DELAY DIFFERENTIAL EQUATIONS AND CONTINUATION

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Abstract. In these lecture notes, we demonstrate how rigorous numerics can help studying the dynamics of delay equations. We present a rigorous continuation method for solutions of finite and infinite dimensional parameter dependent problems, which is applied to compute branches of periodic solutions of a delayed Van der Pol equation and of Wright's equation.

Contents

1. Introduction, Motivation and Examples

The main purpose of these lectures notes is to demonstrate how rigorous numerics can help gaining some understanding in the study of the dynamics of delay differential equations (DDEs). One of the main motivating example we consider in these notes is Wright's equation, essentially because it is one of the simplest looking delay equation and it is arguably the most studied equation in the broad field of DDEs. Moreover, it has been the subject of active research for more than 60 years and has been studied by many different mathematicians (e.g. see [1, 2, 3, 4, 5]). As one will see later, the dynamics of this equation naturally leads to studying branches of periodic solutions parameterized by the parameter in the equation. This is why a large part of the notes is dedicated to the presentation of a rigorous continuation method for solutions of finite and infinite dimensional parameter dependent problems. This part will be independent from delay equations. While in this section, we focus on Wright's equation to introduce some concepts and ideas, the method introduced in these notes is quite general and can be applied to a large class of DDEs. Note however that the notes are not meant to provide a general introduction to the field of DDEs. The interested reader will find great introductory materials in the book of Hale and Verduyn Lunel [6], the book of Diekmann, van Gils, Verduyn Lunel and Walther [7], and in the recent survey paper of Walther [8]. In order to start the discussion, we begin by presenting a quote from R. Nussbaum taken from [9].

An intriguing feature of the global study of nonlinear functional differential equations (FDEs) is that progress in understanding even the simplest-looking FDEs has been slow and has involved a combination of careful analysis of the equation and heavy machinery from functional analysis and algebraic topology. A partial list of tools which have been employed includes fixed point theory and the fixed point index, global bifurcation theorems, a global Hopf bifurcation theorem, the Fuller index, ideas related to the Conley index, and equivariant degree theory. Nevertheless, even for the so-called Wright's equation,

$$y'(t) = -\alpha y(t-1)[1+y(t)], \qquad \alpha \in \mathbb{R} \tag{1}$$

which has been an object of serious study for more than forty-five years, many questions remain open.

Roger Nussbaum, 2002.

This comment is still very true nowadays and is perhaps not surprising, as a large class of FDEs naturally give rise to infinite dimensional nonlinear dynamical systems. In order to understand this, let us consider an initial value problem associated to Wright's equation (1). More precisely, at a given time $t_0 \geq 0$, what kind of initial data guarantees the existence of a unique solution $y(t)$ for all $t > t_0$? Since $y'(t_0)$ is determined by $y(t_0)$ and $y(t_0-1)$, knowing the value of $y(t)$ for all $t > t_0$ requires knowing the value of $y(t)$ on the time interval $[t_0-1, t_0]$. In other words, the *initial condition* is a function $y_0 : [t_0-1, t_0] \to \mathbb{R}$. Shifting time to 0, the initial data is given by $y_0 : [-1, 0] \to \mathbb{R}$. Denote the space of continuous real-valued functions defined on $[-1, 0]$ by

$$C \stackrel{\text{def}}{=} C([-1,0],\mathbb{R}) = \{v : [-1,0] \to \mathbb{R} : v \text{ is continuous}\}.$$

Given $y_0 \in C$, the initial value problem

$$\begin{aligned} y'(t) &= -\alpha y(t-1)[1+y(t)], \quad t \geq 0 \\ y(t) &= y_0(t), \quad \forall\, t \in [-1,0] \end{aligned}$$

has a unique solution (e.g. see Theorem 2.3 of Chapter 2 in [6]), and this naturally leads to an infinite dimensional nonlinear dynamical system. Therefore a *state space* for the solutions of (1) is the infinite dimensional function space C. This is the reason why Wright's equation falls into the class of functional differential equations. In Figure 1, find a cartoon phase portrait of Wright's equation visualized in the function space C. Denote by $y_t \in C$ the solution at time t. As time evolves, the solution y_t of the initial value problem gain more and more regularity, somehow in a similar way that solutions of parabolic partial differential equations (PDEs) gain regularity. However, while the regularizing effect in parabolic PDEs can be instantaneous in time (think for instance of the heat equation), the regularizing process in delay equations is much slower. In fact, this is a *discrete* regularizing process. For instance, if $y_0 \in C = C([-1,0],\mathbb{R})$ and $t_0 \in (0,1)$, then $y'(t_0) = -\alpha y(t_0-1)[1+y(t_0)]$, and so the solution y is differentiable at t_0. In other words, $y_t \in C^1$ for $t \in (0,1]$. Similarly, $y_t \in C^2$ for $t \in (1,2]$, and more generally $y_t \in C^k$ for $t \in (k-1,k]$. This is why we call this a "discrete" regularizing process. At infinity, the solution of the initial value problem is C^∞. As a consequence, this means that bounded solutions of Wright's equations are extremely regular. This a priori knowledge about the regularity of the bounded solutions will be crucial in designing the rigorous numerical methods. As a matter of fact, when studying periodic

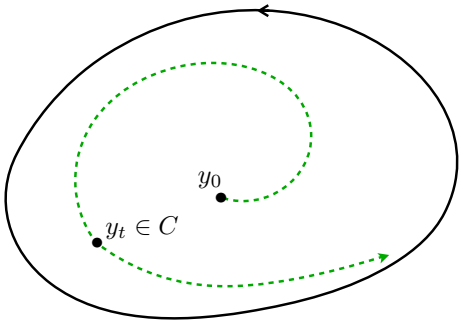

FIGURE 1. A cartoon phase portrait of Wright's equation in the function space $C = C([-1, 0], \mathbb{R})$. A point $y_t \in C$ in the phase portrait is a function.

solutions of delay equations, we can get even analyticity of the solutions, as long as the delay equation itself is analytic [10].

The infinite dimensional nature of the problem comes directly from the presence of the delay in the equation. Suppose for the moment that the delay is absent from the equation, that is consider the scalar ordinary differential equation (ODE)

$$y'(t) = -\alpha y(t)[1 + y(t)]. \tag{2}$$

Then, the phase portrait of (2) is simple and is portrayed in Figure 2. In particular, we get that the equilibrium solution 0 is asymptotically stable for all parameter values $\alpha > 0$.

FIGURE 2. The phase portrait of (2) for any $\alpha > 0$.

Adding a delay severely complicates the behaviour of the solutions of the equation. In fact, we see below that the effect of the delay in Wright's equation leads to a loss of stability of the zero equilibrium solution for all $\alpha > \pi/2$. This property is similar in some sense to Turing instability [11], a phenomenon in which a stable equilibrium solution of an ODE becomes unstable after a diffusion term is added to the ODE. In other words, the steady state loses its stability after the finite dimensional ODE is transformed into an infinite dimensional reaction diffusion PDE.

Let us discuss the history of Wright's equation, following closely the presentation of [12].

At the beginning of the 1950s, the equation

$$y'(t) = -(\log 2)y(t - 1)[1 + y(t)]$$

was brought to the attention of the number theorist Wright (a former Ph.D. student of Hardy at Oxford) because it arose in the application of probability methods to the theory of distribution of prime numbers. In 1955, Wright considered the more general equation (1) and studied the existence of bounded non trivial solutions for different values of $\alpha > 0$ [13]. In 1962, following the pioneer work of Wright, Jones demonstrated in [14] that non trivial periodic solutions of (1) exist for $\alpha > \frac{\pi}{2}$, and using numerical simulations, he remarked in [15]

that a given periodic solution seemed to be globally attractive, that is seemed to attract all initial conditions. In Figure 3 and Figure 4, we reproduced some of the numerical simulations of Jones using the integrator for delay equations `dde23` in MATLAB. The periodic form he referred to is in fact a slowly oscillating periodic solution.

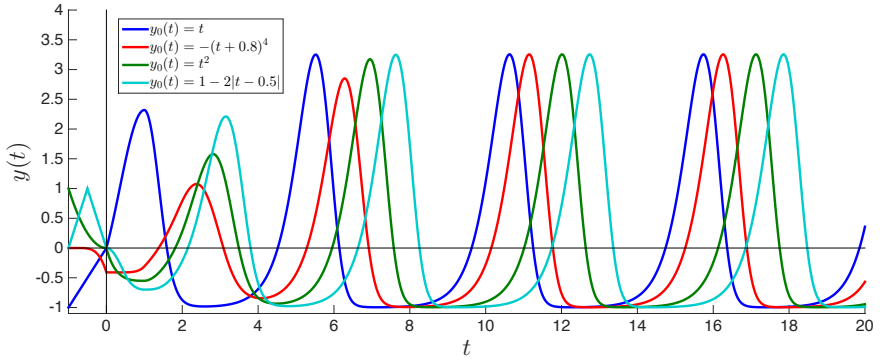

FIGURE 3. Numerical integration of Wright's equation (1) with $\alpha = 2.4$ with different initial conditions y_0 defined on the interval $[-1, 0]$.

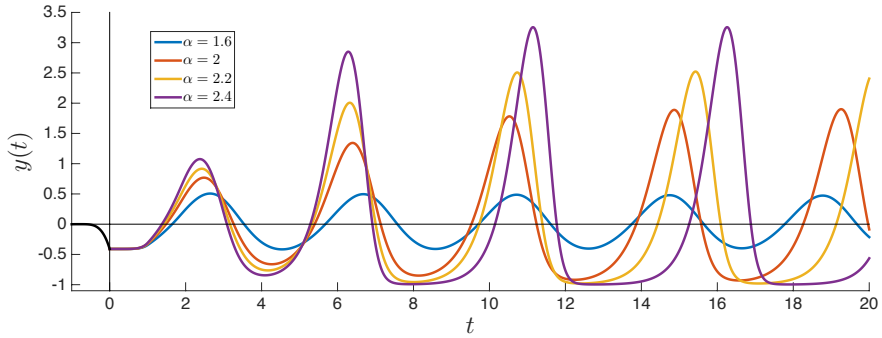

FIGURE 4. Numerical integration of Wright's equation (1) with the initial condition $y_0(t) = -(t + 0.8)^4$ for different parameter values of α.

Definition 1.1. A *slowly oscillating periodic solution* (SOPS) of (1) is a periodic solution $y(t)$ with the following property: there exist $q > 1$ and $p > q + 1$ such that, up to a time translation, $y(t) > 0$ on $(0, q)$, $y(t) < 0$ on (q, p), and $y(t + p) = y(t)$ for all t so that p is the minimal period of $y(t)$.

A geometric interpretation of a SOPS can be found in Figure 5.

After Jones observation in [15], the question of the uniqueness of SOPS in (1) became popular and is still under investigation after more than 50 years. The next conjecture is sometimes called *Jones Conjecture*.

Conjecture 1.2 (Jones, 1962). *For every $\alpha > \frac{\pi}{2}$, (1) has a unique SOPS.*

A result of Walther in [16] shows that if Jones Conjecture is true, then the unique SOPS attracts a dense and open subset of the phase space. A result from Chow and Mallet-Paret

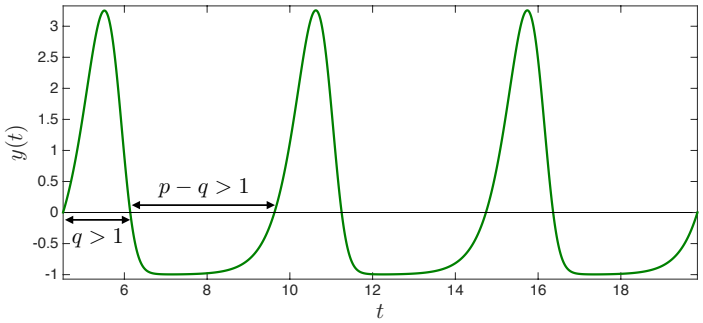

FIGURE 5. A slowly oscillating periodic solution.

from [17] shows that there is a supercritical Hopf bifurcation of SOPS from the trivial solution at $\alpha = \pi/2$. This branch of SOPS which bifurcates (forward in α) from 0 is denoted by $\mathcal{F}_0$. We refer to Figure 6 for a geometric interpretation of bifurcation.

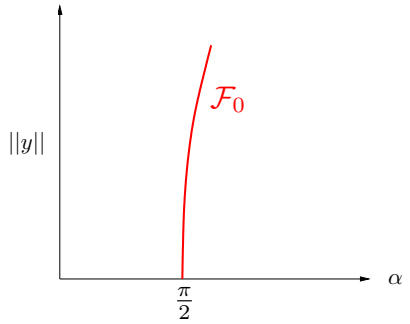

FIGURE 6. A supercritical Hopf bifurcation of SOPS from 0 at $\alpha = \pi/2$.

Regala then proved in his Ph.D. thesis [18] a result that implies that there cannot be any secondary bifurcations from $\mathcal{F}_0$. Hence, $\mathcal{F}_0$ is a regular curve in the (α, y) space. Later, Xie used asymptotic estimates for large α to prove that for $\alpha > 5.67$, Wright's equation has a unique SOPS up to a time translation [19, 20]. Denote by $\mathcal{A}_0 = (\frac{\pi}{2}, 5.67]$ the parameter range not covered by the work of Xie. In [12], it was demonstrated, using the techniques that we introduce in the present notes, that the branch $\mathcal{F}_0$ does not have a fold over the parameter range $[\pi/2 + \varepsilon, 2.3]$, with $\varepsilon = 7.3165 \times 10^{-4}$. Considering the work that has been done in the last 50 years, Jones Conjecture can be reformulated as follows.

Conjecture 1.3 (Jones Conjecture reformulated). *The branch of SOPS $\mathcal{F}_0$ does not have any fold over $\mathcal{A}_0 \setminus [\pi/2+\varepsilon, 2.3]$ and there are no connected components (isolas) of SOPS disjoint from $\mathcal{F}_0$ over $\mathcal{A}_0$.*

Two different scenarios would therefore violate Jones Conjecture. The first scenario is the existence of a fold on $\mathcal{F}_0$ over $\mathcal{A}_0 \setminus [\pi/2+\varepsilon, 2.3]$ which would provide the existence of $\alpha^* \in \mathcal{A}_0$ at which more than one SOPS could co-exist. The second scenario is the existence of an isola $\mathcal{F}_1$ over $\mathcal{A}_0$ which could again force the existence of more than one SOPS. These two scenarios are simultaneously portrayed in Figure 7.

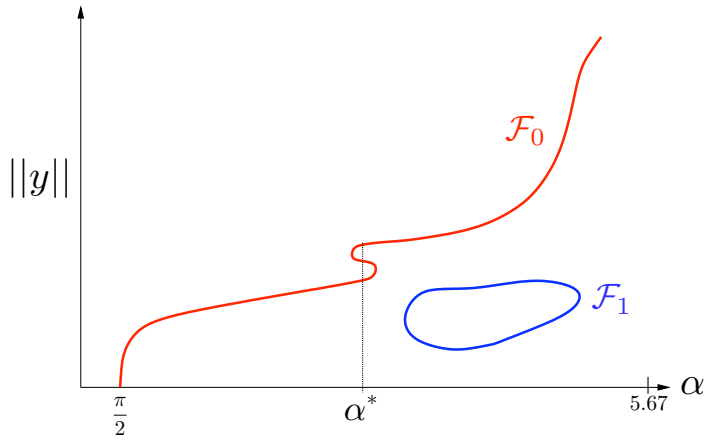

FIGURE 7. Two scenarios which would violate Jones Conjecture: the existence of a fold on $\mathcal{F}_0$ or the existence of an isola in the parameter range $\alpha \in (\frac{\pi}{2}, 5.67]$.

Conjecture 1.3 naturally leads to studying branches of periodic solutions of DDEs. This is the main topic of these lectures notes. More precisely, we introduce a general continuation method to compute global branches of periodic solutions of DDEs using Fourier series and the ideas from rigorous computing (e.g. see [21]). Note that the study of periodic solutions in DDEs is rich [22, 23, 24, 25, 26, 27, 28, 29]. Rather than focussing only on continuation of periodic solutions in DDEs, we present in Section 2 a more general approach to prove existence of branches of solutions for operator equations $F(x, \lambda) = 0$ posed on Banach spaces. In Section 3, we apply the general method to the context of periodic solutions of DDEs.

2. Rigorous Continuation of Solutions

Throughout this section, let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ denote Banach spaces. The vectors spaces X and Y are general and can be either finite or infinite dimensional.

Let $F : X \times \mathbb{R} \to Y$ a C^1 mapping (see Definition 2.2), and consider the general problem of looking for solutions of

$$F(x, \lambda) = 0, \tag{3}$$

where $\lambda \in \mathbb{R}$ is a parameter. The unknown variable x could represent various types of dynamical objects, e.g. a steady state of a PDE, a periodic solution of a DDE, a connecting orbit of an ODE, a minimizer of an action functional, etc. It is important to realize that the *solution set*

$$\mathcal{S} \stackrel{\text{def}}{=} \{(x, \lambda) \in X \times \mathbb{R} \mid F(x, \lambda) = 0\} \subset X \times \mathbb{R}$$

may contain different types of bifurcations and may be complicated (e.g. see Figure 8).

There exists a vast literature on numerical continuation methods to compute solutions of (3). Methods to compute periodic orbits [32, 33], connecting orbits [34, 35, 36] and more generally coherent structures [37] are by now standard, and softwares like AUTO [38] and MATCONT [39] are accessible and well documented. We refer to [40, 41] for more general references on continuation methods. Next we briefly introduce two main algorithms to compute solutions of (3), namely the parameter continuation and the pseudo-arclength continuation. These methods fall into the class of predictor-corrector algorithms.

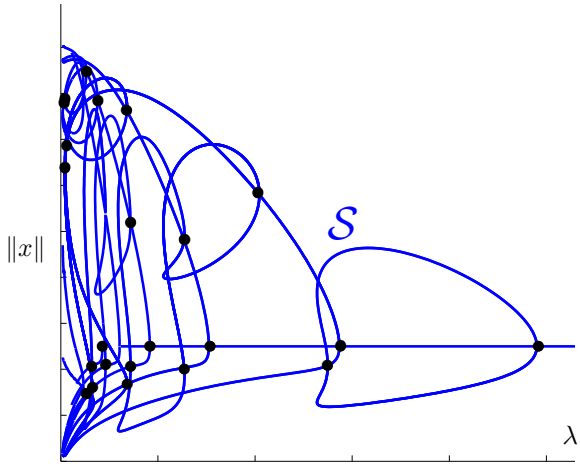

FIGURE 8. Global branches of steady states of a system of reaction-diffusion PDEs introduced in [30] and studied with rigorous numerics in [31].

2.1. **Predictor-Corrector Algorithms.** In this section, we assume that the Banach spaces are finite dimensional and given by $X = Y = \mathbb{R}^n$ ($X = Y = \mathbb{C}^n$ is also an option), we consider a map $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and we study numerically the problem $F(x, \lambda) = 0$. At this point, considering X and Y finite dimensional is not a strong restriction, as any computer algorithm needs to be applied to a problem with a finite resolution. The mapping F could be a finite-dimensional projection of an infinite dimensional operator, e.g. a Galerkin approximation or a discretization scheme. The first predictor-corrector algorithm we introduce is parameter continuation.

2.1.1. *Parameter Continuation.* This method involves a predictor and a corrector step: given, within a prescribed tolerance, a solution x_0 at parameter value λ_0, the predictor step produces an approximate solution $\hat{x}_0$ at nearby parameter value $\lambda_1 = \lambda_0 + \Delta_\lambda$ (for some $\Delta_\lambda \neq 0$), and the corrector step, takes $\hat{x}_1$ as its input and produces with Newton's method, once again within the prescribed tolerance, a solution x_1 at λ_1.

The predictor is obtained by assuming that at the solution (x_0, λ_0), the jacobian matrix $D_x F(x_0, \lambda_0)$ is invertible, which in turns implies by the implicit function theorem that the solution curve is locally parametrized by λ. In this case, close to (x_0, λ_0), we have

$$\frac{\partial}{\partial \lambda}(F(x,\lambda) = 0) \Longleftrightarrow D_x F(x,\lambda)\frac{dx}{d\lambda}(\lambda) + \frac{\partial F}{\partial \lambda}(x,\lambda) = 0 \Longleftrightarrow \frac{dx}{d\lambda}(\lambda) = -D_x F(x,\lambda)^{-1}\frac{\partial F}{\partial \lambda}(x,\lambda).$$

At (x_0, λ_0), a tangent vector to the curve is $\dot{x}_0 \stackrel{\text{def}}{=} \frac{dx}{d\lambda}(\lambda_0)$ and is obtained with the formula

$$\dot{x}_0 = -D_x F(x_0, \lambda_0)^{-1}\frac{\partial F}{\partial \lambda}(x_0, \lambda_0).$$

Once the tangent vector $\dot{x}_0$ is obtained, the predictor is defined by

$$\hat{x}_1 = x_0 + \Delta_\lambda \dot{x}_0.$$

Then, fixing $\lambda_1 = \lambda_0 + \Delta_\lambda$, we *correct* the predictor $\hat{x}_1$ using Newton's method

$$x_1^{(0)} = \hat{x}_1, \qquad x_1^{(n+1)} = x_1^{(n)} - \Big(D_x F(x_1^{(n)}, \lambda_1)\Big)^{-1} F(x_1^{(n)}, \lambda_1), \quad n \geq 0,$$

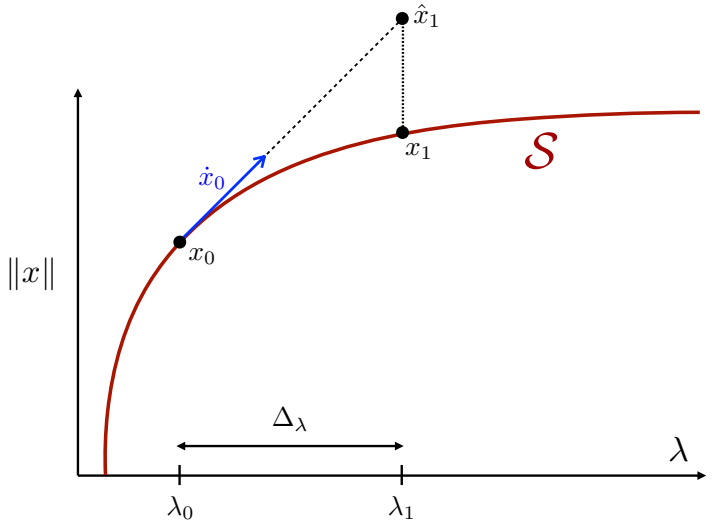

FIGURE 9. Parameter continuation.

to obtain the solution x_1 at λ_1 within the prescribed tolerance. We repeat this procedure iteratively to produce numerically a branch of solutions. We refer to Figure 9 to visualize one step of the parameter continuation algorithm.

Sometimes it may be more natural to parametrize the branches of solutions of (3) by arclength or *pseudo-arclength*, especially when the solution curve is not locally parametrized by λ, for instance at points where the jacobian matrix is singular. This is for instance what is happening when a saddle-node bifurcations (folds) occur. An example of such phenomenon is given by $F(x,\lambda) = x^2 - \lambda = 0$ at the point $(x_0,\lambda_0) = (0,0)$. Pseudo-arclength continuation, as opposed to parameter continuation, allows continuing past folds.

2.1.2. *Pseudo-Arclength Continuation.* In the pseudo-arclength continuation algorithm (e.g. see Keller [42]), the parameter value λ is no longer fixed and instead is left as a variable. The unknown variable is now $X = (x,\lambda)$. Consider the problem $F(X) = 0$ with the map $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$. As before, the process begins with a solution X_0 given within a prescribed tolerance. To produce a predictor, we compute first a unit tangent vector to the curve at X_0, that we denote $\dot{X}_0$, which can be computed using the formula

$$D_X F(X_0)\dot{X}_0 = \left[D_x F(\bar{x}_0, \bar{\lambda}_0) \ \frac{\partial F}{\partial \lambda}(x_0,\lambda_0) \right] \dot{X}_0 = 0 \in \mathbb{R}^n.$$

We now fix a *pseudo-arclength parameter* $\Delta_s > 0$, and set the predictor to be

$$\hat{X}_1 \stackrel{\text{def}}{=} \bar{X}_0 + \Delta_s \dot{X}_0 \in \mathbb{R}^{n+1}.$$

Once the predictor is fixed, we *correct* toward the set $\mathcal{S}$ on the hyperplane perpendicular to the tangent vector $\dot{X}_0$ which contains the predictor $\hat{X}_1$. The equation of this plan is given by

$$E(X) \stackrel{\text{def}}{=} (X - \hat{X}_1) \cdot \dot{X}_0 = 0.$$

Then, we apply Newton's method to the new function

$$X \mapsto \begin{pmatrix} E(X) \\ F(X) \end{pmatrix} \tag{4}$$

with the initial condition $\hat{X}_1$ in order to obtain a new solution X_1 given again within a prescribed tolerance. See Figure 10 for a geometric interpretation of one step of the pseudo-arclength continuation algorithm. At each step of the algorithm, the function defined in (4) changes since the plane $E(X) = 0$ changes. With this method, it is possible to continue past folds. Repeating this procedure iteratively produces a branch of solutions.

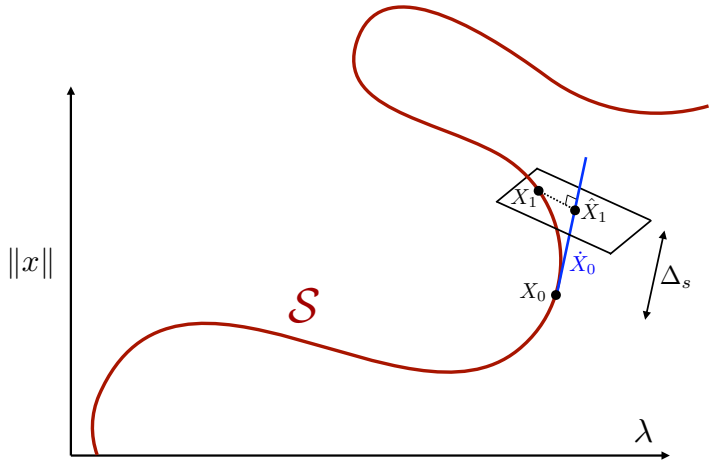

FIGURE 10. Pseudo-arclength continuation.

Remark 2.1. *The above mentioned algorithms do not cover the case of bifurcations of solutions e.g. symmetry-breaking pitchfork bifurcations, branch points, Hopf bifurcations, etc. We refer for instance to the work* [40] *for numerical continuation methods handling bifurcations.*

Now that we have briefly introduced two classical algorithms to numerically compute branches of solutions of the general problem (3), we present an approach that combines the strength of the numerical continuation methods with the ideas of rigorous computing (e.g. see [21]). Before introducing the rigorous continuation method in Section 2.3, we need some background from calculus in general Banach spaces.

2.2. **Background of Calculus in Banach Spaces.** The space of bounded linear operators is defined by

$$B(X,Y) \stackrel{\text{def}}{=} \left\{E\colon X \to Y \mid E \text{ is linear, } \|E\|_{B(X,Y)} < \infty\right\},$$

where $\|\cdot\|_{B(X,Y)}$ denotes the operator norm

$$\|E\|_{B(X,Y)} \stackrel{\text{def}}{=} \sup_{\|x\|_X=1} \|Ex\|_Y.$$

Note that $\left(B(X,Y), \|\cdot\|_{B(X,Y)}\right)$ is a Banach space.

Definition 2.2. A function $F\colon X \to Y$ is *Fréchet differentiable* at $x_0 \in X$ if there exists a bounded linear operator $E\colon X \to Y$ satisfying

$$\lim_{\|h\|_X \to 0} \frac{\|F(x_0+h) - F(x_0) - Eh\|_Y}{\|h\|_X} = 0.$$

The linear operator E is called the *derivative* of F at x_0 and denoted by $E = D_x F(x_0)$. We say that $F\colon X \to Y$ is a C^1 *mapping* if for every $x \in X$, F is Fréchet differentiable at x.

Given a point $x_0 \in X$ and a radius $r > 0$, denote by $B_r(x_0) \subset X$ the closed ball of radius r centered at x_0, that is

$$B_r(x_0) \stackrel{\text{def}}{=} \{x \in X \mid \|x - x_0\|_X \leq r\}.$$

The proof of the following version of the Mean Value Theorem can be found in [43].

Theorem 2.3 (**Mean Value Theorem**). *Let $x_0 \in X$ and suppose that $F\colon B_r(x_0) \subset X \to Y$ is a C^1 mapping. Let*

$$K \stackrel{\text{def}}{=} \sup_{x \in B_r(x_0)} \|D_x F(x)\|_{B(X,Y)}.$$

Then for any $x, y \in B_r(x_0)$ we have that

$$\|F(x) - F(y)\|_Y \leq K\|x - y\|_X.$$

While the following concept could be introduced more generally in the context of metric spaces, we present it in the context of Banach spaces to best suit our needs.

Definition 2.4. Suppose that Λ is a set of parameters. A function $T\colon X \times \Lambda \to X$ is a *uniform contraction* if there exists $\kappa \in [0, 1)$ such that, for all $x, y \in X$ and $\lambda \in \Lambda$,

$$\|T(x, \lambda) - T(y, \lambda)\|_X \leq \kappa\|x - y\|_X.$$

By the Contraction Mapping Theorem if $T\colon X \times \Lambda \to X$ is a uniform contraction, then for every $\lambda \in \Lambda$ there exists a unique $\tilde{x}_\lambda$ such that $T(\tilde{x}_\lambda, \lambda) = \tilde{x}_\lambda$. Thus the function $g\colon \Lambda \to X$ given by $g(\lambda) \stackrel{\text{def}}{=} \tilde{x}_\lambda$ is well defined. As the following theorem indicates this function inherits the same amount of differentiability than T. The proof can be found in [43].

Theorem 2.5 (**Uniform Contraction Theorem**). *Assume that the set of parameters Λ is a Banach space, and consider open sets $U \subset X$ and $V \subset \Lambda$. Assume that $T\colon \overline{U} \times V \to \overline{U}$ is a uniform contraction with contraction constant κ. Define $g\colon V \to \overline{U}$ by $T(g(\lambda), \lambda) = g(\lambda)$. If $T \in C^k(\overline{U} \times V, X)$, then $g \in C^k(V, X)$ for any $k \in \{1, 2, \ldots, \infty\}$.*

2.3. **The Rigorous Continuation Method.** Now that we have introduced some basic notions from calculus in Banach spaces, we are ready to present the general rigorous continuation method. The idea of the proposed approach is to prove the existence of true solution segments of $F(x, \lambda) = 0$ close to piecewise-linear segments of approximations by applying the Uniform Contraction Theorem (Theorem 2.5) over intervals of parameters. This approach has the advantage of being quite general and can be readily generalized to problems depending of several parameters (e.g. see Remark 2.9). However, the rigorous error bounds quickly deteriorate as the width of the interval of parameters (on which the uniform contraction theorem is applied) grows. This is due to the fact that piecewise-linear approximations are coarse approximations of the solution branches of nonlinear problems. Expanding the solutions using high order Taylor approximations in the parameter could for instance increase significantly the error bounds (e.g. see [44, 45]), at the cost of complicating the analysis. This being said, let us mention the existence of a growing literature on rigorous numerical methods to compute branches of parameterized families of solutions [31, 46, 47, 48, 49].

Assume that numerical approximations of (3) have been obtained at two different parameter values λ_0 and λ_1, namely there exists $(\bar{x}_0, \lambda_0)$ and $(\bar{x}_1, \lambda_1)$ such that $F(\bar{x}_0, \lambda_0) \approx 0$ and $F(\bar{x}_1, \lambda_1) \approx 0$. In other words, $(\bar{x}_0, \lambda_0)$ and $(\bar{x}_1, \lambda_1)$ are approximately in the solution set $\mathcal{S}$ (e.g. see Figure 11). The approximations can be computed first by considering a finite dimensional projection of F and then by using one of the two predictor-correctors algorithms presented in Section 2.1. We refer to Section 3.1.3 for an example in the context of periodic

solutions of DDEs. Define the set of *predictors* between the approximations $(\bar{x}_0, \lambda_0)$ and $(\bar{x}_1, \lambda_1)$ by

$$\{(\bar{x}_s, \lambda_s) \mid \bar{x}_s = (1-s)\bar{x}_0 + s\bar{x}_1 \quad \text{and} \quad \lambda_s = (1-s)\lambda_0 + s\lambda_1, \quad s \in [0,1]\}. \tag{5}$$

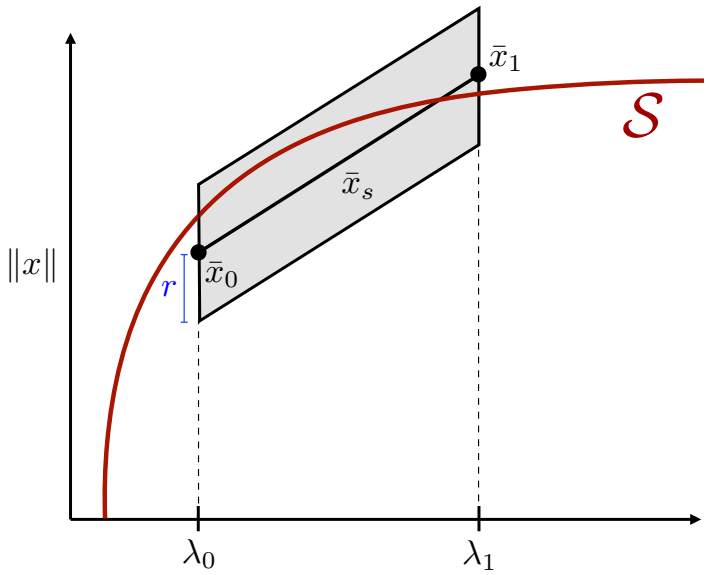

FIGURE 11. The set of predictors $\{(\bar{x}_s, \lambda_s) \mid s \in [0,1]\}$, approximating a segment of the solution set $\mathcal{S}$. The radii polynomial approach, when successful, provides a "tube" of with $r > 0$ (the shaded region) in $X \times \mathbb{R}$, where the true segment of solution curve is guaranteed to exist.

Consider bounded linear operators $A^\dagger \in B(X,Y)$ and $A \in B(Y,X)$. In practice, the operator $A^\dagger$ is chosen to be an approximation of $D_x F(\bar{x}_0, \lambda_0)$ while A is chosen to be an approximate inverse of $D_x F(\bar{x}_0, \lambda_0)$. Assume that A is injective and that

$$AF\colon X \times \mathbb{R} \to X. \tag{6}$$

The following theorem, often called the radii polynomial approach, is a twist of the standard Newton-Kantorovich theorem (e.g. see [50]).

Theorem 2.6 (Radii Polynomial Approach). *Assume that $F \in C^k(X \times \mathbb{R}, Y)$ with $k \in \{1, 2, \ldots, \infty\}$, and let $Y_0, Z_0, Z_1, Z_2 \geq 0$ satisfying*

$$\|AF(\bar{x}_s, \lambda_s)\|_X \leq Y_0, \quad \forall\ s \in [0,1] \tag{7}$$

$$\|I - AA^\dagger\|_{B(X,X)} \leq Z_0 \tag{8}$$

$$\|A[D_x F(\bar{x}_0, \lambda_0) - A^\dagger]\|_{B(X,X)} \leq Z_1, \tag{9}$$

$$\|A[D_x F(\bar{x}_s + b, \lambda_s) - D_x F(\bar{x}_0, \lambda_0)]\|_{B(X,X)} \leq Z_2(r), \quad \forall\ b \in B_r(0) \text{ and } \forall\ s \in [0,1]. \tag{10}$$

Define the radii polynomial

$$p(r) \overset{\text{def}}{=} Z_2(r)r + (Z_1 + Z_0 - 1)r + Y_0. \tag{11}$$

If there exists $r_0 > 0$ such that

$$p(r_0) < 0,$$

then there exists a C^k function

$$\tilde{x} : [0,1] \to \bigcup_{s\in[0,1]} B_{r_0}(\bar{x}_s)$$

such that

$$F(\tilde{x}(s), \lambda_s) = 0, \quad \forall\ s \in [0,1].$$

Furthermore, these are the only solutions in the tube $\bigcup_{s\in[0,1]} B_{r_0}(\bar{x}_s)$.

Proof. Recalling (6), define the operator $T\colon X \times [0,1] \to X$ by

$$T(x,s) = x - AF(x, \lambda_s).$$

We begin by showing that for each $s \in [0,1]$, the operator $T(\cdot, s)$ is a contraction mapping from $B_{r_0}(\bar{x}_s)$ into itself. Now, given $y \in B_{r_0}(\bar{x}_s)$ and applying the bounds (7), (8), (9), and (10), we obtain

$$\begin{aligned}
\|D_x T(y,s)\|_{B(X,X)} &= \|I - AD_xF(y,\lambda_s)\|_{B(X,X)} \\
&\le \|I - AA^\dagger\|_{B(X,X)} + \|A[D_xF(\bar{x}_0,\lambda_0) - A^\dagger]\|_{B(X,X)} \\
&\quad + \|A[D_xF(y,\lambda_s) - D_xF(\bar{x}_0,\lambda_0)]\|_{B(X,X)} \\
&\le Z_0 + Z_1 + Z_2(r_0). \qquad (12)
\end{aligned}$$

We now show that for each $s \in [0,1]$ the operator $T(\cdot, s)$ maps $B_{r_0}(\bar{x}_s)$ into itself. Let $y \in B_{r_0}(\bar{x}_s)$ and apply the Mean Value Theorem (Theorem 2.3) to obtain

$$\begin{aligned}
\|T(y,s) - \bar{x}_s\|_X &\le \|T(y,s) - T(\bar{x}_s,s)\|_X + \|T(\bar{x}_s,s) - \bar{x}_s\|_X \\
&\le \sup_{b\in B_{r_0}(\bar{x}_s)} \|D_xT(b,s)\|_{B(X,X)}\|y - \bar{x}_s\|_X + \|AF(\bar{x}_s,\lambda_s)\|_X \\
&\le (Z_0 + Z_1 + Z_2(r_0))r_0 + Y_0
\end{aligned}$$

where the last inequality follows from (12). Recalling (11) and using the assumption that $p(r_0) < 0$ implies that $\|T(y,s) - \bar{x}_s\|_X < r_0$ for all $s \in [0,1]$, the desired result.

Letting $a, b \in B_{r_0}(\bar{x}_s)$, apply the Mean Value Theorem and (12) to obtain

$$\begin{aligned}
\|T(a,s) - T(b,s)\|_X &\le \sup_{b\in B_{r_0}(\bar{x}_s)} \|D_xT(b,s)\|_{B(X,X)}\|a - b\|_X \\
&\le (Z_0 + Z_1 + Z_2 r_0)\|a - b\|_X. \qquad (13)
\end{aligned}$$

Again, from the assumption that $p(r_0) < 0$, it follows from $Y_0 \ge 0$ that

$$\kappa \overset{\text{def}}{=} Z_0 + Z_1 + Z_2(r_0) < 1 - \frac{Y_0}{r_0} \le 1, \qquad (14)$$

Define the operator

$$\begin{aligned}
\widetilde{T} : B_{r_0}(0) \times [0,1] &\to B_{r_0}(0) \\
(y,s) &\mapsto \widetilde{T}(y,s) \overset{\text{def}}{=} T(y + \bar{x}_s, s) - \bar{x}_s.
\end{aligned}$$

Consider now $x, y \in B_{r_0}(0)$ and $s \in [0,1]$. Then, since $x + \bar{x}_s, y + \bar{x}_s \in B_{r_0}(\bar{x}_s)$, we can use (13) and (14) to get

$$\begin{aligned}
\|\widetilde{T}(x,s) - \widetilde{T}(y,s)\|_X &= \|T(x + \bar{x}_s, s) - T(y + \bar{x}_s, s)\|_X \\
&\le \kappa\|x - y\|_X.
\end{aligned}$$

Since $\kappa < 1$, we conclude that $\widetilde{T} : B_{r_0}(0) \times [0,1] \to B_{r_0}(0)$ is a uniform contraction. By the Uniform Contraction Theorem (Theorem 2.5), there exists $g\colon [0,1] \to B_{r_0}(0)$ by

$$\widetilde{T}(g(s), s) = g(s).$$

Since $F \in C^k(X \times \mathbb{R}, Y)$, then $\widetilde{T} \in C^k(B_{r_0}(0) \times [0,1], B_{r_0}(0))$, and therefore $g \in C^k([0,1], B_{r_0}(0))$. Let

$$\tilde{x}(s) \stackrel{\text{def}}{=} g(s) + \bar{x}_s$$

so that for all $s \in [0,1]$

$$T(\tilde{x}(s), s) = T(g(s) + \bar{x}_s, s) = \widetilde{T}(g(s), s) + \bar{x}_s = g(s) + \bar{x}_s = \tilde{x}(s).$$

Since $T(x, s) = x - AF(x, \lambda_s)$, we get that

$$T(\tilde{x}(s), s) = \tilde{x}(s) - AF(\tilde{x}(s), \lambda_s) = \tilde{x}(s).$$

By assumption that A is injective,

$$F(\tilde{x}(s), \lambda_s) = 0, \quad \forall\ s \in [0,1].$$

It follows from $g \in C^k([0,1], B_{r_0}(0))$ that

$$\tilde{x} : [0,1] \to \bigcup_{s \in [0,1]} B_{r_0}(\bar{x}_s)$$

is a C^k function. Furthermore, it follows from the contraction mapping theorem that these are the only solutions in the *tube* $\bigcup_{s\in[0,1]} B_{r_0}(\bar{x}_s) \times [\lambda_0, \lambda_1]$. □

Theorem 2.6 provides a recipe to compute a local segment of solution curve and to obtain a uniform rigorous error bound r along the set of predictors connecting two numerical approximations $\bar{x}_0$ and $\bar{x}_1$. See Figure 11 for a representation of the region (shaded) where the true segment of solution curve is guaranteed to exist. Assume now that this argument has been repeated iteratively over the set $\{\bar{x}_0, \ldots, \bar{x}_j\}$ of approximations at the parameter values $\{\lambda_0, \ldots, \lambda_j\}$ respectively. For each $i = 0, \ldots, j-1$, this yields the existence of a unique portion of smooth solution curve $\mathcal{S}_i$ in a small *tube* centered at the segment $\{(1-s)\bar{x}_i + s\bar{x}_{i+1} \mid s \in [0,1]\}$. As the following results demonstrates, the set

$$\mathcal{S} \stackrel{\text{def}}{=} \bigcup_{i=0}^{j-1} \mathcal{S}_i$$

is a global smooth solution curve of $F(x, \lambda) = 0$.

Lemma 2.7 (Globalizing the C^k solution branch). *Assume that the radii polynomial approach was successfully applied (via Theorem 2.6) to show the existence of two C^k segments S_0 and S_1 of solution curves parameterized by the parameter λ over the respective parameter intervals $[\lambda_0, \lambda_1]$ and $[\lambda_1, \lambda_2]$. Assume that the sets of predictors are defined by the three points $\bar{x}_0$, $\bar{x}_1$ and $\bar{x}_2$ with $\bar{x}_i \in \mathbb{C}^{2m}$ for some fixed dimension m. Then the new segment of solution curve $S_0 \cup S_1$ is a C^k function of λ.*

Proof. The continuity of $S_0 \cup S_1$ follows from the fact that at the parameter value $\lambda = \lambda_1$, the solution segment S_0 must connect continuously with the solution segment S_1 by the existence and uniqueness result guaranteed by the Contraction Mapping Theorem. Let $p_0(r)$ the radii polynomial built with the predictors generated by $\bar{x}_0$ and $\bar{x}_1$, and defined by the bounds Y_0,

Z_0, Z_1 and Z_2. Let $r_0 > 0$ such that $p_0(r_0) < 0$. By continuity of the radii polynomial p_0, there exists $\delta_0 > 0$ and there exist bounds $\tilde{Y}_0(\delta_0)$ and $\tilde{Z}_2(r, \delta_0)$ such that

$$\|AF(\bar{x}_s, \lambda_s)\|_X \leq \tilde{Y}_0(\delta_0), \quad \forall\ s \in [-\delta_0, 1+\delta_0]$$

$$\|A[D_xF(\bar{x}_s + b, \lambda_s) - D_xF(\bar{x}_0, \lambda_0)]\|_{B(X,X)} \leq \tilde{Z}_2(r, \delta_0), \quad \forall\ b \in B_r(0),\ \forall\ s \in [-\delta_0, 1+\delta_0],$$

and such that

$$\tilde{p}_0(r_0) \overset{\text{def}}{=} \tilde{Z}_2(r_0, \delta_0)r_0 + (Z_1 + Z_0 - 1)r_0 + \tilde{Y}_0(\delta_0) < 0.$$

Then, there exists of a C^k branch of solution curve parameterized by λ over the range $\{(1-s)\lambda_0 + s\lambda_1 \mid s \in [-\delta_0, 1+\delta_0]$, extending smoothly (in fact in a C^k way) the segment S_0 on both sides. Similarly, there exists δ_1 such that the segment S_1 can be extended smoothly over the parameter range $\{(1-s)\lambda_1 + s\lambda_2 \mid s \in [-\delta_0, 1+\delta_0]$. This implies that there is a C^k overlap between S_0 and S_1. □

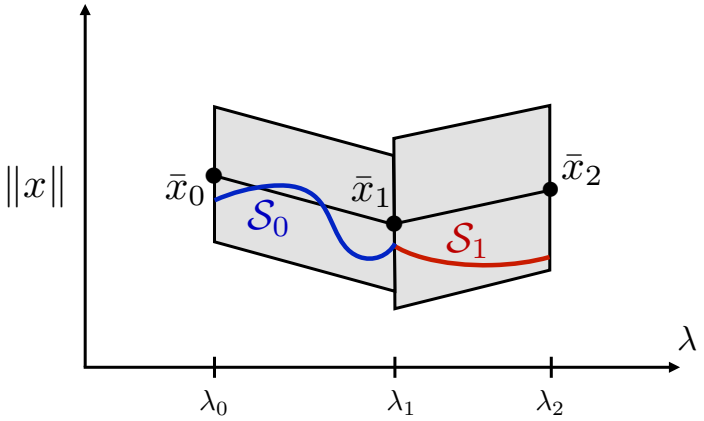

FIGURE 12. Assume that S_0 and S_1 are computed with the radii polynomial approach with predictors defined by three points $\bar{x}_0$, $\bar{x}_1$ and $\bar{x}_2$ with $\bar{x}_i \in \mathbb{C}^{2m}$ for some fixed dimension m. Then the following situation is not possible: a piecewise smooth but not globally smooth piece of solution curve.

Note that argument of using the continuity of the radii polynomial in the proof of Lemma 2.7 is not new and has been used in the previous works [31, 48, 49].

Repeating iteratively the argument of Lemma 2.7 leads to the existence of a smooth solution curve $\mathcal{S}$ of $F = 0$ near the piecewise linear curve of approximations, as portrayed in Figure 13.

Remark 2.8 (**Bifurcations**). *In the present lecture notes we do not discuss how to handle some type bifurcations. Instead, we refer to the lecture notes of Thomas Wanner, where a rigorous computational method to prove existence of saddle-node bifurcations and symmetry-breaking pitchfork bifurcations is presented.*

Remark 2.9 (**Number of Parameters and Multi-parameter Continuation**). *In these lecture notes, we present the ideas in the context of equations depending on a single parameter $\lambda \in \mathbb{R}$. However, the radii polynomial approach (as presented below in Theorem 2.6) works also for problems depending on $p > 1$ parameters. In fact, the method can be trivially extended to prove existence of "solution manifolds" within solutions sets of the form $\{(x, \Lambda) \in X \times \mathbb{R}^p \mid F(x, \Lambda) = 0\}$. The only difference is that the bounds which need to be computed to apply the*

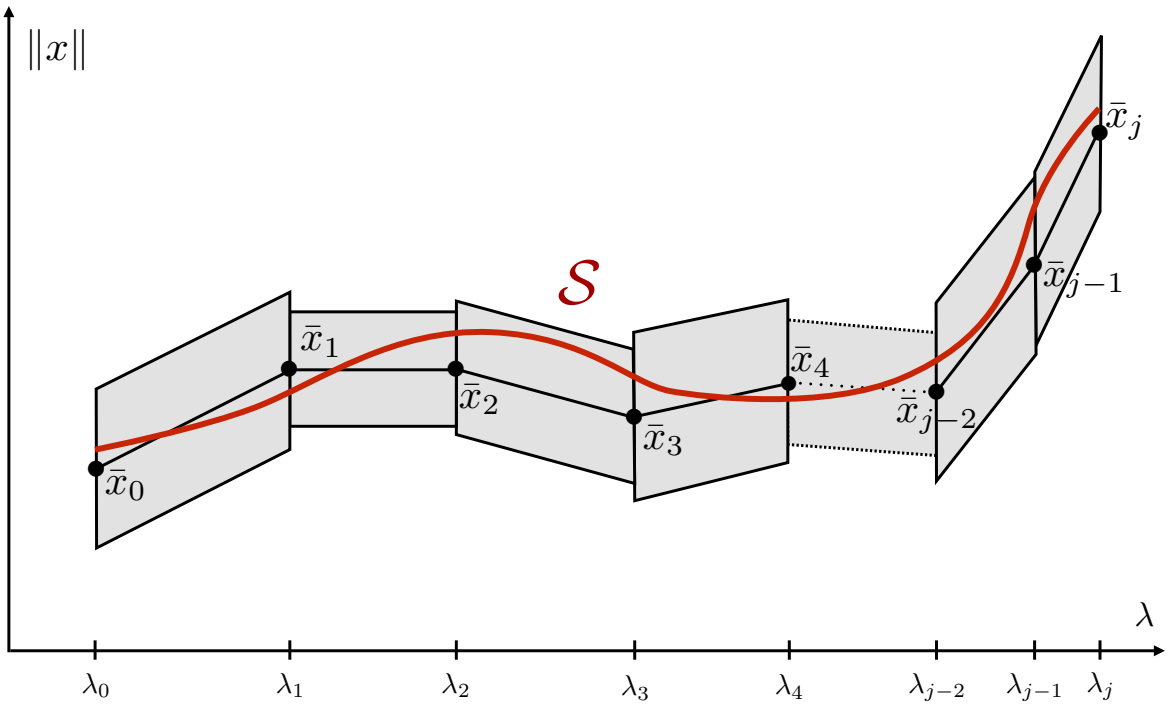

FIGURE 13. Computing rigorously a global branch of solutions.

uniform contraction theorem have to be obtained uniformly over a compact set of parameters in $\mathbb{R}^p$ instead of in $\mathbb{R}$. A more advanced approach based on a rigorous multi-parameter continuation method, generalizing the concept of pseudo-arclength continuation, is introduced in [49] *to compute solutions manifolds and to handle higher dimensional folds.*

Remark 2.10 (**Parameter Continuation vs Pseudo-Arclength Continuation**). *The method of Theorem 2.6 is based on parameter continuation: we compute branches of solutions parametrized by the parameter λ. The method can be extended to pseudo-arclength continuation where solutions are parametrized by pseudo-arclength (e.g. see* [48, 31]*).*

Remark 2.11 (**Computing the bound Y_0**). *To compute the Y_0 bound satisfying* (7)*, denote*

$$\Delta\bar{x} \stackrel{\text{def}}{=} \bar{x}_1 - \bar{x}_0 \quad \textit{and} \quad \Delta\lambda \stackrel{\text{def}}{=} \lambda_1 - \lambda_0,$$

and consider the expansion

$$F(\bar{x}_s, \lambda_s) = F(\bar{x}_0, \lambda_0) + \left[D_x F(\bar{x}_0, \lambda_0) \quad \frac{\partial F}{\partial \lambda}(\bar{x}_0, \lambda_0) \right] \begin{pmatrix} \Delta\bar{x} \\ \Delta\lambda \end{pmatrix} s + \frac{1}{2} \left(\frac{\partial^2}{\partial s^2} F(\bar{x}_s, \lambda_s)\Big|_{s=0} \right) s^2 + h.o.t.$$

Denote

$$y_1 \stackrel{\text{def}}{=} \begin{bmatrix} D_x F(\bar{x}_0, \lambda_0) & \dfrac{\partial F}{\partial \lambda}(\bar{x}_0, \lambda_0) \end{bmatrix} \begin{pmatrix} \Delta \bar{x} \\ \Delta \lambda \end{pmatrix} \tag{15}$$

$$y_2 \stackrel{\text{def}}{=} \frac{1}{2} \left(\frac{\partial^2}{\partial s^2} F(\bar{x}_s, \lambda_s)\Big|_{s=0} \right). \tag{16}$$

Hence,

$$\|AF(\bar{x}_s, \lambda_s)\|_X \leq \|AF(\bar{x}_0, \lambda_0)\|_X + \|Ay_1\|_X + \|Ay_2\|_X + \delta$$

where the extra term $\delta \geq 0$ can be obtained using Taylor remainder's theorem.

2.4. **A Finite Dimensional Example.** In this section, we apply the radii polynomial approach (Theorem 2.6) to prove the existence of branches of solutions of the problem (3) with F a mapping between finite dimensional Banach spaces.

The example we consider is the problem of computing branches of steady states for the atmospheric circulation model introduced by Edward N. Lorenz in [51]

$$\begin{cases} x_1' = -\alpha x_1 - x_2^2 - x_3^2 + \alpha\lambda \\ x_2' = -x_2 + x_1 x_2 - \beta x_1 x_3 + \gamma \\ x_3' = -x_3 + \beta x_1 x_2 + x_1 x_3. \end{cases} \tag{17}$$

Let us fix $\alpha = 0.25$, $\beta = 4$ and $\gamma = 0.5$, and leave λ as a parameter. At these parameter values, equilibria of (17) are solutions of

$$F(x, \lambda) \stackrel{\text{def}}{=} \begin{pmatrix} -\frac{1}{4} x_1 - x_2^2 - x_3^2 + \frac{\lambda}{4} \\ -x_2 + x_1 x_2 - 4 x_1 x_3 + \frac{1}{2} \\ -x_3 + 4 x_1 x_2 + x_1 x_3 \end{pmatrix} = 0. \tag{18}$$

In this case, the Banach spaces are $X = Y = \mathbb{R}^3$ endowed with the sup-norm

$$\|x\| = \max(|x_1|, |x_2|, |x_3|).$$

At $\lambda_0 = 0.8$ and $\lambda_1 = 0.85$, we used Newton's method to compute respectively

$$\bar{x}_0 = \begin{pmatrix} -0.056551859183890 \\ 0.452495729654079 \\ -0.096879200248534 \end{pmatrix} \quad \text{and} \quad \bar{x}_1 = \begin{pmatrix} -0.043505480122129 \\ 0.466188932513298 \\ -0.077744769808933 \end{pmatrix}.$$

We wish to use Theorem 2.6 to prove the existence of a segment of solutions in the solution set $\mathcal{S} = \{(x, \lambda) \in \mathbb{R}^4 \mid F(x, \lambda) = 0\}$. For $s \in [0, 1]$, recall the set of predictors (5) given by $\bar{x}_s = (1 - s)\bar{x}_0 + s\bar{x}_1$ and let $\lambda_s = (1 - s)\lambda_0 + s\lambda_1$. Denote by

$$\Delta \bar{x} \stackrel{\text{def}}{=} \bar{x}_1 - \bar{x}_0 \quad \text{and} \quad \Delta\lambda \stackrel{\text{def}}{=} \lambda_1 - \lambda_0.$$

Recalling the Y_0 bound satisfying (7). Since the vector field is quadratic, recalling (15) and (16), we get the following expansion

$$F(\bar{x}_s, \lambda_s) = F(\bar{x}_0, \lambda_0) + y_1 s + y_2 s^2,$$

where

$$y_2 = \begin{pmatrix} -(\Delta\bar{x})_2^2 - (\Delta\bar{x})_3^2 \\ (\Delta\bar{x})_1(\Delta\bar{x})_2 - 4(\Delta\bar{x})_1(\Delta\bar{x})_3 \\ 4(\Delta\bar{x})_1(\Delta\bar{x})_2 + (\Delta\bar{x})_1(\Delta\bar{x})_3 \end{pmatrix}.$$

The matrix $A \approx DF(\bar{x}_0, \lambda_0)^{-1}$ is computed using MATLAB and is given by

$$A = \begin{pmatrix} -1.031444307007117 & 0.883485538811585 & 0 \\ -1.126473748855484 & 0.059891724595854 & -0.193758400497068 \\ -1.431216390563831 & 1.419669460728974 & -0.904991459308157 \end{pmatrix}.$$

Using the definition of A, y_1 and y_2, we compute $Y_0 = 0.002488451115105$. For this finite dimensional example, we set $A^\dagger = DF(\bar{x}_0, \lambda_0)$, so that $Z_1 = 0$ in (9). Recalling (8), we set $Z_0 = \|I - ADF(\bar{x}_0, \lambda_0)\|_\infty$. In this case, we computed $Z_0 = 2.27 \times 10^{-16}$. To facilitate the computation of Z_2 satisfying (10), consider $c \in B_1(0) \subset \mathbb{R}^3$, that is $\|c\|_\infty \leq 1$, and consider $b \in B_r(0) \subset \mathbb{R}^3$, that is $\|b\|_\infty \leq r$. Then,

$$\begin{aligned}[D_x F(\bar{x}_s + b, \lambda_s) - D_x F(\bar{x}_0, \lambda_0)]c = &\begin{pmatrix} -2b_2c_2 - 2b_3c_3 \\ b_1c_2 - 4b_1c_3 + b_2c_1 - 4b_3c_1 \\ 4b_1c_2 + b_1c_3 + 4b_2c_1 + b_3c_1 \end{pmatrix} \\ &+ s \begin{pmatrix} -2c_2(\Delta\bar{x})_2 - 2c_3(\Delta\bar{x})_3 \\ c_1(\Delta\bar{x})_2 - 4c_1(\Delta\bar{x})_3 + c_2(\Delta\bar{x})_1 - 4c_3(\Delta\bar{x})_1 \\ 4c_1(\Delta\bar{x})_2 + c_1(\Delta\bar{x})_3 + 4c_2(\Delta\bar{x})_1 + c_3(\Delta\bar{x})_1 \end{pmatrix},\end{aligned}$$

and since $|s| \leq 1$, we get the component-wise inequalities

$$|[D_x F(\bar{x}_s + b, \lambda_s) - D_x F(\bar{x}_0, \lambda_0)]c| \leq \begin{pmatrix} 4 \\ 10 \\ 10 \end{pmatrix} r + \begin{pmatrix} 2|(\Delta\bar{x})_2| + 2|(\Delta\bar{x})_3| \\ |(\Delta\bar{x})_2| + 4|(\Delta\bar{x})_3| + 5|(\Delta\bar{x})_1| \\ 4|(\Delta\bar{x})_2| + |(\Delta\bar{x})_3| + 5|(\Delta\bar{x})_1| \end{pmatrix}.$$

Hence, letting

$$Z_2^{(1)} \stackrel{\text{def}}{=} \left\| |A| \begin{pmatrix} 4 \\ 10 \\ 10 \end{pmatrix} \right\|_\infty \quad \text{and} \quad Z_2^{(0)} \stackrel{\text{def}}{=} \left\| |A| \begin{pmatrix} 2|(\Delta\bar{x})_2| + 2|(\Delta\bar{x})_3| \\ |(\Delta\bar{x})_2| + 4|(\Delta\bar{x})_3| + 5|(\Delta\bar{x})_1| \\ 4|(\Delta\bar{x})_2| + |(\Delta\bar{x})_3| + 5|(\Delta\bar{x})_1| \end{pmatrix} \right\|_\infty$$

we can set

$$Z_2(r) = Z_2^{(1)} r + Z_2^{(0)}.$$

Numerically, we obtain $Z_2^{(1)} = 28.971474762626638$ and $Z_2^{(0)} = 0.440592442217924$. Recalling (11) and that $Z_1 = 0$, the radii polynomial is given by

$$\begin{aligned} p(r) &= Z_2(r)r + (Z_0 - 1)r + Y_0 \\ &= Z_2^{(1)} r^2 + (Z_2^{(0)} + Z_0 - 1)r + Y_0 \\ &= 28.971474762626638 r^2 - 0.559407557782076 r + 0.002488451115105. \end{aligned}$$

Note that

$$\mathcal{I} \stackrel{\text{def}}{=} \{r > 0 \mid p(r) < 0\} \supset [0.006949765480451\ ,\ 0.012359143105166].$$

Choosing for instance $r_0 = 0.007 \in \mathcal{I}$, then by Theorem 2.6, there exists a C^∞ function

$$\tilde{x} : [0,1] \to \bigcup_{s \in [0,1]} B_{r_0}(\bar{x}_s)$$

such that $F(\tilde{x}(s), \lambda_s) = 0$ for all $s \in [0,1]$ with F given in (18), and these are the only solutions in the set $\bigcup_{s\in[0,1]} B_{r_0}(\bar{x}_s) \times [\lambda_0, \lambda_1]$.

Also, we applied the method on the intervals of parameters $[\lambda_1, \lambda_2]$ and $[\lambda_2, \lambda_3]$ corresponding respectively to the segments $\{(1-s)\bar{x}_1 + s\bar{x}_2 \mid s \in [0,1]\}$ and $\{(1-s)\bar{x}_2 + s\bar{x}_3 \mid s \in [0,1]\}$, with $\lambda_2 = 0.89$, $\lambda_3 = 0.925$, and

$$\bar{x}_2 = \begin{pmatrix} -0.032746172312211 \\ 0.476481288735590 \\ -0.060432810317938 \end{pmatrix} \quad \text{and} \quad \bar{x}_3 = \begin{pmatrix} -0.022963874112357 \\ 0.484866398590268 \\ -0.043537846136356 \end{pmatrix}.$$

The MATLAB program `script_proof_lorenz.m` available at [67] performs the above computations. It uses the interval arithmetic package INTLAB developed by Siegfried M. Rump [52].

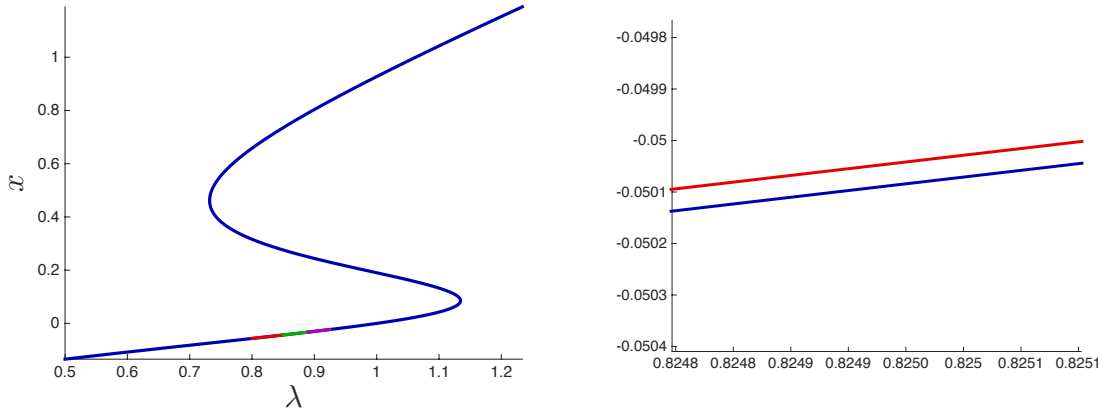

FIGURE 14. **(Left)** A branch of equilibria for the model (17) computed using the pseudo-arclength continuation algorithm as presented in Section 2.1.2. The segments in red, green and purple were rigorously computed with the radii polynomial approach. The respective error bounds between the predictors and the actual solution segments are $r_0 = 7 \times 10^{-3}$ (red), $r_0 = 4.2 \times 10^{-3}$ (green) and $r_0 = 3.9 \times 10^{-3}$ (purple). **(Right)** A zoom-in on the branch where the proof was performed at $\lambda = 0.825$.

Remark 2.12 (**Proofs at fixed parameter values**). *It is important to recall that the purpose of the present section is to introduce a method to compute "branches" of solutions. If however we are interested in proving the existence of a solution at a fixed parameter value, then we can get dramatically better error bounds. Let us do this exercise for model* (17) *with the approximation* $\bar{x}_0$. *In this case,* $\Delta\bar{x} = 0 \in \mathbb{R}^3$, $\Delta\lambda = 0$ *and* $Z_2^{(0)} = 0$, *the radii polynomial is* $p(r) = 28.971474762626652r^2 + (2.195852227948092 \times 10^{-15} - 1)r + 2.449129914171945 \times 10^{-16}$, *and*

$$\mathcal{I} \overset{\text{def}}{=} \{r > 0 \mid p(r) < 0\} \supset \left[2.45 \times 10^{-16}\ ,\ 0.034516710253563\right].$$

Therefore, there exists a unique $\tilde{x}_0 \in B_{2.45\times10^{-16}}(\bar{x}_0)$ *such that* $F(\tilde{x}_0, \lambda_0) = 0$. *In this case, the rigorous error bound is of the order of* 10^{-16}, *as opposed to* 10^{-3} *for the branch of solutions.*

We are now ready to present the main application of the rigorous continuation method.

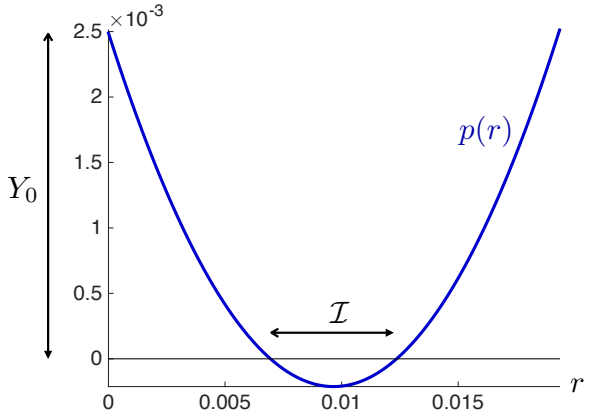

FIGURE 15. The radii polynomial $p(r) = Z_2^{(1)} r^2 + (Z_2^{(0)} + Z_0 - 1)r + Y_0$ associated to the numerical approximations $\bar{x}_0$ and $\bar{x}_1$ as defined above.

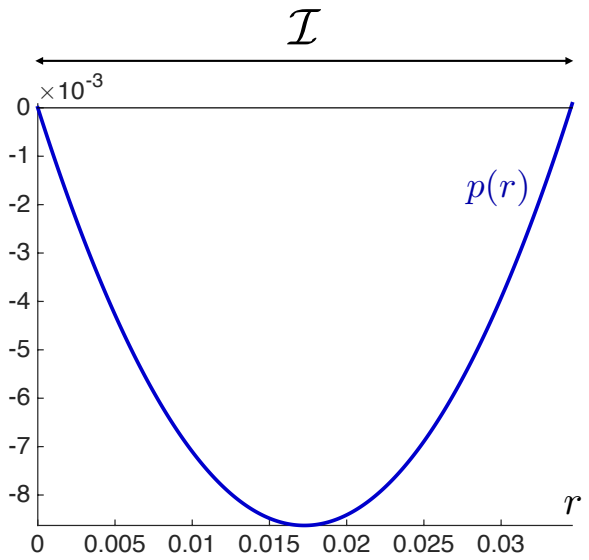

FIGURE 16. The radii polynomial $p(r) = Z_2^{(1)} r^2 + (Z_0 - 1)r + Y_0$ associated to the single numerical approximation $\bar{x}_0$.

3. Computing Branches of Periodic Solutions of Delay Equations

In this section, we show how the radii polynomial approach (Theorem 2.6) can be used to compute rigorously branches of periodic solutions of DDEs. Rather than presenting the ideas for general classes of problems, we focus on presenting the ideas for specific examples, namely for a delayed Van der Pol equation and for Wright's equation. For the delayed Van der Pol equation, we present in details in Section 3.1 all steps, bounds, necessary estimates, choices of function spaces and the explicit coefficients of the radii polynomial, whereas in the case of Wright's equation, we only briefly discuss some results in Section 3.2 and refer to [12] for more details. Note that the ideas presented here should be applicable to systems of N

delay equations of the form

$$y^{(n)}(t) = \mathcal{F}\left(y(t), y(t-\tau_1), \ldots, y(t-\tau_d), \ldots, y^{(n-1)}(t-\tau_1), \ldots, y^{(n-1)}(t-\tau_d)\right), \tag{19}$$

where $y : \mathbb{R} \to \mathbb{R}^N$ and $\mathcal{F} : \mathbb{R}^{N(nd+1)} \to \mathbb{R}^N$ is a multivariate polynomial.

As already mentioned in Section 1 studying rigorously solutions of DDEs is a challenging problem, especially because they are naturally defined on infinite dimensional function spaces. On the other hand, as already mentioned in Section 1, for continuous dynamical systems like (19), individual solutions which exist globally in time are more regular than the typical functions of the natural phase space. That suggests that solving for the Fourier coefficients of the periodic solutions of (19) in a Banach space of fast decaying sequences is a good strategy. In fact periodic solutions of (19) are analytic since $\mathcal{F}$ is analytic (it is polynomial) [10]. We therefore apriori know that the Fourier coefficients of the periodic solutions decay geometrically by the Paley-Wiener Theorem. Before presenting the rigorous numerical method, we briefly describe different methods used to study periodic solutions of (19), following closely the discussion in [53].

Fixed point theory, the fixed point index and global bifurcation theorems are powerful tools to study the existence of solutions of infinite dimensional dynamical systems. To give a few examples in the context of DDEs, the ejective fixed point theorem of Browder [54] and the fixed point index can be used to prove existence of nontrivial periodic solutions [14, 55, 56], and the global bifurcation theorem of Rabinowitz [57] can be used to prove the existence and characterize the (non) compactness of global branches of periodic solutions [58, 59]. This heavy machinery from functional analysis provides powerful existence results about solutions of DDEs, but its applicability may decrease if one asks more specific questions about the solutions of a given equation. For example, it appears difficult in general to use the ejective fixed point theorem to quantify the number of periodic solutions or to use a global bifurcation theorem to conclude about existence of folds, or more generally, of secondary bifurcations. The following Figure 17 taken from [29] shows that global branches of periodic solutions of DDEs may be complicated, as various bifurcations may occur on the branches.

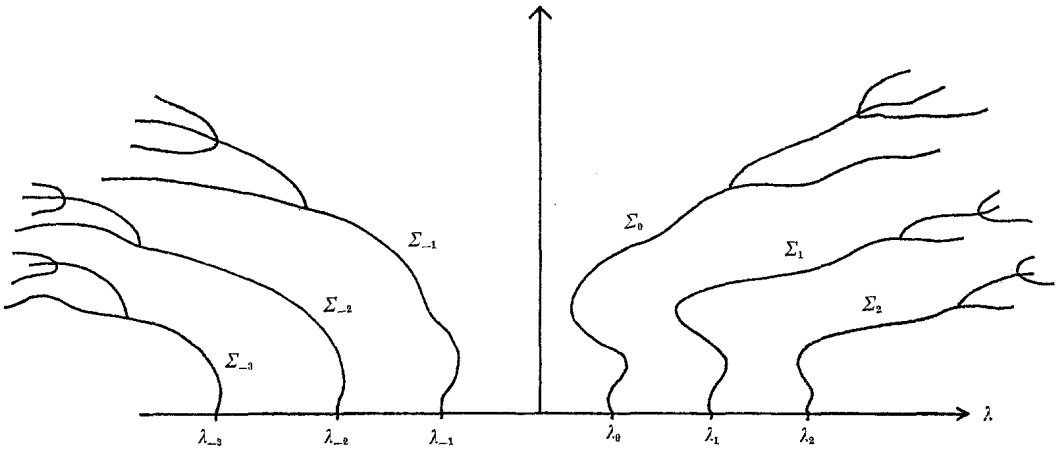

FIGURE 17. Global branches of periodic solutions of delay differential equations. The picture is taken directly from [29].

3.1. **A Delayed Van der Pol Equation.** In the 1920s, the Dutch electrical engineer and physicist Van der Pol proposed his famous *Van der Pol equations* to model oscillations of some electric circuits [60]. Since then, variants of the so-called van der Pol oscillator have been proposed as mathematical models of various real-world processes exhibiting limit cycles when the rate of change of the state variables depend only on their current states. However, there are many processes where this relation is also influenced by past values of the system in question. To model these processes, one may want to consider the use of functional differential equations, see [9, 61, 62].

In [63], Grafton establishes existence of periodic solutions to

$$\ddot{y}(t) - \varepsilon \dot{y}(t)(1 - y^2(t)) + y(t - \tau) = 0, \quad \varepsilon, \tau > 0,$$

a Van der Pol equation with a retarded position variable. His results are based on his periodicity results developed in [64]. In [65], using slightly different notations, Roger Nussbaum considered the more general class of equations

$$\ddot{y}(t) - \varepsilon \dot{y}(t)(1 - y^2(t)) + y(t - \tau) - \lambda y(t) = 0, \quad \varepsilon, \tau > 0, \lambda \in \mathbb{R}, \tag{20}$$

and establishes existence of periodic solution of period greater than 2τ, given that $\lambda < 0$, $-\lambda\tau < \varepsilon$ and $-\frac{1}{2}\lambda\tau^2 \leq 1$. We refer to (20) as Nussbaum's equation. The techniques that Nussbaum uses are sophisticated fixed point arguments, however, as he remarks, he has to restrict the size of $|\lambda|$ in order to guarantee that the zeros of y are at least a distance τ apart (e.g. see p. 287 of [65]). Moreover, he mentions that numerical simulations suggest the existence of periodic solutions to (20) for a large range of $\lambda < 0$.

What we present now is an application of the rigorous continuation method introduced in Section 2, and we establish existence results for periodic solutions to (20) for parameter values outside of the range of parameters accessible with the above results of Nussbaum. The first step is to recast looking for periodic solutions of (20) as one of the form $F(x, \lambda) = 0$.

3.1.1. *Setting up the $F(x, \lambda) = 0$ problem.* Assume that $y(t)$ is a periodic solution of (20) of period $p > 0$. Then

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega t}, \tag{21}$$

where $\omega = \frac{2\pi}{p}$ and the $a_k \in \mathbb{C}$ are the complex Fourier coefficients. Given a complex number $z \in \mathbb{C}$, denote by $\mathrm{conj}(z)$ the complex conjugate of z. During the rigorous continuation approach, we will verify a posteriori that the complex numbers satisfy $a_{-k} = \mathrm{conj}(a_k)$ in order to get that $y \in \mathbb{R}$. As the frequency ω of (21) is not known *a-priori*, it is left as a variable. Formally, using (21)

$$\dot{y}(t) = \sum_{k=-\infty}^{\infty} a_k ik\omega e^{ik\omega t}, \quad \ddot{y}(t) = \sum_{k=-\infty}^{\infty} -a_k k^2\omega^2 e^{ik\omega t} \quad \text{and} \quad y(t-\tau) = \sum_{k=-\infty}^{\infty} a_k e^{-ik\omega\tau} e^{ik\omega t}.$$

Thus (20) becomes

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \left[-k^2\omega^2 - \varepsilon ik\omega - \lambda + e^{-ik\omega\tau}\right] a_k e^{ik\omega t} \\ &\qquad + \varepsilon \sum_{k_1=-\infty}^{\infty} a_{k_1} e^{ik_1\omega t} \sum_{k_2=-\infty}^{\infty} a_{k_2} e^{ik_2\omega t} \sum_{k_3=-\infty}^{\infty} a_{k_3} ik_3\omega e^{ik_3\omega t} = 0. \end{aligned} \tag{22}$$

To obtain the Fourier coefficients in (21), one takes the inner product on both sides of (22) with $e^{ik\omega t}$, yielding that for all $k \in \mathbb{Z}$,

$$g_k \stackrel{\text{def}}{=} \left[-k^2\omega^2 - \varepsilon ik\omega - \lambda + e^{-ik\omega\tau}\right] c_k + i\varepsilon\omega \sum_{k_1+k_2+k_3=k} a_{k_1}a_{k_2}a_{k_3}k_3 = 0.$$

Observing that

$$S_k \stackrel{\text{def}}{=} \sum_{\substack{k_1+k_2+k_3=k\\ k_j\in\mathbb{Z}}} a_{k_1}a_{k_2}a_{k_3}k_3 = \sum_{\substack{k_1+k_2+k_3=k\\ k_j\in\mathbb{Z}}} a_{k_1}a_{k_2}a_{k_3}(k-k_1-k_2) = k\sum_{\substack{k_1+k_2+k_3=k\\ k_j\in\mathbb{Z}}} a_{k_1}a_{k_2}a_{k_3} - 2S_k$$

we get that

$$S_k = \frac{k}{3}\sum_{\substack{k_1+k_2+k_3=k\\ k_j\in\mathbb{Z}}} a_{k_1}a_{k_2}a_{k_3}$$

and therefore

$$g_k = \mu_k a_k + \frac{i\varepsilon k\omega}{3}(a^3)_k, \tag{23}$$

where

$$\mu_k = \mu_k(\omega,\lambda) \stackrel{\text{def}}{=} -k^2\omega^2 - \varepsilon ik\omega - \lambda + e^{-ik\omega\tau} \quad \text{and} \quad (a^3)_k \stackrel{\text{def}}{=} \sum_{\substack{k_1+k_2+k_3=k\\ k_j\in\mathbb{Z}}} a_{k_1}a_{k_2}a_{k_3}.$$

Denote $a = (a_k)_{k\in\mathbb{Z}}$ and $x = (\omega, a)$ the infinite dimensional vector of unknowns. Denote $g(x,\lambda) = (g_k(x,\lambda))_{k\in\mathbb{Z}}$. In order to eliminate any arbitrary time shift, we append a *phase condition* given by

$$\eta(x) = \sum_{|k|<n} c_k = 0, \tag{24}$$

for some $n \in \mathbb{N}$, which ensures that $y(0) \approx 0$. Combining (24) and (23), we let

$$F(x,\lambda) \stackrel{\text{def}}{=} \begin{pmatrix} \eta(x) \\ g(x,\lambda) \end{pmatrix}. \tag{25}$$

Let us now introduce the Banach space in which we look for solutions of $F(x,\lambda) = 0$ with F given in (25).

Since periodic solutions of analytic delay differential equations are analytic (e.g. see [10]), their Fourier coefficients decay geometrically. This will motivate the choice of Banach space in which we will embed the Fourier coefficients. Given a weight $\nu > 1$, define the sequence space

$$\ell^1_\nu = \{a = (a_k)_{k\in\mathbb{Z}} \mid \|a\|_{1,\nu} < \infty\}, \tag{26}$$

where

$$\|a\|_{1,\nu} \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} |a_k|\nu^{|k|}. \tag{27}$$

Define the Banach space

$$X \stackrel{\text{def}}{=} \mathbb{C}\times\ell^1_\nu$$

endowed with the norm

$$\|x\|_X \stackrel{\text{def}}{=} \max\left(|\omega|, \|a\|_{1,\nu}\right).$$

Note that F does not map X into itself. This is because a differential operator is in general unbounded on ℓ^1_ν. In order to overcome this problem we look for an injective linear "smoothing" operator A such that (6) holds, that $AF(x,\lambda) \in X$ for all $x \in X$ and $\lambda \in \mathbb{R}$. The choice of the approximate inverse A is presented in Section 3.1.4. For now we take A as given and define the *Newton-like operator* by

$$T(x,s) = x - AF(x,\lambda_s), \tag{28}$$

for $s \in [0,1]$. Given $s \in [0,1]$, the injectivity of A implies that x is a solution of $F(x,\lambda_s) = 0$ if and only if it is a fixed point of $T(\cdot,s)$. Moreover $T(\cdot,s)$ now maps X back into itself.

3.1.2. *Symmetry of the fixed points of T.* We are interested in showing the existence of a real periodic solution y given by (21) satisfying the symmetry property

$$y(t+p/2) = -y(t), \quad \forall\ t \in \mathbb{R}.$$

In Fourier space, this means that the Fourier coefficients satisfy the relation

$$a_{2j} = 0,\ \forall\ j \in \mathbb{Z}. \tag{29}$$

To do this, we design the method so that fixed points of T are in the *symmetry* space

$$X_{sym} \stackrel{\text{def}}{=} \mathbb{R} \times \tilde{\ell}^1_\nu, \tag{30}$$

where

$$\tilde{\ell}^1_\nu \stackrel{\text{def}}{=} \left\{ a \in \ell^1_\nu \,|\, a_{-k} = \text{conj}(a_k)\ \forall\ k \in \mathbb{Z},\ \text{ and }\ a_{2j} = 0\ \forall\ j \in \mathbb{Z} \right\}. \tag{31}$$

Remark 3.1. *The condition $a_{-k} = \text{conj}(a_k)$ is imposed in the function space ℓ^1_ν because we want y to be a real periodic solution, that is* $\text{conj}(y(t)) = y(t)$.

Lemma 3.2. *Assume that $\bar{x}_s \in X_{sym}$ and consider the closed ball $B_r(\bar{x}_s) \subset X$. Define T as in* (28) *and assume that the approximate inverse A satisfies*

$$AF : X_{sym} \times \mathbb{R} \to X_{sym}. \tag{32}$$

Assume that for every $s \in [0,1]$, $T(\cdot,s) : B_r(\bar{x}_s) \to B_r(\bar{x}_s)$ is a contraction, and let $\tilde{x}_s \in X$ the unique fixed point of $T(\cdot,s)$ in $B_r(\bar{x}_s)$ which exists by the Contraction Mapping Theorem. Then, $\tilde{x}_s \in X_{sym}$ for all $s \in [0,1]$.

Proof. By (32), $T : X_{sym} \times [0,1] \to X_{sym}$. Using that $\bar{x}_s \in X_{sym} \cap B_r(\bar{x}_s)$, and that X_{sym} is a closed subset of X for every $s \in [0,1]$, we obtain that

$$\tilde{x}_s = \lim_{n\to\infty} T^n(\bar{x}_s, s) \in X_{sym}. \qquad \square$$

3.1.3. *Computation of the numerical approximations.* We are ready to compute numerical approximations of $F(x,\lambda) = 0$ with F given in (25). Since the operator F is defined on an infinite dimensional space, this requires considering a finite dimensional projection. Given $a = (a_k)_{k\in\mathbb{Z}} \in \ell^1_\nu$ and a *projection dimension* $m \in \mathbb{N}$, denote by $a^{(m)} = (a_k)_{|k|<m} \in \mathbb{C}^{2m-1}$ a finite part of a of size $2m-1$. Moreover, given $x = (\omega, a) \in X = \mathbb{C} \times \ell^1_\nu$, denote $x^{(m)} = (\omega, a^{(m)}) \in \mathbb{C}^{2m}$. Consider a finite dimensional projection $F^{(m)} : \mathbb{C}^{2m} \times \mathbb{R} \to \mathbb{C}^{2m}$ of (25) given by

$$F^{(m)}(x^{(m)},\lambda) = \begin{pmatrix} \eta(x^{(m)}) \\ g^{(m)}(x^{(m)},\lambda) \end{pmatrix}, \tag{33}$$

where $g^{(m)}(x^{(m)},\lambda) \in \mathbb{C}^{2m-1}$ corresponds to the finite part of g of size $2m-1$, that is $g^{(m)} = \{(g^{(m)})_k\}_{|k|<m}$. More explicitly, given $|k| < m$,

$$g_k^{(m)}(x^{(m)},\lambda) = \mu_k(\omega,\lambda)a_k + \frac{i\varepsilon k\omega}{3} \sum_{\substack{k_1+k_2+k_3=k \\ |k_i|<m}} a_{k_1}a_{k_2}a_{k_3}.$$

We can now apply the parameter continuation method as introduced in Section 2.1.1 to the finite dimensional problem $F^{(m)} : \mathbb{C}^{2m} \times \mathbb{R} \to \mathbb{C}^{2m}$. Assume that at the parameter value $\lambda = \lambda_0$, $\bar{x}_0 = (\bar{\omega}_0, \bar{a}_0) \in \mathbb{C} \times \mathbb{C}^{2m-1}$ is an approximate solution, that is $F^{(m)}(\bar{x}_0,\lambda_0) \approx 0$.

The next step is to introduce an approximate inverse operator A that satisfies (32) and the operator $A^\dagger$, required to apply the radii polynomial approach (Theorem 2.6).

3.1.4. *Definition of the operators A and $A^\dagger$.* We now define an approximate inverse A for $D_xF(\bar{x}_0,\lambda_0)$ so that (6) holds and the operator $A^\dagger$ which approximates $D_xF(\bar{x}_0,\lambda_0)$. Assume that $\bar{x}_0 = (\bar{\omega}_0, \bar{a}_0) \in X_{sym}$. The Fréchet derivative $D_xF(\bar{x}_0,\lambda_0)$ can be visualized by block as

$$D_xF(\bar{x}_0,\lambda_0) = \begin{pmatrix} 0 & D_a\eta(\bar{x}_0) \\ \partial_\omega g(\bar{x}_0,\lambda_0) & D_a g(\bar{x}_0,\lambda_0) \end{pmatrix},$$

since $\partial_\omega F_0(\bar{x}_0) = 0$, and where

$$\begin{cases} \partial_\omega g(\bar{x}_0,\lambda_0) : \mathbb{C} \to \ell^1_\nu, \\ D_a\eta(\bar{x}_0) : \ell^1_\nu \to \mathbb{C} \text{ is a linear functional} \\ D_a g(\bar{x}_0,\lambda_0) : \ell^1_\nu \to \ell^1_{\nu'} \text{ is a linear operator with } \nu' < \nu. \end{cases}$$

We first approximate $D_xF(\bar{x}_0,\lambda_0)$ with the operator

$$A^\dagger \overset{\text{def}}{=} \begin{pmatrix} 0 & A^\dagger_{a,0} \\ A^\dagger_{\omega,1} & A^\dagger_{a,1} \end{pmatrix},$$

which acts on $b = (b_0, b_1) \in X = \mathbb{C} \times \ell^1_\nu$ component-wise as

$$\begin{aligned} (A^\dagger b)_0 &= A^\dagger_{a,0} \cdot b_1 \overset{\text{def}}{=} D_{a^{(m)}}\eta(\bar{x}_0) \cdot b_1^{(m)} \\ (A^\dagger b)_1 &= A^\dagger_{\omega,1} b_0 + A^\dagger_{a,1} b_1 \in \ell^1_{\nu'}, \end{aligned}$$

where $A^\dagger_{\omega,1} = \partial_\omega g^{(m)}(\bar{x}_0,\lambda_0)$ and $A^\dagger_{a,1}b_1 \in \ell^1_{\nu'}$ is defined component-wise by

$$\left(A^\dagger_{a,1}b_1\right)_k = \begin{cases} \left(D_{a^{(m)}}g^{(m)}(\bar{x}_0,\lambda_0)b_1^{(m)}\right)_k, & |k| < m \\ \mu_k(\bar{\omega}_0,\lambda_0)(b_1)_k, & |k| \geq m. \end{cases}$$

Let $A^{(m)}$ a finite dimensional approximate inverse of $D_xF^{(m)}(\bar{x}_0,\lambda_0)$ which is obtained numerically and which has the decomposition

$$A^{(m)} = \begin{pmatrix} A^{(m)}_{\omega,0} & A^{(m)}_{a,0} \\ A^{(m)}_{\omega,1} & A^{(m)}_{a,1} \end{pmatrix} \in \mathbb{C}^{2m\times 2m},$$

where $A^{(m)}_{\omega,0} \in \mathbb{C}$, $A^{(m)}_{a,0} \in \mathbb{C}^{1\times(2m-1)}$, $A^{(m)}_{\omega,1} \in \mathbb{C}^{(2m-1)\times 1}$ and $A^{(m)}_{a,1} \in \mathbb{C}^{(2m-1)\times(2m-1)}$. Assume moreover that $A^{(m)}$ satisfies the following *symmetry* assumptions:

$$
\begin{aligned}
&1.\ (A^{(m)}_{a,0})_{-j} = \operatorname{conj}\left((A^{(m)}_{a,0})_j\right),\ j=-m+1,\dots,m-1,\\
&2.\ (A^{(m)}_{\omega,1})_{-k} = \operatorname{conj}\left((A^{(m)}_{\omega,1})_k\right),\ k=-m+1,\dots,m-1,\\
&3.\ (A^{(m)}_{a,1})_{-k,-j} = \operatorname{conj}\left((A^{(m)}_{a,1})_{k,j}\right),\ k,j=-m+1,\dots,m-1,\\
&4.\ (A^{(m)}_{\omega,1})_{2k} = 0,\ \forall\ 2k \in \{-m+1,\dots,m-1\},\\
&5.\ (A^{(m)}_{a,1})_{2k,2j+1} = 0,\ \forall\ 2k, 2j+1 \in \{-m+1,\dots,m-1\}.
\end{aligned}
\tag{34}
$$

Assumption 1 of (34) implies that $(A^{(m)}_{a,0})_0 \in \mathbb{R}$ while assumption 2 implies that $(A^{(m)}_{\omega,1})_0 \in \mathbb{R}$. We define the approximate inverse A of the infinite dimensional operator $D_xF(\bar{x}_0,\lambda_0)$ by

$$
A \stackrel{\text{def}}{=} \begin{pmatrix} A_{\omega,0} & A_{a,0} \\ A_{\omega,1} & A_{a,1} \end{pmatrix},
$$

where A acts on $b=(b_0,b_1)\in X = \mathbb{C}\times\ell^1_\nu$ component-wise as

$$
\begin{aligned}
(Ab)_0 &= A^{(m)}_{\omega,0} b_0 + A^{(m)}_{a,0} b^{(m)}_1\\
(Ab)_1 &= A^{(m)}_{\omega,1} b_0 + A_{a,1} b_1,
\end{aligned}
$$

where $A^{(m)}_{\omega,1} \in \mathbb{C}^{(2m-1)\times 1}$ is understood to be an element of ℓ^1_ν by *padding* the tail with zeros, and $A_{a,1}b_1 \in \ell^1_\nu$ is defined component-wise by

$$
(A_{a,1}b_1)_k = \begin{cases} \left(A^{(m)}_{a,1} b^{(m)}_1\right)_k, & |k|<m \\ \dfrac{1}{\mu_k(\bar{\omega}_0,\lambda_0)}(b_1)_k, & |k|\geq m. \end{cases}
$$

Let us now verify that (6) holds.

Lemma 3.3. *Let $x\in X$ and $\lambda\in\mathbb{R}$. Then $AF(x,\lambda)\in X$.*

Proof. Consider $x=(\omega,a)\in X$ and let $F(x,\lambda)=(\eta(x),g(x,\lambda))$, with $\eta(x)$ given in (24) and g given component-wise in (23). For sake of simplicity of the presentation, we denote $F_0=\eta(x)$ and $F_1=g(x,\lambda)$.

We need to show that $\|(AF(x))_1\|_\nu<\infty$. Since

$$
\mu_k(\omega,\lambda) = -k^2\omega^2 - \varepsilon i k\omega - \lambda + e^{-ik\omega\tau},
\tag{35}
$$

$$
\lim_{k\to\pm\infty}\frac{|\mu_k(\omega,\lambda)|}{|\mu_k(\bar{\omega}_0,\lambda_0)|} = \lim_{k\to\pm\infty}\frac{|-k^2\omega^2-\varepsilon ik\omega-\lambda+e^{-ik\omega\tau}|}{|-k^2\bar{\omega}_0^2-\varepsilon ik\bar{\omega}_0-\lambda_0+e^{-ik\bar{\omega}_0\tau}|} = \left(\frac{\omega}{\bar{\omega}_0}\right)^2 < \infty,
$$

there exists $C<\infty$ such that

$$
\left|\frac{\mu_k(\omega)}{\mu_k(\bar{\omega}_0)}\right|, \left|\frac{1}{\mu_k(\bar{\omega}_0)}\cdot\frac{i\varepsilon k\omega}{3}\right| < C,\ \text{for all } |k|\geq m.
$$

Then,

$$\begin{aligned}
\|(AF(x))_1\|_\nu &= \sum_{k\in\mathbb{Z}} |((AF(x))_1)_k|\,\nu^{|k|} = \sum_{k\in\mathbb{Z}} \left|\left(A^{(m)}_{\omega,1}F_0 + A_{a,1}F_1\right)_k\right| \nu^{|k|} \\
&\le \sum_{|k|<m} \left|\left(A^{(m)}_{\omega,1}\right)_{k,1}\right| |F_0|\nu^{|k|} + \sum_{|k|<m} \left|\left(A^{(m)}_{a,1}F^{(m)}_1\right)_k\right| \nu^{|k|} \\
&+ \sum_{|k|\ge m} \left|\frac{1}{\mu_k(\bar\omega_0,\lambda_0)}(\mu_k(\omega,\lambda)a_k + \frac{i\varepsilon k\omega}{3}(a^3)_k)\right| \nu^{|k|} \\
&\le \sum_{|k|<m} \left|\left(A^{(m)}_{\omega,1}\right)_{k,1}\right| |F_0|\nu^{|k|} + \sum_{|k|<m} \left|\left(A^{(m)}_{a,1}F^{(m)}_1\right)_k\right| \nu^{|k|} \\
&+ C\sum_{|k|\ge m} |a_k|\,\nu^{|k|} + C\sum_{|k|\ge m} \left|(a^3)_k\right| \nu^{|k|} \\
&\le \sum_{|k|<m} \left|\left(A^{(m)}_{\omega,1}\right)_{k,1}\right| |F_0|\nu^{|k|} + \sum_{|k|<m} \left|\left(A^{(m)}_{a,1}F^{(m)}_1\right)_k\right| \nu^{|k|} \\
&+ C\|a\|_{1,\nu} + C(\|a\|_{1,\nu})^3 < \infty,
\end{aligned}$$

where we used the fact that $\|a^3\|_{1,\nu} \le (\|a\|_{1,\nu})^3$, because ℓ^1_ν is a Banach algebra. □

Let us now show that the operator A satisfies the symmetry assumption (32).

Lemma 3.4. *Let $x \in X_{sym}$ and $\lambda \in \mathbb{R}$. Then $AF(x,\lambda) \in X_{sym}$.*

Proof. Let $x = (\omega, a) \in X_{sym} = \mathbb{R} \times \tilde\ell^1_\nu$, with $\tilde\ell^1_\nu$ as defined in (31). This implies that $a_{-k} = \mathrm{conj}(a_k)$ and $a_{2k} = 0$ for all $k \in \mathbb{Z}$. Again, denote $F_0 = \eta(x)$ and $F_1 = g(x,\lambda)$.

We begin the proof by showing that the operator F preserves the symmetry conditions, that is we show that $F_0 \in \mathbb{R}$, $(F_1)_{-k} = \mathrm{conj}((F_1)_k)$ and $(F_1)_{2k} = 0$.

Recalling the definition of the phase condition (24),

$$F_0 = \sum_{|k|\le n} a_k = a_{-n} + a_{-n+1} + \cdots + a_{n-1} + a_n = \mathrm{conj}(a_n) + \mathrm{conj}(a_{n-1}) + \cdots + a_{n-1} + a_n \in \mathbb{R}.$$

Also, from (35), we see that $\mu_{-k}(\omega,\lambda) = \mathrm{conj}\,(\mu_k(\omega,\lambda))$. Then,

$$\begin{aligned}
(F_1)_{-k} &= \mu_{-k}(\omega,\lambda)a_{-k} - \frac{i\varepsilon k\omega}{3} \sum_{k_1+k_2+k_3=-k} a_{k_1}a_{k_2}a_{k_3} \\
&= \mathrm{conj}\,(\mu_k(\omega,\lambda)a_k) - \frac{i\varepsilon k\omega}{3} \sum_{k_1+k_2+k_3=k} a_{-k_1}a_{-k_2}a_{-k_3} \\
&= \mathrm{conj}\,(\mu_k(\omega,\lambda)a_k) + \mathrm{conj}\left(\frac{i\varepsilon k\omega}{3}\right) \sum_{k_1+k_2+k_3=k} \mathrm{conj}(a_{k_1})\mathrm{conj}(a_{k_2})\mathrm{conj}(a_{k_3}) \\
&= \mathrm{conj}\,((F_1)_k)\,.
\end{aligned}$$

Now,

$$(F_1)_{2k} = \mu_{2k}(\omega,\lambda)a_{2k} + \frac{i\varepsilon 2k\omega}{3}(a^3)_{2k} = \mu_{2k}(\omega,\lambda)0 + \frac{i\varepsilon 2k\omega}{3} \sum_{k_1+k_2+k_3=2k} a_{k_1}a_{k_2}a_{k_3} = 0, \tag{36}$$

since $k_1 + k_2 + k_3 = 2k$ implies that there exists $i \in \{1,2,3\}$ such that k_i is even.

The second part of the proof is to show that AF preserves the symmetry conditions, that is $(AF(x))_0 \in \mathbb{R}$, $((AF(x))_1)_{-k} = \operatorname{conj}(((AF(x))_1)_k)$ and $((AF(x))_1)_{2k} = 0$.

Combining that $A^{(m)}_{\omega,0}, F_0 \in \mathbb{R}$, $(F_1)_{-k} = \operatorname{conj}((F_1)_k)$ and assumption 1 of (34),

$$\begin{aligned}
(AF(x))_0 &= A^{(m)}_{\omega,0}F_0 + A^{(m)}_{a,0}F^{(m)}_1 \\
&= A^{(m)}_{\omega,0}F_0 + \sum_{k=-m+1}^{m-1}(A^{(m)}_{a,0})_k(F_1)_k \\
&= A^{(m)}_{\omega,0}F_0 + \sum_{k=-m+1}^{-1}(A^{(m)}_{a,0})_k(F_1)_k + (A^{(m)}_{a,0})_0(F_1)_0 + \sum_{k=1}^{m-1}(A^{(m)}_{a,0})_k(F_1)_k \\
&= A^{(m)}_{\omega,0}F_0 + (A^{(m)}_{a,0})_{1,0}(F_1)_0 + \sum_{k=1}^{m-1}\left(\operatorname{conj}\left((A^{(m)}_{a,0})_k(F_1)_k\right) + (A^{(m)}_{a,0})_k(F_1)_k\right) \in \mathbb{R}.
\end{aligned}$$

By assumptions 2 and 3 of (34), for $|k| < m$,

$$\begin{aligned}
((AF(x))_1)_{-k} &= \left(A^{(m)}_{\omega,1}F_0 + A_{a,1}F_1\right)_{-k} \\
&= \left(A^{(m)}_{\omega,1}\right)_{-k}F_0 + \left(A^{(m)}_{a,1}F^{(m)}_1\right)_{-k} \\
&= \left(A^{(m)}_{\omega,1}\right)_{-k}F_0 + \sum_{j=-m+1}^{m-1}(A^{(m)}_{a,1})_{-k,j}(F_1)_j \\
&= \left(A^{(m)}_{\omega,1}\right)_{-k}F_0 + \sum_{j=-m+1}^{m-1}(A^{(m)}_{a,1})_{-k,-j}(F_1)_{-j} \\
&= \operatorname{conj}\left(\left(A^{(m)}_{\omega,1}\right)_k F_0\right) + \sum_{j=-m}^{m}\overline{(A^{(m)}_{a,1})_{k,j}(F_1)_j} \\
&= \operatorname{conj}(((AF(x))_1)_k),
\end{aligned}$$

and for $|k| \geq m$,

$$\begin{aligned}
((AF(x))_1)_{-k} &= \left(A^{(m)}_{\omega,1}F_0 + A_{a,1}F_1\right)_{-k} \\
&= (A_{a,1}F_1)_{-k} = \frac{1}{\mu_{-k}(\bar{\omega}_0, \lambda_0)}(F_1)_{-k} = \operatorname{conj}(((AF(x))_1)_k).
\end{aligned}$$

That shows that $((AF(x))_1)_{-k} = \operatorname{conj}(((AF(x))_1)_k)$ for all k. It remains to show that $((AF(x))_1)_{2k} = 0$.

By assumptions 4 and 5 in (34), and using (36), we get that for $|k| < m$,

$$\begin{aligned}
((AF(x))_1)_{2k} &= \left(A^{(m)}_{\omega,1}F_0 + A_{a,1}F_1\right)_{2k} = \left(A^{(m)}_{\omega,1}\right)_{2k}F_0 + \left(A^{(m)}_{a,1}F^{(m)}_1\right)_{2k} \\
&= \sum_{j=-m+1}^{m-1}(A^{(m)}_{a,1})_{2k,j}(F_1)_j = \sum_{\substack{j=-m+1 \\ j \text{ odd}}}^{m-1}(A^{(m)}_{a,1})_{2k,j}(F_1)_j = 0,
\end{aligned}$$

and for $|k| \geq m$,

$$((AF(x))_1)_{2k} = \frac{1}{\mu_{2k}(\bar{\omega}_0, \lambda_0)}(F_1)_{2k} = 0. \qquad \square$$

Having defined A satisfying (32) we can now use the radii polynomial approach to compute a branch of solutions of $F(x, \lambda) = 0$ near the set of predictors $\bar{x}_s = (1-s)\bar{x}_0 + s\bar{x}_1$, for $s \in [0, 1]$. In order to compute all bounds necessary to obtain the coefficient Y_0, Z_0, Z_1 and Z_2 to define the radii polynomial (11), we need some basic background from functional analysis.

3.1.5. *Basic functional analytic background.*

Lemma 3.5. *The function space ℓ^1_ν defined in* (26) *is a Banach algebra under discrete convolution. More precisely, $\left(\ell^1_\nu, \|\cdot\|_{1,\nu}\right)$ is a Banach space with the norm defined in* (27)*, and for any $a, b \in \ell^1_\nu$, $a * b = \{(a*b)_k\}_{k\in\mathbb{Z}}$ defined by*

$$(a*b)_k \stackrel{\text{def}}{=} \sum_{k_1+k_2=k} a_{k_1} b_{k_2} \tag{37}$$

satisfy $a * b \in \ell^1_\nu$ and $\|a*b\|_{1,\nu} \leq \|a\|_{1,\nu}\|b\|_{1,\nu}$.

Proof. We omit the proof that ℓ^1_ν is a Banach space. Let $a, b \in \ell^1_\nu$, that is $\|a\|_{1,\nu}, \|b\|_{1,\nu} < \infty$. Consider $a * b$ defined component-wise by (37). Then,

$$\begin{aligned}
\|a*b\|_{1,\nu} &= \sum_{k\in\mathbb{Z}} |(a*b)_k|\nu^{|k|} = \sum_{k\in\mathbb{Z}} \left| \sum_{\substack{k_1+k_2=k \\ k_1,k_2\in\mathbb{Z}}} a_{k_1} b_{k_2} \right| \nu^{|k|} \\
&\leq \sum_{k\in\mathbb{Z}} \sum_{\substack{k_1+k_2=k \\ k_1,k_2\in\mathbb{Z}}} |a_{k_1}||b_{k_2}|\nu^{|k|} \leq \sum_{k\in\mathbb{Z}} \sum_{\substack{k_1+k_2=k \\ k_1,k_2\in\mathbb{Z}}} |a_{k_1}|\nu^{|k_1|}|b_{k_2}|\nu^{|k_2|} \\
&\leq \left(\sum_{k_1\in\mathbb{Z}} |a_{k_1}|\nu^{|k_1|} \right) \left(\sum_{k_2\in\mathbb{Z}} |b_{k_2}|\nu^{|k_2|} \right) = \|a\|_{1,\nu}\|b\|_{1,\nu}.
\end{aligned}$$

That shows that ℓ^1_ν is a Banach algebra. $\square$

Recall the classical fact that the dual space of ℓ^1_1 is the space ℓ^∞. Similarly if $\nu > 1$ then the dual of ℓ^1_ν is a weighted "ell-infinity" space which we define now. For a bi-infinite sequence of complex numbers $c = \{c_k\}_{k\in\mathbb{Z}}$, the ν-weighted supremum norm is defined by

$$\|c\|_{\infty,\nu^{-1}} \stackrel{\text{def}}{=} \sup_{k\in\mathbb{Z}} \frac{|c_k|}{\nu^{|k|}}. \tag{38}$$

Let

$$\ell^\infty_\nu = \left\{ c = \{c_k\}_{k\in\mathbb{Z}} \mid c_k \in \mathbb{C} \ \forall\ k \in \mathbb{Z}, \text{ and } \|c\|_{\infty,\nu^{-1}} < \infty \right\}. \tag{39}$$

Lemma 3.6. *Suppose that $a \in \ell^1_\nu$ and $c \in \ell^\infty_\nu$. Then*

$$\left| \sum_{k\in\mathbb{Z}} c_k a_k \right| \leq \sum_{k\in\mathbb{Z}} |c_k||a_k| \leq \|c\|^\infty_\nu \|a\|_\nu.$$

The following result provides a useful and explicit bound on the norm of an "eventually diagonal" linear operator on ℓ^1_ν. The proof is omitted.

Corollary 3.7. *Let $\mathcal{M}^{(m)}$ be an $(2m-1)\times(2m-1)$ matrix with complex valued entries, $\{\delta_k\}_{|k|\geq m}$ a bi-infinite sequence of complex numbers and $\delta_m > 0$ a real number such that*

$$|\delta_k| \leq \delta_m, \text{ for all } |k| \geq m.$$

Given $a = (a_k)_{k\in\mathbb{Z}} \in \ell^1_\nu$, denote by $a^{(m)} = (a_{-m+1},\dots,a_{-1},a_0,a_1,\dots,a_{m-1}) \in \mathbb{C}^{2m-1}$. Define the map $\mathcal{M}\colon \ell^1_\nu \to \ell^1_\nu$ by

$$[\mathcal{M}(a)]_k = \begin{cases} [\mathcal{M}^{(m)}a^{(m)}]_k, & |k|<m \\ \delta_k a_k, & |k|\geq m. \end{cases}$$

Then $\mathcal{M}$ is a bounded linear operator and

$$\|\mathcal{M}\|_{B(\ell^1_\nu,\ell^1_\nu)} \leq \max(K,\delta_m),$$

where

$$K \stackrel{\text{def}}{=} \max_{|n|<m} \frac{1}{\nu^{|n|}} \sum_{|k|<m} |\mathcal{M}_{k,n}|\nu^{|k|}. \tag{40}$$

Lemma 3.8. *Given $\nu \geq 1$, $k\in\mathbb{Z}$ and $a\in\ell^1_\nu$, the function $l^k_a\colon \ell^1_\nu \to \mathbb{C}$ defined by*

$$l^k_a(h) \stackrel{\text{def}}{=} (a*h)_k = \sum_{k_1+k_2=k} a_{k_1}h_{k_2}$$

with $h\in\ell^1_\nu$, is a bounded linear functional, and

$$\|l^k_a\| = \sup_{\|h\|_{1,\nu}\leq 1} \left|l^k_a(h)\right| \leq \sup_{j\in\mathbb{Z}} \frac{|a_{k-j}|}{\nu^{|j|}} < \infty. \tag{41}$$

Fix a truncation mode to be m. Given $h\in\ell^1_\nu$, set

$$h^{(m)} \stackrel{\text{def}}{=} (\dots,0,0,h_{-m+1},\dots,h_{m-1},0,0,\dots) \in \ell^1_\nu$$
$$h^{(I)} \stackrel{\text{def}}{=} h - h^{(m)} \in \ell^1_\nu.$$

Corollary 3.9. *Let $N\in\mathbb{N}$ and let $\bar\alpha = (\dots,0,0,\bar\alpha_{-N},\dots,\bar\alpha_N,0,0,\dots) \in \ell^1_\nu$. Suppose that $|k|<m$ and define $\hat l^k_{\bar\alpha} \in (\ell^1_\nu)^*$ by*

$$\hat l^k_{\bar\alpha}(h) \stackrel{\text{def}}{=} (\bar\alpha * h^{(I)})_k = \sum_{k_1+k_2=k} \bar\alpha_{k_1} h^{(I)}_{k_2}.$$

Then, for all $h\in\ell^1_\nu$ such that $\|h\|_{1,\nu}\leq 1$,

$$\left|\hat l^k_{\bar\alpha}(h)\right| \leq \Psi_k(\bar\alpha) \stackrel{\text{def}}{=} \max\left(\max_{k-N\leq j\leq -m} \frac{|\bar\alpha_{k-j}|}{\nu^{|j|}}, \max_{m\leq j\leq k+N} \frac{|\bar\alpha_{k-j}|}{\nu^{|j|}}\right). \tag{42}$$

3.1.6. *Radii polynomial to compute periodic solutions of Nussbaum's equation.* We have now derived all necessary tools to apply the radii polynomial approach (Theorem 2.6). Assume that $\bar x_0$ and $\bar x_1$ are two numerical approximation.

The Y_0 bound. We begin the computation of the coefficients of the radii polynomial with Y_0 with the help of Remark 2.11. Denote

$$\Delta\bar x \stackrel{\text{def}}{=} \bar x_1 - \bar x_0 \quad \text{and} \quad \Delta\lambda \stackrel{\text{def}}{=} \lambda_1 - \lambda_0,$$

and, using the mean value theorem, consider the expansion

$$F(\bar x_s,\lambda_s) = F(\bar x_0,\lambda_0) + \left(D_xF(\bar x_0,\lambda_0) \quad \frac{\partial F}{\partial\lambda}(\bar x_0,\lambda_0)\right)\begin{pmatrix}\Delta\bar x\\ \Delta\lambda\end{pmatrix} s + \frac{1}{2}\left(\frac{\partial^2}{\partial s^2}F(\bar x_s,\lambda_s)\Big|_{s=\sigma}\right)s^2,$$

for some $\sigma \in [0,1]$. As in (15), denote

$$y_1 \stackrel{\text{def}}{=} \left(D_x F(\bar{x}_0, \lambda_0) \quad \frac{\partial F}{\partial \lambda}(\bar{x}_0, \lambda_0) \right) \begin{pmatrix} \Delta \bar{x} \\ \Delta \lambda \end{pmatrix} \quad \text{and} \quad y_2 \stackrel{\text{def}}{=} \frac{1}{2} \left(\frac{\partial^2}{\partial s^2} F(\bar{x}_s, \lambda_s)\Big|_{s=\sigma} \right).$$

Each quantity $F(\bar{x}_0, \lambda_0)$, y_1 and y_2 is a finite sum which can be evaluated using interval arithmetics. Let w_0, w_1 and w_2 such that

$$\begin{aligned} \|AF(\bar{x}_0, \lambda_0)\|_X &\leq w_0 \\ \|Ay_1\|_X &\leq w_1 \\ \|Ay_2\|_X &\leq w_2. \end{aligned}$$

Hence, set Y_0 such that

$$Y_0 \stackrel{\text{def}}{=} w_0 + w_1 + w_2 \tag{43}$$

so that $\|AF(\bar{x}_s, \lambda_s)\|_X \leq Y_0$ for all $s \in [0,1]$.

The Z_0 bound. Let $B \stackrel{\text{def}}{=} I - AA^\dagger$, which we express as

$$B = \begin{pmatrix} B_{\omega,0} & B_{a,0} \\ B_{\omega,1} & B_{a,1} \end{pmatrix}.$$

Since $B \stackrel{\text{def}}{=} I - AA^\dagger$, then $[(Bc)_1]_k = 0$ for $|k| \geq m$ and $c \in X$. Define

$$Z_0^{(0)} \stackrel{\text{def}}{=} |B_{\omega,0}| + \left(\max_{|k|<m} \frac{|(B_{a,0})_k|}{\nu^{|k|}} \right) \tag{44}$$

$$Z_1^{(0)} \stackrel{\text{def}}{=} \sum_{|k|<m} |(B_{\omega,1})_k| \nu^{|k|} + \max_{|n|<m} \frac{1}{\nu^{|n|}} \sum_{|k|<m} |(B_{a,1})_{k,n}| \nu^{|k|}. \tag{45}$$

Now, recalling (38) and Lemma 3.6, we have that for any $c = (c_0, c_1) \in B_1(0) \subset X$,

$$|(Bc)_0| = \left| B_{\omega,0} c_0 + \sum_{k \in \mathbb{Z}} (B_{a,0})_k (c_1)_k \right| \leq |B_{\omega,0}| + \|B_{a,0}\|_{\infty,\nu^{-1}} = Z_0^{(0)}.$$

Recalling Corollary 3.7 and (40), we get that

$$\|(Bc)_1\|_{1,\nu} = \|B_{\omega,1} c_0 + B_{a,1} c_1\|_{1,\nu} \leq \|B_{\omega,1}\|_{1,\nu} + \|B_{a,1}\|_{B(\ell^1_\nu, \ell^1_\nu)} \leq Z_1^{(0)}.$$

Finally, setting

$$Z_0 \stackrel{\text{def}}{=} \max\left(Z_0^{(0)}, Z_1^{(0)} \right) \tag{46}$$

we get that

$$\|I - AA^\dagger\|_{B(X,X)} = \sup_{c \in B_1(0)} \|Bc\|_X = \sup_{c \in B_1(0)} \max\left(|(Bc)_0|, \|(Bc)_1\|_{1,\nu} \right) \leq Z_0.$$

The Z_1 bound. Recall from (9) that Z_1 is a bound satisfying

$$\|A[D_x F(\bar{x}_0, \lambda_0) - A^\dagger]\|_{B(X,X)} \leq Z_1$$

Let $c = (c_0, c_1) \in B_1(0)$ and set

$$z \stackrel{\text{def}}{=} (D_x F(\bar{x}_0, \lambda_0) - A^\dagger) c.$$

Denote $z = (z_0, z_1)$. Then, if $n < m$, we get that $z_0 = 0$ because the phase condition (24) depends only on the Fourier coefficients $(a_k)_{|k|\le n}$. Given $k \in \mathbb{Z}$,

$$((D_xF(\bar{x}_0, \lambda_0)c)_1)_k = \mu_k(\bar{\omega}_0, \lambda_0)(c_1)_k + \frac{\partial \mu_k}{\partial \omega}(\bar{\omega}_0, \lambda_0)c_0(\bar{a}_0)_k + \frac{1}{3}i\varepsilon k(\bar{a}_0^3)_k c_0 + i\varepsilon k\bar{\omega}_0(\bar{a}_0^2 c_1)_k.$$

Therefore, since $(\bar{a}_0)_k = 0$ for all $|k| \ge m$,

$$(z_1)_k = \begin{cases} i\varepsilon k(\bar{a}_0^2 c_1^{(I)})_k\bar{\omega}_0, & |k| < m \\ \frac{1}{3}i\varepsilon k(\bar{a}_0^3)_k c_0 + i\varepsilon k\bar{\omega}_0(\bar{a}_0^2 c_1)_k, & |k| \ge m \end{cases}$$

From Corollary 3.9, for all $|k| < m$

$$|(z_1)_k| \le \zeta_k \stackrel{\text{def}}{=} \varepsilon|k|\bar{\omega}_0\Psi_k(\bar{a}_0^2). \tag{47}$$

Denote $\zeta \stackrel{\text{def}}{=} (\zeta_k)_{|k|<m} \in \mathbb{C}^{2m-1}$. Hence, for $j = 0$,

$$\left|\left(A[D_xF(\bar{x}_0, \lambda_0) - A^\dagger]c\right)_0\right| = |(Az)_0| = |A_{a,0}z_1| \le Z_0^{(1)} \stackrel{\text{def}}{=} |A_{a,0}^{(m)}|\zeta,$$

while for $j = 1$,

$$\left\|\left(A[D_xF(\bar{x}_0, \lambda_0) - A^\dagger]c\right)_1\right\|_{1,\nu} = \|(Az)_1\|_{1,\nu} = \|A_{a,1}z_1\|_{1,\nu}$$

$$\le Z_1^{(1)} \stackrel{\text{def}}{=} \sum_{k=-m+1}^{m-1}\left(|A_{a,1}^{(m)}|\zeta\right)_k \nu^{|k|} + \frac{\varepsilon}{3}\sum_{|k|=m}^{3m-3}\left|\frac{k}{\mu_k(\bar{\omega}_0, \lambda_0)}(\bar{a}_0^3)_k\right| \nu^{|k|} + \frac{\varepsilon\bar{\omega}_0}{m\bar{\omega}_0^2 - \frac{(|\lambda_0|+1)}{m}}(\|\bar{a}\|_{1,\nu})^2,$$

using that $\|\bar{a}_0^2 c_1\|_{1,\nu} \le (\|\bar{a}_0\|_{1,\nu})^2$ and using that

$$\left|\frac{i\varepsilon k\bar{\omega}_0}{\mu_k(\bar{\omega}_0, \lambda_0)}\right| = \frac{1}{|k|}\frac{\varepsilon\bar{\omega}_0}{\left|-\bar{\omega}_0^2 - \varepsilon i\bar{\omega}_0/k - \lambda_0/k^2 + e^{-ik\bar{\omega}_0\tau}/k^2\right|} \le \frac{\varepsilon\bar{\omega}_0}{m\left(\bar{\omega}_0^2 - \frac{(|\lambda_0|+1)}{m^2}\right)}.$$

Finally, setting

$$Z_1 \stackrel{\text{def}}{=} \max\left(Z_0^{(1)}, Z_1^{(1)}\right) \tag{48}$$

$$\|A[D_xF(\bar{x}_0, \lambda_0) - A^\dagger]\|_{B(X,X)} = \sup_{c\in B_1(0)} \max\left(|(Az)_0|, \|(Az)_1\|_{1,\nu}\right) \le \max\left(Z_0^{(1)}, Z_1^{(1)}\right) = Z_1.$$

The Z_2 bound. The computation of the Z_2 bound, while essentially elementary, involves computing derivatives and considering several expansions. The use of a symbolic software like Maple or Mathematica is very useful here. Recall from (10) that the bound Z_2 satisfies

$$\|A[D_xF(\bar{x}_s + b, \lambda_s) - D_xF(\bar{x}_0, \lambda_0)]\|_{B(X,X)} \le Z_2(r),$$

for all $b \in B_r(0)$ and for all $s \in [0,1]$. Given $c = (c_1, c_2) \in B_1(0) \subset X$ and for a fixed $k \in \mathbb{Z}$, denote $h_k(\phi) = D_x g_k(\bar{x}_{\phi s} + \tau b, \lambda_{\phi s})$. For each $k \in \mathbb{Z}$, there exists $t = t(k) \in [0,1]$ such that

$$
\begin{aligned}
h_k'(t) &= h_k(1) - h_k(0) \\
&= \left(D_x g_k(\bar{x}_s + b, \lambda_s) - D_x g_k(\bar{x}_0, \lambda_0)\right) c \\
&= (-2k^2(s\Delta_\omega + b_0)c_0 - k^2 c_0 \tau^2 (s\Delta_\omega + b_0) e^{-ik(st\Delta_\omega + tb_0 + \bar{\omega}_0)\tau})(st\Delta_a + tb_1 + \bar{a}_0)_k \\
&+ (-2k^2(st\Delta_\omega + tb_0 + \bar{\omega}_0)c_0 - i\varepsilon k c_0 - ikc_0\tau e^{-ik(st\Delta_\omega + tb_0 + \bar{\omega}_0)\tau})(s\Delta_a + b_1)_k \\
&+ (-2k^2(st\Delta_\omega + tb_0 + \bar{\omega}_0)(s\Delta_\omega + b_0) - i\varepsilon k(s\Delta_\omega + b_0) - s\Delta_\lambda - ik(s\Delta_\omega + b_0)\tau e^{-ik(st\Delta_\omega + tb_0 + \bar{\omega}_0)\tau})(c_1)_k \\
&+ i\varepsilon k c_0 \left((st\Delta_a + tb_1 + \bar{a}_0)^2 (s\Delta_a + b_1)\right)_k \\
&+ i\varepsilon k (s\Delta_\omega + b_0) \left((st\Delta_a + tb_1 + \bar{a}_0)^2 c_1\right)_k \\
&+ (2i)\varepsilon k (st\Delta_\omega + tb_0 + \bar{\omega}_0) \left((st\Delta_a + tb_1 + \bar{a}_0) c_1 (s\Delta_a + b_1)\right)_k .
\end{aligned}
$$

Then, using that $|b_0| \le r$ and letting $u_1 \stackrel{\text{def}}{=} \frac{1}{r} b_1 \in B_1(0)$

$$
\begin{aligned}
\left|h_k'(t)\right| &\le (2k^2(|\Delta_\omega| + r) + k^2\tau^2(|\Delta_\omega| + r))(|\Delta_a| + |u_1|r + |\bar{a}_0|)_k \\
&+ (2k^2(|\Delta_\omega| + r + \bar{\omega}_0) + \varepsilon|k| + |k|\tau)(|\Delta_a| + |u_1|r)_k \\
&+ (2k^2(|\Delta_\omega| + r + \bar{\omega}_0)(|\Delta_\omega| + r) + \varepsilon|k|(|\Delta_\omega| + r) + |\Delta_\lambda| + |k|(|\Delta_\omega| + r)\tau)|(c_1)_k| \\
&+ \varepsilon|k| \left((|\Delta_a| + |u_1|r + |\bar{a}_0|)^2 (|\Delta_a| + |u_1|r)\right)_k \\
&+ \varepsilon|k|(|\Delta_\omega| + r) \left((|\Delta_a| + |u_1|r + |\bar{a}_0|)^2 |c_1|\right)_k \\
&+ 2\varepsilon|k|(|\Delta_\omega| + r + \bar{\omega}_0) \left((|\Delta_a| + |u_1|r + |\bar{a}_0|)|c_1|(|\Delta_a| + |u_1|r)\right)_k \\
&= (z_2)_k^{(4)} r^3 + (z_2)_k^{(3)} r^2 + (z_2)_k^{(2)} r + (z_2)_k^{(1)},
\end{aligned}
$$

where

$$
\begin{aligned}
(z_2)_k^{(1)} &= k^2 \left(2|\Delta_\omega| + \tau^2|\Delta_\omega|\right)(|\Delta_a| + |\bar{a}_0|)_k + \left(2k^2(|\Delta_\omega| + \bar{\omega}_0) + |k|(\varepsilon + \tau)\right)|(\Delta_a)_k| \\
&+ \left(2k^2(|\Delta_\omega| + \bar{\omega}_0)|\Delta_\omega| + \varepsilon|k||\Delta_\omega| + |\Delta_\lambda| + \tau|k||\Delta_\omega|\right)|(c_1)_k| + \varepsilon|k| \left((|\Delta_a| + |\bar{a}_0|)^2 (|\Delta_a|)\right)_k | \\
&+ \varepsilon|k||\Delta_\omega|| \left((|\Delta_a| + |\bar{a}_0|)^2 c_1\right)_k | + 2\varepsilon|k|(|\Delta_\omega| + \bar{\omega}_0) \left((|\Delta_a| + |\bar{a}_0|)|\Delta_a||c_1|\right)_k \\
(z_2)_k^{(2)} &= 6\varepsilon|k||\Delta_\omega||(\Delta_a c_1 u_1)_k| + 4\varepsilon|k||\Delta_\omega||(\bar{a}_0 c_1 u_1)_k| + 4\varepsilon|k|\bar{\omega}_0|(\Delta_a c_1 u_1)_k| + 2\varepsilon|k|\bar{\omega}_0|(\bar{a}_0 c_1 u_1)_k| \\
&+ \tau|k||(u_1)_k| + 4k^2|\Delta_\omega||(c_1)_k| + 2|(c_1)_k|k^2\bar{\omega}_0 + (\varepsilon + \tau)|k||(c_1)_k| + \varepsilon|k||(u_1)_k| + 4k^2|\Delta_\omega||(u_1)_k| \\
&+ k^2\tau^2(|\bar{a}_0| + |\Delta_a|)_k + 2k^2\bar{\omega}_0|(u_1)_k| + 4k^2|(\Delta_a)_k| + k^2\tau^2|\Delta_\omega||(u_1)_k| + 3\varepsilon|k||(u_1\Delta_a^2)_k| \\
&+ \varepsilon|k||(u_1\bar{a}_0^2)_k| + 3\varepsilon|k||(c_1\Delta_a^2)_k| + \varepsilon|k||(c_1\bar{a}_0^2)_k| + 2k^2|\bar{a}_0| + 4\varepsilon|k||(\Delta_a\bar{a}_0 u_1)_k| + 4\varepsilon|k||(\Delta_a\bar{a}_0 c_1)_k| \\
(z_2)_k^{(3)} &= k^2\tau^2|(u_1)_k| + 4k^2|(u_1)_k| + 2k^2|(c_1)_k| + 3\varepsilon|k||(\Delta_a u_1^2)_k| + 2\varepsilon|k||(\bar{a}_0 u_1^2)_k| + 3\varepsilon|k||\Delta_\omega||(c_1 u_1^2)_k| \\
&+ 6\varepsilon|k||(\Delta_a c_1 u_1)_k| + 4\varepsilon|k||(\bar{a}_0 c_1 u_1)_k| + 2\varepsilon|k|\bar{\omega}_0|(c_1 u_1^2)_k| \\
(z_2)_k^{(4)} &= \varepsilon|k| \left(3|(c_1 u_1^2)_k| + |(u_1^3)_k|\right).
\end{aligned}
$$

Let us define the operators $\tilde{A}_{a,0}$, $\tilde{B}_{a,0}$, $\tilde{A}_{a,1}$ and $\tilde{B}_{a,1}$ by

$$
\begin{aligned}
\tilde{A}_{a,0} &= \left\{|k||(A_{a,0}^{(m)})_k|\right\}_{|k|<m} \qquad & \tilde{B}_{a,0} &= \left\{k^2|(A_{a,0}^{(m)})_k|\right\}_{|k|<m} \\
\tilde{A}_{a,1} &= \left\{|j||(A_{a,1})_{k,j}|\right\}_{k,j\in\mathbb{Z}} \qquad & \tilde{B}_{a,1} &= \left\{j^2|(A_{a,1})_{k,j}|\right\}_{k,j\in\mathbb{Z}}.
\end{aligned}
$$

Given $j = 0, 1$, denote $\|\cdot\|_j$ to be $|\cdot|$ if $j = 0$ and $\|\cdot\|_{1,\nu}$ if $j = 1$. Moreover, given $j = 0, 1$, denote $\|\cdot\|_{B(j,j)}$ to be $\|\cdot\|_{\infty,\nu^{-1}}$ if $j = 0$ and $\|\cdot\|_{B(\ell^1_\nu,\ell^1_\nu)}$ if $j = 1$. Then, for $j = 0, 1$,

$$\begin{aligned}
\|A_{a,j}z_2^{(1)}\|_j &\le Z_2^{(1,j)} \\
&\stackrel{\text{def}}{=} \left(2+\tau^2\right)|\Delta_\omega|\|\tilde{B}_{a,j}(|\Delta_a|+|\bar{a}_0|)\|_j + 2(|\Delta_\omega|+\bar{\omega}_0)\|\tilde{B}_{a,j}|\Delta_a|\|_j + (\varepsilon+\tau)\|\tilde{A}_{a,j}|\Delta_a|\|_j \\
&\quad + 2(|\Delta_\omega|+\bar{\omega}_0)|\Delta_\omega|\|\tilde{B}_{a,j}\|_{B(j,j)} + (\varepsilon+\tau)|\Delta_\omega|\|\tilde{A}_{a,j}\|_{B(j,j)} + |\Delta_\lambda|\|A_{a,j}\|_{B(j,j)} \\
&\quad + \varepsilon\|\tilde{A}_{a,j}\|_{B(j,j)}\||\Delta_a|+|\bar{a}_0|\|^2_{1,\nu}\|\Delta_a\|_{1,\nu} + \varepsilon|\Delta_\omega|\|\tilde{A}_{a,j}\|_{B(j,j)}\||\Delta_a|+|\bar{a}_0|\|^2_{1,\nu} \\
&\quad + 2\varepsilon(|\Delta_\omega|+\bar{\omega}_0)\|\tilde{A}_{a,j}\|_{B(j,j)}\||\Delta_a|+|\bar{a}_0|\|_{1,\nu}\|\Delta_a\|_{1,\nu} \\
\|A_{a,j}z_2^{(2)}\|_j &\le Z_2^{(2,j)} \\
&\stackrel{\text{def}}{=} 6\varepsilon|\Delta_\omega|\|\tilde{A}_{a,j}\|_{B(j,j)}\|\Delta_a\|_{1,\nu} + 4\varepsilon|\Delta_\omega|\|\tilde{A}_{a,j}\|_{B(j,j)}\|\bar{a}_0\|_{1,\nu} + 4\varepsilon\bar{\omega}_0\|\tilde{A}_{a,j}\|_{B(j,j)}\|\Delta_a\|_{1,\nu} \\
&\quad + 2\varepsilon\bar{\omega}_0\|\tilde{A}_{a,j}\|_{B(j,j)}\|\bar{a}_0\|_{1,\nu} + \tau\|\tilde{A}_{a,j}\|_{B(j,j)} + 4|\Delta_\omega|\|\tilde{B}_{a,j}\|_{B(j,j)} + 2\bar{\omega}_0\|\tilde{B}_{a,j}\|_{B(j,j)} \\
&\quad + (\varepsilon+\tau)\|\tilde{A}_{a,j}\|_{B(j,j)} + \varepsilon\|\tilde{A}_{a,j}\|_{B(j,j)} + 4|\Delta_\omega|\|\tilde{B}_{a,j}\|_{B(j,j)} + \tau^2\|\tilde{B}_{a,j}\|_{B(j,j)}\||\Delta_a|+|\bar{a}_0|\|_{1,\nu} \\
&\quad + 2\bar{\omega}_0\|\tilde{B}_{a,j}\|_{B(j,j)} + 4\|\tilde{B}_{a,j}\|_{B(j,j)}\|\Delta_a\|_{1,\nu} + \tau^2|\Delta_\omega|\|\tilde{B}_{a,j}\|_{B(j,j)} + 6\varepsilon\|\tilde{A}_{a,j}\|_{B(j,j)}\|\Delta_a\|^2_{1,\nu} \\
&\quad + 2\varepsilon\|\tilde{A}_{a,j}\|_{B(j,j)}\|\bar{a}_0\|^2_{1,\nu} + 2\|\tilde{B}_{a,j}\|_{B(j,j)}\|\bar{a}_0\|_{1,\nu} + 4\varepsilon\|\tilde{A}_{a,j}\|_{B(j,j)}\|\Delta_a\|_{1,\nu}\|\bar{a}_0\|_{1,\nu} \\
\|A_{a,j}z_2^{(3)}\|_j &\le Z_2^{(3,j)} \\
&\stackrel{\text{def}}{=} \tau^2\|\tilde{B}_{a,j}\|_{B(j,j)} + 4\|\tilde{B}_{a,j}\|_{B(j,j)} + 2\|\tilde{B}_{a,j}\|_{B(j,j)} + 3\varepsilon\|\tilde{A}_{a,j}\|_{B(j,j)}\|\Delta_a\|_{1,\nu} + 2\varepsilon\|\tilde{A}_{a,j}\|_{B(j,j)}\|\bar{a}_0\|_{1,\nu} \\
&\quad + 3\varepsilon|\Delta_\omega|\|\tilde{A}_{a,j}\|_{B(j,j)} + 6\varepsilon\|\tilde{A}_{a,j}\|_{B(j,j)}\|\Delta_a\|_{1,\nu} + 4\varepsilon\|\tilde{A}_{a,j}\|_{B(j,j)}\|\bar{a}_0\|_{1,\nu} + 2\varepsilon\bar{\omega}_0\|\tilde{A}_{a,j}\|_{B(j,j)} \\
\|A_{a,j}z_2^{(4)}\|_j &\le Z_2^{(4,j)} \\
&\stackrel{\text{def}}{=} 4\varepsilon\|\tilde{A}_{a,j}\|_{B(j,j)}.
\end{aligned}$$

For $i = 1, 2, 3, 4$, set

$$Z_2^{(j)} = \max\left(Z_2^{(j,0)}, Z_2^{(j,1)}\right)$$

so that we can define

$$Z_2(r) \stackrel{\text{def}}{=} Z_2^{(4)}r^3 + Z_2^{(3)}r^2 + Z_2^{(2)}r + Z_2^{(1)}. \tag{49}$$

From this, we conclude that

$$\begin{aligned}
|A_{a,0}\left(D_xg(\bar{x}_s+b,\lambda_s) - D_xg(\bar{x}_0,\lambda_0)\right)c| &\le Z_2(r) \\
\|A_{a,1}\left(D_xg(\bar{x}_s+b,\lambda_s) - D_xg(\bar{x}_0,\lambda_0)\right)c\|_{1,\nu} &\le Z_2(r),
\end{aligned}$$

and therefore

$$\begin{aligned}
\|A[D_xF(\bar{x}_s+b,\lambda_s) - D_xF(\bar{x}_0,\lambda_0)]c\|_X &= \left\|\begin{pmatrix} A_{\omega,0} & A_{a,0} \\ A_{\omega,1} & A_{a,1}\end{pmatrix}\begin{pmatrix} 0 \\ \left(D_xg(\bar{x}_s+b,\lambda_s) - D_xg(\bar{x}_0,\lambda_0)\right)c\end{pmatrix}\right\|_X \\
&= \max\big(\,|A_{a,0}\left(D_xg(\bar{x}_s+b,\lambda_s) - D_xg(\bar{x}_0,\lambda_0)\right)c|\,, \\
&\qquad \|A_{a,1}\left(D_xg(\bar{x}_s+b,\lambda_s) - D_xg(\bar{x}_0,\lambda_0)\right)c\|_{1,\nu}\big) \\
&\le Z_2(r).
\end{aligned}$$

Combining (43), (46), (48) and (49), the radii polynomial is defined by

$$p(r) \stackrel{\text{def}}{=} Z_2^{(4)}r^4 + Z_2^{(3)}r^3 + Z_2^{(2)}r^2 + \left(Z_2^{(1)} + Z_1 + Z_0 - 1\right)r + Y_0. \tag{50}$$

3.1.7. *Proof of existence of periodic solutions for Nussbaum's equation.* Using the radii polynomial approach we could prove the following two theorems.

Theorem 3.10. *Fix $\tau = 2$ and $\varepsilon = 0.15$. Then there is a branch of periodic solution of Nussbaum's equation (20) parameterized by the parameter λ. Given a periodic solution $y(t)$ on the branch, letting p its period, $y(t)$ satisfies the symmetry property*

$$y(t + p/2) = -y(t), \qquad \text{for all } t \in \mathbb{R}. \tag{51}$$

The continuous range of parameter for which the proof of existence is performed is $\lambda \in [-3.8521, 0.65]$. The global branch is a C^∞ function of the parameter λ. The continuous range of periods of the periodic solutions over the branch contains the interval $p \in [3.7, 21.5]$.

Proof. The MATLAB program `script_proof1.m` available at [67] computes the coefficients of the radii polynomial $p(r)$ given by (50), and as a parameter continuation is performed, the code adapts the step size Δ_λ and applies successfully the radii polynomial approach (Theorem 2.6) to show the existence of a global C^∞ branch of periodic solutions. Throughout the whole continuation, the number of Fourier coefficients used for the proof is fixed to be $m = 110$. Along the branch, we make sure that $\bar{x}_s \in X_{sym}$ and that the approximate inverse A satisfies the condition (32), that is $AF : X_{sym} \times \mathbb{R} \to X_{sym}$. By Lemma 3.2, at a given parameter value λ, the solution $\tilde{x} = (\tilde{\omega}, \tilde{a}) \in X_{sym}$. Hence, $\tilde{\omega} \in \mathbb{R}$ (yielding the period $\tilde{p} = \frac{2\pi}{\tilde{\omega}} \in \mathbb{R}$) and the Fourier coefficients $\tilde{a} = (\tilde{a}_k)_{k\in\mathbb{Z}}$ satisfy that $\tilde{a}_{2k} = 0$ for all $k \in \mathbb{Z}$. Consider the periodic solution $\tilde{y}(t)$ associated to $(\tilde{\omega}, \tilde{a})$

$$\tilde{y}(t) = \sum_{\substack{k\in\mathbb{Z} \\ k \text{ odd}}} \tilde{a}_k e^{ik\frac{2\pi}{\tilde{p}}t}.$$

Hence,

$$\tilde{y}(t + \tilde{p}/2) = \sum_{\substack{k\in\mathbb{Z} \\ k \text{ odd}}} \tilde{a}_k e^{ik\frac{2\pi}{\tilde{p}}(t+\tilde{p}/2)} = \sum_{\substack{k\in\mathbb{Z} \\ k \text{ odd}}} \tilde{a}_k (-1)^k e^{ik\frac{2\pi}{\tilde{p}}t} = -\sum_{\substack{k\in\mathbb{Z} \\ k \text{ odd}}} \tilde{a}_k e^{ik\frac{2\pi}{\tilde{p}}t} = -\tilde{y}(t).$$

The global branch of periodic solution is a C^∞ function of λ by Lemma 2.7. □

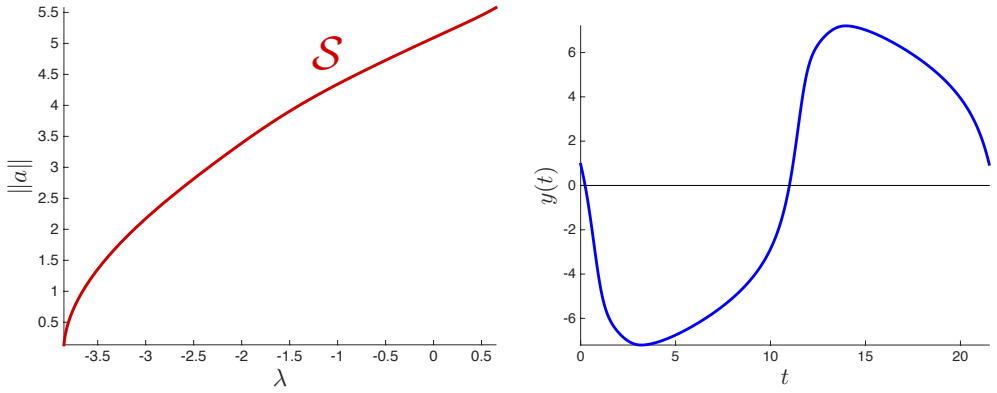

FIGURE 18. **(Left)** As proven in Theorem 3.10, a branch of periodic solutions of Nussbaum's equation (20) parameterized by the parameter λ. **(Right)** The graph of the most right periodic solution on the branch plotted over its period.

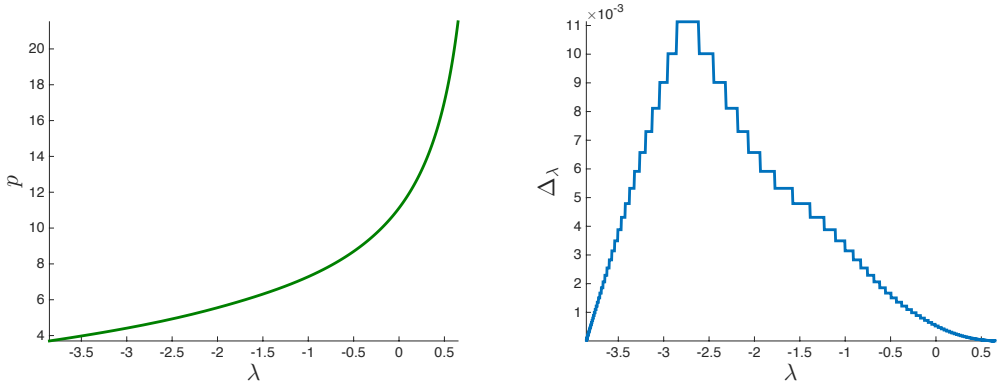

FIGURE 19. **(Left)** Continuous range of periods of the periodic solutions as a function of λ along the branch of Theorem 3.10. **(Right)** The adaptive value of Δ_λ as the parameter λ varies along the branch of Theorem 3.10.

Theorem 3.11. *Fix $\tau = 5$ and $\varepsilon = 0.25$. Then there is a branch of periodic solution of Nussbaum's equation* (20) *parameterized by the parameter λ. Given a periodic solution $y(t)$ on the branch, letting p its period, $y(t)$ satisfies the symmetry property* (51)*.The continuous range of parameter for which the proof of existence is performed is $\lambda \in [-4.8234, -0.476]$. The global branch is a C^∞ function of the parameter λ. The continuous range of periods of the periodic solutions over the branch contains the interval $p \in [3.15942, 5.253625]$.*

Proof. The proof is similar as the proof of Theorem 3.10. In this case, the number of Fourier coefficients used to perform the proof is $m = 20$. The proof is computer-assisted and terminates by executing the MATLAB program `script_proof2.m` available at [67]. □

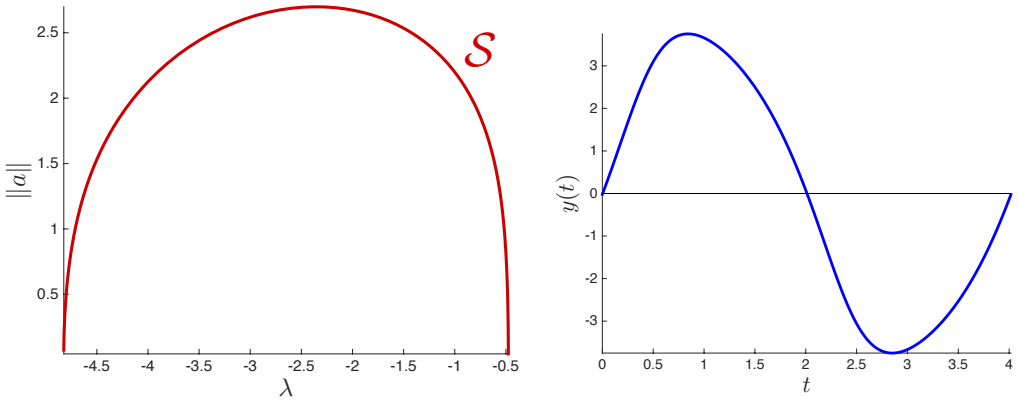

FIGURE 20. **(Left)** As proven in Theorem 3.11, a branch of symmetric periodic solutions of Nussbaum's equation (20) parameterized by the parameter λ. **(Right)** The graph of a periodic solution on the branch over the interval $[0, p]$ with p its period.

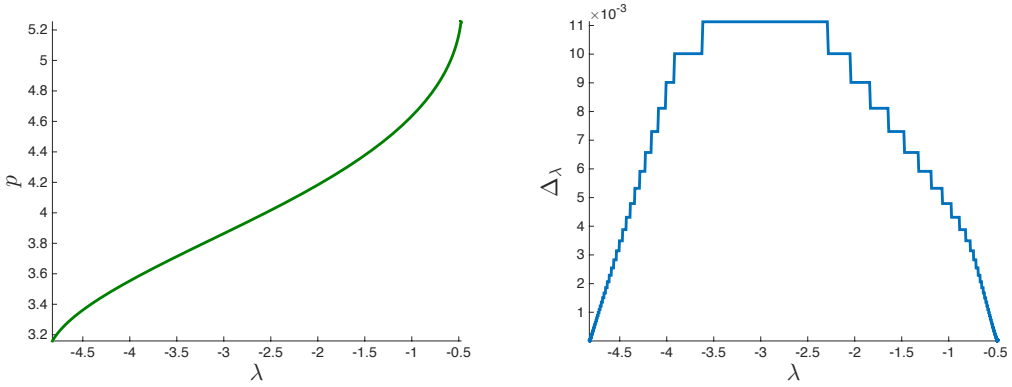

FIGURE 21. **(Left)** Continuous range of periods of the periodic solutions as a function of λ along the branch of Theorem 3.11. **(Right)** The adaptive value of Δ_λ as the parameter λ varies along the branch of Theorem 3.11.

In [66], similar results were obtained for Nussbaum's equation (20). The differences are that in [66], proofs were performed in a weighted ℓ^∞ space (as opposed to a weighted ℓ^1 here), and the proofs were obtained at discrete parameter values of λ.

3.2. **Wright's Equation.** As mentioned in Section 1, Jones Conjecture reformulated as in Conjecture 1.3 requires studying a global continuous branch of periodic solutions of Wright's equation (1). Studying periodic solutions of (1) boils down in this case to study the zeros of the map $g = (g_k)_{k\in\mathbb{Z}}$ given by

$$g_k \overset{\text{def}}{=} \left(ik\omega + \alpha e^{-ik\omega}\right) a_k + \alpha \sum_{k_1+k_2=k} e^{-ik_1\omega} a_{k_1} a_{k_2}.$$

In this case the parameter is set to be $\lambda = \alpha$. Using the radii polynomial approach, a rigorous continuation method was used in [12] to study rigorously the behaviour of the solutions on the branch $\mathcal{F}_0$. The proofs were performed in a different Banach space, that is a weighted ℓ^∞ space. This means that the estimates are different than the one presented above in the weighted ℓ^1 space. Recall from Section 1 the notation $\mathcal{F}_0$, which denotes the branch of periodic solutions of Wright's equation (1) which bifurcates out of a supercritical Hopf bifurcation from the trivial solution at $\alpha = \pi/2$. Using a rigorous continuation method, as the one introduced in details for Nussbaum's equation in Section 3.1, here is the main result of [12].

Theorem 3.12. *Let $\varepsilon = 7.3165 \times 10^{-4}$. Then $\mathcal{F}_0$ is a branch parameterized by λ over the range $\alpha \in \left[\frac{\pi}{2} + \varepsilon, 2.3\right]$, and so $\mathcal{F}_0$ does not have any fold over $\alpha \in \left[\frac{\pi}{2} + \varepsilon, 2.3\right]$.*

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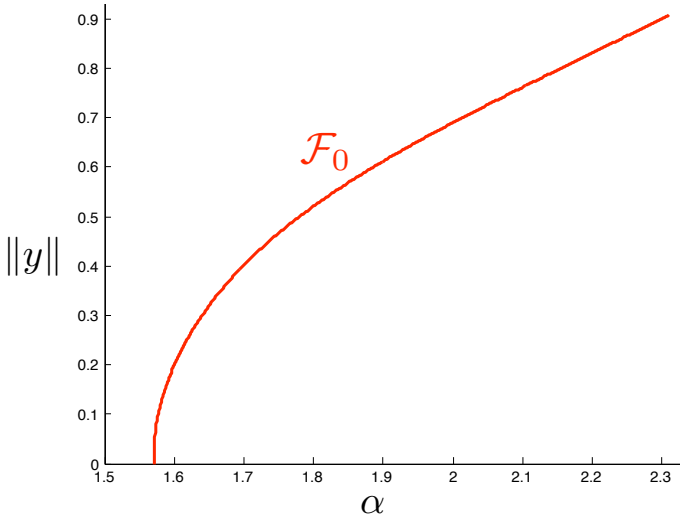

FIGURE 22. Rigorous study of a part of $\mathcal{F}_0$.

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