

BOUNDED ERGODIC CONSTRUCTIONS, DISJOINTNESS, AND WEAK LIMITS OF POWERS

V. V. RYZHIKOV

ABSTRACT. This paper is devoted to the disjointness property of powers of a totally ergodic bounded construction of rank 1 and some generalizations of this result. We look at applications to the problem when the Möbius function is independent of the sequence induced by a bounded construction.

Interest in the subject matter of this paper is related to the following observation. Bounded constructions of rank 1, under the condition that all their nonzero powers are ergodic, have nontrivial weak limits of powers. This implies that the powers of the constructions are disjoint (in the sense of [1]) and, in view of the results in [2], this results in bounded constructions being independent of the Möbius function. Thus, the problem of disjointness of powers of transformations, which had previously been regarded by specialists as a problem within the framework of self-joining theory, has an interesting application.

Sarnak's well-known conjecture [3] states that a strictly ergodic homeomorphism $S: X \rightarrow X$ with zero topological entropy has the property

$$\sum_{i=1}^N f(S^i x) \mu(i) = o(N),$$

where $f \in C(X)$ and μ is the Möbius function. In [3], this property was called being independent of the Möbius function (in the literature cited above this property was called the disjointness or orthogonality property).

In [4] Bourgain studied Riesz products and confirmed Sarnak's conjecture for bounded constructions of rank 1. In [5], using the method of weak limits of powers, it was shown that the positive powers of a totally ergodic bounded construction are disjoint. This simplified the proof of Bourgain's result above. In [6] the authors proved the spectral disjointness of powers of a weakly mixing construction of rank 1 that have so-called nonflat bounded recurrence. We note that the spectral disjointness of powers of a transformation can sometimes be established by the method of weak limits even when there are no nontrivial limits (see [7]). For our purposes we only need to establish disjointness of powers. Note that recently Prikhod'ko constructed a flow of rank 1 with a Lebesgue spectrum [8]. The transformations contained in such a flow have spectrally isomorphic powers. However, as was noted in [9], their powers are disjoint in the sense of [1].

Following the ideas in [5] and [6], the purpose of this note is to give a simple proof that powers of the constructions given above are disjoint. We shall also show how to

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reduce the ergodic case to the totally ergodic case, which lets us prove that bounded constructions of rank 1 are independent of the Möbius function.

§1. A SUFFICIENT CONDITION FOR DISJOINTNESS OF POWERS OF A TRANSFORMATION

From now on, we shall call invertible measure-preserving transformations of a Lebesgue probability space (X, ν) *transformations*. In what follows we shall use the terminology of Markov (stochastic) intertwining operators (see, for example, [9]). Transformations S and T (as operators in the space $L_2(X, \nu)$, $\nu(X) = 1$) are said to be *disjoint* if for any Markov operator J the equation $SJ = JT$ implies $J = \Theta$, where Θ is the orthoprojection onto the space of constants in $L_2(X, \nu)$. This definition of disjointness is equivalent to the classical definition in [1], which uses the concept of joining.

A transformation T is said to be *totally ergodic* if all of its nonzero powers are ergodic. This is equivalent to the fact that for any Markov operator J and any $i > 0$ the equation $T^i J = J$ implies $J = \Theta$.

A Markov operator J satisfying the intertwining condition $SJ = JT$ is said to be *indecomposable* if it cannot be represented in the form of a convex sum of different Markov operators intertwining S and T .

Lemma 1. *Let S, T be totally ergodic transformations, and let $J \neq \Theta$ be an indecomposable operator satisfying the intertwining condition $S^q J = JT^p$, where q, p are coprime. If*

$$Q(S)J = JP(T),$$

where

$$Q(S) = \sum_i a_i S^i, \quad P(T) = \sum_j b_j T^j, \quad \sum_i a_i = 1 = \sum_j b_j, \quad a_i, b_j \geq 0,$$

then the series Q and P are (p/q) -similar: there exists a series R such that

$$Q(S) = R(S^q), \quad P(T) = R(T^p).$$

Proof. Note that operators of the form $S^k J$, JT^m are indecomposable, since S^k , T^m are invertible and J is indecomposable. Then the equation $Q(S)J = JP(T)$ implies that the terms of the series $\sum_i a_i S^i J$ coincide with the terms of the series $\sum_j b_j JT^j$. Hence, for $a_i \neq 0$ we have $b_j \neq 0$ and $S^i J = JT^j$ for some j . If $i \neq 0$, then $j \neq 0$. This follows because, as S^i is ergodic, if $S^i J = J$, then $J = \Theta$. We have $S^i J = JT^j$, $ij \neq 0$. Thus $S^{iq} J = JT^{jq}$, and $S^q J = JT^p$ implies that $S^{iq} J = JT^{ip}$. Hence, $J = JT^{jq-ip}$. But the transformation T is totally ergodic, therefore, $jq = ip$. Since q, p are coprime, we obtain $i = qr$, $j = pr$. Thus we have shown that

$$a_i = b_{iq/p}, \quad \sum_r a_{qr} = 1, \quad \sum_r b_{pr} = 1.$$

This means that the series Q and P are (p/q) -similar. □

The next lemma follows direction from Lemma 1.

Lemma 2. *Suppose that a totally ergodic transformation T has weak limits of the form $T^{qn_i} \rightarrow Q(T)$, $T^{pn_i} \rightarrow P(T)$. If the series $Q(T)$ and $P(T)$ are not (p/q) -similar, then T^q and T^p are disjoint.*

§2. DISJOINTNESS OF POWERS OF PARTIALLY BOUNDED CONSTRUCTIONS

A construction of rank 1 is defined by the following parameters: a number h_1 (the height of the initial tower), a sequence r_j (the number of columns into which the tower is cut at the j th stage), and a sequence of arrays of the heights of the spacers over the columns $(s_j(1), s_j(2), \dots, s_j(r_j - 1), s_j(r_j))$. The height of the j th tower is calculated by the formula

$$h_{j+1} + 1 = (h_j + 1)r_j + \sum_{i=1}^{r_j} s_j(i).$$

A detailed description of constructions of rank 1 can be found in the papers cited. If all the parameters r_j and $s_j(i)$ are bounded, then the construction is said to be *bounded*.

Theorem 1. *Let T be a totally ergodic bounded construction. Then the powers T^p and T^q are disjoint for all positive numbers $q \neq p$.*

A simple proof of this theorem was given in [5, §7]. In [6] the authors considered the following interesting generalization of bounded constructions. We say that a construction is *partially bounded* if the parameters of the construction are bounded at the j th stage for j in some subset of positive integers containing arbitrarily large integer intervals. From this system of intervals we can choose a subsystem of intervals $J_k = \{j_k, j_k + 1, \dots, j_k + l_k\}$ such that the parameters of the construction are the same for all stages of the form $j_k + m$ for $k \geq m$. In [6] an equivalent property was called *recurrent boundedness*. If, however, the heights of spacers stabilize at some constant value, that is, $s_j(i) = s$ for all $j \in J_k$ and $i \leq r_j$, then such constructions are said to be *recurrent-flat*.

We consider interconnections between weak limits of powers of recurrent-bounded transformations. In what follows we use the following notation:

$$H_j = -h_j - s_j, \quad s_j = \min\{s_j(1), s_j(2), \dots, s_j(r_j - 1)\}$$

and

$$P_{d,m}(T) := \lim_{k \rightarrow \infty} T^{dH_{j_k+m}}.$$

We fix numbers p, q that are coprime. We assume without loss of generality that $r_j \geq r > p, q$ (without changing the construction, considering if necessary a thinned out set of stages, we increase the number of cuttings into columns). Let $d < r$; standard calculations of weak limits of powers of transformations of rank 1 give rise to the following relations:

$$(*) \quad \begin{aligned} P_{d,m}(T) &= \cdots + a_{d,m} T^{N(d,m)} P_{1,m+1}, \\ P_{1,m}(T) &= aI + \cdots, \quad a > 0. \end{aligned}$$

We will show that these relations imply unboundedness or flat behavior of spacers. For $P(T) = \sum_z a_z T^z$ we define $\mathbf{P} \subset \mathbf{Z}$ by setting

$$\mathbf{P} = \{z \in \mathbf{Z} : a_z > 0\}.$$

From Lemma 2 we have

$$\mathbf{P}_{p,m} \subset p\mathbf{Z}.$$

Note that from $P_{1,m}(T) = aI + \cdots$ (thus, $0 \in \mathbf{P}_{1,m+1}$) it follows from $(*)$ that

$$\mathbf{P}_{1,m+1} \subset p\mathbf{Z}.$$

Now any sum of the $s_j(i)$ is divisible by p if each $s_j(i)$ is divisible by p , and so

$$\mathbf{P}_{q,m+1} \subset (\mathbf{P}_{1,m+1} + \mathbf{P}_{1,m+1} + \cdots + \mathbf{P}_{1,m+1})$$

(q summands). This implies that

$$\mathbf{P}_{q,m+1} \subset p\mathbf{Z}.$$

Lemma 2 implies that

$$\mathbf{P}_{p,m+1} = \frac{p}{q} \mathbf{P}_{q,m+1} \subset p^2 \mathbf{Z}.$$

Thus,

$$\mathbf{P}_{1,m+2} \subset p^2 \mathbf{Z};$$

repeating the argument we obtain

$$\mathbf{P}_{1,m+3} \subset p^3 \mathbf{Z}, \quad \mathbf{P}_{1,m+4} \subset p^4 \mathbf{Z}, \quad \dots$$

Thus for partially bounded constructions we have shown that for any positive integer m , for a sufficiently large k the bounded differences $s_{j_k+m}(i) - s_{j_k+m}('i)$ are divisible by p^m . But the parameters of the construction are bounded, which means this cannot hold for large values of p^m .

Thus we have shown that for all m such that $p^m > s$, the spacers exhibit flat behavior:

$$s_{j_k+m}(1) = s_{j_k+m}(2) = \dots = s_{j_k+m}(r_{j_k+m} - 1) := s_{k,m}.$$

It is easy to see that for fixed m, m' the local constants are glued together: $s_{k,m} = s_{k,m'}$ for all large k . Indeed, if the constants are different, then a nontrivial bounded polynomial appears in the weak closure, which is impossible by the above. We now summarize.

Theorem 2. *A totally ergodic partially bounded construction in which the powers T^p and T^q are not disjoint for some coprime numbers q, p is a recurrent-flat construction.*

This theorem follows from the results of [6]. We have presented a simplified proof.

In the case of flat constructions an additional possibility arises: after a series of flat spacers, unbounded spacers are allowed. For a wide class of such spacers, we can obtain a relation of the form

$$(\mathbf{I}, \Theta) \quad ((1 - q\varepsilon)I + q\varepsilon\Theta)J = J((1 - p\varepsilon)I + p\varepsilon\Theta)$$

from the intertwining condition $T^q J = JT^p$, where $\varepsilon > 0$ is some small number. But for $q \neq p$ this obviously results in the powers T^q and T^p being disjoint, that is, it leads to the equation $J = \Theta$.

There are weak mixing transformations of rank 1 with isomorphic powers [10], [11], [12]. Such transformations cannot have the weak limits of powers that occur in the formulae (\mathbf{I}, Θ) and (\mathbf{I}, \mathbf{P}) (see below).

§3. INDEPENDENCE OF THE MÖBIUS FUNCTION AND A BOUNDED CONSTRUCTION

In [4] Bourgain proved the following assertion.

Theorem 3. *Bounded constructions of rank 1 are independent of the Möbius function.*

Proof. A bounded construction belongs to one of the following classes:

- 1) systems with discrete rational spectrum (odometers),
- 2) flat weakly mixing constructions,
- 3) nonflat weakly mixing constructions,
- 4) nonflat constructions with a finite compact factor (some power of this construction is the direct sum of weakly mixing bounded constructions).

Thus, the proof is divided into four parts.

1. That odometers are independent of the Möbius function follows from the well-known fact that

$$\sum_{i=1}^N \mu(pi) = o(N).$$

The fact is that for any n an odometer is a cyclic permutation of sets $E, TE, \dots, T^{p-1}E$, where $p > n$, whose indicator functions in the form of a linear combination are close to a given continuous function. Therefore the problem reduces to looking at the indicator functions of these sets, χ_E . These are fully understood: for $x \in E$,

$$\sum_{i=1}^N \chi_E(T^i x) \mu(i) = \sum_{0 < i \leq N/p} \mu(pi) = o(N).$$

2. In [5, §7] it was proved that in the case when a bounded construction is not an odometer but has flat spacers (is flat-recurrent), nontrivial polynomial limits of the form $(1 - m\varepsilon)I + m\varepsilon P$ again appear. Namely, from the intertwining condition $T^q J = JT^p$ we obtain

$$(\mathbf{I}, \mathbf{P}) \quad ((1 - q\varepsilon)I + q\varepsilon P)J = J((1 - p\varepsilon)I + p\varepsilon P)$$

for some polynomial $P \neq I$ and small $\varepsilon > 0$, which gives $J = JT^k$, $k \neq 0$; consequently, if T^k is an ergodic power, then $J = \Theta$. By [2], this implies that the ‘process’ T is independent of the Möbius function.

3. In this case we apply Theorem 2.

4. It remains to examine the case where some power of the transformation T is not ergodic. If the bounded construction T is not an odometer and is not totally ergodic, then T as an operator has an eigenvalue λ such that $\lambda^d = 1$ holds for some (smallest) number $d > 0$, and λ generates the group of all eigenvalues of the operator T . This is a consequence of the fact that a nontrivial polynomial

$$P(T) = \sum_i a_i T^i \neq I$$

exists as a weak limit of powers. Indeed, from

$$\left| \sum_i a_i \lambda^i \right| = 1,$$

it follows that for $a_k \neq 0$ we have $\lambda^k = 1$. We set

$$d = \min\{k - k' : a_k a_{k'} \neq 0, k \neq k'\}.$$

A group of eigenvalues of this type only arises if the automorphism T permutes certain sets $E, TE, \dots, T^{d-1}E$ cyclically; these sets are such that the restriction $S = T^d|E$ is a weakly mixing automorphism (has continuous spectrum) on the space E .

The number d must divide the heights of columns with stories built over them, that is, d divides the numbers $h_j + s_j(i)$ for all sufficiently large j . Indeed, suppose that a story of a tower at an ‘infinitely large’ step j with infinitely small relative error term is contained in the set E . If a column with index i exists such that $h_j + s_j(i) = dM + q$, $0 < q < d$, then part of the set E of measure greater than $\nu(E)/r - \varepsilon$ (the number r bounds the number of columns) turns out to lie outside the set E . But this contradicts the invariance of E under the power T^d .

Thus, $S = T^d|E$ is a bounded weakly mixing construction. But we have already proved above that its powers are disjoint. Therefore S is independent of the Möbius function: for $f \in C(E)$ (we regard R as a segment) and $n > 0$ we have

$$\sum_{i=1}^N f(S^{ni}x) \mu(i) = o(N).$$

We regard f as a function on X after extending it by zero outside the set E . Let d be a prime. The Möbius function satisfies $\mu(d) = -1$, $\mu(d^2k) = 0$, and $\mu(dk) = \mu(d)\mu(k)$ for

$k \perp d$. For $x \in E$ we obtain

$$\begin{aligned} \sum_{i=1}^N f(T^i x) \mu(i) &= \sum_{0 < k \leq N/d} f(T^{dk} x) \mu(dk) \\ &= \sum_{0 < k \leq N/d} f(S^k x) \mu(d) \mu(k) - \sum_{0 < m \leq N/d^2} f(S^{dm} x) \mu(d) \mu(dm) \\ &= \frac{o(N)}{d} + \sum_{0 < m \leq N/d^2} f(S^{dm} x) \mu(dm) \\ &= \frac{o(N)}{d} + \sum_{0 < m \leq N/d^2} f(S^{dm} x) \mu(d) \mu(m) + \sum_{0 < n \leq N/d^3} f(S^{d^2 n} x) \mu(dn) \\ &\leq \frac{o(N)}{d} + \frac{o(N)}{d^2} + \frac{N \|f\|}{d^3}. \end{aligned}$$

Similarly, for any M , for all sufficiently large N we obtain

$$\left| \sum_{i=1}^N f(T^i x) \mu(i) \right| \leq \frac{N \|f\|}{d^M}; \quad \sum_{i=1}^N f(T^i x) \mu(i) = o(N).$$

We now represent $F \in C(X)$ as $F = \sum_{i=0}^{d-1} f_i$, where $\text{supp } f_i \subset T^i E$. We obtain

$$\sum_{i=1}^N F(T^i x) \mu(i) = o(N).$$

We have shown that all prime extensions preserve the property of being independent of the Möbius function. If d is not a prime we can perform a series of prime extensions consecutively, in accordance with the decomposition of the number d into prime factors.

□

From Theorem 2 and the reduction of the ergodic case to the totally ergodic one we obtain the following fact [6]: *a partially bounded construction is flat-recurrent or has the property of being independent of the Möbius function.*

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MOSCOW STATE UNIVERSITY
E-mail address: vryzh@mail.ru

Translated by E. KHUKHRO