Deterministic Evolutionary Game Dynamics

Josef Hofbauer

1 November 2010

1 Introduction: Evolutionary Games

We consider a large population of players, with a finite set of pure strategies $\{1, \dots, n\}$. x_i denotes the frequency of strategy i. $\Delta_n = \{x \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^n x_i = 1\}$ is the $(n-1)$–dimensional simplex. (The index n will often be omitted).

The payoff to strategy i in a population x is $a_i(x)$, with $a_i : \Delta \to \mathbb{R}$ a continuous function (population game). The most important special case is that of a symmetric two person game with $n \times n$ payoff matrix $A = (a_{ij})$; with random matching this leads to the linear payoff function $a_i(x) = \sum_j a_{ij} x_j = (Ax)_i$.

$\hat{x} \in \Delta$ is a Nash equilibrium iff

$$\hat{x} \cdot a(\hat{x}) \geq x \cdot a(\hat{x}) \quad \forall x \in \Delta. \tag{1}$$

Occasionally I will also look at bimatrix games (played between two player populations), with $n \times m$ payoff matrices A, B, or N person games.

Evolutionarily stable strategies

According to Maynard Smith [31], a mixed strategy $\hat{x} \in \Delta$ is an *evolutionarily stable strategy* (ESS) if

$$\begin{array}{lccll} \text{(i)} & x \cdot A\hat{x} & \leq & \hat{x} \cdot A\hat{x} & \forall x \in \Delta, \quad \text{and} \\ \text{(ii)} & x \cdot Ax & < & \hat{x} \cdot Ax & \text{for } x \neq \hat{x}, \text{ if there is equality in (i).} \end{array}$$

The first condition (i) is simply Nash's definition (1) for an equilibrium. It is easy to see that $\hat{x}$ is an ESS, iff $\hat{x} \cdot Ax > x \cdot Ax$ holds for all $x \neq \hat{x}$ in a neighbourhood of $\hat{x}$. For an interior equilibrium $\hat{x}$, the equilibrium condition $\hat{x} \cdot A\hat{x} = x \cdot A\hat{x}$ for all $x \in \Delta$ together with (ii) implies $(\hat{x} - x) \cdot A(x - \hat{x}) > 0$ for all x and hence

$$z \cdot Az < 0 \qquad \forall z \in \mathbb{R}^n_0 = \{z \in \mathbb{R}^n : \sum_i z_i = 0\} \quad \text{with} \quad z \neq 0. \tag{2}$$

Condition (2) says that the mean payoff $x \cdot Ax$ is a strictly concave function on Δ. Conversely, games satisfying (2) have a unique ESS (possibly on the boundary) which is also the unique Nash equilibrium of the game. The slightly weaker condition

$$z \cdot Az \leq 0 \qquad \forall z \in \mathbb{R}^n_0 \tag{3}$$

includes also the limit case of zero–sum games and games with an interior equilibrium that is a 'neutrally stable' strategy (i.e., equality is allowed in (ii)). Games satisfying (3) need no longer have a unique equilibrium, but the set of equilibria is still a nonempty convex subset of Δ.

For the *rock–scissors–paper* game with (a cyclic symmetric) pay-off matrix

$$A = \begin{pmatrix} 0 & -b & a \\ a & 0 & -b \\ -b & a & 0 \end{pmatrix} \quad \text{with } a, b > 0 \tag{4}$$

with the unique NE $E = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ we obtain the following: with $z \in \mathbb{R}^3_0$, $z_1 + z_2 + z_3 = 0$,

$$z \cdot Az = (a - b)(z_1 z_2 + z_2 z_3 + z_1 z_3) = \frac{b-a}{2}[z_1^2 + z_2^2 + z_3^2].$$

Hence for $0 < b < a$, the game is negative definite, and E is an ESS. On the other hand, if $0 < a < b$, the game is positive definite:

$$z \cdot Az > 0 \quad \forall z \in \mathbb{R}^n_0 \setminus \{0\} \tag{5}$$

the equilibrium E is not evolutionarily stable, indeed the opposite, and might be called an 'anti–ESS'.

2 Game Dynamics

In this section I present 6 special (families of) game dynamics. As we will see they enjoy a particularly nice property: Interior ESS are globally asymptotically stable. Phase portraits of these dynamics for the RSP game (4) are shown in the lecture notes of Sandholm. The presentation follows [18, 20, 24].

1. Replicator dynamics
2. Best response dynamics
3. Logit dynamics
4. Brown–von Neumann–Nash dynamics
5. Payoff comparison dynamics
6. payoff projection dynamics

Replicator dynamics

$$\dot{x}_i = x_i \left(a_i(x) - x \cdot a(x) \right), \quad i = 1, \ldots, n \qquad \text{(REP)} \tag{6}$$

In the zero-sum version $a = b$ of the RSP game, all interior orbits are closed, circling around the interior equilibrium E, with $x_1 x_2 x_3$ as a constant of motion.

Theorem 1 *In a negative definite game satisfying* (2)*, the unique Nash equilibrium* $p \in \Delta$ *is globally asymptotically stable for (REP). In particular, an interior ESS is globally asymptotically stable.*
On the other hand, in a positive definite game satisfying (5) *with an interior equilibrium* p *(anti-ESS):* p *is a global repellor. All orbits except* p *go to the boundary* $bd\Delta$*.*

The proof uses $V(x) = \prod x_i^{p_i}$ as a Liapunov function.
For this and further results on (REP) see Sigmund's notes.

Best response dynamics

In the best response dynamics[1] [10, 30, 15] one assumes that in a large population, a small fraction of the players revise their strategy, choosing best replies[2] $BR(x)$ to the current population distribution x.

$$\dot{x} \in BR(x) - x. \tag{7}$$

Since best replies are in general not unique, this is a differential *inclusion* rather than a differential equation. For continuous payoff functions $a_i(x)$ the right hand side is a non-empty convex, compact subset of Δ which is upper semi-continuous in x. Hence solutions exist, and they are Lipschitz functions $x(t)$ satisfying (7) for almost all $t \geq 0$, see [1].

For games with linear payoff, solutions can be explicitely constructed as piecewise linear functions, see [15, 23], or Sigmund's talk.

For interior NE of linear games we have the following stability result [15].

Let $\mathcal{B} = \{b \in \mathrm{bd}\Delta_n : (Ab)_i = (Ab)_j \text{ for all } i, j \in \mathrm{supp}(b)\}$ denote the set of all rest points of (REP) on the boundary. Then the function

$$w(x) = \max\left\{ \sum_{b\in\mathcal{B}} b^T Ab\, u(b) : u(b) \geq 0, \sum_{b\in\mathcal{B}} u(b) = 1, \sum_{b\in\mathcal{B}} u(b) b = x \right\} \tag{8}$$

can be interpreted in the following way. Imagine the population in state x being decomposed into subpopulations of size $u(b)$ which are in states $b \in \mathcal{B}$, and call this a *$\mathcal{B}$–segregation of b*. Then $w(x)$ is the maximum mean payoff the population x can obtain by such a $\mathcal{B}$–segregation. It is the smallest concave function satisfying $w(b) \geq b^T Ab$ for all $b \in \mathcal{B}$.

Theorem 2 *The following three conditions are equivalent:*
(a) There is a vector $p \in \Delta_n$, such that $p^T Ab > b^T Ab$ holds for all $b \in \mathcal{B}$.
(b) $V(x) = \max_i (Ax)_i - w(x) > 0$ for all $x \in \Delta_n$.
(c) There exist a unique interior equilibrium $\hat{x}$, and $\hat{x}^T A\hat{x} > w(\hat{x})$.
These conditions imply:
The equilibrium $\hat{x}$ is reached in finite and bounded time by any BR path.

The proof consists in showing that the function V from (b) decreases along the solutions of the BR dynamics (7).

In the rock–scissors–paper game [9], the set $\mathcal{B}$ reduces to the set of pure strategies, and the Lyapunov function is simply $V(x) = \max_i (Ax)_i$.

If $p \in \mathrm{int}\,\Delta$ is an interior ESS then condition (a) holds not only for all $b \in \mathcal{B}$ but for all $b \neq p$. In this case the simpler Lyapunov function $V(x) = \max_i (Ax)_i - x^T Ax \geq 0$ can also be used. This leads to

Theorem 3 [18] *For a negative semidefinite game* (3) *the convex set of its equilibria is globally asymptotically stable for the best–response dynamics.*[3]

---

[1] For bimatrix games, this dynamics is closely related to the 'fictitious play' by Brown [4, 5], see Sorin's notes.

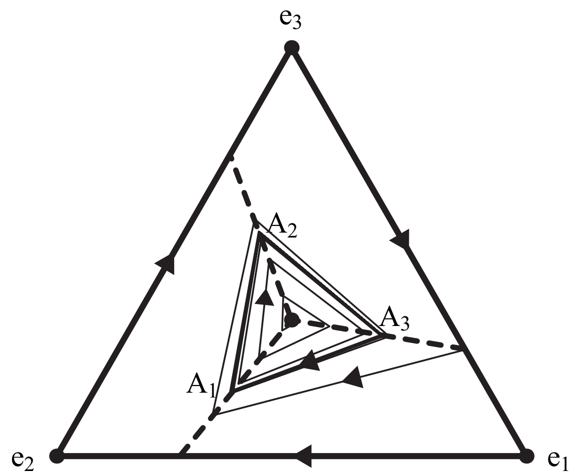
[2] Recall the set of best replies $\mathrm{BR}(x) = \mathrm{Argmax}_{y\in\Delta}\; y\cdot a(x) = \{y \in \Delta : y\cdot a(x) \geq z\cdot a(x) \forall z \in \Delta\} \subseteq \Delta$.

[3] Using tools from Sorin's notes this implies also global convergence of (discrete time) fictitious play.

Proof: The Liapunov function $V(x) = \max_i (Ax)_i - x^T Ax \geq 0$ satisfies $\dot{V} = \dot{x}{\cdot}A\dot{x} - \dot{x}{\cdot}Ax < 0$ along piecewise linear solutions.

A similar result holds for nonlinear payoff functions, see [20].

For positive definite RSP games ($b > a$) the NE is a repeller and orbits converge to a limit cycle (called the Shapley triangle of the game in honor of [41]), see the figure.



Smoothed best replies

The BR dynamics can be approximated by smooth dynamics such as the *logit dynamics* (first considered in [3, 8, 27])

$$\dot{x} = L(\frac{Ax}{\varepsilon}) - x \tag{9}$$

with

$$L : \mathbb{R}^n \to \Delta, \quad L_k(u) = \frac{e^{u_k}}{\sum_j e^{u_j}}.$$

with $\varepsilon > 0$. As $\varepsilon \to 0$, this approaches the best reply dynamics, and every family of rest points[4] $\hat{x}_\varepsilon$ accumulates in the set of Nash equilibria.
There are (at least) two ways to motivate and generalize this 'smoothing'.

Whereas $BR(x)$ is the set of maximizers of the linear function $z \mapsto \sum_i z_i a_i(x)$ on Δ, consider $b_{\varepsilon v}(x)$, the unique maximizer of the function $z \mapsto \sum_i z_i a_i(x) + \varepsilon v(z)$ on $\operatorname{int}\Delta$, where $v : \operatorname{int}\Delta \to \mathbb{R}$ is a strictly concave function such that $|v'(z)| \to \infty$ as z approaches the boundary of Δ. If v is the entropy $-\sum z_i \log z_i$, the corresponding smoothed best reply dynamics

$$\dot{x} = b_{\varepsilon v}(x) - x \tag{10}$$

reduces to the logit dynamics (9) above [8]. Another choice[5] is $v(x) = \sum_i \log x_i$ used by Harsányi [13] in his logarithmic games.

[4]These are the quantal response equilibria of McKelvey and Palfrey [32].

[5]A completely different approximate best reply function appears already in Nash's Ph.D. thesis [35], in his first attempt to prove existence of equilibria by Brouwer's fixed point theorem.

Another way to perturb best replies are stochastic perturbations. Let ε be a random vector in $\mathbb{R}^n$ distributed according to some positive density function. For $z \in \mathbb{R}^n$, let

$$C_i(z) = \text{Prob}(z_i + \varepsilon_i \geq z_j + \varepsilon_j \quad \forall j), \tag{11}$$

and $b(x) = C(a(x))$. It can be shown [19] that each such stochastic perturbation can be represented by a deterministic perturbation as described before. The main idea is that there is a potential function $W : \mathbb{R}^n \to \mathbb{R}$, with $\frac{\partial W}{\partial a_i} = C_i(a)$ which is convex, and has $-v$ as its Legendre transform. If the (ε_i) are i.i.d. with the extreme value distribution $F(x) = \exp(-\exp(-x/\varepsilon))$ then $C(a) = L(a)$ is the logit choice function and we obtain (9).

Theorem 4 [18] *In a negative semidefinite game* (3)*, the smoothed BR dynamics* (10) *(including the logit dynamics) has a unique equilibrium $\hat{x}_\varepsilon$. It is globally asymptotically stable.*

The proof uses the Liapunov function

$$V(x) = \pi_{\varepsilon v}(b_{\varepsilon v}(x), x) - \pi_{\varepsilon v}(x, x) \geq 0 \quad \text{with} \quad \pi_{\varepsilon v}(z, x) = z \cdot a(x) + \varepsilon v(z).$$

The Brown–von Neumann–Nash dynamics

The *Brown–von Neumann–Nash dynamics* (BNN) is defined as

$$\dot{x}_i = \hat{a}_i(x) - x_i \sum_{j=1}^{n} \hat{a}_j(x), \tag{12}$$

where

$$\hat{a}_i(x) = [a_i(x) - x^T a(x)]_+ \tag{13}$$

(with $u_+ = \max(u, 0)$) denotes the positive part of the excess payoff for strategy i. This dynamics is closely related to the continuous map $f : \Delta \to \Delta$ defined by

$$f_i(x) = \frac{x_i + h\hat{a}_i(x)}{1 + \sum_{j=1}^{n} h\hat{a}_j(x)} \tag{14}$$

which Nash [36] used (for $h = 1$) to prove the existence of equilibria, by applying Brouwer's fixed point theorem: It is easy to see that $\hat{x}$ is a fixed point of f iff it is a rest point of (12) iff $\hat{a}_i(\hat{x}) = 0$ for all i, i.e. iff $\hat{x}$ is a Nash equilibrium of the game.

Rewriting the Nash map (14) as a difference equation, and taking the limit $\lim_{h \to 0} \frac{f(x) - x}{h}$ returns (12). This differential equation was considered earlier by Brown and von Neumann [6] in the special case of zero–sum games, for which they proved global convergence to the set of equilibria.

In contrast to the best reply dynamics (12) is Lipschitz (if payoffs are Lipschitz) and hence has unique solutions.

Equation (12) defines an 'innovative better reply' dynamics. A strategy not present that is best (or at least better) reply against the current population will enter the population.

Theorem 5 [6, 37, 18] *For a negative semidefinite game* (3)*, the convex set of its equilibria is globally asymptotically stable for the BNN dynamics* (12)*.*

The proof uses the Liapunov function $V = \frac{1}{2} \sum_i \hat{a}_i(x)^2$, since $V(x) \geq 0$ with equality at NE, and

$$\dot{V} = \dot{x} \cdot A\dot{x} - \dot{x} \cdot Ax \sum_i \hat{a}_i(x) < 0.$$

Dynamics based on pairwise comparison

The BNN dynamics is a prototype of an innovative dynamics. A more natural way to derive innovative dynamics is the following,

$$\dot{x}_i = \sum_j x_j \rho_{ji} - x_i \sum_j \rho_{ij} \tag{15}$$

in the form of an input–output dynamics. Here $x_i \rho_{ij}$ is the flux from strategy i to strategy j, and $\rho_{ij} = \rho_{ij}(x) \geq 0$ is the rate at which an i player switches to the j strategy. The matrix ρ is called the *revision protocol*, see Sandholm's notes.

A natural assumption on the revision protocol ρ is

$$\rho_{ij} > 0 \Leftrightarrow a_j > a_i, \quad \text{and} \quad \rho_{ij} \geq 0.$$

Here switching to *any* better reply is possible, as opposed to the BR dynamics where switching is only to the optimal strategies (usually there is only one of them), or the BNN dynamics where switching occurs only to strategies better than the population average.

An important special case is when the switching rate depends on the payoff difference only, i.e.,

$$\rho_{ij} = \phi(a_j - a_i) \tag{16}$$

where ϕ is a function with $\phi(u) > 0$ for $u > 0$ and $\phi(u) = 0$ for $u \leq 0$. The resulting dynamics (15) is called *pairwise comparison dynamics*. The canonical choice seems $\phi(u) = u_+$, given by the proportional rule

$$\rho_{ij} = [a_j - a_i]_+. \tag{17}$$

The resulting *pairwise difference dynamics* (PD)

$$\dot{x}_i = \sum_j x_j [a_i - a_j]_+ - x_i \sum_j [a_j - a_i]_+ \tag{18}$$

was introduced by Michael J. Smith [43] in the transportation literature as a dynamic model for congestion games. He also proved the following global stability result.

Theorem 6 [43] *For a negative semidefinite game* (3)*, the convex set of its equilibria is globally asymptotically stable for the PD dynamics* (18).

The proof uses the Lyapunov function $V(x) = \sum_{i,j} x_j [a_i(x) - a_j(x)]_+^2$, by showing $V(x) \geq 0$ and $V(x) = 0$ iff x is a NE, and

$$2\dot{V} = \dot{x} \cdot A\dot{x} + \sum_{k,j} x_k \rho_{kj} \sum_i (\rho_{ji}^2 - \rho_{ki}^2) < 0.$$

This result extends to pairwise comparison dynamics (15,16), see [20].

The payoff projection dynamics

A more recent proof of existence of Nash equilibria, due to *Gül–Pearce–Stacchetti* [11] uses the payoff projection map

$$P_h x = \Pi_\Delta(x + ha(x))$$

Here $h > 0$ is fixed and $\Pi_\Delta : \mathbb{R}^n \to \Delta$ is the projection onto the simplex Δ, assigning to each vector $u \in \mathbb{R}^n$ the point in the compact convex set Δ which is closest to u. Now $\Pi_\Delta(z) = y$ iff for all $x \in \Delta$, the angle between $x - y$ and $z - y$ is obtuse, i.e., iff $(x - y)\cdot(z - y) \le 0$ for all $x \in \Delta$. Hence, $P_h\hat{x} = \hat{x}$ iff for all $x \in \Delta$, $(x - \hat{x})\cdot a(\hat{x}) \le 0$, i.e., iff $\hat{x}$ is a Nash equilibrium. Since the map $P_h : \Delta \to \Delta$ is continuous (it is even Lipschitz) Brouwer's fixed point theorem implies the existence of a Nash equilibrium.

Writing this map as a difference equation, we obtain in the limit $h \to 0$

$$\dot{x} = \lim_{h\to 0} \frac{\Pi_\Delta(x + ha(x)) - x}{h} = \Pi_{T(x)} a(x) \tag{19}$$

with

$$T(x) = \{\xi \in \mathbb{R}^n : \sum_i \xi_i = 0, \xi_i \ge 0 \text{ if } x_i = 0\}$$

being the cone of feasible directions at x into Δ.

This is the *payoff projection dynamics* of Lahkar and Sandholm [29]. The latter equality in (19) and its dynamic analysis use some amount of convex analysis, in particular the Moreau decomposition, see [1, 29].

For $x \in \operatorname{int}\Delta$ we obtain

$$\dot{x}_i = a_i(x) - \frac{1}{n}\sum_k a_k(x)$$

which, for a linear game, is simply a linear dynamics, which appears in many places as a suggestion for a simple game dynamics, but how to treat this on the boundary has been rarely dealt with. Indeed, the vector field (19) is discontinuous on $\operatorname{bd}\Delta$. However, essentially because P_h is Lipschitz, solutions exist for all $t \ge 0$ and are unique (at least in forward time). This can be shown by rewriting (19) as a viability problem in terms of the normal cone ([1, 29])

$$\dot{x} \in a(x) - N_\Delta(x), \quad x(t) \in \Delta.$$

Theorem 7 [29] *In a negative definite game* (2)*, the unique NE is globally asymptotically stable for the payoff projection dynamics* (19)*.*

The proof uses as Liapunov function the distance to the equilibrium $V(x) = \sum_i (x_i - \hat{x}_i)^2$.

Summary

As we have seen many of the special dynamics are related to maps that have been used to prove existence of Nash equilibria. The best response dynamics, the perturbed best response dynamics, and the BNN dynamics correspond to the three proofs given by Nash himself: [34, 35, 36]. The payoff projection dynamics is related to [11]. Even the replicator dynamics can be used to provide such a proof, if only after adding a mutation term, see [22, 23], or Sigmund's notes:

$$\dot{x}_i = x_i\,(a_i(x) - x{\cdot}a(x)) + \varepsilon_i - x_i \sum_j \varepsilon_j, \quad i = 1, \dots, n \tag{20}$$

with $\varepsilon_i > 0$ describing mutation rates.

Moreover, we obtain a result similar to (10).

Theorem 8 *For a negative semidefinite game* (3)*, and any $\varepsilon_i > 0$,* (20) *has a unique rest point $\hat{x}(\varepsilon) \in \Delta$. It is globally asymptotically stable for* (20)*, and for $\varepsilon \to 0$ it approaches the set of NE of the game.*

For the proof, use first Brouwer's fixed point theorem to find an $\hat{x}(\varepsilon)$. Then use $\sum_i \hat{x}_i(\varepsilon) \log x_i$ as Liapunov function for global stability and hence uniqueness of $\hat{x}(\varepsilon)$.
Thus also the replicator dynamics is part of a 'canonical' family.

The six special dynamics described above enjoy the following common property:

1. The unique NE of a negative definite game (e.g., any interior ESS) is globally asymptotically stable.

2. Interior NE of a positive definite game ('anti-ESS') are repellors.

Two of these six dynamics stand out: *only (REP) and (BR) eliminate strictly dominated strategies.* For the other dynamics there are games with a strictly dominated strategy that survives in the long run, see [2, 29, 21].

Because of the nice behaviour of negative (semi-)definite games with respect to these 'canonical' dynamics, Sandholm suggested to christen them **stable games**.

For nonlinear games in a single population these are games whose payoff function $a : \Delta \to \mathbb{R}^n$ satisfies

$$(a(x) - a(y))(x - y) \le 0 \qquad \forall x, y \in \Delta \tag{21}$$

or equivalently, if a is smooth,

$$z \cdot a'(x) z \le 0 \qquad \forall x \in \Delta, z \in \mathbb{R}^n_0$$

Examples are congestion games, the war of attrition, the sex–ratio game, or the habitat selection game.

For the general case of N populations see [20, 39].

3 Bimatrix games

The replicator dynamics for an $n \times m$ bimatrix game (A, B) reads

$$\dot{x}_i = x_i\big((Ay)_i - x{\cdot}Ay\big), \quad i = 1, \dots n$$

$$\dot{y}_j = y_j\big((B^T x)_j - x{\cdot}By\big) \quad j = 1, \dots m$$

For its properties see [22, 23] and especially [17]. N person games are treated in [46].

The best reply dynamics for bimatrix games reads

$$\dot{x} \in BR^1(y) - x \qquad \dot{y} \in BR^2(x) - y \tag{22}$$

See Sorin's notes for more information.

For bimatrix games, stable games include zero-sum games, but not much more. We call an $n \times m$ bimatrix game (A, B) a *rescaled zero-sum game* [22, 23] or *linearly equivalent to a zero sum game* if

$$\exists c > 0 : \quad u \cdot Av = -cu \cdot Bv \qquad \forall u \in R_0^n, v \in \mathbb{R}_0^m \tag{23}$$

or equivalently, there exists an $n \times m$ matrix C, $\alpha_i, \beta_j \in \mathbb{R}$ and $\gamma > 0$ s.t.

$$a_{ij} = c_{ij} + \alpha_j, b_{ij} = -\gamma c_{ij} + \beta_i, \quad \forall i = 1, \dots n, j = 1, \dots, m$$

For 2×2 games, this defines an open set of payoff matrices, corresponding to games with a cyclic best reply structure (as in the 'matching pennies' game), or equivalently, those with a unique, interior Nash equilibrium. For larger n, m this is a thin set of games, e.g. for 3×3 games, this set has codimension 3.

For such rescaled zero-sum games, the set of Nash equilibria is stable for (REP), (BR) and the other 'canonical' dynamics.

One of the main open problems in evolutionary game dynamics concerns the converse.

Conjecture 9 *Let (p, q) be an isolated interior equilibrium of a bimatrix game (A, B), which is stable for the BR dynamics or for the replicator dynamics. Then $n = m$ and (A, B) is a rescaled zero sum game.*

4 Supermodular Games and Monotone Flows

An interesting class of games are the *supermodular games* (also known as games with strict strategic complementarities) which are defined by

$$(a_{i+1,j+1} - a_{i+1,j} - a_{i,j+1} + a_{i,j})(x) > 0 \qquad \forall x \in \Delta, \forall i, j \tag{24}$$

where $a_{i,j} = \frac{\partial a_i}{\partial x_j}$ for short. This means that for any $i < n$, $a_{i+1,k} - a_{ik}$ increases strictly with k. *Stochastic dominance* defines a partial order on the simplex Δ_n:

$$p \succeq p' \quad \Leftrightarrow \quad \sum_{k=1}^{m} p_k \leq \sum_{k=1}^{m} p'_k \quad \forall\, m = 1, \dots, n-1. \tag{25}$$

If all inequalities in (25) are strict, we write $p \succ p'$. The pure strategies are totally ordered: $\mathbf{e}_1 \prec \mathbf{e}_2 \cdots \prec \mathbf{e}_n$.

The crucial property of supermodular games is the monotonicity of the best reply correspondence: If $x \preceq y$, $x \neq y$ then $\max BR(x) \leq \min BR(y)$. This property was used in [28] to prove convergence of fictitious play, and in [19] the result was extended to perturbed best response maps.

Theorem 10 *For every supermodular game*

$$x \preceq y, x \neq y \quad \Rightarrow \quad C(\mathbf{a}(x)) \prec C(\mathbf{a}(y))$$

holds if the choice function $C : \mathbb{R}^n \to \Delta_n$ *is* C^1 *and the partial derivatives* $C_{i,j} = \frac{\partial C_i}{\partial x_j}$ *satisfy for all* $1 \leq k, l < n$

$$\sum_{i=1}^{k} \sum_{j=1}^{l} C_{i,j} > 0, \tag{26}$$

and for all $1 \leq i \leq n$,

$$\sum_{j=1}^{n} C_{i,j} = 0. \tag{27}$$

The conditions (26, 27) on C hold for every stochastic choice model (11), since there $C_{i,j} < 0$ for $i \neq j$. As a consequence, the perturbed best reply dynamics

$$\dot{x} = C(a(x)) - x \tag{28}$$

generates a strongly monotone flow: If $x(0) \preceq y(0), x(0) \neq y(0)$ then $x(t) \prec y(t)$ for all $t > 0$. The theory of monotone flows developped by Hirsch and others (see [42]) implies that almost all solutions of (28) converge to a rest point of (28).

5 Partnership games and general adjustment dynamics

We consider now games with a symmetric payoff matrix $A = A^T$ ($a_{ij} = a_{ji}$ for all i, j). Such games are known as *partnership games* [22, 23] and *potential games* [33]. The basic population genetic model of Fisher and Haldane is equivalent to the replicator dynamics for such games, which is then a gradient system with respect to the Shahshahani metric and the mean payoff $x \cdot Ax$ as potential, see e.g. [22, 23]. The resulting increase of mean fitness or mean payoff $x \cdot Ax$ in time is often referred to as the *Fundamental Theorem of Natural Selection*. This statement about the replicator dynamics generalizes to the other dynamics considered here.

The generalization is based on the concept, defined by Swinkels [45], of a (myopic) adjustment dynamics which satisfies $\dot{x} \cdot Ax \geq 0$ for all $x \in \Delta$, with equality only at equilibria. If $A = A^T$ then the mean payoff $x \cdot Ax$ is increasing for every adjustment dynamics since $(x \cdot Ax)^{\cdot} = 2\dot{x} \cdot Ax \geq 0$. It is obvious that the best response dynamics (7) is an adjustment dynamics and it is easy to see that the other special dynamics from section 2 are as well.

As a consequence, we obtain the following result.

Theorem 11 [16, 18] *For every partnership game* $A = A^T$, *the potential function* $x \cdot Ax$ *increases along trajectories. Hence every trajectory of every adjustment dynamics (in particular* (6), (7), (12), *etc.) converges to (a connected set of) equilibria.*

For the perturbed dynamics (10) and (20) there are similar results: A suitably perturbed potential function is a Liapunov function.

Theorem 12 [18, 22] *For every partnership game $A = A^T$:*
the function $P(x) = \frac{1}{2}x \cdot Ax + \varepsilon v(x)$ increases monotonically along solutions of (10),
the function $P(x) = \frac{1}{2}x \cdot Ax + \sum_i \varepsilon_i \log x_i$ is a Liapunov function for (20).
Hence every solution converges to a connected set of rest points.

For some dynamics the common payoff function is indeed a potential function in the sense that the dynamics is its gradient vector field with respect to a suitably chosen Riemannian metric on Δ.

For bimatrix games the adjustment property is defined as

$$\dot{x} \cdot Ay \geq 0, \quad x \cdot B\dot{y} \geq 0.$$

A bimatrix game is a potential game if $A = B$, i.e. if both players obtain the same payoff. Then the potential $x(t) \cdot Ay(t)$ increases monotonically along every solution of every adjustment dynamics.

For the general situation of potential games between N populations with nonlinear payoff functions see [39].

6 A universal Shapley example

The simplest example of persistent cycling in a game dynamics is probably the RSP game (4) with $b > a$ for the BR dynamics (7) which leads to a triangular shaped limit cycle, see the figure on p. 4. Historically, Shapley [41] gave the first such example in the context of 3×3 bimatrix games (but less easy to visualize because of the 4d state space). Our six 'canonical' dynamics show a similar cycling behavior for positive definite RSP games.

But given the huge pool of adjustment dynamics, we now ask[6]: Is there an evolutionary dynamics, which converges for each game from each initial condition to an equilibrium?

Such a dynamics is assumed to be given by a differential equation

$$\dot{x} = f(x, a(x)) \tag{29}$$

such that f depends continuously on the strategy profile x and the payoff function a.

For N population binary games (each player chooses between two strategies only) general evolutionary dynamics are easy to describe:
The better of the two strategies increases, the other one decreases, i.e.,

$$\dot{x}_{i1} = -\dot{x}_{i2} > 0 \Leftrightarrow a^i(1, x^{-i}) > a^i(2, x^{-i}) \tag{30}$$

holds for all i at all (interior) states. Here x_{ij} denotes the frequency of strategy j used by player i, and $a^i(j, x^{-i})$ his payoff. In a common interest game where each player has the same payoff function $P(x)$, along solutions $x(t)$, $P(x(t))$ increases monotonically:

$$\dot{P} = \sum_{i=1}^{N} \sum_{k=1}^{2} a^i(k, x^{-i})\dot{x}_{ik} = \sum_{i=1}^{N} \left[a^i(1, x^{-i}) - a^i(2, x^{-i})\right] \dot{x}_{i1} \geq 0. \tag{31}$$

[6]This section follows [25].

A familiy of $2 \times 2 \times 2$ games

Following [25], we consider 3 players, each with 2 pure strategies. The payoffs are summarized in the usual way as follows.

-1, -1, -1	0, 0, ε
0, ε, 0	ε, 0, 0

ε, 0, 0	0, ε, 0
0, 0, ε	-1, -1, -1

The first player (upper left payoff) chooses the row, the second chooses the column, the third (lower right payoff) chooses one of the matrices. For $\varepsilon \neq 0$, this game has a unique equilibrium $E = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ at the centroid of the state space, the cube $[0,1]^3$. This equilibrium is regular for all ε. For $\varepsilon > 0$, this game has a best response cycle among the pure strategy profiles $122 \to 121 \to 221 \to 211 \to 212 \to 112 \to 122$.

For $\varepsilon = 0$, this game is a potential game: Every player gets the same payoff $P = -x_{11}x_{21}x_{31} - x_{21}x_{22}x_{32}$.

The minimum value of P is -1 which is attained at the two pure profiles 111 and 222. At the interior equilibrium E, its value is $P(E) = -\frac{1}{4}$. P attains its maximum value 0 at the set Γ of all profiles, where two players use opposite pure strategies, whereas the remaining player may use any mixture. All points in Γ are Nash equilibria. Small perturbations in the payoffs ($\varepsilon \neq 0$) can destroy this component of equilibria.

For every natural dynamics, $P(x(t))$ increases. If $P(x(0)) > P(E) = -\frac{1}{4}$ then $P(x(t)) \to 0$ and $x(t) \to \Gamma$. Hence Γ is an attractor (an asymptotically stable invariant set) for the dynamics, for $\varepsilon = 0$.

For small $\varepsilon > 0$, there is an attractor Γ_ε near Γ whose basin contains the set $\{x : P(x) > -\frac{1}{4} + \gamma(\varepsilon)\}$, with $\gamma(\varepsilon) \to 0$, as $\varepsilon \to 0$.

This follows from the fact that attractors are upper–semicontinuous against small perturbations of the dynamics.

But for $\varepsilon > 0$, the only equilibrium is E.

For each dynamics satisfying the assumptions (30) and continuity in payoffs, there is an open set of games and an open set of initial conditions $x(0)$ such that $x(t)$ stays away from the set of NE.

Similar examples can be given as 4×4 symmetric one population games, see [23], and 3×3 bimatrix games, see [25]. The proofs follow the same lines: For $\varepsilon = 0$ these are potential games, the potential maximizer is a quadrangle or a hexagon, and this component of NE disappears for $\varepsilon \neq 0$.

A different general nonconvergence result is due Hart and Mas-Colell [14].

For specific dynamics there are many examples with cycling and even chaotic behavior: Starting with Shapley [41] there are [7, 9, 12, 38, 44] for the best response dynamics. For other examples and a more complete list of references see [39].

References

[1] Aubin, J.P. and A. Cellina: *Differential Inclusions.* Springer, Berlin (1984).

[2] Berger, U. and Hofbauer, J.: Irrational behavior in the Brown–von Neumann–Nash dynamics. *Games and Economic Behavior* **56** (July 2006), 1–6.

[3] Blume, L.E.: *The statistical mechanics of strategic interaction,* Games Econ. Behav. **5** (1993), 387-424.

[4] Brown, G. W. (1949): Some notes on computation of games solutions. RAND Report P-78.

[5] Brown, G. W. (1951): Iterative solution of games by fictitious play. In: *Activity Analysis of Production and Allocation*, pp. 374–376. Wiley. New York.

[6] Brown, G. W., von Neumann, J. (1950): Solutions of games by differential equations. *Ann. Math. Studies* **24**, 73–79.

[7] Cowan, S.: *Dynamical systems arising from game theory*, Dissertation (1992), Univ. California, Berkeley.

[8] Fudenberg, D., Levine, D. K. (1998): *The Theory of Learning in Games.* MIT Press.

[9] Gaunersdorfer, A. and J. Hofbauer: *Fictitious play, Shapley polygons and the replicator equation*, Games Econ. Behav. **11** (1995), 279-303.

[10] Gilboa, I., Matsui, A. (1991): Social stability and equilibrium. *Econometrica* **59**, 859–867.

[11] Gül, F., D. Pearce and E. Stacchetti: A bound on the proportion of pure strategy equilibria equilibria in generic games. Math. Operations Research **18** (1993) 548–552.

[12] Hahn, M.: Shapley polygons in 4×4 games. *Games* **1** (2010) 189–220.

[13] Harsanyi, J. C.: Oddness of the number of equilibrium points: a new proof. *Int. J. Game Theory* **2** (1973) 235–250.

[14] Hart S. and A. Mas-Colell: Uncoupled dynamics do not lead to Nash equilibrium. *American Economic Review* **93** (2003) 1830–1836.

[15] Hofbauer, J.: Stability for the best response dynamics. Preprint (1995).

[16] Hofbauer, J.: Imitation dynamics for games. Preprint (1995).

[17] Hofbauer, J.: Evolutionary dynamics for bimatrix games: A Hamiltonian system? J. Math. Biol. **34** (1996) 675–688.

[18] Hofbauer, J.: From Nash and Brown to Maynard Smith: Equilibria, dynamics and ESS. *Selection* **1** (2000), 81-88.

[19] Hofbauer, J. and W. H. Sandholm: On the global convergence of stochastic fictitious play. Econometrica **70** (2002), 2265–2294.

[20] Hofbauer, J. and W. H. Sandholm: Stable games and their dynamics. *J. Economic Theory* **144** (2009) 1665–1693.

[21] Hofbauer, J. and W. H. Sandholm: Survival of Dominated Strategies under Evolutionary Dynamics. *Theor. Economics* (2010).

[22] Hofbauer, J. and K. Sigmund: *The Theory of Evolution and Dynamical Systems*. Cambridge UP (1988).

[23] Hofbauer, J. and K. Sigmund: *Evolutionary Games and Population Dynamics*. Cambridge University Press (1998).

[24] Hofbauer, J. and K. Sigmund: Evolutionary Game Dynamics. *Bull. Amer. Math. Soc.* **40** (2003) 479–519.

[25] Hofbauer, J. and J. Swinkels: *A universal Shapley example*. Preprint. 1995

[26] Hofbauer, J. and J. W. Weibull: *Evolutionary selection against dominated strategies*. J. Econ. Theory, **71** (1996), 558–573

[27] Hopkins, E. (1999): A note on best response dynamics. *Games Econ. Behav.* **29**, 138–150.

[28] Krishna, V.: *Learning in games with strategic complementarities*, Preprint 1992.

[29] Lahkar, R. and W .H. Sandholm: The projection dynamic and the geometry of population games. Games Econ. Behav. **64** (2008), 565–590.

[30] Matsui, A.: Best response dynamics and socially stable strategies. *J. Econ. Theory* **57** (1992), 343-362.

[31] Maynard Smith, J. (1982): *Evolution and the Theory of Games*. Cambridge University Press.

[32] McKelvey, R. D. and T. D. Palfrey: Quantal response equilibria for normal form games. *Games Econ. Behav.* **10** (1995), 6–38.

[33] Monderer, D. and L. Shapley: *Potential games*, Games Econ. Behav. **14** (1996), 124-143

[34] Nash, J. (1950a): Equilibrium points in N–person games. *Proc. Natl. Ac. Sci.* **36**, 48–49.

[35] Nash, J. (1950b): *Non-cooperative games*. Dissertation, Princeton University, Dept. Mathematics.

[36] Nash, J. (1951): Non-cooperative games. *Ann. Math.* **54**, 287–295.

[37] Nikaido, H. (1959): Stability of equilibrium by the Brown–von Neumann differential equation. *Econometrica* **27**, 654–671.

[38] Rosenmüller, J.: Über Periodizitätseigenschaften spieltheoretischer Lernprozesse. *Z. Wahrscheinlichkeitstheorie Verw. Geb.* **17** (1971) 259–308.

[39] Sandholm, W. H.: *Population Games and Evolutionary Dynamics.* MIT Press, Cambridge. 2010.

[40] Schlag, K.H.: *Why imitate, and if so, how? A boundedly rational approach to multi-armed bandits,* J. Econ. Theory **78** (1997) 130–156.

[41] Shapley, L.: *Some topics in two-person games*, Ann. Math. Studies **5** (1964), 1–28.

[42] Smith, H.: *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems.* Amer. Math. Soc. Math. Surveys and Monographs, Vol.41 (1995).

[43] Smith, M. J.: The stability of a dynamic model of traffic assignment—an application of a method of Lyapunov. Transportation Science **18** (1984) 245–252.

[44] Sparrow, C., S. van Strien and C. Harris: Fictitious play in 3×3 games: the transition between periodic and chaotic bahavior. *Games and Economic Behavior* **63** (2008) 259–291.

[45] Swinkels, J. M. (1993): Adjustment dynamics and rational play in games. *Games Econom. Behav.* **5**, 455–484.

[46] Weibull, J. W. (1995): *Evolutionary Game Theory.* MIT Press.