

## ANALYTIC CLASSIFICATION OF SADDLE NODES

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**ABSTRACT.** Isolated degenerate elementary singular points (saddle nodes) of germs of holomorphic vector fields in  $(\mathbb{C}^2, 0)$  are studied. An analytic classification of them is obtained; it is shown that the analytic classification has two times more moduli (numeric and functional) than the orbital analytic classification. A theorem on sectorial normalization is proved.

### 1. INTRODUCTION

Let  $\mathcal{V}$  be the class of germs of holomorphic vector fields in  $(\mathbb{C}^2, 0)$  with an isolated elementary degenerate singular point 0 (that is, such that the linear part of the germ at the origin is degenerate, but not all its eigenvectors vanish).

Germs  $v$  and  $\tilde{v}$  in  $\mathcal{V}$  are said to be *analytically (formally) equivalent* if there exists a local holomorphic (formal) change of coordinates  $H$  in  $(\mathbb{C}^2, 0)$  converting one of the germs into the other:  $H' \cdot v = \tilde{v} \circ H$ . Germs  $v$  and  $\tilde{v}$  in  $\mathcal{V}$  are said to be *orbitally analytically (formally) equivalent* if there exists a local holomorphic change of coordinates taking the phase portrait of one germ to the phase portrait of the other (if there exists a formal change of coordinates  $H$  and a formal power series  $k$  with non-zero free term such that  $H' \cdot v = k \cdot \tilde{v} \circ H$ ).

Orbital analytic and formal classifications of germs in  $\mathcal{V}$  are well known ([2]; see also [1, pp. 29, 33]). In this paper we obtain an analytic classification of germs in  $\mathcal{V}$ ; it turns out that this classification has two times more moduli (both numeric and functional) than the orbital analytic classification.

We note that the same relation between the moduli of analytic [7], [8] and orbital analytic ([3], see also [1]) classifications was obtained for resonance saddles.

For typical degenerate elementary singular points an analytic classification was considered in [9]: in that paper a complete description is given; however, the proofs are only given in outline. In this paper we consider the general situation and give complete proofs of all results.

Finally, in the general case a similar result was recently obtained in [4] and [5], but by another method. Namely, in those papers for each germ  $v$  in  $\mathcal{V}$  an analytic classification of germs proportional to  $v$  was studied; taking into account the well-known results of [2] on the orbital analytic classification of germs in  $\mathcal{V}$ , this enables one to obtain an analytic classification of germs of the entire class  $\mathcal{V}$ . In the present paper the main result is the theorem on sectorial normalization of germs in  $\mathcal{V}$ . We use this to construct a normalizing atlas (see [1]) for the germ  $v \in \mathcal{V}$ , and then determine invariants of the analytic classification of  $v$  in terms of the transition functions of this atlas. We note that the theorem on sectorial normalization is of importance in its own right and is a generalization of the analogous theorem for orbital equivalence (see [1]).

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**1.1. Formal classification. The classes  $\mathcal{V}_{p,\lambda,a}$  and  $\mathcal{V}_{p,\lambda,a}^N$ .** In the papers [1] and [2] it is proved that a germ in  $\mathcal{V}$  is formally orbitally equivalent to one of the germs of the form

$$v_{p,\lambda} = x \frac{\partial}{\partial x} + \frac{y^{p+1}}{1 + \lambda y^p} \frac{\partial}{\partial y}, \quad \lambda \in \mathbb{C}.$$

We denote by  $\mathcal{V}_{p,\lambda}$  the class of germs formally orbitally equivalent to  $v_{p,\lambda}$ .

**Theorem 1** (Formal classification). *Each germ in  $\mathcal{V}_{p,\lambda}$  is formally equivalent to one of the germs*

$$v_{p,\lambda,a} = v_{p,\lambda} \cdot a(y), \text{ where } a(y) = \sum_{k=0}^p a_k y^k, \quad a_0 \neq 0, \quad a_k \in \mathbb{C}, \quad k = 0, 1, \dots, p.$$

We denote by  $\mathcal{V}_{p,a,\lambda}$  the class of germs in  $\mathcal{V}$  formally equivalent to  $v_{p,\lambda,a}$ .

**Corollary 1.** *For any  $N \in \mathbb{N}$  and any  $v \in \mathcal{V}_{p,\lambda,a}$  there exists  $\tilde{v} \in \mathcal{V}_{p,\lambda,a}$  such that  $\tilde{v}$  is analytically equivalent to  $v$  and  $j_0^N \tilde{v} = j_0^N v_{p,\lambda,a}$ .*

Theorem 1 and its corollary can be significantly improved.

**Definition 1.** A power series  $\sum_{k=0}^{\infty} f_k(x)y^k$  whose coefficients are holomorphic in some neighbourhood of the origin (the same neighbourhood for all  $k$ ) is called a *semiformal series*; a map whose components are semiformal series is called a *semiformal map*.

We denote by  $\mathcal{V}'_{p,\lambda,a}$  the class of germs in  $\mathcal{V}$  with normalized  $(2p+1)$ -jet:

$$v \in \mathcal{V}'_{p,\lambda,a} \Leftrightarrow j_0^{2p+1} v = j_0^{2p+1} v_{p,\lambda,a}.$$

**Theorem 2.** *For any germ  $v \in \mathcal{V}'_{p,\lambda,a}$  there exists a unique formal normalizing change of coordinates  $\hat{H}$  such that  $j_0^{p+1} \hat{H} = \text{id}$ . This change of coordinates is a semiformal map conjugating the germs  $v_{p,\lambda,a}$  and  $v$ :*

$$\hat{H}' \cdot v_{p,\lambda,a} = v \circ \hat{H}.$$

**Corollary 2.** *For any germ  $v \in \mathcal{V}_{p,\lambda,a}$  and for any  $N \in \mathbb{N}$ , there exists a germ  $\tilde{v}$  that is analytically equivalent to  $v$  and is such that*

$$(1) \quad \tilde{v}(x, y) - v_{p,\lambda,a}(x, y) = \mathcal{O}(y^N) \frac{\partial}{\partial x} + \mathcal{O}(y^{N+p}) \frac{\partial}{\partial y}, \quad y \rightarrow 0.$$

**Definition 2.** The semiformal change of coordinates  $\hat{H}$  in Theorem 2 is said to be *normalized*; we denote by  $\mathcal{V}_{p,\lambda,a}^N$  (for  $N > p+1$ ) the class of germs  $\tilde{v}$  satisfying condition (1).

*Remark 1.* In Theorem 2 one says that the only ‘bad’ variable among the variables  $(x, y)$  is the variable  $y$ : the formal normalizing change of coordinates is analytic with respect to  $x$ . Accordingly, Corollary 2.2 enables one to normalize the  $N$ -jet of the field  $v \in \mathcal{V}'_{p,\lambda,a}$  via an analytic change of coordinates not only at the point 0 (as the corollary to Theorem 1 asserts), but also at all points of the separatrix  $\{y = 0\}$ .

*Remark 2.* Note that  $\mathcal{V}'_{p,\lambda,a} \subset \mathcal{V}_{p,\lambda,a}$  and, since  $N > p+1$ , it follows that  $\mathcal{V}_{p,\lambda,a}^N \subset \mathcal{V}'_{p,\lambda,a}$ . Therefore, by Theorem 2, there exists for each  $v \in \mathcal{V}_{p,\lambda,a}^N$  a unique normalized, normalizing change of coordinates  $\hat{H}$  for the germ  $v$ . Furthermore, for the semiformal map  $\hat{H}$  we can obtain from Theorem 2 the somewhat more refined asymptotics:

$$(2) \quad \hat{H}(x, y) = (x, y) + O(y^{N-p}, y^N).$$

*Proof.* A full proof of Theorem 1 can be found in [10]. The same proof applied to the class  $\mathcal{V}'_{p,\lambda,a}$  also gives the proof of Theorem 2 along with the uniqueness assertion for the normalized semiformal normalizing change of coordinates and the estimate (2) for it.  $\square$

**1.2. Sectorial normalization.** Let  $\alpha \in (\frac{\pi}{2p}, \frac{\pi}{p})$ . For  $j \in \mathbb{N}$ ,  $1 \leq j \leq 2p$ , we consider the region

$$\Omega_j = \left\{ (x, y) \in \mathbb{C}^2 : |x| < \varepsilon, 0 < |y| < \varepsilon, \left| \arg y + \frac{\pi}{2p} - \frac{\pi j}{p} \right| < \alpha \right\}.$$

We call the system of regions  $\{\Omega_j\}$  ( $j = 1, \dots, 2p$ ) a *good covering* (see [1]) of the region  $\{|x| < \varepsilon, 0 < |y| < \varepsilon\}$ ; the parameters  $\alpha$  and  $\varepsilon$  are respectively called the *opening* and *radius* of the good covering.

**Definition 3.** Let  $U$  be a neighbourhood of the origin in  $\mathbb{C}$  and let  $S \subset \mathbb{C}$  be a sector of finite radius with vertex at the origin. We call the domain  $\Omega = U \times S$  a *sectorial domain*. A semiformal map  $\widehat{H} = \sum_{k=0}^{\infty} f_k(x)y^k$  with holomorphic coefficients in  $U$  is said to be *asymptotic* for the holomorphic map  $H : \Omega \rightarrow \mathbb{C}^2$  on the sectorial domain  $\Omega = U \times S$  if for each partial sum  $H_n = \sum_{k=0}^n f_k(x)y^k$  we have

$$H(x, y) - H_n(x, y) = o(y^n) \quad \text{for } (x, y) \in \Omega, \quad y \rightarrow 0.$$

**Theorem 3** (on sectorial normalization). *For any germ  $v \in \mathcal{V}_{p,\lambda,a}^N$  and any good covering  $\Omega = \{\Omega_j\}$  with a given opening and sufficiently small radius there exists a unique system of holomorphic maps  $H_j : \Omega_j \rightarrow H_j(\Omega_j) \subset \mathbb{C}^2$  such that:*

1°.  *$H_j$  conjugates the germ  $v$  with its formal normal form  $v_{p,\lambda,a}$   $v$   $v_{p,\lambda,a}$  on  $\Omega_j$ :*

$$(3) \quad H'_j \cdot v_{p,\lambda,a} = v \circ H_j \quad \text{on } \Omega_j.$$

2°. *The normalized formal normalizing change of coordinates  $\widehat{H}$  of  $v$  is asymptotic for  $H_j$  on  $\Omega_j$ .*

Section 2 is devoted to the proof of the theorem on sectorial normalization.

**1.3. Analytic classification.** Along with analytic equivalence we shall also consider strict equivalence.

**Definition 4.** We call germs  $v, \tilde{v} \in \mathcal{V}_{p,\lambda,a}^N$  *strictly equivalent* if they are equivalent and the change of coordinates conjugating them has the form  $H(x, y) = (x + o(1), y + o(y^{p+1}))$ .

*Remark 3.* Strict equivalence is more convenient than equivalence because the formal normalizing change of coordinates is unique. Each germ in  $\mathcal{V}_{p,\lambda,a}^N$  is not only formally equivalent to its formal normal form  $v_{p,\lambda,a}$  but is also strictly formally equivalent to it.

Let  $\mathcal{M}_{p,\lambda}$  be the space of all tuples  $(c, \phi, \psi)$  such that  $c \in \mathbb{C}^p$ ;  $\varphi = (\varphi_1, \dots, \varphi_p)$ ,  $\psi = (\psi_1, \dots, \psi_p)$ ,  $\varphi_j$  and  $\psi_j$  are holomorphic in  $(\mathbb{C}, 0)$ ;  $\varphi_j(0) = \psi_j(0) = 0$ ,  $\varphi'_k(0) = 1$   $\forall k < p$ ,  $\varphi'_p(0) = \exp\{2\pi i \lambda\}$ .

Let  $p_a$  be the greatest common divisor of  $p$  and all those  $k \in \{1, \dots, p\}$  for which  $a_k \neq 0$ ,  $n_a = p/p_a$ . Two tuples  $(c, \phi, \psi)$  and  $(\tilde{c}, \tilde{\phi}, \tilde{\psi})$  in  $\mathcal{M}_{p,\lambda}$  are said to be *equivalent* if for some  $C \in \mathbb{C}_*^p$ ,  $C = (C_1, \dots, C_p)$ , and some  $s \in \mathbb{Z}$ ,  $0 \leq s < p_a$ ,

$$(4) \quad \tilde{c}_{j+s n_a} = C_j \cdot c_j, \quad \tilde{\varphi}_{j+s n_a}(z) \equiv C_{j+1+s n_a} \varphi_j(C_j^{-1} z), \quad \tilde{\psi}_{j+s n_a}(z) = \psi(C_j^{-1} z)$$

(the numbering is cyclic). Let  $M_{p,\lambda,a}$  be the space of equivalence classes in  $\mathcal{M}_{p,\lambda}$ .

**Theorem 4** (Analytic classification of germs in  $\mathcal{V}_{p,\lambda,a}$ ). *There exists a map*

$$m : \mathcal{V}_{p,\lambda,a} \rightarrow M_{p,\lambda,a}, \quad m : v \mapsto m_v,$$

such that the following assertions are true:

1°. *Equivalence and equimodality.*  $v \sim \tilde{v} \Leftrightarrow m_v = m_{\tilde{v}}$ .

2°. *Realization.* For each  $m \in M_{p,\lambda,a}$  there exists  $v \in \mathcal{V}_{p,\lambda,a}$  such that  $m = m_v$ .

3°. *Analytic dependence.* For each analytic family  $v_\varepsilon$  of germs in  $\mathcal{V}_{p,\lambda,a}$  some representatives  $\mu_\varepsilon$  of the moduli  $m_{v_\varepsilon}$  also form an analytic family.

This theorem is a refined analogue of the well-known theorem on the orbital analytic classification of germs in  $\mathcal{V}_{p,\lambda}$  ([2]; [1, §3, p. 33]). Note that the number of moduli in the problem of analytic classification is twice the number of moduli in the problem of orbital analytic classification. In fact, the orbital analytic classification has  $p+1$  numeric (one formal modulus  $\lambda$  and  $p$  moduli  $c = (c_1, \dots, c_p)$  of the analytic classification) and  $p$  functional moduli  $\{\varphi_j\}$ ; the analytic classification has  $2p+2$  numeric ( $p+2$  formal moduli  $\lambda, a_0, \dots, a_p$  and  $p$  analytic moduli of the tuple  $c$ ) and  $2p$  functional moduli  $\{\varphi_j\}, \{\psi_j\}$ .

For brevity we can replace all three parts of Theorem 4 by the following single sentence: “The space  $M_{p,\lambda,a}$  is the space of moduli of the analytic classification of germs of the class  $\mathcal{V}_{p,\lambda,a}^N$ .” We then have the following result.

**Theorem 5.** *The space  $\mathcal{M}_{p,\lambda}$  is the moduli space of the strict analytic classification of germs of the class  $\mathcal{V}_{p,\lambda,a}^N$ .*

Theorems 4 and 5 are proved in subsections 3.2–3.5, where Theorem 4 will be obtained from Theorem 5. The non-uniqueness of the normalizing change of coordinates noted above in the problem of the analytic classification explains the interconnection between the spaces  $\mathcal{M}_{p,\lambda}$  and  $M_{p,\lambda,a}$ : the space  $M_{p,\lambda,a}$  is obtained from the space  $\mathcal{M}_{p,\lambda}$  by factoring with respect to the equivalence (4) resulting from this non-uniqueness.

## 2. PROOF OF THE THEOREM ON SECTORIAL NORMALIZATION

In this section we prove Theorem 3.

**2.1. Outline of the proof.** We seek a solution  $H_j$  of the functional equation (3) with data  $v = v_{p,\lambda,a} + \Delta$  in the form  $H_j = \text{id} + \mathbf{h}$ ,  $\mathbf{h} = (h, g)$ ; the vectors  $\Delta$  and  $\mathbf{h}$  are called the *discrepancy* and *correction*, respectively. Singling out in (3) all the terms that are linear with respect to the correction and collecting them on the left-hand side of the equation we can rewrite this equation in the form

$$(5) \quad \mathcal{L}\mathbf{h} = \mathcal{R}(\mathbf{h}, \Delta)$$

with a linear operator  $\mathcal{L}$  on the left-hand side and a non-linear operator  $\mathcal{R}$  on the right-hand side. Neglecting the second order terms (with respect to the discrepancy-correction) in equation (5) we obtain the so-called homological equation

$$(6) \quad \mathcal{L}\mathbf{h} = \Delta.$$

Let  $\Phi = \mathcal{L}^{-1}$  be the operator solving equation (6); then equation (5) with data  $\Delta$  reduces to the equation

$$(7) \quad \mathbf{h} = S(\mathbf{h}),$$

where  $S(\mathbf{h}) = \Phi \circ \mathcal{R}(\mathbf{h}, \Delta)$ . It turns out that, under a suitable choice of the metric for the space  $\mathcal{M} = \{\mathbf{h}\}$ , the operator  $S$  acts from  $\mathcal{M}$  to  $\mathcal{M}$  and is contracting. This ensures the existence and uniqueness of the solution  $\mathbf{h} \in \mathcal{M}$  of equation (7), which proves the first part of Theorem 3. Here the condition  $\mathbf{h} \in \mathcal{M}$  ensures that certain estimates on  $\mathbf{h}$  analogous to (2) hold. Varying  $N$ , from these estimates and the uniqueness of the equation, we obtain the proof of the second part of Theorem 3.

A complete proof of the theorem is given in §§2.2–2.11 (case  $j = 1$ ) and in §2.12 ( $j > 1$ ). In §2.2 we derive the homological equation and describe the operators  $\mathcal{L}$  and  $\mathcal{R}$  in explicit form. The homological equation is studied in §§2.3–2.7. In §2.8 we obtain certain estimates which are then used in §2.9 for estimating the operator  $\mathcal{R}$  (its “norm” and its Lipschitz property). The contracting property of the operator  $S$  (and the proof of the first part of Theorem 3) are obtained in §2.10. The second part of the theorem is proved in §2.11.

**2.2. The homological equation.** Let  $w(y) = \frac{y^{p+1}}{1+\lambda y^p}$ . In the notation of §2.1 the functional equation (5) for the correction  $\mathbf{h} = (h, g)$  has the form

$$(8) \quad \begin{cases} (x(1+h'_x) + w(y)h'_y)a(y) = (x+h)a(y+g) + \tilde{\Delta}_1, \\ (xg'_x + w(y)(1+g'_y))a(y) = \omega(y+g)a(y+g) + \tilde{\Delta}_2, \end{cases}$$

where  $\tilde{\Delta}_j = \Delta_j(x+h, y+g)$ ,  $\Delta = (\Delta_1, \Delta_2)$ ,  $h = h(x, y)$ ,  $g = g(x, y)$ .

Collecting terms that are linear with respect to the correction  $\mathbf{h} = (h, g)$  on the left-hand side, we rewrite the system in the form

$$(9) \quad \begin{cases} (xh'_x + w(y)h'_y - h)a(y) - x a'(y)g = \mathcal{R}_1[\Delta_1, h, g], \\ (xg'_x + w(y)g'_y)a(y) - q'(y)g = \mathcal{R}_2[\Delta_2, h, g], \end{cases}$$

where  $q(y) = a(y)w(y)$ , and  $\mathcal{R}_1, \mathcal{R}_2$  contain all the non-linear components:

$$(10) \quad \begin{aligned} \mathcal{R}_1[\Delta_1, h, g] &= \tilde{\Delta}_1 + \mathcal{R}_{11} + \mathcal{R}_{12}, \\ \mathcal{R}_{11} &= h \cdot (a(y+g) - a(y)), \\ \mathcal{R}_{12} &= x \cdot (a(y+g) - a(y) - a'(y)g), \\ \mathcal{R}_2[\Delta_2, h, g] &= \tilde{\Delta}_2 + \mathcal{R}_{22}, \\ \mathcal{R}_{22} &= q(y+g) - q(y) - q'(y)g. \end{aligned}$$

The corresponding homological equation (6) in 2.1 is written in expanded form as the system

$$(11) \quad \begin{cases} (xh'_x + w(y)h'_y - h)a(y) - x a'(y)g = \Delta_1, \\ (xg'_x + w(y)g'_y)a(y) - q'(y)g = \Delta_2. \end{cases}$$

Thus the linear operator  $\mathcal{L}$  in equations (5), (6) is defined by the left-hand side of the system (11), while the non-linear operator  $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$  in (5) is defined by the formulae (10).

In the next subsection we will investigate the system (11).

**2.3. Reduction of the homological equation to the auxiliary equation.** To begin with we consider the second equation of the system (11). Note that the function  $q(y)$  clearly satisfies the corresponding homogeneous equation. Setting  $g = q\varphi$  and substituting it in the second equation of (11), we obtain the equation

$$(12) \quad x\varphi'_x + \omega\varphi'_y = \delta_2$$

for the unknown function  $\varphi$ , where  $\delta_2 = \Delta_2/(aq)$ . In the last equation we make the change of variable  $\xi = A(y)$ , where  $A(y) = -\frac{1}{py^p} + \lambda \ln y$ . Since  $A'(y) = \frac{1}{w(y)}$ , it follows that  $w(y)\frac{d}{dy} = \frac{d}{d\xi}$ , and equation (12) takes the form

$$(13) \quad x\tilde{\varphi}'_x + \tilde{\varphi}'_\xi = \tilde{\delta},$$

where  $\tilde{\varphi}(x, \xi) = \varphi(x, y)$ ,  $\tilde{\delta}(x, \xi) = \delta_2(x, y)$ ,  $\xi = A(y)$ .

We call this equation the *auxiliary equation*; we shall study it in subsection 2.4.

The first equation of the system (11) also reduces to the auxiliary equation. Indeed, the substitution  $h = \psi \cdot e^{A(y)}$  reduces it to the equation

$$(14) \quad x\psi'_x + \omega\psi'_y = \delta_{12},$$

with right-hand side

$$\delta_{12} = \frac{(\Delta_1 + xa'(y)g)e^{A(y)}}{a},$$

where  $g$  is the solution of the second equation of the system (11). The same substitution brings this equation to an equation of the form (13).

**2.4. Existence and uniqueness of the solution of the auxiliary equation.** By the *upper sectorial domain* with parameters  $\beta$ ,  $\varepsilon$ ,  $R$ ,  $\beta \in (\frac{\pi}{2p}, \frac{\pi}{p})$ ,  $\varepsilon > 0$ ,  $R > 1$  we mean the domain  $\tilde{\Omega} = \tilde{\Omega}(\beta, \varepsilon, R) = \{|x| < \varepsilon\} \times \tilde{D}(\beta, R)$ , where

$$\tilde{D}(\beta, R) = \{\xi : |\arg \xi - \frac{\pi}{2}| < p\beta, \max\{|\operatorname{Re} \xi|, |\operatorname{Im} \xi|\} > R\}.$$

The next lemma establishes the existence and uniqueness of the solution of the auxiliary equation on a domain of this form.

**Lemma 1.** *Suppose that the function  $\tilde{\delta}$  is holomorphic in some upper sectorial domain  $\tilde{\Omega}$  and satisfies the following estimate for some  $s > 1$ :*

$$(15) \quad \tilde{\delta}(x, \xi) = \mathcal{O}((\operatorname{Im} \xi)^{-s}) \quad \text{as } \operatorname{Im} \xi \rightarrow +\infty, \quad (x, \xi) \in \tilde{\Omega}.$$

*Then in this domain there exists a unique solution of equation (13) normalized by the condition*

$$(16) \quad \tilde{\varphi}(x, \xi) = o(1) \quad \text{as } \operatorname{Im} \xi \rightarrow +\infty, \quad (x, \xi) \in \tilde{\Omega}.$$

*This solution is given by the formula*

$$(17) \quad \tilde{\varphi}(x, \xi) = - \int_0^{+i\infty} \tilde{\delta}(xe^t, \xi + t) dt.$$

*Proof.* First we prove uniqueness. For this we set

$$u(t) = \tilde{\varphi}(xe^t, \xi + t).$$

It follows from (13) that

$$\frac{d}{dt} u(t) = \tilde{\delta}(xe^t, \xi + t).$$

Integrating this equation along the ray  $[0, +i\infty]$  and taking (16) into account we obtain the representation (17) for the solution  $\tilde{\varphi}$  of the equation (13).

On the other hand, under the condition (15) the integral on the right-hand side of the equality (17) converges, so that the function  $\tilde{\varphi}$  is well defined. Using Cauchy's estimate in (15) it is easy to see that we have uniform convergence on compact subsets of  $\tilde{\Omega}$  of the integrals of the derivatives; applying the standard theorems on differentiability with respect to parameters we see that  $\varphi$  is holomorphic on  $\tilde{\Omega}$  and one can differentiate under the integral sign. After this the verification of equality (13) becomes trivial. The estimate (16) is a trivial consequence of (15), since  $s > 1$ .  $\square$

*Remark 4.* The somewhat unusual form of the domain  $\tilde{\Omega}$  in this subsection is simple to explain: with such domains there is no problem with our choice of the path of integration.

**2.5. Estimates of solutions of the auxiliary equation.** Let  $\tilde{\Omega} = \tilde{\Omega}(\beta, \varepsilon, R)$  be the domain in §2.4. We consider the space  $\tilde{\mathcal{B}}_s$ , consisting of functions  $\tilde{\delta}$  that are holomorphic on  $\tilde{\Omega}$  and for which the norm  $\|\tilde{\delta}\|_s = \sup_{\tilde{\Omega}} |\tilde{\delta}(x, \xi)| \cdot |\xi|^s$  is finite. We define the operator  $\tilde{\Phi}_2 : \tilde{\delta} \mapsto \tilde{\varphi}$  by the equality (17).

**Lemma 2.** *The operator  $\tilde{\Phi}_2$  acts from  $\tilde{\mathcal{B}}_s$  to  $\tilde{\mathcal{B}}_{s-1}$  and is bounded:*

$$\exists C = C(\beta, s) : \|\tilde{\Phi}_2 \tilde{\delta}\|_{s-1} \leq C \|\tilde{\delta}\|_s.$$

*Proof.* Let  $\|\tilde{\delta}\|_s = 1$ ; then  $|\delta(x, \xi)| \leq |\xi|^{-s}$  and it follows from (17) that

$$(18) \quad |\tilde{\varphi}(x, \xi)| \leq \int_0^{+\infty} \frac{d\tau}{|\xi + i\tau|^s}.$$

Let  $\xi = u + iv$ ,  $u, v \in \mathbb{R}$ . In the case when  $\arg \xi \in [\pi/4, 3\pi/4]$  we have  $\sqrt{2}v > |\xi|$  and the integral in (18) has the following estimate:

$$\int_0^{+\infty} \frac{d\tau}{|\xi + i\tau|^s} \leq \int_0^{+\infty} \frac{d\tau}{|v + \tau|^s} = \frac{1}{s-1} v^{1-s} \leq \frac{2^{\frac{s-1}{2}}}{s-1} |\xi|^{1-s}.$$

Otherwise  $|\arg \xi - \pi/2| \in (\pi/4, p\beta)$ , so that  $|u| > c|\xi|$  for some constant  $c = c(\beta)$ . Then the integral in (18) can be estimated as follows:

$$\int_0^{+\infty} \frac{d\tau}{|\xi + \tau|^s} = \int_0^{+\infty} \frac{d\tau}{(u^2 + (v + \tau)^2)^{s/2}} = |u|^{1-s} \int_{v/|u|}^{+\infty} \frac{dr}{(1 + r^2)^{s/2}} \leq \text{const} \cdot |\xi|^{1-s},$$

where

$$\text{const} = c^{1-s} \int_{-\infty}^{+\infty} \frac{dr}{(1 + r^2)^{s/2}}.$$

It follows from these two estimates that the product  $|\tilde{\varphi}(x, \xi)| \cdot |\xi|^{1-s}$  is bounded by some constant that depends only on  $s$  and  $\beta$ , as required.  $\square$

**2.6. Properties of the rectifying map  $A$ . The standard domains  $\Omega_{\varepsilon, R}$ .** Let  $a(y)$ ,  $w(y) = \frac{y^{p+1}}{1+\lambda y^p}$  be parameters of a formal normal form and, as before, let  $q(y) = a(y)w(y)$ . We choose some  $\rho_0 > 0$  such that the functions  $\frac{1}{a(y)}$  and  $w(y)$  are holomorphic on the disc  $\{|y| \leq \rho_0\}$ . Then for some  $c_1, c_2 > 0$  we have the estimates

$$(19) \quad c_1 \leq |a(y)| < c_2, \quad c_1|y|^{p+1} \leq |q(y)| \leq c_2|y|^{p+1}, \quad |y| < \rho_0.$$

Let  $\mathcal{D}_{\varepsilon, \alpha} = \{y : |y| < \varepsilon, \arg y \in (\frac{\pi}{2p} - \alpha, \frac{\pi}{2p} + \alpha)\}$  be a sectorial domain. Then  $\xi = A(y) = -\frac{1}{py^p} + \lambda \ln y$  is a rectifying map on  $\mathcal{D}_{\varepsilon, \alpha}$  for some vector field  $\omega = w(y) \frac{\partial}{\partial y}$ :  $A'(y)\omega = \frac{\partial}{\partial \xi}$ .

For a given  $\alpha \in (\frac{\pi}{2p}, \frac{\pi}{p})$  we choose  $\alpha'$ ,  $\beta \in (\alpha, \frac{\pi}{p})$  such that  $\alpha' > \beta$ ; let  $\tilde{\mathcal{D}}(\beta, R)$  be the domain in §2.4.

**Lemma 3.** 1.  $\exists \rho_1 \in (0, \rho_0)$ ,  $R_0 > 1$  such that  $A$  is injective on  $\mathcal{D}_0 := \mathcal{D}_{\rho_1, \alpha'}$  and

$$A(\mathcal{D}_0) \supset \tilde{\mathcal{D}}_{\beta, R_0} =: \tilde{\mathcal{D}}_0$$

(and, in particular, the inverse map  $A^{-1} : \tilde{\mathcal{D}}_0 \rightarrow \mathcal{D}_0$  is defined on  $\tilde{\mathcal{D}}_0$ ).

2.  $\forall R \geq R_0 \ \exists \rho = \rho(R) > 0 : A^{-1}(\tilde{\mathcal{D}}_{\beta, R}) \supset \mathcal{D}_{\rho, \alpha}$ .
3. We have the following estimates for some positive constants  $c_3, c_4$ :

$$(20) \quad \begin{aligned} c_3|y|^{-p} &\leq |A(y)| \leq c_4|y|^{-p} \text{ on } \mathcal{D}_0, \\ c_3|\xi|^{-1/p} &\leq |A^{-1}(\xi)| \leq c_4|\xi|^{-1/p} \text{ on } \tilde{\mathcal{D}}_0. \end{aligned}$$

*Proof.* The proof is purely technical.  $\square$

Starting from this point we shall regard the parameters  $p, \lambda, a_k$  (that is, the coefficients of the polynomial  $a$ ) of the formal normal form as fixed. We also fix the opening  $\alpha$  of the sectorial domains and the corresponding parameters  $\alpha'$ ,  $\beta$ ,  $R_0$  and  $\rho_1$  in the last lemma. We denote by  $\Omega_{\varepsilon, R}$  the product  $\{|x| < \varepsilon\} \times \mathcal{D}_R$ , where  $\mathcal{D}_R = A^{-1}(\tilde{\mathcal{D}}(\beta, R))$ ,  $R \geq R_0$ ,  $\varepsilon > 0$ . It follows from the second part of Lemma 3 that for any  $\varepsilon > 0$ ,  $R > R_0$  the domain  $\Omega_{\varepsilon, R}$  contains a sectorial domain of type  $\Omega_j$  (see §1.2) for  $j = 1$  of opening  $\alpha$  and sufficiently small radius. It is therefore sufficient to prove Theorem 3 for domains of the form  $\Omega_{\varepsilon, R}$ , which we shall call *standard*.

**2.7. Solution of the homological equation in the domain  $\Omega_{\varepsilon,R}$ .** Let  $\Omega = \Omega_{\varepsilon,R}$  be a standard domain. For any  $N > 0$  we consider the space  $\mathcal{B}_N$  consisting of functions  $\varphi$  that are holomorphic on  $\Omega$  with finite norm  $\|\varphi\|_N = \sup_{\Omega} |\varphi(x, y)| \cdot |y|^{-N}$ . Let  $\tilde{\Omega} = \{(x, A(y)) : (x, y) \in \Omega\}$ ; we consider the “change of variable” operator  $Z : \varphi = \varphi(x, y) \mapsto \tilde{\varphi} = \tilde{\varphi}(x, \xi) = \varphi(x, A^{-1}(\xi))$  and the multiplication operators  $Q_1 : \varphi \mapsto \varphi q$ ,  $Q_2 : \varphi \mapsto a\varphi q$ . The next lemma is trivial in view of the estimates (20) and (19):

- Lemma 4.**
1.  $Q_i$  is a bijection from  $\mathcal{B}_N$  onto  $\mathcal{B}_{N+1+p}$ ,  $i = 1, 2$ .
  2.  $Z$  is a bijection from  $\mathcal{B}_N$  onto  $\tilde{\mathcal{B}}_{N/p}$ .
  3. The operators  $Q_i : \mathcal{B}_N \rightarrow \mathcal{B}_{N+p+1}$ ,  $Z : \mathcal{B}_N \rightarrow \tilde{\mathcal{B}}_{N/p}$ , as well as their inverses, are bounded; the norms of all these operators do not exceed some constant depending only on  $N$  (and the parameters of the formal normal form).

**2.7.1. Solution of the second equation of the system of homological equations.** In accordance with the reduction scheme in §2.3, by Lemma 1, the solution  $g$  of the second equation of the system (11) exists on the domain  $\Omega$  if  $\delta_2 \in \mathcal{B}_{N+p}$ ,  $N > p + 1$ , and it is unique if one seeks it in the class  $\mathcal{B}_N$ . This solution is determined by the chain  $\Delta_2 \mapsto \delta_2 = Q_2^{-1}\Delta_2 \mapsto \tilde{\delta} = Z\delta_2 \mapsto \tilde{\varphi} = \tilde{\Phi}_2\delta \mapsto \varphi = Z^{-1}\tilde{\varphi} \mapsto g = Q_1\varphi$ , where  $\tilde{\Phi}_2$  is the operator in §2.5. We denote by  $\Phi_2$  the operator solving the second equation of the system (11), so that  $\Phi_2 : \Delta_2 \mapsto g$ . Then  $\Phi_2 = Q_1 \circ Z^{-1} \circ \tilde{\Phi}_2 \circ Z \circ Q_2^{-1}$ , and using Lemmas 2 and 4 we obtain the following result.

**Lemma 5.** *The operator  $\Phi_2$  acts from  $\mathcal{B}_{N+p}$  to  $\mathcal{B}_N$  and is bounded:*

$$\exists C = C(N) : \forall \Delta_2 \in \mathcal{B}_{N+p} \quad \|\Phi_2\Delta_2\|_N \leq C\|\Delta_2\|_{N+p}.$$

**2.7.2. Solution of the first equation of the system of homological equations.** Suppose that in the system (11),  $\Delta_1 \in \mathcal{B}_N$ ,  $\Delta_2 \in \mathcal{B}_{N+p}$ . By Lemma 5 the solution  $g$  of the second equation is of class  $\mathcal{B}_N$ , so that the function  $\Delta_{12} = (\Delta_1 + kg)/a$ ,  $k = xa'(y)$  is also of class  $\mathcal{B}_N$ . We define the functions  $\psi$  and  $\delta_{12}$  in accordance with §2.3 and in (14) we make the change of variables  $\xi = A(y)$ . Then we obtain the following equation for the functions  $\tilde{\Psi} = Z\psi$  and  $\tilde{\delta}_{12} = Z\delta_{12}$ :

$$(21) \quad x\tilde{\Psi}'_x + \tilde{\Psi}'_\xi = \tilde{\delta}_{12},$$

which is solved in the same way as equation (13). Unfortunately, the functions  $\tilde{\Psi}$  and  $\tilde{\delta}_{12}$  do not belong to any of the classes  $\tilde{\mathcal{B}}_s$  in general. However, we note that the functions  $\tilde{h} = Zh$  and  $\tilde{\Delta}_{12}$  satisfy the equalities

$$(22) \quad \tilde{h}(x, \xi) = \tilde{\Psi}(x, \xi)e^\xi, \quad \tilde{\Delta}_{12}(x, \xi) = \tilde{\delta}_{12}(x, \xi)e^\xi.$$

Since in the asymptotic equalities (15), (16) the presence of the factor  $e^\xi$  has no effect, it follows by Lemma 1, that there exists (assuming that  $\Delta_1 \in \mathcal{B}_N$ ,  $\Delta_2 \in \mathcal{B}_{N+p}$ ,  $N > p + 1$ ) a solution  $h$  of the first equation of the system (11), and also this solution is unique (in the class  $\mathcal{B}_N$ ), and we have a formula for calculating it (it is obtained from (17), after replacing  $\varphi$  by  $\psi$  and  $\delta$  by  $\delta_{12}$ ). Using (22), we finally obtain for the function  $H = Zh$  the formula

$$(23) \quad \tilde{H}(x, \xi) = - \int_0^{i\infty} \tilde{\Delta}_{12}(xe^t, \xi + t)e^{-t} dt,$$

where  $\tilde{\Delta}_{12} = Z\Delta_{12}$ . We define the operator  $\tilde{\Phi}_1 : \tilde{\Delta}_{12} \mapsto \tilde{H}$  by formula (23).

**Lemma 6.** *The operator  $\tilde{\Phi}_1$  acts from  $\mathcal{B}_s$  to  $\mathcal{B}_{s-1}$  and is bounded.*

*Proof.* The proof is carried out in exactly the same way as that of Lemma 2, since  $|e^t| = 1$  for  $\operatorname{Re} t = 0$ .  $\square$

Finally, suppose that  $\widehat{\Phi}_1 : \Delta_{12} \mapsto h$  is not an operator solving the equation  $xh'_x + \omega h'_y = \Delta_{12}$ . Since  $\widehat{\Phi}_1 = Z^{-1}\widetilde{\Phi}_1 Z$ , the following result is an immediate consequence of Lemmas 4 and 6.

**Corollary 3.** *The operator  $\widehat{\Phi}_1$  acts from  $\mathcal{B}_N$  to  $\mathcal{B}_{N-p}$  and is bounded.*

2.7.3. *The operator  $\Phi = \mathcal{L}^{-1}$ .* We consider the operator  $\Sigma$ , acting from  $\mathcal{B}_N \times \mathcal{B}_N$  to  $\mathcal{B}_N$  in accordance with the formula  $\Sigma(\Delta_1, g) = (\Delta_1 + x a'(y) g)/a$ . Finally, we define the operator  $\Phi_1$  by the formula

$$\Phi_1(\Delta_1, \Delta_2) = \widehat{\Phi}_1 \circ \Sigma(\Delta_1, \Phi_2(\Delta_2)).$$

In accordance with all the preceding constructions the operator

$$\Phi = (\Phi_1, \Phi_2) : (\Delta_1, \Delta_2) \mapsto (h, g)$$

solves the system of homological equations (11) (that is, it is the inverse of the operator  $\mathcal{L}$  in §2.1). For this operator we have the following result.

**Proposition 1.** *The operator  $\Phi$  acts from  $\mathcal{B}_N \times \mathcal{B}_{N+p}$  to  $\mathcal{B}_{N-p} \times \mathcal{B}_N$  and is bounded:*

$$(24) \quad \begin{aligned} \exists C = C(N) : \forall \Delta_1 \in \mathcal{B}_N, \Delta_2 \in \mathcal{B}_{N+p} \\ \max\{|h||_{N-p}, |g||_N\} \leq C \cdot \max\{|\Delta_1||_N, |\Delta_2||_{N+p}\}. \end{aligned}$$

2.8. **Preliminary estimates.** In this subsection we give some estimates that we require in what follows to study the non-linear operator  $\mathcal{R}$ .

2.8.1. *Estimates of functions defined by the parameters of the formal normal form.* Let  $\rho_0$  in §2.6 be such that  $\omega, 1/a$  are holomorphic at the points of the disc  $\{|y| \leq \rho_0\}$ , and the estimates (19) hold. The following lemmas are obvious.

**Lemma 7.** *Let*

$$\begin{aligned} r_{11}(y, h, g) &= h(a(y+g) - a(y)), \quad r_{12}(x, y, g) = x(a(y+g) - a(y) - a'(y)g), \\ r_{22}(y, g) &= q(y+g) - q(y) - q'(y)g_0. \end{aligned}$$

*Then there exists a constant  $C_1$  such that for all  $x, y$ ,  $|x| < 1$ ,  $|y| < \rho_0/2$ , and any  $\delta$ ,  $0 < \delta < \rho_0/2$ , we have:*

1. *It follows from the condition  $|g| < \delta$ ,  $|h| < \delta$  that*

$$(25) \quad |r_{11}(y, h, g)| < C_1|h| \cdot |g|, \quad |r_{12}(x, y, g)| < C_1|g|^2, \quad |r_{22}(y, g)| < C_1|g|^2.$$

2. *It follows from the condition  $|g_i| < \delta$ ,  $|h_i| < \delta$ ,  $i = 1, 2$ , that*

$$(26) \quad \begin{aligned} |r_{11}(y, h_1, g_1) - r_{11}(y, h_2, g_2)| &\leq C_1(|h_1 - h_2| \cdot |g_1| + |h_2| \cdot |g_1 - g_2|), \\ |r_{12}(x, y, g_1) - r_{12}(x, y, g_2)| &\leq C_1(|g_1|^2 - |g_2|^2), \\ |r_{22}(y, g_1) - r_{22}(y, g_2)| &\leq C_1(|g_1|^2 - |g_2|^2). \end{aligned}$$

2.8.2. *Estimates for the discrepancy.* Let  $v \in \mathcal{V}_{p,\lambda,a}^N$ ,  $v - v_{p,\lambda,a} = \Delta = (\Delta_1, \Delta_2)$ . In accordance with the definition of the class  $\mathcal{V}_{p,\lambda,a}$  this means that  $\Delta_1(x, y) = \mathcal{O}(y^N)$ ,  $\Delta_2(x, y) = \mathcal{O}(y^{N+p})$  as  $y \rightarrow 0$ . Hence it follows that there exist constants  $C_2$ ,  $\varepsilon_0$ , such that for all  $(x, y) \in \mathbb{C}^2$ ,  $|x| < \varepsilon_0$ ,  $|y| < \varepsilon_0$  we have the estimates

$$(27) \quad |\Delta_1(x, y)| \leq C_2|y|^N, \quad |\Delta_2(x, y)| \leq C_2|y|^{N+p},$$

$$(28) \quad \left| \frac{\partial}{\partial x} \Delta_1(x, y) \right| \leq C_2|y|^N, \quad \left| \frac{\partial}{\partial x} \Delta_2(x, y) \right| \leq C_2|y|^{N+p},$$

$$(29) \quad \left| \frac{\partial}{\partial y} \Delta_1(x, y) \right| \leq C_2|y|^{N-1}, \quad \left| \frac{\partial}{\partial y} \Delta_2(x, y) \right| \leq C_2|y|^{N+p-1}.$$

Note that the constants  $\varepsilon_0$ ,  $C_2$  depend only on the germ  $v \in \mathcal{V}_{p,\lambda,a}^N$ .

**2.9. Estimates for the operator  $\mathcal{R}$ .** In this subsection we prove that the operator  $\mathcal{R}$  is “bounded” and Lipschitzian. Let  $\Omega = \Omega_{\varepsilon, R}$  be the standard domain, and  $\mathcal{B}_N$  the normed space in §2.7. We set  $\rho = \rho(R) = \sup\{|y| : (x, y) \in \Omega_{\varepsilon, R}\}$ ; in accordance with Lemma 3, we have

$$(30) \quad \rho(R) \rightarrow 0 \quad \text{as } R \rightarrow +\infty.$$

We set  $\varepsilon^* = \min\{\varepsilon_0, \rho_0, 1\}$ , where  $\varepsilon_0$  is the quantity in §2.8.2 and  $\rho_0$  is the quantity in §2.6. Let  $\varepsilon, \rho \in (0, \varepsilon^*/2)$ ,  $\Omega = \Omega_{\varepsilon, R}$ ,  $\rho = \rho(R)$ . We consider the space  $\mathcal{B}_N^* = \mathcal{B}_{N-p} \times \mathcal{B}_N$  with norm

$$\|(h, g)\|_N = \max\{|h|_{N-p}, |g|_N\}.$$

Let  $\mathcal{M}_d^N = \{\mathbf{h} \in \mathcal{B}_N^* : \|\mathbf{h}\|_N \leq d\}$  be the closed ball in  $\mathcal{B}_N^*$  of radius  $d$ .

**Lemma 8.** *Let  $N > p + 1$  and suppose that the parameters  $d$  and  $\rho$  are such that*

$$(31) \quad \delta = d\rho < \frac{\varepsilon}{2}.$$

*Then there exist constants  $C_k > 0$ ,  $k = 3, 4, 5, 6$ , not depending on  $\rho$  or  $d$  such that*

1. *Let  $d' = C_3 + C_4 d^2 \rho$ . Then*

$$(32) \quad \mathcal{R}(\mathcal{M}_d^N) \subset \mathcal{M}_{d'}^{N+p}.$$

2. *The map  $\mathcal{R}$  on the ball  $\mathcal{M}_d^N$  is Lipschitzian with constant  $L = \rho(C_5 + C_6 d)$ :*

$$(33) \quad \forall \mathbf{h}^1, \mathbf{h}^2 \in \mathcal{M}_d^N : \quad \|\mathcal{R}(\mathbf{h}^1) - \mathcal{R}(\mathbf{h}^2)\|_{N+p} \leq L \|\mathbf{h}^1 - \mathbf{h}^2\|_N.$$

*Proof.* For  $(x, y) \in \Omega$  we have  $|y| \leq \rho < 1$ , so that

$$\begin{aligned} |h(x, y)| &\leq \|h\|_{N-p} |y|^{N-p} \leq d\rho^{N-p} \leq d\rho = \delta < \frac{\varepsilon}{2}, \\ |g(x, y)| &\leq \|g\|_N |y|^N \leq d\rho^{N-1} \leq d\rho = \delta |\rho|. \end{aligned}$$

Consequently,  $|x + h(x, y)| \leq \varepsilon^*$ ,  $|y + g(x, y)| \leq \varepsilon^*$ , and also  $|y + g(x, y)| \leq 2|y|$ . Hence we can use formula (27) for estimating the terms  $\tilde{\Delta}_j$ , and formula (25) for estimating the terms  $\mathcal{R}_{ij}$ . Applying them we obtain

$$\begin{aligned} \|\mathcal{R}_1(h, g)\|_N &\leq C_2 2^N + C_1 d^2 \rho^{N-p} + C_1 d^2 \rho^N, \\ \|\mathcal{R}_2(h, g)\|_{N+p} &\leq C_2 2^{N+p} + C_1 d^2 \rho^{N-1} \end{aligned}$$

for the components  $\mathcal{R}_1, \mathcal{R}_2$  of the operator  $\mathcal{R}$ , so that it suffices to set  $C_3 = C_2 \cdot 2^{N+p}$ ,  $C_4 = 2C_1$ . Similarly, to prove the Lipschitzian property, for any two points  $\mathbf{h}^1, \mathbf{h}^2 \in \mathcal{M}_d^N$ ,  $\mathbf{h}^i = (h_i, g_i)$ ,  $i = 1, 2$ , we obtain  $|h_j| < \delta$ ,  $|g_j| < \delta$ ,  $|x + h_j(x, y)| < \varepsilon_0$ ,  $|y + g_j(y)| < \varepsilon_0$ . Then from (28) and (29) we obtain estimates of the differences  $|\tilde{\Delta}_j^1 - \tilde{\Delta}_j^2|$ , where  $\tilde{\Delta}_j^i = \Delta_j \cdot \mathcal{H}^i$ ,  $\mathcal{H}^i = \text{id} + \mathbf{h}^i$ ; and from (26) we obtain estimates of the differences  $\mathcal{R}_{ij}(\mathbf{h}^1) - \mathcal{R}_{ij}(\mathbf{h}^2)$ . As a result we obtain (33) with constant  $L = \rho(C_5 + C_6 d)$ , where  $C_5 = 2C_2$ ,  $C_6 = 3C_1$ .  $\square$

**2.10. Contracting property of the operator  $S$ . Existence of a solution  $\mathbf{h}$  of the system (5).** Recall that for a given discrepancy  $\Delta$  the operator  $S$  is defined by  $S : \mathbf{h} \mapsto \Phi \circ \mathcal{R}(\Delta, \mathbf{h})$ , where  $\Phi$  is the operator solving the homological equation (see §2.1). The following lemma is a trivial consequence of Lemma 7 and Proposition 1.

**Lemma 9.** *Suppose that the parameters  $\rho$  and  $d$  satisfy (31). Then the operator  $S$  maps the ball  $\mathcal{M}_d^N$  into the ball  $\mathcal{M}_{d''}^N$ , where  $d'' = Cd'$ , and is Lipschitzian on  $\mathcal{M}_d^N$  with constant  $L' = CL$ .*

We recall that  $d' = C_3 + C_4d^2\rho$ ,  $L = \rho(C_5 + C_6d)$ . We set  $d = CC_3 + 1$  and choose  $\rho$  such that (31) holds and in addition,  $Cd^2C_4\rho < 1$  and  $CL < 1$ . Then  $d'' \leq d$ , so that  $S(\mathcal{M}_d^N) \subset \mathcal{M}_d^N$ , and  $S$  is contracting on  $\mathcal{M}_d^N$ : its Lipschitz constant  $L' = LC < 1$ . Note that the normed space  $\mathcal{B}_N^*$  is a Banach space, and therefore the metric space  $\mathcal{M} = \mathcal{M}_d^N$  (with metric  $\text{dist}(\mathbf{h}^1, \mathbf{h}^2) = ||\mathbf{h}^1 - \mathbf{h}^2||_N$ ) is complete. Hence, by the Contraction Mapping Theorem there exists a unique solution  $\mathbf{h} = (h, g)$  of the equation (5) satisfying the estimates

$$(34) \quad |h(x, y)| \leq d|y|^{N-p}, \quad |g(x, y)| \leq d|y|^N.$$

Since the domain  $\Omega_{\varepsilon, R}$  contains a sectorial domain of type  $\Omega_1$  (of opening  $\alpha$  and sufficiently small radius), this means that we have proved the first part of Theorem 3.

*Remark 5.* In this discussion  $d$  can, in fact, be taken to be an arbitrary constant, greater than  $CC_3$ , and the inequalities  $d'' \leq d$  (31) and  $L' < 1$  can be ensured by choosing  $\rho$  suitably. Consequently, by the uniqueness theorem for analytic functions, our arguments also prove the uniqueness of the sectorial normalizing map  $H_1 = \text{id} + (h, g)$  in the broader class of functions for which the estimate (34) is replaced by the weaker asymptotic estimate

$$(35) \quad h(x, y) = \mathcal{O}(y^{N-p}), \quad g(x, y) = \mathcal{O}(y^N), \quad y \rightarrow 0.$$

**2.11. Proof of the second part of Theorem 3.** Let  $v \in \mathcal{V}_{p, \lambda, a}^N$ ,  $N > p + 1$ , and let  $H = \text{id} + (h, g)$  be a normalizing map on the sectorial domain  $\Omega$  constructed above. Let  $\hat{H}$  be a normalized semiformal normalizing change of coordinates of the germ  $v$ ,  $\hat{H}(x, y) = \sum_{k=0}^{\infty} H_k(x)y^k$ . We denote by  $\hat{H}_n$  the  $n$ th partial sum of the series  $\hat{H}$ . Suppose that the holomorphic change of coordinates  $\hat{H}_n$  takes the germ  $v$  to the germ  $v_n$ :  $\hat{H}'_n v = v_n \circ \hat{H}_n$ ; then  $v_n \in \mathcal{V}_{p, \lambda, a}^{n-p+1}$ . Let  $\mathcal{H}_n$  be the sectorial normalizing change of coordinates for the germ  $v_n$  defined in accordance with §§2.2–2.10. Then

$$(36) \quad \mathcal{H}_n(x, y) = \text{id} + o(y^{n-2p}), \quad y \rightarrow 0.$$

But then the change of coordinates  $\tilde{H} = \mathcal{H}_n \circ \hat{H}_n$  is normalizing for the germ  $v$ . For the change of coordinates  $H$  we have the asymptotic formulae (35). But these formulae are in fact valid (for  $n > N + p$ ) for  $\tilde{H}$  as well, in view of (2). Hence by Remark 5 the maps  $H$  and  $\tilde{H}$  coincide (on some sectorial domain):  $H = \mathcal{H}_n \circ \hat{H}_n$ . It then follows from (36) that

$$H(x, y) - \hat{H}_{n-2p}(x, y) = o(y^{n-2p}), \quad y \rightarrow 0.$$

Since  $n$  is arbitrary, this means that  $\hat{H}$  is an asymptotic map for  $H$ .

**2.12. Conclusion of the proof of Theorem 3.** In §§2.1–2.11 we proved the existence of a sectorial normalizing map on a sectorial domain of type  $\Omega_1$ , we showed that  $\hat{H}$  is its asymptotic map, and we proved the uniqueness of the sectorial normalizing map in the class of maps with asymptotics (35) (Remark 5). Since we have the analogous asymptotics (2) for the normalizing semiformal change of coordinates  $\hat{H}$ , this means that we have established the uniqueness of a sectorial map satisfying the conditions 1°, 2° of the theorem on sectorial normalization. This completes the proof of the theorem for the domain  $\Omega_j$  for  $j = 1$ .

For odd  $j > 1$  the proof is precisely the same: one merely has to make sure that the branch of the logarithm in the definition of the map  $A$  is defined in accordance with the restrictions on the argument of the variable  $y$ .

For even  $j$  the proof is again similar: one merely has to replace the upper standard domains by the “lower” domains analogous to them, to carry the integration in (17) between the limits  $(-i\infty, 0)$ , and change the sign there.

Thus the theorem on sectorial normalization is completely proved.

### 3. PROOF OF THE THEOREM ON ANALYTIC CLASSIFICATION

In this section we prove Theorems 4 and 5.

**3.1. Outline of the proof.** The theorem on sectorial normalization enables one to construct for each germ  $v \in \mathcal{V}_{p,\lambda,a}^N$  and some regular covering  $\omega$  a uniquely defined family of maps  $\mathcal{H}_v = \{H_j\}$  (the so-called normalizing atlas): in the charts of the normalizing atlas the formal normal form  $v_{p,\lambda,a}$  coincides with  $v$ . Let us now consider the transition functions of this atlas. It turns out that the system of transition functions is an invariant of strict analytic equivalence. The system of transition functions of the normalizing atlas has the following properties: the transition functions preserve the formal normal form; the identity map is asymptotic for them. It turns out that any system of maps with such properties is a collection of transition functions of the normalizing atlas of some germ in  $\mathcal{V}_{p,\lambda,a}$ . Therefore the space  $R\mathcal{F}$  of all such systems is the moduli space of a strict analytic equivalence of germs in  $\mathcal{V}_{p,\lambda,a}$ . It remains to give a description of this space.

We parametrize the phase curves of the formal normal form  $v_{p,\lambda,a} = v_{p,\lambda} \cdot a$  by the values of the first integral  $J = x e^{-A(y)}$  of the field  $v_{p,\lambda}$  on them. The transition functions preserve the formal normal form  $v_{p,\lambda,a}$ , and therefore they take the phase curves  $v_{p,\lambda}$  to the phase curves  $v_{p,\lambda}$ . Thus, corresponding to each transition function there is a well-defined map from the parameter space to itself. It turns out that half of these maps are shifts on the parameter space (this gives us the set of constants  $c$  of the modulus  $\mu = (c, \varphi, \psi) \in \mathcal{M}_{p,\lambda}$ , on which the shifts take place); the second half gives us a collection of maps  $\varphi$ . Note that these two collections  $(c, \varphi)$  are precisely the Martinet–Ramis moduli [2]. Finally, the system of functions  $\psi = \{\psi_j\}$  of the modulus  $\mu$  consists of functions  $\psi_j = \psi_j(z)$  showing to what extent the transition functions shift the phase curves of the field  $v_{p,\lambda,a}$  by comparison with some “standard” shift corresponding to the families  $(c, \varphi)$ .

Thus the moduli space  $R\mathcal{F}$  of strict equivalence can be identified with  $\mathcal{M}_{p,\lambda}$ , which proves Theorem 5. Theorem 4 is obtained from Theorem 5 as indicated in the introduction.

The full proof is carried out below in §§3.2–3.5. In §3.2 we give the precise definition of the normalizing atlas and consider the properties of its transition functions. In §3.3 we give a precise definition of the space  $R\mathcal{F}$  and prove that it is the moduli space of a strict analytic equivalence. A description of the elements of  $R\mathcal{F}$  and the construction of the one-to-one correspondence between the elements of  $R\mathcal{F}$  and  $\mathcal{M}_{p,\lambda}$  are carried out in §3.4; this completes the proof of Theorem 5. The proof of Theorem 4 is carried out in §3.5.

**3.2. The normalizing atlas and its transition functions.** Let  $\omega = \{\Omega_j\}$  be a regular covering. We call any regular covering  $\omega' = \{\Omega'_j\}$  with smaller opening and radius a *constriction* of  $\omega$ ; we also call the domains  $\Omega'_j$  constrictions of the domains  $\Omega_j$ .

**Definition 5.** A system of maps  $\{H_j\}$ ,  $H_j : \Omega_j \rightarrow \mathbb{C}^2$  is called a *normalizing atlas* for the germ  $v$  (and the regular covering  $\omega = \{\Omega_j\}$ ) if conditions 1°, 2° of Theorem 3 hold and, in addition, the following two conditions hold:

- 3°. The  $H_j$  are injective on  $\Omega_j$ .
- 4°. For some constriction  $\omega' = \{\Omega'_j\}$  of the regular covering  $\omega$  we have the inclusions  $H_j(\Omega_j) \supset \Omega'_j$ .

**Proposition 2.** *For any germ  $v \in \mathcal{V}_{p,\lambda,a}^N$ ,  $N > p + 1$  a normalizing atlas exists. The normalizing atlas is unique in the following sense: any two normalizing atlases of  $v$  coincide on some regular covering.*

*Proof.* A normalizing atlas can be constructed from the family of sectorial normalizing maps of Theorem 3 by restricting their domains of definition. Indeed, by part 2° of Theorem 3, the maps  $H_j : \Omega_j \rightarrow \mathbb{C}^2$  have as asymptotics on  $\Omega_j$  some semiformal map  $\widehat{H}$  with asymptotics (2). But  $N > p + 1$ , and therefore we have the following estimates for the components  $(h, g) = H_j - \text{id}$  on  $\Omega_j$ :

$$(37) \quad h(x, y) = \mathcal{O}(y^2), \quad g(x, y) = \mathcal{O}(y^2), \quad y \rightarrow 0.$$

Hence by Cauchy's inequality it is easy to obtain estimates of the derivatives

$$\begin{aligned} h'_x(x, y) &= \mathcal{O}(y^2), \quad h'_y(x, y) = \mathcal{O}(y), \\ g'_x(x, y) &= \mathcal{O}(y^2), \quad g'_y(x, y) = \mathcal{O}(y), \quad y \rightarrow 0 \end{aligned}$$

on any restriction  $\widetilde{\Omega}_j$  of the domain  $\Omega_j$ . Hence the derivative  $(H_j - \text{id})_*$  is small on the domain  $\widetilde{\Omega}_j$  (provided that its radius  $\rho$  is sufficiently small), which ensures the injectivity of  $H_j$  on  $\widetilde{\Omega}_j$ .

Condition 4° holds automatically provided that the radius  $\rho$  of the constriction  $\omega'$  is sufficiently small. Indeed, consider the homotopy  $H^t = \text{id} + t(h, g)$ , joining the maps  $H^0$  and  $H^1 = H_j$ . In view of the estimates (37) for each  $(x', y') \in \widetilde{\Omega}_j$  the degree of the map  $H^t$  on  $\overline{\Omega}_j$  with respect to the point  $(x', y')$  is the same for all  $t \in [0, 1]$  (provided that  $\rho$  is sufficiently small). But the degree of  $H^0$  on  $\overline{\Omega}_j$  with respect to  $(x', y')$  is equal to 1, since  $(x', y') \in \Omega_j$ . Consequently, by the property of the degree,  $(x', y') \in H^1(\Omega_j)$ , as required. The uniqueness assertion is a direct consequence of the corresponding uniqueness assertion in Theorem 3.  $\square$

Let  $\omega = \{\Omega_j\}$  be an arbitrary covering of radius  $\varepsilon$  and opening  $\alpha \in (\frac{\pi}{2p}, \frac{\pi}{p})$ . We consider the system  $\delta\omega = \{S_j\}$ , consisting of the domains

$$S_j = \{|x| < \varepsilon\} \times \left\{ |y| < \varepsilon, \left| \arg y - \frac{\pi j}{p} \right| < \alpha - \frac{\pi}{2p} \right\}.$$

As is easily seen, this system consists of pairwise-intersecting domains of the good covering  $\omega : S_j = \Omega_j \cap \Omega_{j+1}$  (for  $p > 1$ ). The numbering here is cyclic, that is,  $\Omega_{2p+1} = \Omega_1$ .

*Remark 6.* For  $p = 1$  the good covering consists of two domains  $\Omega_1$  and  $\Omega_2$  whose intersection splits into two connected components  $S_1$  and  $S_2$ . To keep the notation uniform, even in this case, by the intersection of  $\Omega_1$  and  $\Omega_2$  we shall mean the domain  $S_1$ , while  $S_2$  will be understood to be the intersection of  $\Omega_2$  and  $\Omega_3$  (although  $\Omega_3$  is the same as  $\Omega_1$ ).

**Definition 6.** Let  $\mathcal{H} = \{H_j\}$  be the normalizing atlas on the regular covering  $\omega = \{\Omega_j\}$  for the germ  $v \in \mathcal{V}_{p,\lambda,a}^N$ . We call the family of maps  $\delta\mathcal{H} = \{\Phi_j\}_{j=1}^{2p}$ ,  $\Phi_j = H_{j+1}^{-1} \circ H_j$  the *family of transition functions* of the normalizing atlas  $\mathcal{H}$ .

**Lemma 10.** *There exists a regular covering  $\omega' = \{\Omega'_j\}$  for the germ  $v \in \mathcal{V}_{p,\lambda,a}^N$ , such that the system of domains  $\delta\omega' = \{S'_j\}$  satisfies the following conditions:*

- 1°. *The maps  $\Phi_j$  and  $\Phi_j^{-1}$  are holomorphic and injective on  $S'_j$ .*
- 2°.  *$\Phi_j$  preserves the formal normal form  $v_{p,\lambda,a}$ :*

$$(38) \quad \Phi'_j \cdot v_{p,\lambda,a} = v_{p,\lambda,a} \circ \Phi_j \quad \text{on } S'_j.$$

- 3°. *The identity map is asymptotic for  $\Phi_j$  on  $S'_j$ .*

*Proof.* The proof of the first part of the lemma is similar to that of the last part of Proposition 2. The second and third parts are immediate consequences of the properties 1°, 2° of the normalizing atlas.  $\square$

**3.3. The moduli space  $R\mathcal{F}$ .** We denote by  $\mathcal{F}$  the space of all systems of maps  $\{\Phi_j\}$  satisfying the conditions 1°–3° of Lemma 10. We say that two systems  $\{\Phi_j\}$  and  $\{\tilde{\Phi}_j\}$  in  $\mathcal{F}$  are *equivalent* if for some regular covering  $\omega = \{\Omega_j\}$  for all  $j$  the maps  $\Phi_j$  and  $\tilde{\Phi}_j$  are defined and coincide on the domain  $S'_j$  in the system of domains  $\delta\omega' = \{S'_j\}$ .

Classes of equivalent systems in  $\mathcal{F}$  are usually called germs; we denote the equivalence class containing the system  $\Phi \in \mathcal{F}$  by  $\Phi$ . Suppose finally that  $R\mathcal{F}$  is the space of all germs of systems in  $\mathcal{F}$ .

As was shown in the previous subsection, there is a well-defined map  $\Pi_1 : \mathcal{V}_{p,\lambda,a}^N \rightarrow R\mathcal{F}$ , associating the germ  $v \in \mathcal{V}_{p,\lambda,a}^N$  with the germ  $\Phi_v$  of the system  $\delta\mathcal{H} = \{\Phi_j\}$  of transition functions of some normalizing atlas  $\mathcal{H}$  of  $v$ ,  $\Pi_1 : v \rightarrow \Phi_v$ . The next proposition asserts that  $\Pi_1$  is a “map of moduli of strict equivalence”.

**Proposition 3.** 1. The germs  $v, \tilde{v} \in \mathcal{V}_{p,\lambda,a}^N$  are strictly analytically equivalent if and only if their germs of transition functions coincide:  $\Phi_v = \Phi_{\tilde{v}}$ .  
2. For each  $\Phi \in R\mathcal{F}$  there exists  $v \in \mathcal{V}_{p,\lambda,a}^N$  such that  $\Phi = \Phi_v$ .

*Proof.* 1. ( $\Rightarrow$ ) Suppose that the germs  $v$  and  $\tilde{v}$  are strictly analytically equivalent, that is, there exists a holomorphic change of coordinates  $H$ ,  $H(x, y) = (x + o(1), y + o(y^{p+1}))$ , such that  $H'v = \tilde{v} \circ H$ . Let  $\{H_j\}$  and  $\{\tilde{H}_j\}$  be normalizing atlases for  $v$  and  $\tilde{v}$ , respectively. In this case the family of maps  $\{G_j\}$ ,  $G_j = H \circ H_j$ , is also a normalizing atlas for  $\tilde{v}$  for some regular covering. It follows from the uniqueness of normalizing atlases (see Proposition 2) that the maps  $G_j$  and  $\tilde{H}_j$  coincide on the domains  $\Omega_j$  of some regular covering  $\omega = \{\Omega_j\}$ . But then  $\tilde{\Phi}_j = (\tilde{H}_{j+1})^{-1} \circ \tilde{H}_j = (H \circ H_{j+1})^{-1} \circ (H \circ H_j) = (H_{j+1})^{-1} \circ H_j = \Phi_j$ . Hence the germs  $\Phi_v$  and  $\Phi_{\tilde{v}}$  coincide.

( $\Leftarrow$ ) Suppose that the germs  $\Phi_v$  and  $\Phi_{\tilde{v}}$  coincide. Then on the domains  $S_j$  the family of intersections  $\delta\omega$  of some regular covering  $\omega$  of the maps  $\Phi_j = (H_{j+1})^{-1} \circ H_j$  and  $\tilde{\Phi}_j = (\tilde{H}_{j+1})^{-1} \circ \tilde{H}_j$  (constructed from the normalizing atlases  $\{H_j\}$  and  $\{\tilde{H}_j\}$  of the germs  $v$  and  $\tilde{v}$ , respectively) coincide. We set  $G_j = \tilde{H}_j \circ H_j^{-1}$ ; the composites  $G_j$  are defined on the domains  $\tilde{\Omega}_j$  of some constriction  $\tilde{\omega}$  of the covering  $\omega$ . Since  $H_j$  and  $\tilde{H}_j$  conjugate the formal normal forms  $v_{p,\lambda,a}$  with  $v$  and  $\tilde{v}$ , respectively, it follows that  $G_j$  conjugates  $v$  and  $\tilde{v}$ :  $G'_j v = \tilde{v} \circ G_j$  on  $\tilde{\Omega}_j$ . Since  $\Phi_j = \tilde{\Phi}_j$ , it follows that  $G_j = G_{j+1}$  on the intersection of the sectorial domains  $\tilde{\Omega}_j$  and  $\tilde{\Omega}_{j+1}$ . Consequently, the map  $G$ , which coincides with  $G_j$  on  $\tilde{\Omega}_j$ , is well defined on  $\bigcup \tilde{\Omega}_j = \Omega$ ; here  $G'v = \tilde{v} \circ G$  on  $\Omega$ . Let  $\widehat{H}$  and  $\widetilde{\widehat{H}}$  be formal normalized changes of coordinates for  $v$  and  $\tilde{v}$ ; then it follows from property 2° of normalizing atlases that the series  $\widehat{G} = \widehat{H} \circ (\widetilde{\widehat{H}})^{-1}$  is asymptotic for  $G$ . Since the series  $\widehat{H}$  and  $\widetilde{\widehat{H}}$  are normalized, it follows that the series  $\widehat{G}$  is normalized:

$$\widehat{G}(x, y) = (x, y) + (o(y^{N-p-1}), o(y^{N-1})).$$

Hence it immediately follows that the limit  $\lim_{y \rightarrow 0} G(x, y) = (x, 0)$  exists. This enables us to extend the definition of the map  $G$  to the line  $\{y = 0\}$ , thereby obtaining the required holomorphic change of coordinates establishing the strict equivalence of  $v$  and  $\tilde{v}$ .

2. Let  $\Phi \in R\mathcal{F}$ , let  $\{\Phi_j\}$  be a representative of the germ  $\Phi$ ,  $\Phi_j : S_j \rightarrow \mathbb{C}^2$ ,  $\{S_j\} = \delta\omega$ , where  $\omega = \{\Omega_j\}$  is a regular covering of opening  $\alpha$  and radius  $\varepsilon$ . We consider the topological space  $\mathcal{M}$ , obtained from the domain  $\Omega_j$  by gluing via the map  $\Phi_j$ : the points

of  $\mathcal{M}$  are the points in  $\Omega_j$ : and for all  $j = 1, \dots, 2p$  the points  $P \in \Omega_j$  and  $Q \in \Omega_{j+1}$  ( $P \in S_j$ ,  $Q = \Phi_j(P)$ ) are identified. Let  $i_j : \Omega_j \rightarrow \mathcal{M}$  be the natural embedding  $\tilde{\Omega}_j = i_j(\Omega_j)$ ,  $\pi_j = i_j^{-1} : \tilde{\Omega}_j \rightarrow \Omega_j$ . The topology on  $\mathcal{M}$  is defined in the standard fashion: a set  $U \subset \mathcal{M}$  is open if and only if the sets  $\pi_j(U \cap \tilde{\Omega}_j)$  are open. As shown in [6], the parameters  $\alpha, \varepsilon$  can be chosen so that  $\mathcal{M}$  is Hausdorff; therefore, the charts  $(\tilde{\Omega}_j, \pi_j)$  define on  $\mathcal{M}$  a complex-manifold structure. Furthermore, it is shown there that this manifold is biholomorphically equivalent to some domain  $\tilde{V}$  obtained from a neighbourhood of the origin in  $\mathbb{C}^2$  by removing the points with zero  $y$ -coordinate. This means that there exists a holomorphic map  $\mathcal{H} : \mathcal{M} \rightarrow \tilde{V}$  that is a bijection; as  $\mathcal{H}$  is holomorphic, so are the maps  $H_j = \mathcal{H} \circ i_j : \Omega_j \rightarrow \mathbb{C}^2$ , and the maps  $H_j$  and  $H_{j+1} \circ \Phi_j$  coincide on the domains  $S_j$ . It follows from the bijectivity of  $\mathcal{H}$  that the maps  $H_j$  are injective. In [6] it is also shown that for any  $N \in \mathbb{N}$  the parameters  $(\alpha, \varepsilon)$  and the map  $\mathcal{H}$  can be chosen so that the maps  $H_j$  are “ $N$ -planar with respect to  $y$ ”:

$$(39) \quad H_j(x, y) = o(y^N) \quad \text{as } y \rightarrow 0, \quad (x, y) \in \Omega_j.$$

Consider the vector field  $\tilde{v}$  on  $\mathcal{M}$  that is the image of the vector field  $v_{p,\lambda,a}$  under the maps  $i_j$ :

$$\tilde{v} \circ i_j = (i_j)_* \cdot v_{p,\lambda,a} \quad \text{on } \Omega_j,$$

where the subscript  $*$  denotes the tangent map: since  $(i_{j+1})^{-1} \circ i_j|_{S_j} = \Phi_j$  and the maps  $\Phi_j$  preserve the vector field  $v_{p,\lambda,a}$ , it follows that the field  $\tilde{v}$  is well defined and is a holomorphic vector field on  $\mathcal{M}$ . Let  $v$  be the image of the vector field  $\tilde{v}$  under the map  $\mathcal{H}$ :

$$\mathcal{H}_* \tilde{v} = v \circ \mathcal{H} \quad \text{on } \mathcal{M}.$$

Then the maps  $H_j$  convert the field  $v_{p,\lambda,a}$  into the field  $v$ :

$$(40) \quad (H_j)_* v_{p,\lambda,a} = v \circ H_j \quad \text{on } \Omega_j.$$

Since the maps  $H_j$  are “ $N$ -planar with respect to  $y$ ”, it follows that on the domains  $V_j = H_j(\Omega_j)$  we have the asymptotic equalities  $v(x, y) = v_{p,\lambda,a}(x, y) + o(y^N)$  as  $y \rightarrow 0$ ,  $(x, y) \in V_j$ . Consequently, the same asymptotic equality holds on the entire domain  $\tilde{V} = \bigcup_{j=1}^{2p} V_j$ :

$$(41) \quad v(x, y) = v_{p,\lambda,a}(x, y) + o(y^N) \quad \text{as } y \rightarrow 0, \quad (x, y) \in \tilde{V}.$$

This asymptotic equality shows that for each fixed  $x$  the point  $y = 0$  is a removable singular point for the holomorphic vector-valued function  $v(x, y)$ . We extend the definition of  $v$  to the points of the line  $\{y = 0\}$ , by setting

$$v(x, 0) = \lim_{y \rightarrow 0} v(x, y) = v_{p,\lambda,a}(x, 0) = x \frac{\partial}{\partial x}.$$

We retain the notation  $v$  for this extended vector-valued function. Then the vector-valued function  $v$  is defined on some neighbourhood of the origin  $V \subset \mathbb{C}^2$  so that  $\tilde{V} = V \setminus \{y = 0\}$ ;  $v(x, y)$  is holomorphic with respect to  $y$  for each  $x$  and is holomorphic with respect to  $x$  for each fixed  $y$ . Hence by Hartogs's theorem the field  $v$  is holomorphic on  $V$ . Taking (41) into account, we find that  $v \in \mathcal{V}_{p,a,\lambda}^N$ , and therefore  $v$  is formally equivalent to the field  $v_0$ . It follows from (39) and (40) that the maps  $H_j$  are sectorial normalizing maps for the field  $v$ . Since  $(H_{j+1})^{-1} \circ H_j|_{S_j} = \Phi_j$ , it follows that the system of maps  $\{\Phi_j\}$  is the system of transition functions for the normalizing atlas  $\{H_j\}$  of  $v$ , which means that the germs  $\Phi$  and  $\Phi_v$  coincide. The second part of the theorem is proved.  $\square$

**3.4. Construction of the bijection  $\Pi_2 : R\mathcal{F} \rightarrow \mathcal{M}_{p,\lambda}$ .** Let  $\{\Phi_j\}$  be representatives of the germ  $\Phi \in R\mathcal{F}$ ,  $\Phi_j : S_j \rightarrow \mathbb{C}^2$ ,  $\{S_j\} = \delta\omega$ , and let  $\omega$  be a regular covering. Let

$$\Phi_j = (\alpha_j, \beta_j), \quad a(y) = \sum_{k=0}^p a_k y^k, \quad w(y) = \frac{y^{p+1}}{1 + \lambda y^p}.$$

In expanded form (38) becomes

$$(42) \quad \begin{cases} a(y)(x\alpha_{j,x} + w(y)\alpha_{j,y}) = \alpha_j \cdot a \circ \beta_j, \\ a(y)(x\beta_{j,x} + w(y)\beta_{j,y}) = a \circ \beta_j \cdot \omega \circ \beta_j. \end{cases}$$

**3.4.1. Martinet–Ramis moduli.** Let  $J = J(x, y) = xe^{-A(y)}$ . Since  $A'(y) = \frac{1}{w(y)}$ ,  $J$  is a first integral for  $v_{p,\lambda}$  and therefore also for  $v_{p,\lambda,a}$ . We set  $\Psi_j = J \circ \Phi_j$ , and let  $\tilde{\Psi}_j(y, z) = \Psi_j(ze^{A(y)}, y)$ . Then it follows immediately from (42) that  $\frac{d}{dy}\tilde{\Psi}_j(y, z) = 0$ . This means that the function  $\tilde{\Psi}_j$  depends only on  $z$ :  $\tilde{\Psi}_j(y, z) = \hat{\Psi}_j(z)$  for some holomorphic function  $\hat{\Psi}_j(z)$  of one variable. Here, by the uniqueness theorem,  $\tilde{\Psi}_j(z)$  can be analytically continued to the entire region  $W_j = J(S_j)$ . Repeating these arguments for  $\Phi_j^{-1}$  (and, possibly, replacing the original covering  $\omega$  by a constriction of it), we find that the map  $\hat{\Psi}_j$  is invertible (which means that it is injective on  $W_j$ ). Note that for odd  $j$  the region  $W_j$  is the entire plane (and then  $\hat{\Psi}_j$  is linear), while for even  $j$  the region  $W_j$  is a punctured neighbourhood of the origin.

Next, the identity map is asymptotic for  $\Phi_j$  (part 3° of Lemma 10):

$$(43) \quad \forall n \quad \alpha_j(x, y) = o(y^n), \quad \beta_j(x, y) = y + o(y^n), \quad y \rightarrow 0, \quad (x, y) \in S_j.$$

Hence it follows that

$$\Psi_j(x, y) = (x + o(y^n)) \cdot \exp\left\{-\frac{1}{y^p}(1 + o(y^n))\right\} = z \cdot (1 + x^{-1}o(y^n)) \cdot (1 + o(y^{n-p}))$$

for  $z = x \cdot e^{-A(y)}$ ,  $y \rightarrow 0$ ,  $x \neq 0$ .

It follows that for  $j = 2k$  we have  $\hat{\Psi}_{2k}(z) = z + o(z)$ ,  $z \rightarrow 0$  (and hence  $\hat{\Psi}_{2k}(z)$  coincides on  $(\mathbb{C}_*, 0)$  with some function  $\varphi_k$  that is holomorphic in  $(\mathbb{C}, 0)$  with  $\varphi_k(0) = 0$ ,  $\varphi'_k(0) = 1$ ). Hence it follows that for odd  $j = 2k - 1$  we have  $\hat{\Psi}_{2k-1}(z) = z + o(z)$ ,  $z \rightarrow \infty$ ; this means that  $\hat{\Psi}_{2k}(z) = z + c_k$  for some constant  $c_k$ .

*Remark 7.* In the construction of the family  $c, \varphi$  we had a certain freedom of choice for the branch of the multivalued map

$$A(y) = -\frac{1}{py^p} + \lambda \ln y,$$

used in the definition of the first integral  $J(x, y) = xe^{-A(y)}$ . We now do away with this freedom of choice by selecting the branch of  $\ln y$  on the region  $S_j$  in accordance with the restrictions on  $\arg y$  featured in its definition. Finally, for  $j = 2p$ , along with the multivalued function  $J$  we shall use the branch  $\tilde{J} = J \cdot \nu$  of it ( $\nu = \exp\{2\pi i\lambda\}$ ); and we define  $\hat{\Psi}_{2p}$  by the equality  $\hat{\Psi}_{2p} \circ J = \tilde{J} \circ \Phi_{2p}$ . Then all the above arguments remain true and the validity of the condition  $\varphi'_p(0) = \nu$  is ensured.

Accordingly, we can construct the invariants  $c = \{c_k\}$  and  $\varphi = \{\varphi_k\}$  (the Martinet–Ramis invariants [2]).

3.4.2. *Construction of the “temporal part”  $\psi$  of the invariant  $\mu_v$ .* Let  $B(y)$  be a primitive of the function  $\frac{1}{a(y)w(y)}$ , that is,  $B'(y) = \frac{1}{a(y)w(y)}$ . We consider the function  $b_j(x, y) = B \circ \beta_j(x, y) - B(y)$ , and we set  $\tilde{b}_j(y, z) = b_j(ze^{A(y)}, y)$ . It follows from the second equation of the system (42) that

$$\frac{\partial \tilde{b}_j}{\partial y} = \frac{1}{a \circ \beta_j \cdot \omega \circ \beta_j} \cdot \left( \frac{x \beta'_j x}{w(y)} + \beta'_j y \right) \Big|_{x=ze^{A(y)}} - \frac{1}{a(y)w(y)} = 0.$$

This means that the function  $\tilde{b}_j(y, z)$  actually depends only on  $z$ :  $\tilde{b}_j(y, z) = \hat{b}_j(z)$  for some holomorphic function  $\hat{b}_j$ . As above, we find that the function  $\hat{b}_j$  can be continued analytically to the entire plane  $W_j = J(S_j)$ . Furthermore, it follows from the asymptotic equalities (43) that for any  $n$  we have  $b_j(x, y) = o(y^n)$  as  $y \rightarrow 0$ ,  $(x, y) \in S_j$ ; hence the function  $\hat{b}_j(z)$  satisfies the asymptotic equality

$$(44) \quad \hat{b}_j(x e^{-A(y)}) = o(y^n) \quad \text{as } y \rightarrow 0, \quad (x, y) \in S.$$

For odd  $j$  the domain  $W_j$  coincides with  $\mathbb{C}$ , and it follows from (44) that  $b_j(z) \rightarrow 0$  as  $z \rightarrow \infty$ . Hence in this case  $\hat{b}_j \equiv 0$ .

For even  $j = 2k$  the domain  $W_j$  is a punctured neighbourhood of the origin and it follows from (44) that  $\hat{b}_{2k}(z) \rightarrow 0$  as  $z \rightarrow 0$ . This means that the point 0 is a removable singularity for the function  $\hat{b}_{2k}(z)$ : the function  $\hat{b}_{2k}$  coincides on  $(\mathbb{C}_*, 0)$  with a function  $\psi_k$  that is holomorphic in  $(\mathbb{C}, 0)$  and is such that  $\psi_k(0) = 0$ . We have constructed the last component  $\psi = \{\psi_k\}$  of the element  $\mu_v \in \mathcal{M}_{p,\lambda}$ .

Thus, corresponding to each germ  $\Phi \in R\mathcal{F}$  there is a uniquely determined element  $\mu = (c, \varphi, \psi) \in \mathcal{M}_{p,\lambda}$ . We denote the map defined in this way by  $\Pi_2$ ,  $\Pi_2 : R\mathcal{F} \rightarrow \mathcal{M}_{p,\lambda}$ .

3.4.3. *Bijectivity of the map  $\Pi_2$ .* Let  $\mu = (c, \varphi, \psi) \in \mathcal{M}_{p,\lambda}$ ,  $c = \{c_k\}$ ,  $\varphi = \{\varphi_k\}$ ,  $\psi = \{\psi_k\}$ , where  $\psi_k$  and  $\varphi_k$  are holomorphic in  $(\mathbb{C}, 0)$ ,  $\varphi_k(0) = \psi_k(0) = 0$ ,  $\varphi'_k(0) = 1$ ,  $k = 1, 2, \dots, p-1$ ,  $\varphi'_p = \nu = \exp\{2\pi i \lambda\}$ . Let  $\omega$  be a regular covering of small radius,  $\delta\omega = \{S_j\}$ . Let  $g_{v_{p,\lambda,a}}^t$  be the phase flow of the formal normal form  $v_{p,\lambda,a}$ .

For even  $j = 2k$  we consider the maps

$$G_{2k} : (x, y) \mapsto (\varphi_k(x e^{-A(y)}) e^{A(y)}, y), \quad F_{2k}(x, y) = g_{v_{p,\lambda,a}}^{\psi_k(z)}(x, y) \Big|_{z=x e^{-A(y)}},$$

and we set  $\Phi_{2k} = G_{2k} \circ F_{2k}$  (the branches of the multivalued function  $A(y)$  are chosen in accordance with Remark 7).

For odd  $j = 2k-1$  we consider the map  $\Phi_{2k-1}(x, y) = (x + c_k e^{A(y)}, y)$ . Finally, we denote the family of maps  $\{\Phi_j\}$  that we have constructed by  $\Phi_\mu$ . The following lemma is proved by direct calculations.

**Lemma 11.** 1. For each  $\mu \in \mathcal{M}_{p,\lambda}$ , we have  $\Phi_\mu \in \mathcal{F}$ .  
2.  $\Pi_2([\Phi_\mu]) = \mu$ .

**Corollary 4.** The map  $\mu \mapsto [\Phi_\mu]$  is inverse to  $\Pi_2$  (which means that  $\Pi_2$  is a bijection from  $R\mathcal{F}$  onto  $\mathcal{M}_{p,\lambda}$ ).

3.4.4. *Conclusion of the proof of Theorem 5.* We consider the map

$$\Pi : \mathcal{V}_{p,\lambda,a} \rightarrow \mathcal{M}_{p,\lambda}, \quad \Pi : v \rightarrow \mu_v = \Pi_2(\Phi_v).$$

We call the element  $\mu_v \in \mathcal{M}$  the modulus (of the strict analytic equivalence) of the germ  $v$ .

The assertions concerning the “equivalence and equimodality” as well as the “realization” (see the statement of Theorem 4) follow immediately from Proposition 3 and the corollary to Lemma 11. The analytic dependence of the modulus on the parameter

is obtained by applying the standard theorems on differentiability with respect to the parameter at each stage of the construction of the modulus.

**3.5. Proof of Theorem 4.** We note that the following maps preserve the formal normal form  $v_{p,\lambda,a}$ : the dilations  $\Lambda_k : (x, y) \mapsto (kx, y)$ ; the shifts  $T_t : (x, y) \mapsto g_{v_{p,\lambda,a}}^t(x, y)$  for fixed time  $t \in \mathbb{C}$ ; the rotations  $R_m : (x, y) \mapsto (x, \epsilon^m y)$ , where  $\epsilon = \exp\left\{\frac{2\pi i}{p_a}\right\}$  is the  $p_a$ th root of unity (recall that  $p_a$  is the greatest common divisor of  $p$  and those  $k \in \{1, \dots, p\}$  for which  $a_k \neq 0$ ) and, finally, the composites  $S_{k,t,m}$  of such maps. In the class of formal changes of variables the list of changes of variables preserving the formal normal form (the so-called symmetries) is exhausted by the above list (see [10]). Therefore any map  $H$  conjugating a pair of germs in  $\mathcal{V}_{p,\lambda,a}^N$  is uniquely representable as a composite  $S \circ H_0$ , where  $S$  is a symmetry and  $H_0$  is a change of variables normalized by condition (2). Therefore the analytic equivalence classes of germs in  $\mathcal{V}_{p,\lambda,a}^N$  are obtained from the strict analytic equivalence classes by combining all the classes in the orbit of the action of the group  $\mathcal{S} = \{S\}$  of all symmetries  $S$  of the formal normal form. The action of the group of symmetries  $\mathcal{S}$  on the space  $\mathcal{V}_{p,\lambda,a}^N$  is uniquely defined:  $S(v) = \tilde{v} \Leftrightarrow S'v = \tilde{v} \circ S$ , and induces an action of the group  $\mathcal{S}$  on the space of normalizing atlases:  $S(\{H_j\}) = \{S \circ H_j \circ S^{-1}\}$ . This defines an action of the group  $\mathcal{S}$  on the space of transition functions  $\mathcal{F}$ :

$$(45) \quad S(\{\Phi_j\}) = \{\tilde{\Phi}_j\} \quad \text{for } \tilde{\Phi}_j = S^{-1} \circ \Phi_{j+2mn_a} \circ S, \quad S = S_{k,t,m}, \quad n_a = \frac{p}{p_a}.$$

(The symmetry  $S_{k,t,m} = \Lambda_k \circ T_k \circ R_m$  takes the sectorial domain  $S_j$  of the system  $\delta\omega = \{S_J\}$  to a domain close to  $S_{j+2mn_a}$ .) Therefore, in accordance with Proposition 3, the space of orbits of the action of the group  $\mathcal{S}$  on  $R\mathcal{F}$  also gives a moduli space of analytic equivalence of the germs in  $\mathcal{V}_{p,\lambda,a}^N$ . Finally, by transferring the equivalence relation (45) via the bijection  $\Pi_2$  onto the space  $\mathcal{M}_{p,\lambda}$  (taking into account Remark 7), we obtain the equivalence relation (4) on the space  $\mathcal{M}_{p,\lambda}$ . This completes the proof of Theorem 4.

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