g, *[Niyogi, Smale, Weinberger, 2006], [Chazal and Oudot 2008]*
- C^{α} homotopy equivalent to P^{α}
  - Nerve Lemma
- $H(C^{\alpha})$ interleaves $H(X^{\alpha})$ at homology level
  - *[Chazal and Oudot 2008]*
- R^{α} and C^{α} interleave
- Derive homology inference from the interleaving sequence of homomorphisms

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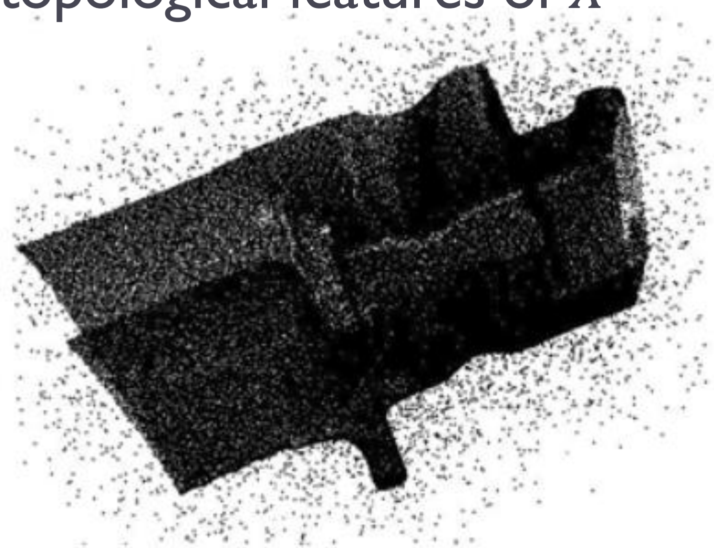
Handling of Noise

Noise

- Previous approach can handle Hausdorff type noise
  - Where noise is within a tubular neighborhood of X
- How about more general noise?
  - E.g, Gaussian noise, background noise
- Some approaches
  - Bound the probability that input samples fall in Hausdorff model
  - Denoise so that fall in Hausdorff model
  - E.g, *[Niyogi, Smale, Weinberger 2006, 2008]*
- Distance to measure framework
  - *[Chazal, Cohen-Steiner, Mérigot, 2011]*

Overview

- Input:
  - A set of points P sampled from a probabilistic measure μ on R^d potentially concentrated on a hidden compact (e.g, manifold) X.
- Goal:
  - Approximate topological features of X



Courtesy of Chazal et al 2011

Main Idea

- The work of *[Chazal, Cohen-Steiner, Mérigot, 2011]*
- A new distance function $d_{\mu,m}$ to a probability μ measure (ie., distance to measures) to replace the role of distance function d_X.
- Show that the two distance fields are close (in L_∞ norm)
- Topological inference follows from some stability results or the interleaving sequences

Definitions

- μ: a probability measure on R^d; $\mu\left(R^d\right) = 1$
- $0 < \ m, m_0 < 1$: *mass* parameters
- $\delta_{\mu,m}$: a pseudo-distance function such that

$$\delta_{\mu,m}(\mathrm{x}) \coloneqq \inf\{\mathrm{r} > 0; \mu\left(\bar{B}(x,r)\right) > m\ \}$$

where $\bar{B}(x,r)$ is the closed Euclidean ball at x

- That is, $\delta_{\mu,m}(\mathrm{x})$ is the radius of the ball necessary in order to enclose mass m
- *Distance to measure* d_{μ,m_0}:

$$d^2{}_{\mu,m_0}(x) = \frac{1}{m_0}\int_0^{m_0} \delta_{\mu,m}(x)^2 \, dm$$

Distance to Measures

- *Distance to measure* d_{μ,m_0}:

$$d^2_{\mu,m_0}(x) = \frac{1}{m_0}\int_0^{m_0} \delta_{\mu,m}(x)^2 \, dm$$

- Intuitive, $d_{\mu,m}(\mathrm{x})$ averages distance within a range and is more robust to noise.
- Examples.
- Wasserstein distance:

Wasserstein Distance

- A *transport plan* between two probability measures μ, ν on R^d is a probability measure π on $R^d \times R^d$ such that for every $A, B \subseteq R^d$, $\pi\left(A \times R^d\right) = \mu(A)$ and $\pi\left(R^d \times B\right) = \mu(B)$.
- The p-cost of a transport plan π is:

$$C_p(\pi) = \left(\int_{R^d \times R^d} \|x - y\|^p d\pi(x, y)\right)^{\frac{1}{p}}$$

- The *Wasserstein distance* of order p between μ, ν on R^d with finite p-moment
  - $W_p(\mu, \nu) =$ the minum p-cost $C_p(\pi)$ of any transport plan π between μ and ν.

Properties

- **Theorem [Distance-Likeness] [Chazal et al 2011]**

  d^2_{μ,m_0} is distance like. That is:

  - The function d_{μ,m_0} is 1-Lipschitz.
  - The function d^2_{μ,m_0} is 1-semiconcave, meaning that the map $x \to d^2_{\mu,m_0}(x) - \|x\|^2$ is concave.

- **Theorem [Stability] [Chazal et al 2011]**

  Let μ, μ' be two probability measures on R^d and $m_0 > 0$. Then

$$\left\| d_{\mu,m_0} - d_{\mu',m_0} \right\|_\infty \leq \frac{1}{\sqrt{m_0}} W_2(\mu, \mu')$$

Distance-Like Function

- There are many interesting consequences of [Theorem Distance-Likeness Theorem].
  - E.g, one can define critical points, weak feature size, etc.

- Theorem [Isotopy Lemma]:

  Let ϕ be a distance-like function and $r_1 < r_2$ be two positive numbers such that ϕ has no critical points in the subset $\phi^{-1}([r_1, r_2])$. Then all the sublevel sets $\phi^{-1}([0, r])$ are isotopic for $r \in [r_1, r_2]$.

Relation to Distance Function

- Now suppose P is sampled from, not compact X, but a probabilistic measure μ on R^d concentrated X.
  - Consider P as a noisy sample of X
- Let ν_X denote the uniform measure on X

- Theorem [Approximation Distance]:

$$\left\| d_X - d_{\mu,m_0} \right\|_\infty \leq C(X)^{-\frac{1}{k}} m_0^{\frac{1}{k}} + \frac{1}{\sqrt{m_0}} W_2(\mu, \nu_X)$$

where X is a k-dimensional smooth manifold and $C(X)$ is a quantity depending on X and k.

Computational Aspect

- Distance to measures can be approximated efficiently for a set of points P
  - *[Guibas, Mérigot and Morozov, DCG 2013]*

- Can be extended to metric spaces, and build weighted Cech / Rips complexes for reconstruction and homology inference
  - *[Buchet, Chazal, Oudot, Sheehy, 2013]*

Outline

- From PCDs to simplicial complexes
  - Delaunay, Cech, Vietoris-Rips, witness complexes
  - Graph induced complex
- Sampling conditions
  - Local feature size, and homological feature size
- Topology inference
  - Homology inference
  - Handling noise
  - *Approximating cycles of shortest basis of the first homology group*
  - *Approximating Reeb graph*

Two Additional Examples

- Previously:
  - Homology inference: topological information of a space
- Approximating cycles of shortest basis of the first homology group
  - As an example of combining geometry and topology
- Approximating Reeb graph
  - As an example of approximating the topology of a scalar field
- See separate slides for these two additional topics.

Summary

- One example of the pipeline of approximating certain topological structure from discrete samples
- The components in the pipeline are quite generic
- Many other issues too:
  - Stability
  - Efficiency
  - Sparsification
  - etc

Summary – cont.

- Starting to have more interaction with statistics and probability theory
  - E.g, *[Balakrishnan et al., AISTATS 2012], [Bendich et al., SoDA 2012]*
- How to develop algorithms that integrating computational geometry / topology ideas with statistics, especially in data analysis?

References

- *Weak witnesses for Delaunay triangulations of submanifolds*. D. Attali, H. Edelsbrunner, and Y. Mileyko. Proc. ACM Sympos. on Solid and Physical Modeling, 143–150 (2007)
- *Manifold reconstruction in arbitrary dimensions using witness complexes*. D. Boissonnat and L. J. Guibas and S.Y. Oudot. SoCG 194–203, (2007)
- *Efficient and Robust Topological Data Analysis on Metric Spaces*. M. Buchet, F. Chazal, S.Y. Oudot, D. Sheehy:. CoRR abs/1306.0039 (2013)
- *A sampling theory for compacts in Euclidean space* . F. Chazal, D. Cohen-Steiner, and A. Lieutier. Discrete Comput. Geom. 41, 461--479 (2009).
- *Geometric Inference for Probability Measures*. F. Chazal, D. Cohen-Steiner, Q. Mérigot. Foundations of Computational Mathematics 11(6): 733-751 (2011)
- *Stability and Computation of Topological Invariants of Solids in R^n*. F. Chazal, A. Lieutier. Discrete & Computational Geometry 37(4): 601-617 (2007)
- *Towards persistence-based reconstruction in Euclidean spaces*. F. Chazal and S. Oudot. Proc. Annu. Sympos. Comput. Geom. (2008), 232--241.

References – cont.

- *Proximity of persistence modules and their diagrams*. F. Chazal, D. Cohen-Steiner, M. Glisse, L. Guibas, and S. Oudot. Proc. 25th Annu. Sympos. Comput. Geom. (2009), 237-246.
- *A weak definition of Delaunay triangulation.* Vin de Silva. CoRR cs.CG/0310031 (2003)
- *Topological estimation using witness complexes.* V. de Silva, G. Carlsson, Symposium on Point-Based Graphics,, 2004
- *Topological estimation using witness complexes.* Vin de Silva and Gunnar Carlsson. Eurographics Sympos. Point-Based Graphics (2004).
- *Curve and Surface Reconstruction :Algorithms with Mathematical Analysis*. Tamal. K. Dey. Cambridge University Press, 2006.
- *Graph induced complex on point data.* T. K. Dey, F. Fan, and Y. Wang. to appear in 29th Annual Sympos. Comput. Geom. (SoCG) 2013.

References – cont.

- *Witnessed k-Distance*. L. J. Guibas, D. Morozov, Q. Mérigot. Discrete & Computational Geometry 49(1): 22-45 (2013)
- *Reconstructing using witness complexes*, L. J. Guibas and S.Y. Oudot. Discrete Comput. Geom. 30, 325–356 (2008).
- *Any open bounded subset of R^n has the same homotopy type as its medial axis*. A. Lieutier. Computer-Aided Design 36(11): 1029-1046 (2004)
- *Finding the homology of submanifolds with high confidence from random samples*. P. Niyogi, S. Smale, and S. Weinberger. Discrete Comput. Geom., 39, 419--441 (2008).

References – cont.

- *Minimax rates for homology inference*. S. Balakrishnan, A. Rinaldo, D. Sheehy, A. Singh, L. A. Wasserman. AISTATS 2012: 64-72
- *Local Homology Transfer and Stratification Learning*. P. Bendich, B. Wang and S. Mukherjee. ACM-SIAM Symposium on Discrete Algorithms, (2012).
- *Greedy optimal homotopy and homology generators*. J. Erickson and K. Whittlesey. Proc. Sixteenth ACM-SIAM Sympos. Discrete Algorithms (2005), 1038--1046.
- *Approximating cycles in a shortest basis of the first homology group from point data*. T. K. Dey, J. Sun, and Y. Wang. Inverse Problems, 27 (2011).
- *Reeb graphs: Approximation and Persistence*. T. K. Dey and Y. Wang., Discrete Comput. Geom (DCG) 2013.