Quantum Knots & Mosaics

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This talk is based on the paper
Lomonaco, Samuel, and Louis Kauffman,
Quantum Knots and Mosaics,
http://arxiv.org/abs/0805.0339

This talk was motivated by a number of papers


Lomonaco, Samuel J., Jr., The modern legacies of Thomson’s atomic vortex theory in classical electrodynamics, AMS PSAPM/51, Providence, RI (1996), 145 – 166.

Classical Vortices in Plasmas

Lomonaco, Samuel J., Jr., The modern legacies of Thomson’s atomic vortex theory in classical electrodynamics, AMS PSAPM/51, Providence, RI (1996), 145 – 166.
Knots Naturally Arise in the Quantum World as Dynamical Processes

Examples of dynamical knots in quantum physics:
- Knotted vortices
  - In supercooled helium II
  - In the Bose-Einstein Condensate
  - In the Electron fluid found within the fractional quantum Hall effect

Reason for current intense interest:
A Natural Topological Obstruction to Decoherence

Objectives
- We seek to create a quantum system that simulates a closed knotted physical piece of rope.
- We seek to define a quantum knot in such a way as to represent the state of the knotted rope, i.e., the particular spatial configuration of the knot tied in the rope.
- We also seek to model the ways of moving the rope around (without cutting the rope, and without letting it pass through itself.)

Rules of the Game
Find a mathematical definition of a quantum knot that is
- Physically meaningful, i.e., physically implementable, and
- Simple enough to be workable and useable.

Aspirations
We would hope that this definition will be useful in modeling and predicting the behavior of knotted vortices that actually occur in quantum physics such as
- In supercooled helium II
- In the Bose-Einstein Condensate
- In the Electron fluid found within the fractional quantum Hall effect

Overview
Part 1. Mosaic Knots
We reduce tame knot theory to a formal system of string manipulation rules, i.e., string rewriting rules.

Part 2. Quantum Knots
We then use mosaic knots to build a physically implementable definition of quantum knots.
Mosaic Knots

Mosaic Tiles

Let $T^{(u)}$ denote the following set of 11 symbols, called mosaic (unoriented) tiles:

Please note that, up to rotation, there are exactly 5 tiles.

Definition of an $n$-Mosaic

An $n$-mosaic is an $n \times n$ matrix of tiles, with rows and columns indexed $0, 1, \ldots, n - 1$.

An example of a 4-mosaic

Tile Connection Points

A connection point of a tile is a midpoint of an edge which is also the endpoint of a curve drawn on a tile. For example,

Contiguous Tiles

Two tiles in a mosaic are said to be contiguous if they lie immediately next to each other in either the same row or the same column.

Suitably Connected Tiles

A tile in a mosaic is said to be suitably connected if all its connection points touch the connection points of contiguous tiles. For example,
A knot mosaic is a mosaic with all tiles suitably connected. For example,

**Knot Mosaics**

- **Figure Eight Knot 5-Mosaic**
- **Hopf Link 4-Mosaic**
- **Borromean Rings 6-Mosaic**

**Notation**

- $M^{(n)} = \text{Set of } n\text{-mosaics}$
- $K^{(n)} = \text{Subset of knot } n\text{-mosaics}$

**Planar Isotopy Moves**
Non-Deterministic Tiles

We use the following tile symbols to denote one of two possible tiles:

For example, the tile denotes either or

Planar Isotopy (PI) Moves on Mosaics

It is understood that each of the above moves depicts all moves obtained by rotating the \(2 \times 2\) sub-mosaics by 0, 90, 180, or 270 degrees.

For example, represents each of the following 4 moves:

Terminology: k-Submosaic Moves

Def. A \(k\)-submosaic move on a mosaic \(M\) is a mosaic move that replaces one \(k\)-submosaic in \(M\) by another \(k\)-submosaic.

All of the PI moves are examples of \(2\)-submosaic moves. I.e., each PI move replaces a \(2\)-submosaic by another \(2\)-submosaic.

For example,

Planar Isotopy (PI) Moves on Mosaics

Each of the PI \(2\)-submosaic moves represents any one of the \((n-k+1)^2\) possible moves on an \(n\)-mosaic

Planar Isotopy (PI) Moves on Mosaics

Each PI move acts as a local transformation on an \(n\)-mosaic whenever its conditions are met. If its conditions are not met, it acts as the identity transformation.

Ergo, each PI move is a permutation of the set of all knot \(n\)-mosaics \(K^{(n)}\)

In fact, each PI move, as a permutation, is a product of disjoint transpositions.
Reidemeister Moves

More Non-Deterministic Tiles

We also use the following tile symbols to denote one of two possible tiles:

For example, the tile \[ \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\end{array} \]
denotes either\[
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\end{array}
\]
or\[
\begin{array}{c}
\text{\textbullet} \\
\end{array}
\]

Synchronized Non-Deterministic Tiles

Non-deterministic tiles labeled by the same letter are synchronized:

Just like each PI move, each R move is a permutation of the set of all
knot n-mosaics \( A^{(n)} \)

In fact, each R move, as a permutation, is a product of disjoint transpositions.
The Ambient Group $A(n)$

We define the ambient isotopy group $A(n)$ as the subgroup of the group of all permutations of the set $K^{(n)}$ generated by the all PI moves and all Reidemeister moves.

Knot Type

The Mosaic Injection $\iota : M^{(n)} \to M^{(n+1)}$

We define the mosaic injection $\iota : M^{(n)} \to M^{(n+1)}$ as:

$$M^{(n+1)}_{i,j} = \begin{cases} 
M^{(n)}_{i,j} & \text{if } 0 \leq i, j < n \\
\text{otherwise} & 
\end{cases}$$

Mosaic Knot Type

Def. Two $n$-mosaics $M$ and $M'$ are of the same knot $n$-type, written $M^n \sim M'^n$, provided there exists an element of the ambient group $A(n)$ that transforms $M$ into $M'$.

Two $n$-mosaics $M$ and $M'$ are of the same knot type if there exists a non-negative integer $k$ such that $i^k M \sim i^k M'$.
There are 29 oriented tiles, and 9 tiles up to rotation. Rotationally equivalent tiles have been grouped together.
Let $\mathcal{H}$ be the 11 dimensional Hilbert space with orthonormal basis labeled by the tiles $T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9$. We define the Hilbert space $\mathcal{M}^{(n)}$ of $n$-mosaics as

$$\mathcal{M}^{(n)} = \bigotimes_{k=0}^{n-1} \mathcal{H}$$

This is the Hilbert space with induced orthonormal basis

$$\left\{ \bigotimes_{k=0}^{n-1} |T_{R(k)}\rangle : 0 \leq \ell(k)<11 \right\}$$

We identify each basis element $\bigotimes_{k=0}^{n-1} |T_{R(k)}\rangle$ with the mosaic labeled ket $|M\rangle$ via the bijection $T_{ij} \leftrightarrow M_{ij}$ using row major order.

For example, in the 3-mosaic Hilbert space $\mathcal{M}^{(3)}$, the basis ket $|T_1\rangle \otimes |T_2\rangle \otimes |T_3\rangle \otimes |T_4\rangle \otimes |T_5\rangle \otimes |T_6\rangle$ is identified with the 3-mosaic labeled ket $|T_1 T_2 T_3\rangle$.

The Hilbert space $\mathcal{K}^{(n)}$ of quantum knots is defined as the sub-Hilbert space of $\mathcal{M}^{(n)}$ spanned by all orthonormal basis elements labeled by knot n-mosaics.

An Example of a Quantum Knot

$$|K\rangle = \frac{1}{\sqrt{2}} \left( |\text{mosaic 1}\rangle + |\text{mosaic 2}\rangle \right)$$
Since each element $g \in A(n)$ is a permutation, it is a linear transformation that simply permutes basis elements.

Hence, under this identification, the ambient group $A(n)$ becomes a discrete group of unitary transfs on the Hilbert space $\mathcal{K}^{(n)}$.

**The Quantum Knot System $\left(\mathcal{K}^{(n)}, A(n)\right)$**

**Def.** A quantum knot system $\left(\mathcal{K}^{(n)}, A(n)\right)$ is a quantum system having $\mathcal{K}^{(n)}$ as its state space, and having the Ambient group $A(n)$ as its set of accessible unitary transformations.

The states of quantum system $\left(\mathcal{K}^{(n)}, A(n)\right)$ are quantum knots. The elements of the ambient group $A(n)$ are quantum moves.

Choosing an integer $n$ is analogous to choosing a length of rope. The longer the rope, the more knots that can be tied.

The parameters of the ambient group $A(n)$ are the “knobs” one turns to spacially manipulate the quantum knot.

**Quantum Knot Type**

**Def.** Two quantum knots $\left| K_1 \right>$ and $\left| K_2 \right>$ are of the same knot $n$-type, written $\left| K_1 \right> \sim_n \left| K_2 \right>$, provided there is an element $g \in A(n)$ s.t.

$g \left| K_1 \right> = \left| K_2 \right>$

They are of the same knot type, written $\left| K_1 \right> \sim \left| K_2 \right>$, provided there is an integer $m \geq 0$ such that

$\left| K_1 \right> \sim_{nm} \left| K_2 \right>$
Two Quantum Knots NOT of the Same Knot Type

\[ |K_1\rangle = \begin{array}{c} \text{\textbullet} \\
\end{array} \]

\[ |K_2\rangle = \begin{array}{c} \text{\textbullet} \\
\end{array} + \begin{array}{c} \text{\textbullet} \\
\end{array} \]
\[ \sqrt{2} \]

Hamiltonians of the Generators of the Ambient Group

Hamiltonians for \( A(n) \)

Each generator \( g \in A(n) \) is the product of disjoint transpositions, i.e.,

\[ g = (K_{\alpha_1}, K_{\beta_1})(K_{\alpha_2}, K_{\beta_2}) \cdots (K_{\alpha_s}, K_{\beta_s}) \]

Choose a permutation \( \eta \) so that

\[ \eta^{-1}g\eta = (K_{\alpha_1}, K_{\beta_1})(K_{\alpha_2}, K_{\beta_2}) \cdots (K_{\alpha_s}, K_{\beta_s}) \]

Hence,

\[ \begin{pmatrix} 
\sigma_1 & 0 \\
0 & \sigma_1 \\
0 & \sigma_{L-1} \\
\end{pmatrix} \]

, where \( \sigma_i = \begin{pmatrix} 1 & 0 \\
0 & 1 \\
\end{pmatrix} \)

Also, let \( \sigma_s = \begin{pmatrix} 0 & 1 \\
0 & 0 \\
\end{pmatrix} \), and note that

\[ \ln(\sigma_s) = \frac{i\pi}{2} (2s+1)(\sigma_s - \sigma_s) \]

For simplicity, we always choose the branch \( s = 0 \).

\[ H_s = -i\eta \ln(\eta^{-1}g\eta)\eta \]

Some Miscellaneous Unitary Transformations Not in \( A(n) \)

Using the Hamiltonian for the Reidemeister 2 move

and the initial state

we have that the solution to Schroedinger's equation for time \( t \) is

\[ e^{\frac{-i\pi}{2}} = \begin{pmatrix} 
\sigma_1 & 0 \\
0 & \sigma_1 \\
0 & \sigma_{L-1} \\
\end{pmatrix} \]

\[ -i\eta \left( I_t \otimes (\sigma_s - \sigma_s) \otimes 0 \right) \eta^{-1} \]
Misc. Transformations

The crossing tunneling transformation
\[ \tau_0 = \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} \]

The mirror image transformation
\[ \mu = \prod_{i,j} \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} \]

The hyperbolic transformation
\[ \eta_0 = \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} \]

The elliptic transformation
\[ \varepsilon_0 = \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} \]

Observables which are Quantum Knot Invariants

Question. What do we mean by a physically observable knot invariant?

Let \( (\mathcal{K}^{(n)}, A(n)) \) be a quantum knot system. Then a quantum observable \( \Omega \) is a Hermitian operator on the Hilbert space \( \mathcal{K}^{(n)} \).

Question. But which observables \( \Omega \) are actually knot invariants?

Def. An observable \( \Omega \) is an invariant of quantum knots provided \( U\Omega U^{-1} = \Omega \) for all \( U \in A(n) \).

Question. But how do we find quantum knot invariant observables?

Theorem. Let \( (\mathcal{K}^{(n)}, A(n)) \) be a quantum knot system, and let
\[ \mathcal{K}^{(n)} = \bigoplus_{\ell} W_{\ell} \]
be a decomposition of the representation \( A(n) \times \mathcal{K}^{(n)} \to \mathcal{K}^{(n)} \) into irreducible representations.

Then, for each \( \ell \), the projection operator \( P_{\ell} \) for the subspace \( W_{\ell} \) is quantum knot observable.
Theorem. Let \((\mathcal{X}, A(n))\) be a quantum knot system, and let \(\Omega\) be an observable on \(\mathcal{X}\). Let \(S_t(\Omega)\) be the stabilizer subgroup for \(\Omega\), i.e.,

\[
S_t(\Omega) = \{ U \in A(n) : U\Omega U^{-1} = \Omega \}
\]

Then the observable

\[
\sum_{U \in A(n)/S_t(\Omega)} U\Omega U^{-1}
\]

is a quantum knot invariant, where the above sum is over a complete set of coset representatives of \(S_t(\Omega)\) in \(A(n)\).

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**Future Directions & Open Questions**

- What is the structure of the ambient group \(A(n)\) and its direct limit \(A = \lim_{n \to \infty} A(n)\)? Can one find a presentation of this group?

- Unlike classical knots, quantum knots can exhibit the non-classical behavior of quantum superposition and quantum entanglement. Are quantum and topological entanglement related to one another? If so, how?

- How does one find a quantum observable for the Jones polynomial?

- How does one create quantum knot observables that represent other knot invariants such as, for example, the Vassiliev invariants?

- What is gained by extending the definition of quantum knot observables to POVMs?

- What is gained by extending the definition of quantum knot observables to mixed ensembles?
**Future Directions & Open Questions**

**Def.** We define the **mosaic number** of a knot $k$ as the smallest integer $n$ for which $k$ is representable as a knot $n$-mosaic.

- The mosaic number of the trefoil is 4. In general, how does one compute the mosaic number of a knot?
- Is the mosaic number related to the crossing number of a knot?

**Future Directions & Open Questions**

- Can quantum knot systems be used to model and predict the behavior of quantum vortices in supercooled helium 2?
- Quantum vortices in the Bose-Einstein Condensate
- Fractional charge quantification that is manifest in the fractional quantum Hall effect

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**Weird !!!**