DYNAMICS AND CHAOS FOR MAPS AND THE CONLEY INDEX

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Discrete-time dynamical systems modeled by iteration of continuous maps exhibit a wide variety of interesting behaviors. One illustrative example is the one-dimensional logistic model. For the logistic model, chaotic dynamics may be proven via a topological conjugacy onto an appropriate subshift of finite type, a symbolic system that acts as a catalogue of dynamics and for which a proof of chaos is attainable. Analysis and proofs of dynamics for other models, especially in dimensions larger than one, often prove to be more challenging. In this course, we examine methods for constructing *outer approximations*, finite representations of discrete-time models that are amenable to computational studies and computer-assisted proofs. These methods rely heavily on *Conley index theory*, an algebraic topological generalization of Morse Theory. Both theory and algorithms will be presented with sample results for the Hénon map shown for illustration.

Contents

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These notes cover some basic definitions and ideas from dynamical systems and computational Conley index theory. Sample computations for the Hénon map will be used for illustration. Additional references are listed throughout and include [LM95, Rob95] for dynamics, [Con78, MM02] for Conley index theory, and [TKM04, DJM05, DJM04, Day03a, DFT08] for computational Conley index theory.

1 Symbolic Dynamics and Topological Entropy

The computational framework described in these notes may be used in computer-assisted proofs of dynamics ranging from fixed points to chaos for discrete-time systems governed by iterated maps. In this approach, we aim to construct a symbolic dynamical system that is topologically semi-conjugate to the system under study. The topological semi-conjugacy links the two systems, establishing the constructed symbolic system as a "lower bound" on the dynamics of the system under study in that dynamics in the symbolic system correspond to dynamics in the original system. Furthermore, the symbolic system acts as a catalogue of dynamics – fixed points, periodic orbits, connecting orbits, and topological entropy are all readily available from a directed graph representation of the symbolic system as self-loops and cycles, connecting paths, and the log of the spectral radius (of the associated adjacency matrix for the graph) respectively. The sources [LM95, Rob95] contain more detailed descriptions of symbolic dynamics and topological entropy.

The symbolic systems we focus on here are *sofic shifts*. These are defined as invariant subsystems of full shifts as follows. Given a set of symbols $\mathcal{A}$, also referred to as an *alphabet*, we define the (one-sided) *full shift* to be the set of all infinite symbol sequences

$$\mathcal{A}^{\mathbb{N}} = \{a_0 a_1 a_2 \ldots | a_i \in \mathcal{A}\}$$

together with the shift map $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$

$$\sigma(a_0 a_1 a_2 \ldots) := a_1 a_2 a_3 \ldots .$$

A *subshift* is given by a set $\Sigma \subset \mathcal{A}^{\mathbb{N}}$ that is invariant under σ, that is, $\sigma(\Sigma) \subseteq \Sigma$ so that $\sigma : \Sigma \to \Sigma$ is a well-defined subsystem. There are various presentations for a subshift $\sigma : \Sigma \to \Sigma$. One method for defining a subshift is creating a list of *prohibited words*, or blocks of symbols, and allowing all infinite symbol sequences that do not contain a prohibited word. This list is finite in the case of *subshifts of finite type* and methods described in [DFT08] focus on the construction of subshifts of this form. Graphical presentations, including *vertex shifts* and *edge shifts*, are also useful. See [LM95] for more details. In what follows we focus on vertex shift presentations.

Definition 1 *Let G be a directed graph with vertex set $V(G)$ and edge set $E(G)$. Given an alphabet $\mathcal{A}$ and vertex labeling $\ell : V(G) \to \mathcal{A}$, we define the corresponding collection of label sequences of infinite walks in G by*

$$\Sigma_G = \{\ell(v_0)\ell(v_1)\ell(v_2)\ldots \mid (v_i, v_{i+1}) \in E(G)\} \subseteq \mathcal{A}^{\mathbb{N}}.$$

We have $\sigma(\Sigma_G) \subset \Sigma_G$ since each $\boldsymbol{a} = \ell(v_0)\ell(v_1)\ell(v_2)\ldots \in \Sigma_G$ has a corresponding walk $v_0v_1v_2\ldots$ in G, and $\sigma(\boldsymbol{a})$ will be given as the label sequence for the walk $v_1v_2v_3\ldots$ in G obtained by removing the first vertex/edge. A graph G is a vertex presentation *of a subshift $\Sigma \subseteq \mathcal{A}^{\mathbb{N}}$ if $\Sigma_G = \Sigma$. If, furthermore, for each node $v \in V(G)$ and pair of edges $(v, u_1), (v, u_2) \in E(G)$ out of v, the labels on u_i satisfy $\ell(u_1) \neq \ell(u_2)$, then we say that G is* right-resolving.

A subshift $\Sigma \subseteq \mathcal{A}^{\mathbb{N}}$ is *sofic* if it has a right-resolving vertex presentation with finitely many vertices. During our construction of sofic subshifts, we also consider when one subshift is contained in another. The language of *admissible blocks* allows us to present an alternative way of checking that $\Sigma \subset \Sigma'$.

Definition 2 *Given a subshift $\Sigma \subseteq \mathcal{A}^{\mathbb{N}}$, the* language $\mathcal{B}(\Sigma)$ *of Σ, also called the set of* admissible n-blocks*, is the set of finite words appearing in points of Σ. Formally, given $\boldsymbol{a} \in \mathcal{A}^{\mathbb{N}}$, let $[\boldsymbol{a}]_n = a_0a_1\cdots a_{n-1}$ and $\mathcal{B}_n(\Sigma) = \{[\boldsymbol{a}]_n : \boldsymbol{a} \in \Sigma\}$. Then $\mathcal{B}(\Sigma) = \bigcup_{n\in\mathbb{N}_+} \mathcal{B}_n(\Sigma) = \{[\boldsymbol{a}]_n : \boldsymbol{a} \in \Sigma, n \in \mathbb{N}_+\}$.*

It is important to note that for an appropriate choice of metric on $\mathcal{A}^{\mathbb{N}}$ (and hence on Σ_G), σ is a continuous map and $\sigma : \Sigma_G \to \Sigma_G$ is a dynamical system (see e.g. [Rob95]). Sofic subshifts given with a (finite) vertex shift presentation G are particularly nice for extracting dynamics. For example, if one is looking for a period n orbit, then one checks that there is a symbol sequence $\boldsymbol{a}^* = (a_0, a_1, \ldots) \in \Sigma_G$ such that $a_{i+n} = a_i$ for all $i = 0, 1, \ldots$. This periodic symbol sequence corresponds to a cycle in G.

One method for quantifying the complexity of a given dynamical system is to compute its *topological entropy*. The following is based on Bowen's definition of topological entropy in [Bow71].

Definition 3 *Let $f : S \to S$ be a continuous map. A set $W \subset S$ is called (n, ϵ, f)-separated if for any two different points x, $y \in W$ there is an integer j with $0 \leq j < n$ so that the distance between $f^j(x)$ and $f^j(y)$ is greater than ϵ. Let $s(n, \epsilon, f)$ be the maximum cardinality of any (n, ϵ, f)-separated set. The* topological entropy *of f is the number*

$$h_{\text{top}}(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log(s(n, \epsilon, f))}{n}. \tag{1}$$

As a measurement of chaos, we say that a map f for which $h_{\text{top}}(f) > 0$ is chaotic, and, if $h_{\text{top}}(f) > h_{\text{top}}(g)$, then f is exhibits more complexity than g.

Once again, we can turn to symbolic dynamics in order to perform concrete computations. See [Rob95] for more details.

Theorem 4 *Let G be vertex shift presentation of the sofic subshift $\sigma : \Sigma_G \to \Sigma_G$. Then*

$$h_{\text{top}}(\sigma|_{\Sigma_G}) = \lim_{N \to \infty} \frac{|\mathcal{B}_N(\Sigma_G)|}{N} = \log(\text{sp}(G))$$

where $sp(G)$ is the spectral radius of the adjacency matrix A with

$$a_{ij} = \begin{cases} 1 & \textit{if } (v_i, v_j) \in E(G) \\ 0 & \textit{otherwise} \end{cases}$$

for G.

In essence, (n, ϵ, σ)-separation is encoded in the representation of the system and may be computed directly from the vertex shift presentation G.

2 Itineraries and Topological Semi-conjugacy

Topological conjugacies and topological semi-conjugacies link two systems, preserving information about dynamics.

The *itinerary function* defined below allows us to re-write a system $f : S \to S$ as a subshift. Our methods are designed to ensure that the itinerary function serves as a topological semi-conjugacy between the two systems.

Definition 5 *A continuous map $\rho : X \to Y$ is a* topological semi-conjugacy *from $\psi : Y \to Y$ to $\phi : X \to X$ if $\rho \circ \phi = \psi \circ \rho$ and ρ is surjective (onto). If, in addition, ρ is injective (one-to-one), then ρ is a* topological conjugacy*.*

Topological conjugacies preserve many properties of dynamical systems. One such example is the following theorem. (For more details, see [Dev89].)

Theorem 6 *Let ρ be a topological conjugacy between $\phi : X \to X$ and $\psi : Y \to Y$. Then $y \in Y$ is a periodic point of period n under ψ (i.e. $\psi^n(y) = y$) if and only if $\rho^{-1}(y)$ is a periodic point of period n under ϕ. In addition, $h_{\text{top}}(\phi) = h_{\text{top}}(\psi)$.*

If $f : S \to S$ is topologically conjugate to a sofic subshift, then we have a convenient list of trajectories of f given by the subshift. Indeed, in this case, the topological conjugacy acts as a coordinate transformation of the original system onto a decipherable (symbolic) system.

Example 1 *There is a topological conjugacy from the logistic map $f(x) = rx(1-x)$, $r =$?? to the full shift on two symbols Σ_G, where G is the complete graph on two vertices. Defining properties of chaos including density of periodic points, sensitive dependence on initial conditions, and topological transitivity may all be proven for $\sigma : \Sigma_G \to \Sigma_G$ and, therefore, for the logistic map. We also have that $h_{\text{top}}(f) = h_{\text{top}}(\sigma|_{\Sigma_G}) = \log(2)$.*

See [**?**]... for further details. In practice, such a complete description may be beyond our reach and we instead construct sofic subshifts that we prove are topologically semi-conjugate to $f : S' \to S'$ for some appropriately defined subset S'. The topologically semi-conjugate system ($\psi = \sigma$) acts as a lower bound for dynamics under $\phi = f$ resulting in $h_{\text{top}}(\phi) \geq h_{\text{top}}(\psi)$.

Definition 7 *Suppose $N \subset X$ may be decomposed into $m < \infty$ disjoint, closed subsets ($N = \cup_{i=1,\ldots m} N_i$, $N_i \cap N_j = \emptyset$ for all $i \neq j$) and let $\mathcal{A} = \{1, 2, \ldots, m\}$ be the associated alphabet of symbols. Let S be the maximal invariant set in N (i.e. S is the largest set such that $S \subset N$ and $f(S) = S$). Then $f^j(S) \subset N$ for all $j = 0, 1, \ldots$. The* itinerary function *$\rho : S \to \mathcal{A}^{\mathbb{N}}$ is given by $\rho(x) = s_0 s_1 \ldots$, where $s_j = i$ for $f^j(x) \in N_i$.*

The itinerary function is continuous under the appropriate choice of metrics and naturally satisfies the commutativity condition required for topological semi-conjugacy (that is, $\rho \circ f = \sigma \circ \rho$). (See [Dev89], [Rob95] for more details.) In what follows, we describe a procedure based on Conley index theory that allows us to construct a sofic subshift $\sigma : \Sigma_G \to \Sigma_G$ with vertex presentation G and $\Sigma_G \subset \mathcal{A}^{\mathbb{N}}$ so that for some $S' \subseteq S$, $\rho : S' \to \Sigma_G$ is surjective. The surjectivity condition completes the proof that ρ is a topologically semi-conjugacy from $\sigma : \Sigma_G \to \Sigma_G$ to $f : S' \to S'$.

Since if g is topologically semi-conjugate to f then $h_{top}(f) \geq h_{top}(g)$, the following corollary allows us to use a semi-conjugate sofic subshift to obtain a lower bound for the topological entropy of the system under study. (See also [LM95] for further details.)

Corollary 8 *Suppose that the itinerary function ρ is a semi-conjugacy from $\sigma : \Sigma_G \to \Sigma_G$ to $f : S' \to S'$ for some $S' \subset X$ and sofic subshift with vertex shift presentation G. Then*

$$h_{\mathrm{top}}(f) \geq \log(\mathrm{sp}(G))$$

where $sp(G)$ is the spectral radius of the adjacency matrix for G.

3 Conley Index Theory and Surjectivity

We now present some of the topological tools used to build the sofic subshift and prove surjectivity of the itinerary map ρ. These tools are based on Conley index theory for which we now give definitions, facts, and theorems. (See [TKM04, DJM05, DJM04, Day03a, DFT08] for more details.) A discussion of the implementation of these ideas in a computational framework follows in Section 5.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. A *trajectory through $x \in \mathbb{R}^n$* is a sequence

$$\gamma_x := (\ldots, x_{-1}, x_0, x_1, \ldots) \tag{2}$$

such that $x_0 = x$ and $x_{n+1} = f(x_n)$ for all $n \in \mathbb{Z}$. The *invariant set relative to $N \subset \mathbb{R}^n$* is

$$\mathrm{Inv}(N, f) := \{x \in N |\ \text{there exists a trajectory } \gamma_x \text{ with } \gamma_x \subset N\}. \tag{3}$$

One example of a relative invariant set is the domain $S = \mathrm{Inv}(N, f)$ on which we defined the itinerary function ρ in Definition 7.

We are now ready to present some of the basic structures in Conley index theory.

Definition 9 *A compact set* $N \subset \mathbb{R}^n$ *is an* isolating neighborhood *if*

$$\mathrm{Inv}(N, f) \subset \mathrm{int}(N) \tag{4}$$

where $\mathrm{int}(N)$ *denotes the interior of* N*.* S *is an* isolated invariant set *if* $S = \mathrm{Inv}(N, f)$ *for some isolating neighborhood* N*.*

We use the next two definitions to encode the dynamics on an isolating neighborhood.

Definition 10 *Let* $P = (P_1, P_0)$ *be a pair of compact sets with* $P_0 \subset P_1 \subset X$*. The map induced on the pointed quotient space* $(P_1/P_0, [P_0])$ *is*

$$f_P(x) := \begin{cases} f(x) & \text{if } x, f(x) \in P_1 \setminus P_0 \\ [P_0] & \text{otherwise.} \end{cases} \tag{5}$$

Definition 11 *([RS88]) The pair of compact sets* $P = (P_1, P_0)$ *with* $P_0 \subset P_1 \subset X$ *is an* index pair *for* f *provided that*

1. *the induced map,* f_P*, is continuous,*
2. $\overline{P_1 \setminus P_0}$*, the closure of* $P_1 \setminus P_0$*, is an isolating neighborhood.*

In this case, we say that P *is an index pair for the isolated invariant set* $S = \mathrm{Inv}(\overline{P_1 \setminus P_0}, f)$*.*

The following definition is required for the definition of the Conley index.

Definition 12 *Two group homomorphisms,* $\phi : G \to G$ *and* $\psi : G' \to G'$ *on abelian groups* G *and* G' *are* shift equivalent *if there exist group homomorphisms* $r : G \to G'$ *and* $s : G' \to G$ *and a constant* $m \in \mathbb{N}$ *(referred to as the 'lag') such that*

$$r \circ \phi = \psi \circ r, \quad s \circ \psi = \phi \circ s, \quad r \circ s = \psi^m, \quad \text{and} \quad s \circ r = \phi^m.$$

The shift equivalence class of ϕ*, denoted* $[\phi]_s$*, is the set of all homomorphisms* ψ *such that* ψ *is shift equivalent to* ϕ*.*

Definition 13 *Let* $P = (P_1, P_0)$ *be an index pair for the isolated invariant set* $S = \text{Inv}(\overline{P_1 \setminus P_0}, f)$ *and let* $f_{P*} : H_*(P_1, P_0) \to H_*(P_1, P_0)$ *be the map induced on the relative homology groups* $H_*(P_1, P_0)$ *from the map* f_P*. The* Conley index *of* S *is the shift equivalence class of* f_{P*}

$$\text{Con}(S, f) := [f_{P*}]_s. \tag{6}$$

The Conley index for the isolated invariant set S given in Definition 13 is well-defined, namely, every isolated invariant set has an index pair, and the corresponding shift equivalence class remains invariant under different choices for this index pair (see e.g. [MM02, TKM04]).

So far we have passed from continuous maps to induced maps on relative homology. Our overall goal, however, is to describe the dynamics of our original map. Here we present measurements based on the map on homology that may give us information about the original map. The first theorem is Ważewski's Principle in the context of Conley index theory.

Theorem 14 *If* $\text{Con}(S, f) \neq [0]_s$*, then* $S \neq \emptyset$*.*

The Lefschetz Theorem may be used to prove the existence of a fixed point.

Theorem 15 *If the* Lefschetz number

$$L(S, f) := \sum_k (-1)^k \text{tr}\,(f_{Pk}) \tag{7}$$

is nonzero, then S *contains a fixed point.*

By requiring additional structure in the isolating neighborhood N of S, we can use modifications of Theorems 14 and 15 to study finer structure in S. This follows closely the construction in [Szy97].

Corollary 16 *Let* $N \subset X$ *be the union of pairwise disjoint, compact sets* $N_1, \ldots, N_m$ *and let* $S := \text{Inv}(N, f)$ *be the isolated invariant set relative to* N*. For* $B_n = a_0 a_1 \ldots a_{n-1}$*, an* n*-block of symbols from the alphabet* $\mathcal{A} = \{1, \ldots, m\}$*, set*

$$f^{B_n} := f_{N_{a_{n-1}}} \circ \cdots \circ f_{N_{a_0}}$$

where $f_{N_{a_i}}$ *denotes the restriction of the map* f *to the region* N_{a_i}*. Then for* $S' = \text{Inv}(N, f^{B_n}) \subset S$*, if*

$$\text{Con}(S', f^{B_n}) \neq [0]_s, \tag{8}$$

then $S' \neq \emptyset$*. More specifically, there exists a point in* S *whose trajectory under* f *travels through the regions* $N_{a_0}, \ldots, N_{a_{n-1}}$ *in the prescribed order. Similarly, if*

$$L(S', f^{B_n}) \neq 0, \tag{9}$$

then S' *contains a periodic point whose trajectory travels through the regions* $N_{a_0}, \ldots, N_{a_{n-1}}$ *in the prescribed order.*

We note here that given the hypotheses of Corollary 16, we may obtain the index of S' given the computed index map f_{P*}, where $P = (P_1, P_0)$ is an index pair with $N = \overline{P_1 \setminus P_0}$. Using an approach developed by Szymczak in [Szy95], we set

$$f_P^{ij}(x) := \begin{cases} f(x) & \text{if } x \in N_i \text{ and } f(x) \in N_j \\ [P_0] & \text{otherwise} \end{cases} \tag{10}$$

Then $f_{P*}^{ij} : H_*(P_1, P_0 \cup (\cup_{l \neq i} N_l)) \to H_*(P_1, P_0 \cup (\cup_{l \neq j} N_l))$. Given f_{Pk} in matrix form representing the linear map on $H_k(P_1, P_0)$, we may label the columns/rows by location of the associated relative homology generators in the subgroups $H_k(P_1, P_0 \cup (\cup_{l \neq 1} N_l)), \ldots H_k(P_1, P_0 \cup (\cup_{l \neq n} N_l))$. To simplify notation, we say that generator g is in region N_i if $g \in H_k(P_1, P_0 \cup (\cup_{l \neq i} N_l))$. Then f_{Pk}^{ij} is the $n_j \times n_i$ submatrix with n_i columns corresponding to the n_i generators in N_i and n_j rows corresponding to the n_j generators in N_j. Furthermore, for $B_n = a_0 a_1 \cdots a_{n-1}$, $(P_1, P_0 \cup (\cup_{a \neq a_0} N_a))$ is an index pair for the isolated invariant set $S' = \text{Inv}(N_{a_0}, f_{N_{a_{n-1}}} \circ \cdots \circ f_{N_{a_0}})$ with index map $f_{P*}^{B_n} := f_{P*}^{a_{n-1} a_0} \circ \cdots \circ f_{P*}^{a_0 a_1} : H_*(P_1, P_0 \cup (\cup_{l \neq a_0} N_l)) \to H_*(P_1, P_0 \cup (\cup_{l \neq a_0} N_l))$. Therefore,

$$\text{Con}(S', f^{B_n}) = [f_{P*}^{B_n}]_s = [f_{P*}^{a_{n-1} a_0} \circ \cdots \circ f_{P*}^{a_0 a_1}]_s \tag{11}$$

with representative shift equivalence class representative $f_{P*}^{a_{n-1} a_0} \circ \cdots \circ f_{P*}^{a_0 a_1}$ computed as the product of submatrices of f_{P*}.

The Lefschetz number (Theorem 15) is computable given a representative $f_{P*}^{B_n}$. However, using Ważewski's Principle (Theorem 14) directly is less immediate. In general, determining whether the linear map $f_{P*}^{B_n}$ is not shift equivalent to 0 may be difficult. We here focus on a computable, sufficient condition. Nilpotency is preserved by shift equivalence, and so if $f_{Pk}^{B_n}$ is not nilpotent for some k then $\mathrm{Con}(S, f^{B_n}) \neq [0]_s$. Algorithms that identify symbol sequences for which no power of the corresponding matrix product may be 0 are presented in [DFT08, DF15]. These conditions may be checked by hand for the sample results presented in Section 6.

Recall that by construction, the remaining condition to check in order for $\rho : S \to \Sigma_G$ to be a topological semiconjugacy is that it is surjective. For systems Σ_G constructed as the closure of a set of periodic points, we first show that $\mathrm{Con}(S', f^{B_n}) \neq [0]_s$ for every block B_n corresponding to a cycle in G. Since ρ is continuous and S is compact, ρ must then map onto Σ_G, the closure of the set of periodic points. This approach is summarized in the following corollary.

Corollary 17 *If $f_{P*}^{B_n}$ is not nilpotent for every block B_n corresponding to a cycle in G and Σ_G is the closure of the set of periodic points in Σ_G, then $\rho : S \to \Sigma_G$ is surjective and is, therefore, a topological semiconjugacy.*

4 Combinatorial Outer Approximation

Now that we have the relevant tools from Conley index theory, we can begin applying them algorithmically to extract information about the dynamical system $f : X \to X$. In this section, we describe the construction of a *combinatorial representation* of f. This combinatorial representation is an example of a *multivalued map* F that will be used to incorporate error bounds in the representation. See [TKM04, DJM05, DJM04, Day03a, DFT08] for more details.

Definition 18 *The* multivalued map $F : X \rightrightarrows X$ *is a map from X to its power set, i.e. for all $x \in X$, $F(x) \subset X$. If for a continuous, single-valued map $f : X \to X$, $f(x) \in F(x)$ and $F(x)$ is acyclic (i.e. has the homology of a point) for all $x \in X$, then f is a* continuous selector *of F and F is an* outer approximation *of f.*

In what follows, we discuss how to construct a *(combinatorial) outer approximation* of the map under study. The purpose of the outer approximation is to incorporate round-off and other errors that occur in computations. This construction requires rigorous, small error bounds in order to create an outer approximation whose images are not so large as to obscure relevant dynamics. Given an appropriate outer approximation, the topological tools from Section 3 may be used to uncover dynamics of the underlying map. Furthermore, there are algorithms for both the construction of the outer approximation and the computation of the Conley index. These algorithms require a further step – discretizing the domain in order to store it in the computer as a finite list of objects.

We begin by using a subdivision procedure to create a grid $\mathcal{G}$ on a compact (rectangular) region in X. In practice, the region chosen for representation is usually determined either experimentally through non-rigorous numerical simulations or analytically given special structure or symmetry for the system (e.g. a compact attracting region). Although variations exist, the simplest is a *uniformly-subdivided cubical grid* of a rectangular set $W = \prod_{k=1}^{n}[x_k^-, x_k^+] \subset \mathbb{R}^n$ given by

$$\mathcal{G}^{(d)} := \left\{ \prod_{k=1}^{n} \left[x_k^- + \frac{i_k r_k}{2^d}, x_k^- + \frac{(i_k+1) r_k}{2^d} \right] \middle| \, i_k \in \{0, \ldots, 2^d - 1\} \right\}$$

where $r_k = x_k^+ - x_k^-$ is the radius of W in the kth coordinate and the depth d is a nonnegative integer. We call an element of the grid, $B =$

$\prod_{k=1}^{n}\left[x_k^- + \frac{i_k r_k}{2^d}, x_k^- + \frac{(i_k+1) r_k}{2^d}\right]$, a *box*. For a collection of boxes, $G \subset \mathcal{G} = \mathcal{G}^{(d)}$, define the *topological realization* of G as $|G| := \cup_{B\in G} B \subset \mathbb{R}^n$.

Constructing a useful combinatorial outer approximation involves bounding all round-off and other errors. In our study of the Hénon map in Section 6, we use interval arithmetic to bound $f(|G|)$. This produces a bounding box, $\tilde{f}(|G|)$, for the image $f(|G|)$, which is then intersected with the grid $\mathcal{G}$ to produce the combinatorial image

$$\mathcal{F}(G) := \{G' \in \mathcal{G} : |G'| \cap \tilde{f}(|G|) \neq \emptyset\}.$$

The combinatorial map, $\mathcal{F} : \mathcal{G} \rightrightarrows \mathcal{G}$, yields an outer approximation $F = |\mathcal{F}|$ of f in the following way: define $|\mathcal{F}| : W \rightrightarrows W$, where $W = \cup_{G\in\mathcal{G}} |G|$,

$$|\mathcal{F}|(x) := \bigcup_{G\in\mathcal{G} \text{ with } x\in|G|} |\mathcal{F}(G)|. \tag{12}$$

Dynamics of a continuous selector f forces structure in $\mathcal{F}$. Viewing $\mathcal{F}$ as the directed graph $G(\mathcal{F}) = (V, E)$ where $V = \mathcal{G} = \{G\}$ and $(G_i, G_j) \in E$ whenever $G_j \in \mathcal{F}(G_i)$,

- $(G, G) \in E$ whenever $f(x) = x$ for some $x \in |G|$,
- periodic points of f must be contained in cycles of $G(\mathcal{F})$,
- recurrent sets under f must be contained in strongly connected components of $G(\mathcal{F})$,
- connecting orbits under f are contained in connecting paths in $G(\mathcal{F})$,
- vertices for which all forward paths terminate in a trapping region ($\mathcal{S}$ such that $\mathcal{F}(\mathcal{S}) \subseteq \mathcal{S}$) correspond to points in the basin of attraction for that trapping region,
- the maximal invariant set under f is contained in the maximal invariant set of $G(\mathcal{F})$.

The above list is helpful for determining candidate regions for dynamics of f. For example, self loops $(G, G) \in E$ yield regions, $|G|$ containing possible fixed points for f. However, a box with a self-loop may not contain a fixed point of f. The extreme example is the combinatorial outer approximation

$\mathcal{F} : \mathcal{G} \rightrightarrows \mathcal{G}$ given by $\mathcal{F}(G) = \mathcal{G}$ for every $G \in \mathcal{G}$. Every box $G \in \mathcal{G}$ has a self-loop, but this could just be due to large image bounds that are not able to resolve finer dynamics. We use Conley index theory In order to *prove* that dynamics observed in $\mathcal{F}$ exists for a continuous selector f. In what follows, we focus on an algorithmic approach for computing Conley indices from combinatorial outer approximations.

5 Computational Conley Index Theory

In this section, we list sample algorithms for computing isolating neighborhoods, index pairs, and Conley indices. These algorithms are shown in more detail in [DJM05, DFT08].

Definition 19 *A* combinatorial trajectory *of a combinatorial outer approximation $\mathcal{F}$ through $G \in \mathcal{G}$ is a bi-infinite sequence $\gamma_G = (\ldots, G_{-1}, G_0, G_1, \ldots)$ with $G_0 = G$, $G_n \in \mathcal{G}$, and $G_{n+1} \in \mathcal{F}(G_n)$ for all $n \in \mathbb{Z}$.*

Definition 20 *The* combinatorial invariant set *in $\mathcal{N} \subset \mathcal{G}$ for a combinatorial outer approximation $\mathcal{F}$ is*

$$\operatorname{Inv}(\mathcal{N}, \mathcal{F}) := \{G \in \mathcal{G} : \text{ there exists a trajectory } \gamma_G \subset \mathcal{N}\}.$$

Definition 21 *The* combinatorial neighborhood *of $\mathcal{B} \subset \mathcal{G}$ is*

$$o(\mathcal{B}) := \{G \in \mathcal{G} : |G| \cap |\mathcal{B}| \neq \emptyset\}.$$

This set, $|o(\mathcal{B})|$, sometimes referred to as a *one box neighborhood of $\mathcal{B}$ in $\mathcal{G}$*, is the smallest representable neighborhood of $|B|$ in the grid $\mathcal{G}$.

While there are different characterizations of isolation in the setting of combinatorial outer approximations, we chose the following for simplicity and illustration purposes.

Definition 22 *If*

$$o(\operatorname{Inv}(\mathcal{N}, \mathcal{F})) \subset \mathcal{N}$$

then $\mathcal{N} \subset \mathcal{G}$ is a combinatorial isolating neighborhood *under $\mathcal{F}$.*

Note that by construction, the topological realization $|\mathcal{N}|$ of a combinatorial isolating neighborhood $\mathcal{N}$ is an isolating neighborhood for any continuous selector $f \in |\mathcal{F}|$. This definition is stronger than what is actually required to guarantee isolation on the topological level. It is, however, the definition that we will use in this work and is computable using the following approach.

Let $\mathcal{S} \subset \mathcal{G}$. Set $\mathcal{N} = \mathcal{S}$ and let $o(\mathcal{N})$ be the combinatorial neighborhood of $\mathcal{N}$ in $\mathcal{G}$. If $\mathrm{Inv}(o(\mathcal{N}), \mathcal{F}) = \mathcal{N}$, then $\mathcal{N}$ is isolated under $\mathcal{F}$. If not, set $\mathcal{N} := \mathrm{Inv}(o(\mathcal{N}), \mathcal{F})$ and repeat the above procedure. In this way, we grow the set $\mathcal{N}$ until either the isolation condition is met, or the set grows to intersect the boundary of $\mathcal{G}$ in which case the algorithm fails to locate an isolating neighborhood in $\mathcal{G}$. This procedure is outlined in more detail in the following algorithm from [DJM04].

Algorithm 1: Grow Isolating Neighborhood

```
INPUT:  grid 𝒢, combinatorial outer approximation ℱ on 𝒢, set 𝒮 ⊂ 𝒢
OUTPUT: a combinatorial isolating neighborhood 𝒩 containing 𝒮
        or 𝒩 = ∅ if the isolation condition is not met

𝒩 = grow_isolating_neighborhood(𝒢, ℱ, 𝒮)
    𝒢_boundary:= {G ∈ 𝒢 : |G| ∩ ∂|𝒢| ≠ ∅};
    𝒩 := 𝒮;
    while Inv(o(𝒩), ℱ)) ⊄ 𝒩 and 𝒩 ∩ 𝒢_boundary= ∅,
        𝒩 := Inv(o(𝒩), ℱ);
    end
    if 𝒩 ∩ 𝒢_boundary= ∅, return 𝒩;
    else return ∅;
    end
```

Once we have an isolating neighborhood for f, our next goal is to compute a corresponding index pair. The following definition of a *combinatorial index pair* again emphasizes our goal of using the combinatorial outer approximation to compute structures for f.

Definition 23 *A pair $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_0)$ of cubical sets is a* combinatorial index pair *for a combinatorial outer approximation $\mathcal{F}$ if the corresponding topo-*

logical realization $P = (P_1, P_0)$, where $P_i := \overline{|\mathcal{P}_i|}$, is an index pair for any continuous selector $f \in |\mathcal{F}|$. Namely, $\overline{P_1 \setminus P_0} = |\mathcal{P}_1 \setminus \mathcal{P}_0|$ is an isolating neighborhood under f and the map f_P, as defined in Definition 10, is continuous.

The following algorithm produces a combinatorial index pair associated to a combinatorial isolating neighborhood produced via Algorithm 1.While there are other algorithms for producing combinatorial index pairs, this algorithm works well with later index computations. For more details, see the description of *modified combinatorial index pairs* in [Day03b].

Algorithm 2: Build Index Pair

```
INPUT:  grid 𝒢, combinatorial outer approximation ℱ on 𝒢,
        combinatorial isolating neighborhood 𝒩 produced by Algorithm 1
OUTPUT: combinatorial index pair 𝒫 = (𝒫1, 𝒫0) with 𝒫1 \ 𝒫0 = 𝒩

[𝒫1, 𝒫0] = build_index_pair(𝒢, ℱ, 𝒩);
    𝒫0 := ∅;
    New := ℱ(𝒩) ∩ o(𝒩) \ 𝒩;
    while New ≠ ∅
        𝒫0 := 𝒫0 ∪ New;
        New := (ℱ(𝒫0) ∩ o(𝒩)) \ 𝒫0;
    end
    𝒫1 := 𝒩 ∪ 𝒫0;
    return [𝒫1, 𝒫0];
```

We now have an isolating neighborhood $|\mathcal{N}|$ and corresponding index pair $P := (|\mathcal{P}_1|, |\mathcal{P}_0|)$ for f. What remains in computing the Conley index for the associated isolated invariant set, $S := \mathrm{Inv}(|\mathcal{N}|, f)$, is to compute the map $f_{P*} : H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \to H_*(|\mathcal{P}_1|, |\mathcal{P}_0|)$. Once again, the combinatorial outer approximation offers the appropriate computational framework and we may use a computational homology software program such as *homcubes* in [Pil98] to compute f_{P*}. This step is outlined in Algorithm 3.

Algorithm 3: Compute Index Map

```
INPUT:  grid 𝒢, combinatorial enclosure ℱ on 𝒢,
        combinatorial index pair 𝒫 = (𝒫₁, 𝒫₀) produced by Algorithm 2
OUTPUT: relative homology groups H_*(|𝒫₁|, |𝒫₀|),
        the induced index map f_{P*} : H_*(|𝒫₁|, |𝒫₀|) → H_*(|𝒫₁|, |𝒫₀|),
        and the induced submaps {f^{ij}_{Pk}} on connected components

[f_{P*} H_*(|𝒫₁|, |𝒫₀|) {f^{ij}_{Pk}}] = compute_index_map(𝒢, ℱ, 𝒫₁, 𝒫₀)
    𝒬₁ = ℱ(𝒫₁);
    𝒬₀ = ℱ(𝒫₀);
    [f_{P*} H_*(|𝒫₁|, |𝒫₀|) {f^{ij}_{Pk}}] := homcubes(𝒫₁, 𝒫₀, 𝒬₁, 𝒬₀, ℱ);
    return [f_{P*} H_*(|𝒫₁|, |𝒫₀|) {f^{ij}_{Pk}}];
```

Algorithm 5 produces a sequence of matrices for the maps $f_{P0}, f_{P1}, \ldots, f_{Pn}$ where n is the dimension of the phase space X. For $k > n$, $f_{Pk} = 0$. The associated Conley index is $\mathrm{Con}_*(S) = [f_{P*}]_s$, for $S := \mathrm{Inv}(|\mathcal{P}_1 \setminus \mathcal{P}_0|, f)$. The submaps $f^{ij}_{Pk} : H_k(|\mathcal{P}_1|, |\mathcal{P}_0| \cup_{l\neq i} |\mathcal{N}_l|) \to H_k(|\mathcal{P}_1|, |\mathcal{P}_0| \cup_{l\neq j} |\mathcal{N}_l|)$, where $|\mathcal{N}_1|, \ldots |\mathcal{N}_n|$ are the connected components of $|\mathcal{P}_1 \setminus \mathcal{P}_0|$, are given as submatrices of f_{Pk}. These are the maps required for Corollary 16. In the following section, we present sample computations to illustrate the methods.

6 The Hénon Map: Sample Results

The Hénon map [Hén76] is a simplified model of stretching and folding. Here we study a rescaled version $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 - ax^2 + y/5 \\ 5bx \end{pmatrix}$$

at the standard parameter values $a = 1.3$ and $b = 0.2$. Results using these methods on the Hénon map are also given in [Day03a, DJM05, DFT08]. For the application of these methods to other systems and extensions of the methods to families of systems and infinite-dimensional systems, see e.g. [Day03a, AKK+09, DJM04, DK13].

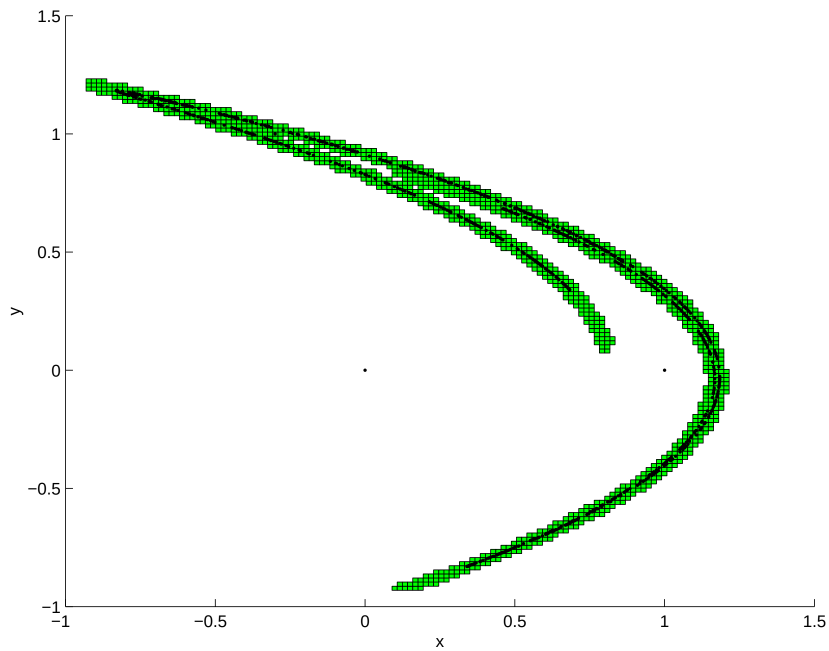


Figure 1: An initial simulation (in black) and box covering of the maximal invariant set (in green/grey).

In what follows, we use the software package GAIO [DFJ01] and homcubes [Pil98] to produce sample results. In the lectures, I will also mention two recently developed packages that may be used for these computations. For the first sample result, we use a uniformly subdivided cubical grid at depth $d = 7$ constructed by subdividing the box $W = [-0.9311, 1.2845]^2$ 14 times (7 times in each of the two directions). Keeping only the boxes covering the maximal invariant set results in 1188 boxes, each of box radius $[0.0087, 0.0087]$. This box covering of the maximal invariant set is shown in Figure 1. Outward rounding interval arithmetic for computing image bounds results in a combinatorial outer approximation $\mathcal{F}$, on these boxes.

As an initial study, we look for a 2-cycle in $G(\mathcal{F})$. We use the algorithms in Section 5 to produce a combinatorial index pair shown in Figure 2.

The relative homology of the index pair depicted in Figure 2 is

$$H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \cong (0,\ \mathbb{Z}^2,\ 0,\ 0,\ \ldots).$$

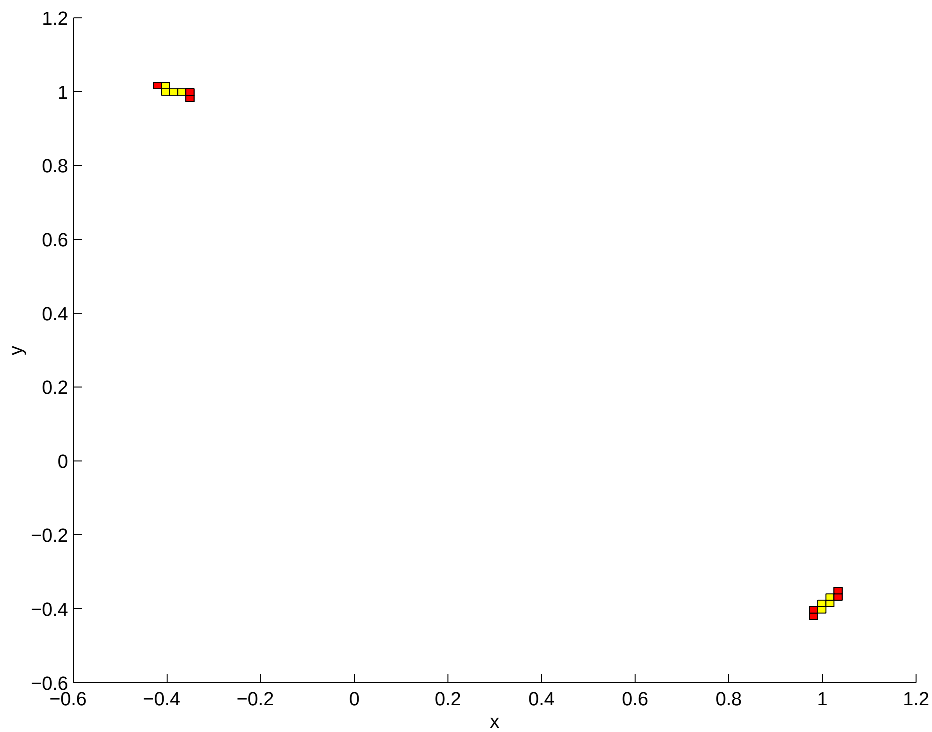


Figure 2: A combinatorial index pair for a 2-cycle. The combinatorial isolating neighborhood is shown in yellow (lighter grey) while the exit set, $\mathcal{P}_0$, is shown in red (darker grey).

and the only nontrivial map induced on homology is

$$f_{P1} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Sample Result 1 *There exists a periodic orbit of minimal period two for the Hénon map with one point in each of the disjoint connected components of the isolating neighborhood shown in Figure 2.*

Proof. The Lefschetz number $L(S, f^2) = -\text{tr}\,(f_{P1}^2) = -2$ is nonzero so $\mathcal{S}$ contains a periodic point of period two. Furthermore, this orbit has minimal period two since $\mathcal{F}$ prohibits there being a fixed point in $\mathcal{S}$. (Equivalently, there are no self-loops in $G(\mathcal{F}|_{\mathcal{S}})$.)

We now search for an orbit connecting a period 2 orbit to a period 4 orbit in $\mathcal{F}$ at depth $d = 9$. The union of a 2-cycle, 4-cycle and connecting path from the 2-cycle to 4-cycle in $G(\mathcal{F})$ serves as the input to the algorithms in Section 5. The resulting index pair is depicted in Figure 3.

The computed relative homology for the index pair in Figure 3 is

$$H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \cong (0,\ \mathbb{Z}^{10},\ 0,\ 0,\ \ldots).$$

and the map on $H_1(|\mathcal{P}_1|, |\mathcal{P}_0|)$ is given by

$$f_{P1} := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

In this case, there is one generator per region and, through careful labeling, f_{P1}^{ij} is entry (i, j) of f_{P1}.

Sample Result 2 *There exists an orbit for the Hénon map which limits in forward time to a box neighborhood of a period 4 orbit, and in backward time to a box neighborhood containing a period 2 orbit.*

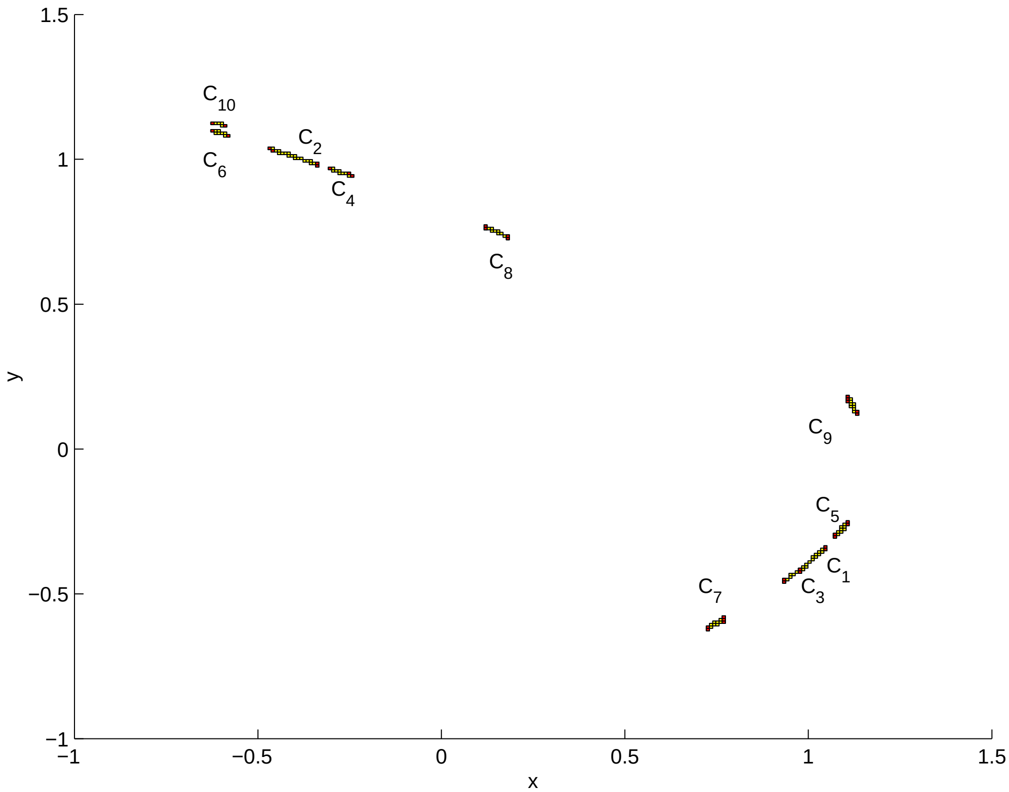


Figure 3: A combinatorial index pair for a 2-cycle, 4-cycle and a connecting path between the two. The combinatorial isolating neighborhood is shown in yellow (lighter grey) while the exit set, $\mathcal{P}_0$, is shown in red (darker grey). Labels for the disjoint components of the isolating neighborhood are shown as well.

Proof. Since tr $f^{12}_{P1} = -1$ and tr $f^{B_4}_{P1} = -1$ for B_4 giving the 4-cycle through generators $7, 8, 9, 10$, Lefschetz number computations confirm that period 2 and period 4 orbits exist in the given regions.

Next, note that f_{P*} is not shift equivalent to $f_{C_1 \cup C_2 *} \oplus f_{\cup_{i\in\{7,8,9,10\}} C_i *}$ where $f_{C_1 \cup C_2 *}$ are the induced maps on $(P_1, P_0 \cup (\cup_{l \neq 1,2} N_l))$ and $(P_1, P_0 \cup (\cup_{l \neq 7,8,9,10} N_l))$ giving the indices of the two periodic regions respectively. Therefore, the Conley indices of $\mathrm{Inv}(S, f)$ and $\mathrm{Inv}(S \cup_{i\in\{1,2,7,8,9,10\}} C_i, f) = \mathrm{Inv}(S \cup_{i\in\{1,2\}} C_i, f) \cup \mathrm{Inv}(S \cup_{i\in\{7,8,9,10\}} C_i, f)$ are different and $\mathrm{Inv}(C, f) \neq \mathrm{Inv}(\cup_{i\in\{1,2,7,8,9,10\}} C_i, f)$

This, in addition to the combinatorial outer approximation only allowing trajectories to move from the neighborhood of the period 2 orbit to the neighborhood of the period 4 orbit, leads to the conclusion that there is the sought connecting orbit.

Finally, we consider the input set from the previous example with this addition of a connecting path from the 4-cycle back to the 2-cycle. The computed isolating neighborhood and index pair are shown in Figure 4.

The map on the relative homology of the index pair is

$$H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \cong (0,\ \mathbb{Z}^{15},\ 0,\ 0,\ \ldots).$$

while the induced map f_{P1} is shown in Figure 5.

Sample Result 3 *[DJM05] There is a set contained in the isolating neighborhood shown in Figure 4, on which f is topologically semi-conjugate to the subshift Σ_G with vertex shift presentation given in Figure 5(a).*

Proof. By construction, the diagram shown in Figure 1 commutes. It remains to show that the itinerary function $\rho : S \to \Sigma_G$ is surjective. It is not difficult to see that $f^{B_n}_{P1} = \pm 1$ for all B_n corresponding to cycles in G. By Corollary 17, ρ is a semi-conjugacy.

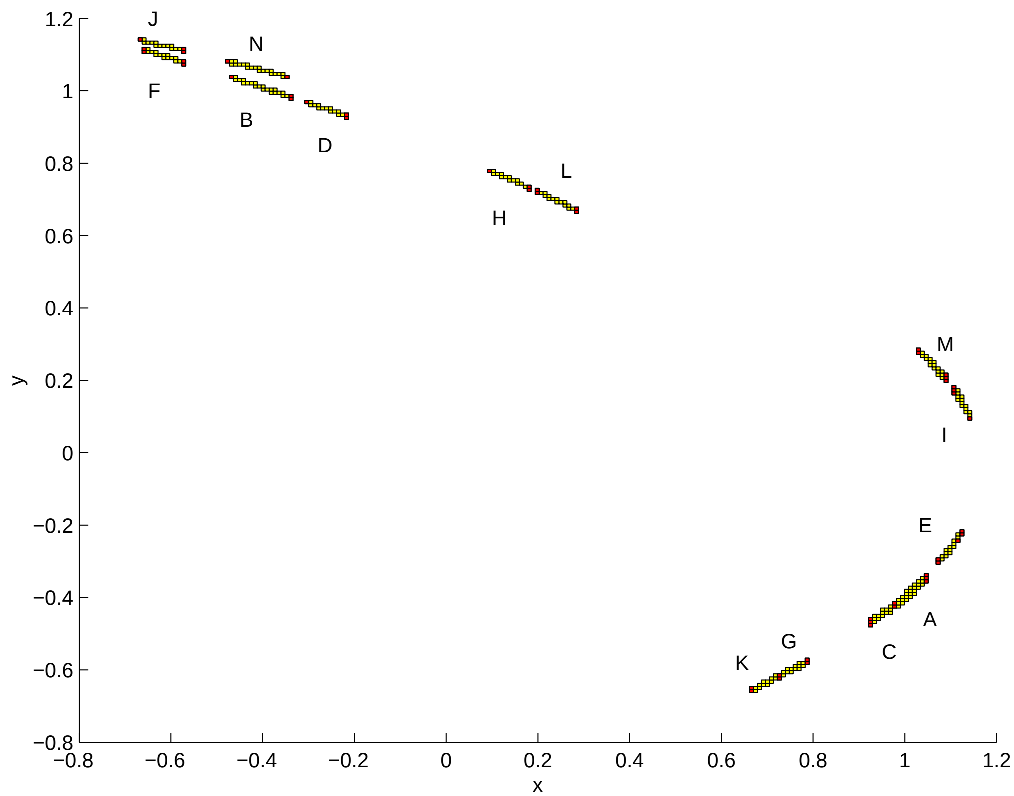
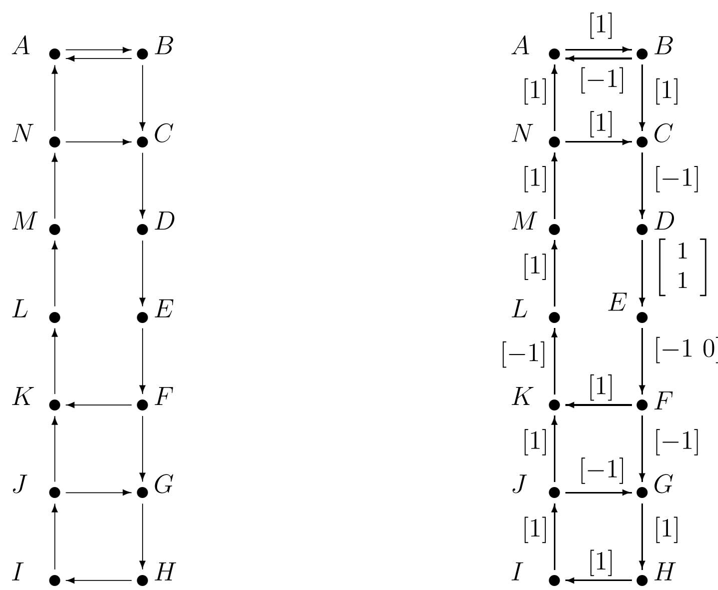


Figure 4: An index pair for a 2-cycle, 4-cycle, and connecting paths between the two. The combinatorial isolating neighborhood is shown in yellow while the exit set ($\mathcal{P}_0$) is shown in red. Labels for the disjoint components of the isolating neighborhood are shown as well.



(a) (b)

Figure 5: (a) Transition graph G on labeled regions. (b) The induced map on homology with nontrivial maps $f_{P1}^{s_i,s_j}$ shown on the edge (s_i, s_j).

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