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ON POSITIVE SOLUTIONS OF ONE CLASS OF NONLINEAR INTEGRAL EQUATIONS OF HAMMERSTEIN–NEMYTSKIĬ TYPE ON THE WHOLE AXIS

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Abstract. This paper is devoted to studying one class of nonlinear integral equations of Hammerstein–Nemytskiĭ type on the whole axis, which occurs in the theory of transfer in inhomogeneous medium. It is proved that these equations can be solved in various function spaces, and the asymptotic behaviour at infinity of the solutions that are constructed is studied.

§ 1. Introduction

Nonlinear integral equations of the form

$$x(t)=\int_{-\infty}^{+\infty}K(t,s)\mu_0(s,x(s))\,ds+\mu_1(t,x(t)),\quad t\in\mathbb{R}, \tag{1}$$

describe a number of physical processes in an inhomogeneous medium. In particular, equations of the form (1) occur in the theory of radiative transfer, in the kinetic theory of gases, in biology, in optimal control theory and in economics (see, for example, [1–10]). Furthermore, when the kernel K depends on the difference of its arguments, the class of equations under consideration is a natural nonlinear generalization of the linear integral convolution equation on the whole axis. The corresponding linear equations of convolution type were studied in numerous papers by both Armenian and foreign authors (see [11–15] and the references therein). The corresponding nonlinear integral equations on a half-axis were considered in [16,17]. For example, in the recent paper [16] the author studied solvability in the space $L_1(\mathbb{R}^+)$ of the nonlinear integral equation of Hammerstein–Nemytskiĭ type

$$f(t)=\int_{0}^{\infty}K(t,s)B(s,f(s))\,ds+A(t,f(t)),\quad t\in\mathbb{R}^+, \tag{2}$$

in the case where the kernel K is majorized by a difference conservative kernel, while the corresponding nonlinear operator is noncompact. In [17], a similar question was studied for equation (2), but under the assumption that the corresponding nonlinear Hammerstein operator be compact, while the Nemytskiĭ operators $\widehat{B}$ and $\widehat{A}$ (generated by the functions B and A) are continuous maps of the space $L_1(\mathbb{R}^+)$ into itself. This paper made substantial use of Krasnosel'skiĭ's fixed point theorem and the theorem in [4] concerning a continuous map of the operator $\widehat{B}$ acting in $L_1(\mathbb{R}^+)$, in the case where $|B(s,u)|\le a(s)+b|u|$ with $a\in L_1(\mathbb{R}^+)$, $b\ge 0$.

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This paper is devoted to studying the integral equation (1) in various function spaces without assuming that the corresponding nonlinear operator is compact, and this is crucial. In § 2 we present some auxiliary results from the linear theory of conservative integral convolution equations. Section 3 is devoted to studying one class of nonlinear integral equations of Hammerstein type on the whole axis. In this section we give nonlinear analogues of the theorems presented in § 2 and also some auxiliary lemmas. In § 4 we prove the existence of a positive and bounded solution of the original equation (1) and analyse the asymptotic behaviour of the solution we obtain at $\pm\infty$, when a linear function is a local majorant for the function $\mu_0(s,\tau)$. In § 5 we study the solvability of equation (1) in the space $L_1^0(\mathbb{R})\cap L_\infty(\mathbb{R})$ when a nonlinear function with certain properties is a local majorant for the function $\mu_0(s,\tau)$ (recall that $L_1^0(\mathbb{R})$ is the space of integrable functions on $\mathbb{R}$ with zero limit at $\pm\infty$). To end § 5 we give some special cases of the functions $\mu_j(s,\tau)$, $j=0,1$, which satisfy the hypotheses of the theorems stated in the paper.

§ 2. Some auxiliary results from the linear theory of integral equations of convolution type

We consider the homogeneous integral equation of convolution type

$$B(t)=\lambda(t)\int_{-\infty}^{+\infty}\mathring{K}(t-s)B(s)\,ds,\quad t\in\mathbb{R}, \tag{3}$$

with respect to a measurable function $B(t)$. Here, λ and $\mathring{K}$ are measurable functions defined on the set $(-\infty,+\infty)$ which satisfy the following conditions:

$$0\le\lambda(t)\le 1,\quad \mathring{K}(t)\ge 0,\ t\in\mathbb{R},\quad \int_{-\infty}^{+\infty}\mathring{K}(t)\,dt=1, \tag{4}$$

$$\nu(\mathring{K})\equiv\int_{-\infty}^{+\infty}z\mathring{K}(z)\,dz\ne 0,\quad \int_{-\infty}^{+\infty}z^2\mathring{K}(z)\,dz<+\infty. \tag{5}$$

The following theorem was proved in [18] and will be used below.

Theorem 1. *Suppose that conditions* (4) *and* (5) *hold.*

a) If $1-\lambda\in L_1^0(\mathbb{R}^+)$ and $\nu(\mathring{K})<0$, then equation (3) *has a non-negative nonzero bounded solution $B\le 1$ such that*

$$\int_0^t(1-B(s))\,ds=o(t),\quad t\to+\infty. \tag{6}$$

b) If $1-\lambda\in L_1^0(\mathbb{R}^-)$ and $\nu(\mathring{K})>0$, then equation (3) *has a nonnegative nonzero bounded solution $B\le 1$ such that*

$$\int_0^t(1-B(s))\,ds=o(t),\quad t\to-\infty. \tag{7}$$

Here, $L_1^0(\mathbb{R}^\pm)$ is the space of functions in $L_1(\mathbb{R}^\pm)$ that have zero limit at $+\infty$ and at $-\infty$, respectively.

Using the iterative process

$$B_{n+1}(t)=\lambda(t)\int_{-\infty}^{+\infty}\mathring{K}(t-s)B_n(s)\,ds,\qquad (8)$$
$$B_0(t)\equiv 1,\quad n=0,1,2,\dots,\quad t\in\mathbb{R},$$

we can verify that if $\lambda\uparrow$ on $\mathbb{R}$, then equation (3) also has a nonnegative nonzero bounded solution

$$B^*(t)\geq B(t),\qquad \lim_{n\to\infty}B_n(t)=B^*(t)\leq 1,$$

which is monotonically increasing.

In view of (4), it follows from Theorem 1 that

$$B(t)\leq\lambda(t),\quad t\in\mathbb{R}.\qquad (9)$$

Below we shall apply the following property of the convolution operation.

Lemma 1 (see [18])**.** *Let φ and ψ be arbitrary functions in $L_1(\mathbb{R})\cap L_\infty(\mathbb{R})$. Then the convolution $g=\varphi*\psi$ of these functions*

$$g(t)=(\varphi*\psi)(t)=\int_{-\infty}^{+\infty}\varphi(t-s)\psi(s)\,ds,\quad t\in\mathbb{R},\qquad (10)$$

satisfies the limit relations $g(t)\to 0$ as $t\to\pm\infty$.

§ 3. A nonlinear analogue of Theorem 1. Auxiliary lemmas

We consider the nonlinear integral equation of Hammerstein type

$$y(t)=\lambda(t)\int_{-\infty}^{+\infty}\mathring{K}(t-s)G(y(s))\,ds,\quad t\in\mathbb{R},\qquad (11)$$

with respect to the unknown measurable function $y(t)$. Here, the functions λ and $\mathring{K}$ satisfy the hypotheses of Theorem 1, while G is a real measurable function defined on $(-\infty,+\infty)$ for which there exists $\eta>0$ such that the following hold:

$$G(\tau)\geq\tau,\ \tau\in[0,\eta],\ G(\eta)=\eta,\qquad (12)$$

$$G\in C[0,\eta],\ G\uparrow\ \text{with respect to } \tau \text{ on } [0,\eta].\qquad (13)$$

The following lemma is a nonlinear analogue of Theorem 1.

Lemma 2. *Suppose that conditions* (4), (5), (12) *and* (13) *hold.*

a) If $1-\lambda\in L_1^0(\mathbb{R}^+)$ and $\nu(\mathring{K})<0$, then equation (11) *has a nonnegative nonzero bounded solution $y\leq\eta$ such that*

$$\int_0^t(\eta-y(s))\,ds=o(t),\quad t\to+\infty.$$

b) If $1-\lambda\in L_1^0(\mathbb{R}^-)$ and $\nu(\mathring{K})>0$, then equation (11) *has a nonnegative nonzero bounded solution $y\leq\eta$ such that*

$$\int_0^t(\eta-y(s))\,ds=o(t),\quad t\to-\infty.$$

Proof. We introduce successive approximations for equation (11):

$$y_{n+1}(t) = \lambda(t) \int_{-\infty}^{+\infty} \overset{\circ}{K}(t-s) G(y_n(s))\, ds, \quad y_0 = \eta, \quad n = 0, 1, 2, \dots, \quad t \in \mathbb{R}. \tag{14}$$

By induction on n it is easy to verify that

$$y_n(t) \downarrow \text{ with respect to } n, \quad y_n(t) \geq B_\eta(t) \equiv \eta B(t), \quad n = 0, 1, 2, \dots, \tag{15}$$

where $B(t)$ is a solution of equation (3) that has property (6) (or (7)). Consequently, the sequence of functions $\{y_n(t)\}_{n=0}^{\infty}$ has a pointwise limit as $n \to +\infty$. We denote this limit by $y(t)$. It follows from relations (15) that

$$B_\eta(t) \leq y(t) \leq \eta, \quad t \in \mathbb{R}. \tag{16}$$

Applying Levi's theorem (see [10]) we find that $y(t)$ is a solution of equation (11). Taking relations (6) (or (7)) we complete the proof of the lemma. □

Remark. If $\lambda \uparrow$ with respect to t on $\mathbb{R}$, then $y(t) \uparrow$ with respect to t on $\mathbb{R}$.

Indeed, writing the iterations (14) in the form

$$y_{n+1}(t) = \lambda(t) \int_{-\infty}^{+\infty} \overset{\circ}{K}(u) G(y_n(t-u))\, du, \quad y_0 = \eta, \ n = 0, 1, 2, \dots, \tag{17}$$

we can prove by induction on n that

$$y_n(t) \uparrow \text{ with respect to } t, \ n = 0, 1, 2, \dots .$$

Consequently, $\lim_{n\to\infty} y_n(t) = y(t) \uparrow$ with respect to t on $\mathbb{R}$.

We now give several examples of the function G:

1) $G(u) = e^{u-1}$, $\eta = 1$;
2) $G(u) = u^q$, $q \in (0,1)$, $\eta = 1$;
3) $G(u) = u + \sin^2 u$, $\eta = \pi k$, $k \in \mathbb{N}$;
4) $G(u) = \sqrt{ue^{u-1}}$, $\eta = 1$.

The following lemma also holds.

Lemma 3. *Suppose that $Q(\tau)$ is a measurable function defined on $\mathbb{R}$ for which there exists a number $\xi > 0$ such that*

i_1) *$Q \uparrow$ on $[0, \xi]$,*

i_2) *$Q(0) = 0$, $Q(\xi) = \xi$,*

i_3) *Q satisfies a Lipschitz condition on the closed interval $[0, \xi]$; that is, there exists a positive number L such that the inequality $|Q(\tau_1) - Q(\tau_2)| \leq L|\tau_1 - \tau_2|$ holds for any $\tau_1, \tau_2 \in [0, \xi]$.*

Then the function Q generates a one-parameter family of functions $\{\widetilde{Q}_\alpha\}_{\alpha \in I}$ with the following properties:

j_1) *$\widetilde{Q}_\alpha \in C[0, \xi]$ and $\widetilde{Q}_\alpha \uparrow$ on $[0, \xi]$ for all $\alpha \in I \equiv \left(0, \min\left(1, \frac{1}{L}\right)\right)$;*

j_2) *$\widetilde{Q}_\alpha(0) > 0$ and $\widetilde{Q}_\alpha(\xi) = \xi$ for all $\alpha \in I$;*

j_3) *for every $\alpha \in I$ there exists a positive number θ (which is unique) such that $\widetilde{Q}_\alpha(\theta) = 2\theta$, $2\theta < \xi$;*

j_4) *$\widetilde{Q}_\alpha$ is a contracting map on the closed interval $[0, \xi]$ for every $\alpha \in I$.*

Proof. Consider the family of functions

$$\widetilde{Q}_\alpha(\tau) = \xi - \alpha Q(\xi - \tau), \quad \alpha \in I, \ \tau \in [0, \xi]. \tag{18}$$

We will verify that every function $\widetilde{Q}_\alpha$ has properties j_1)–j_4). Properties j_1) and j_2) follow directly from (18). We will prove j_3) and j_4). To do this we first verify that there exists $\beta \in (0,1)$ such that

$$\big|\widetilde{Q}_\alpha(\tau_1) - \widetilde{Q}_\alpha(\tau_2)\big| \le \beta|\tau_1 - \tau_2| \tag{19}$$

for all $\tau_1, \tau_2 \in [0,\xi]$. In view of condition i_3) we have

$$\big|\widetilde{Q}_\alpha(\tau_1) - \widetilde{Q}_\alpha(\tau_2)\big| = \alpha|Q(\xi-\tau_2) - Q(\xi-\tau_1)| \le \alpha L|\tau_1-\tau_2| = \beta|\tau_1-\tau_2|,$$

where $\beta = \alpha L < 1$, since $\alpha \in I$. Hence $\widetilde{Q}_\alpha$ is a contracting map on the closed interval $[0,\xi]$ for every $\alpha \in I$, and so we have proved j_4).

Consider the function

$$\Psi_\alpha(\tau) = \widetilde{Q}_\alpha(\tau) - 2\tau, \quad \tau \in [0,\xi],\ \alpha \in I. \tag{20}$$

Obviously, $\Psi_\alpha \in C[0,\xi]$, $\Psi_\alpha(0) = \widetilde{Q}_\alpha(0) > 0$, and $\Psi_\alpha(\xi) = -\xi < 0$. Consequently, there exists a number $0 < \theta < \xi$ such that

$$\Psi_\alpha(\theta) = 0. \tag{21}$$

We will now prove that the solution to $\Psi_\alpha(\tau) = 0$ on the closed interval $[0,\xi]$ is unique. Suppose the opposite: there exist numbers $\theta_1, \theta_2 \in [0,\xi]$, $\theta_1 \ne \theta_2$, such that $\Psi_\alpha(\theta_j) = 0$, $j = 1,2$. We assume without loss of generality that $\theta_1 > \theta_2$. Then by (19) and (20) we obtain

$$\begin{aligned}\Psi_\alpha(\theta_1) - \Psi_\alpha(\theta_2) &= \widetilde{Q}_\alpha(\theta_1) - \widetilde{Q}_\alpha(\theta_2) - 2(\theta_1-\theta_2)\\ &\le \beta(\theta_1-\theta_2) - 2(\theta_1-\theta_2) = (\beta-2)(\theta_1-\theta_2) < 0,\end{aligned}$$

because $\theta_1 > \theta_2$, $\beta < 1$. Hence, $\Psi_\alpha(\theta_1) < \Psi_\alpha(\theta_2)$. This contradiction proves property j_3).

Thus, the lemma is proved. □

We consider the integral equation of Hammerstein type

$$f(t) = \lambda(t)\int_{-\infty}^{+\infty} \overset{\circ}{K}(t-s)\widetilde{Q}_\alpha(f(s))\,ds, \quad t \in \mathbb{R},\ \alpha \in I, \tag{22}$$

with respect to the measurable function $f(t)$. The following lemma holds.

Lemma 4. *Suppose that all the hypotheses of Lemma 3 hold. Suppose that*

$$\frac{1}{2} \le \lambda(\tau) \le 1,\ \tau \in \mathbb{R}, \quad \overset{\circ}{K}(u) \ge 0, \quad \int_{-\infty}^{+\infty} \overset{\circ}{K}(u)du = 1,\ u \in \mathbb{R}. \tag{23}$$

Then equation (22) *has a positive bounded solution $f(t)$. Moreover, if $1-\lambda \in L_1^0(\mathbb{R})$, then $\xi - f \in L_1^0(\mathbb{R})$.*

Proof. Consider the following successive approximations:

$$\begin{gathered} f_{n+1}(t) = \lambda(t)\int_{-\infty}^{+\infty} \overset{\circ}{K}(t-s)\widetilde{Q}_\alpha(f_n(s))\,ds,\\ f_0(t) \equiv \xi, \quad t \in \mathbb{R},\ \alpha \in I,\ n = 0,1,2,\dots. \end{gathered} \tag{24}$$

Taking (23) into account and applying Lemma 3, it is easy to verify by induction on n that the following facts are true:

a) $f_n(t) \downarrow$ with respect to $n, t \in \mathbb{R}$, $n = 0,1,2,\dots$.

b) $f_n(t) \ge \theta$, $t \in \mathbb{R}$, $n = 0,1,2,\dots$.

Consequently, the sequence of functions $\{f_n(t)\}_{n=0}^{\infty}$ has a pointwise limit as $n \to \infty$: $\lim_{n\to\infty} f_n(t) = f(t) \le \xi$, $t \in \mathbb{R}$, and this limit satisfies equation (22) by Levi's theorem.

Now suppose that $1-\lambda \in L_1^0(\mathbb{R})$. We will prove that $\xi - f \in L_1^0(\mathbb{R})$. To do this, we consider the nonlinear integral equation

$$\rho(t) = \xi(1-\lambda(t)) + \lambda(t) \int_{-\infty}^{+\infty} \overset{\circ}{K}(t-s)\big(\xi - \widetilde{Q}_\alpha(\xi - \rho(s))\big)\, ds, \quad t \in \mathbb{R}, \tag{25}$$

with respect to the function $\rho(t)$. We introduce the following iterations:

$$\begin{gathered} \rho_{n+1}(t) = \xi(1-\lambda(t)) + \lambda(t) \int_{-\infty}^{+\infty} \overset{\circ}{K}(t-s)\big(\xi - \widetilde{Q}_\alpha(\xi - \rho_n(s))\big)\, ds, \\ \rho_0 \equiv 0, \quad n = 0,1,2,\dots, \quad t \in \mathbb{R}. \end{gathered} \tag{26}$$

Using induction on n it is easy to verify that

$$\rho_n(t) \uparrow \text{ with respect to } n, \quad t \in \mathbb{R}, \tag{27}$$

$$\rho_n \in L_1(\mathbb{R}), \quad n = 0,1,2\dots. \tag{28}$$

First we verify that the following inequalities hold:

$$\int_{-\infty}^{+\infty} \rho_n(t)\, dt \le \xi(1-\beta)^{-1} \|1-\lambda\|_{L_1(\mathbb{R})}, \quad n = 0,1,2\dots. \tag{29}$$

For $n = 0$ inequality (29) is obvious. Assuming that (29) holds for some $n \in \mathbb{N}$ and taking conditions (23), the fact that $1-\lambda \in L_1^0(\mathbb{R})$ and Lemma 3 into account, from (26) and (28) we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} \rho_{n+1}(t)\, dt &\le \xi\|1-\lambda\|_{L_1(\mathbb{R})} + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overset{\circ}{K}(t-s)\big(\widetilde{Q}_\alpha(\xi) - \widetilde{Q}_\alpha(\xi-\rho_n(s))\big)\, ds\, dt \\ &\le \xi\|1-\lambda\|_{L_1(\mathbb{R})} + \beta \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overset{\circ}{K}(t-s)\rho_n(s)\, ds\, dt \\ &\le \xi\|1-\lambda\|_{L_1(\mathbb{R})} + \frac{\xi\beta}{1-\beta}\|1-\lambda\|_{L_1(\mathbb{R})} = \xi(1-\beta)^{-1}\|1-\lambda\|_{L_1(\mathbb{R})}. \end{aligned}$$

Thus, it follows from Levi's theorem that the sequence of functions $\{\rho_n(t)\}_{n=0}^{\infty}$ has a limit

$$\lim_{n\to\infty} \rho_n(t) = \rho(t) \in L_1(\mathbb{R});$$

furthermore,

$$\rho(t) \ge 0, \quad \int_{-\infty}^{+\infty} \rho(t)\, dt \le \xi(1-\beta)^{-1}\|1-\lambda\|_{L_1(\mathbb{R})} \tag{30}$$

and $\rho(t)$ satisfies equation (25). However, it follows from (25) and (26) that $\rho(t) \le \xi$, and so $\rho \in L_1(\mathbb{R}) \cap \Omega_\xi$, where $\Omega_\xi \equiv \{\varphi \in L_\infty(\mathbb{R});\ 0 \le \varphi(t) \le \xi,\ t \in \mathbb{R}\}$. Since $\beta < 1$, it is easy to prove that the solution of equation (25) is unique in the class of functions Ω_ξ.

On the other hand, by a straightforward substitution one can verify that the function $\rho(t) \equiv \xi - f(t) \in \Omega_\xi$ is a solution of equation (25). Thus, the unique solution of equation (25) is $\rho(t) = \xi - f(t) \in L_1(\mathbb{R}) \cap \Omega_\xi$. From (25), in view of Lemma 3 we have

$$0 \le \xi - f(t) \le \xi(1 - \lambda(t)) + \beta \int_{-\infty}^{+\infty} \overset{\circ}{K}(t-s)(\xi - f(s))\, ds. \tag{31}$$

Since $1 - \lambda \in L_1^0(\mathbb{R})$, $\overset{\circ}{K} \in L_1(\mathbb{R})$ and $\xi - f \in L_1(\mathbb{R})$, taking (31) into account we find from Lemma 1 that $\xi - f \in L_1^0(\mathbb{R}) \cap \Omega_\xi$. The lemma is proved. □

The following lemma is proved in a similar fashion.

Lemma 5. *Suppose that all the hypotheses of Lemma* 3 *hold and condition* (23) *holds. If* $2\lambda - 1 \in L_1^0(\mathbb{R})$, *then a bounded solution of equation* (22) *has the following additional property*: $f - \theta \in L_1^0(\mathbb{R})$.

To end this section we give several examples of the functions Q and λ.

1_Q) $Q(\tau) = \tau^p$, $p > 0$,

2_Q) $Q(\tau) = \tau + \sin\tau$.

An example of the function λ for Lemma 4: $\lambda(t) = 1 - \varepsilon e^{-|t|}$, $\varepsilon \in \left(0, \frac{1}{2}\right]$.

An example of the function λ for Lemma 5: $\lambda(t) = \frac{1+\delta e^{-t^2}}{2}$, $\delta \in (0, 1]$.

§ 4. Solvability of equation (1) in the case when a linear function is a local majorant for the function $\mu_0(s,z)$

In this section we prove the following theorem on solvability of equation (1) in the case when a linear function is a majorant for the function $\mu_0(s,z)$ on some closed interval $[0,\eta]$.

Theorem 2. *Suppose that conditions* (4) *and* (5) *hold, and the kernel* $K(t,s)$ *satisfies the following relation*:

$$0 \le K(t,s) \le \lambda(t)\overset{\circ}{K}(t-s) \quad \forall (t,s) \in \mathbb{R} \times \mathbb{R}. \tag{32}$$

Suppose that the functions $\mu_j(t,z)$ *are defined on the set* $\mathbb{R} \times \mathbb{R}$ *and satisfy the following conditions*: *there exist positive numbers* $\eta > 0$ *and* $\eta_0 \in (0,\eta)$ *such that*

γ_1) $\mu_j(t,z) \uparrow$ *with respect to* z *on the closed interval* $[0,\eta]$ *for every fixed* $t \in \mathbb{R}$, $j = 0, 1$;

γ_2) *the functions* $\mu_j(t,z)$ $(j = 0,1)$ *satisfy the Carathéodory condition on the set* $\mathbb{R} \times [0,\eta]$ *with respect to the argument* z; *that is, for every fixed* $z \in [0,\eta]$ *the functions* $\mu_j(t,z)$ *are measurable with respect to* t *and are continuous in* z *on the closed interval* $[0,\eta]$ *for almost all* $t \in \mathbb{R}$;

γ_3) *the following inequalities hold*:

$$0 \le \mu_0(t,z) \le z \quad \forall (t,z) \in \mathbb{R} \times [0,\eta], \tag{33}$$

$$\mu_1(t, \varphi_{\eta_0}(t)) \ge \varphi_{\eta_0}(t), \quad \mu_1(t,\eta) \le \varphi_\eta(t) \quad \forall t \in \mathbb{R}, \tag{34}$$

where

$$\varphi_\delta(t) = \delta(1 - \lambda(t)), \quad \delta > 0, \quad t \in \mathbb{R}. \tag{35}$$

Then

a) *if* $1 - \lambda \in L_1^0(\mathbb{R}^+)$ *and* $\nu(\overset{\circ}{K}) < 0$, *then equation* (1) *has a nonnegative nonzero bounded solution* $x(t)$ *such that*

$$\int_0^t x(s)\, ds = o(t), \quad t \to +\infty;$$

b) if $1-\lambda \in L_1^0(\mathbb{R}^-)$ *and* $\nu(\overset{\circ}{K}) > 0$, *then equation* (1) *has a nonnegative nonzero bounded solution* $x(t)$ *such that*

$$\int_0^t x(s)\,ds = o(t), \quad t \to -\infty.$$

Proof. We introduce the following iterations for equation (1):

$$x_{n+1}(t) = \int_{-\infty}^{+\infty} \overset{\circ}{K}(t,s)\mu_0(s, x_n(s))\,ds + \mu_1(t, x_n(t)), \quad t \in \mathbb{R}, \tag{36}$$
$$x_0(t) = \varphi_{\eta_0}(t), \quad n = 0, 1, 2, \dots .$$

Using induction on n we can verify that

$$x_n(t) \uparrow \text{ with respect to } n, \quad t \in \mathbb{R}. \tag{37}$$

In fact (37) follows in a straightforward way from the obvious inequalities

$$0 \le \varphi_{\eta_0}(t) \le \varphi_\eta(t) \le \eta, \quad t \in \mathbb{R}. \tag{38}$$

Below we use induction on n to verify that the following inequalities are true:

$$x_n(t) \le \eta(1 - B(t)), \quad n = 0, 1, 2, \dots, \quad t \in \mathbb{R}, \tag{39}$$

where $B(t) \le 1$ is a solution of equation (3) that is bounded and nonnegative and has integral asymptotics of the form (6) (or (7)).

When $n = 0$ inequality (39) follows from (9):

$$x_0(t) = \varphi_{\eta_0}(t) \le \eta(1 - \lambda(t)) \le \eta(1 - B(t)).$$

Suppose that (39) holds for some $n \in \mathbb{N}$. Then, taking (32), (33), (34) and condition γ_1) into account, from (36) we obtain

$$\begin{aligned} x_{n+1}(t) &\le \lambda(t) \int_{-\infty}^{+\infty} \overset{\circ}{K}(t-s)\mu_0\big(s, \eta(1 - B(s))\big)\,ds + \mu_1\big(t, \eta(1 - B(t))\big) \\ &\le \lambda(t) \int_{-\infty}^{+\infty} \overset{\circ}{K}(t-s)(\eta - B_\eta(s))\,ds + \mu_1(t, \eta) \\ &\le \eta\lambda(t) - B_\eta(t) + \varphi_\eta(t) = \eta(1 - B(t)). \end{aligned}$$

Thus, it follows from (37) and (39) that the sequence of functions $\{x_n(t)\}_{n=0}^\infty$ has a pointwise limit as $n \to \infty$: $\lim_{n\to\infty} x_n(t) = x(t)$. Since condition γ_2) holds, by Levi's theorem this limit satisfies equation (1), and

$$\varphi_{\eta_0}(t) \le x(t) \le \eta(1 - B(t)), \quad t \in \mathbb{R}. \tag{40}$$

This completes the proof, in view of Theorem 1. □

In the next section we deal with the construction of a nonnegative nonzero solution of equation (1) in the case when $\alpha Q(z)$ is a local majorant of the functions $\mu_0(s, z)$.

§ 5. Solvability of equation (1) in the case when a nonlinear function of the form $\alpha Q(z)$ is a local majorant for the function $\mu_0(s,z)$

The following theorem holds.

Theorem 3. *Suppose that conditions* (23) *and* (32) *hold and that* $\lambda(t)$ *is such that* $1-\lambda\in L_1^0(\mathbb{R})$. *Suppose that the functions* $\mu_j(t,z)$ *are defined on the set* $\mathbb{R}\times\mathbb{R}$ *and satisfy the following conditions*:

$\varkappa_1$) $\mu_j(t,z)\uparrow$ *with respect to* z *on the closed interval* $[0,\xi]$ *for every fixed* $t\in\mathbb{R}$, $j=0,1$;

$\varkappa_2$) *the functions* $\mu_j(t,z)$ $(j=0,1)$ *satisfy the Carathéodory condition on the set* $\mathbb{R}\times[0,\xi]$ *with respect to the argument* z;

$\varkappa_3$) *there exists a number* $\alpha\in I$ *such that*

$$0\le\mu_0(t,z)\le\alpha Q(z)\quad\forall(t,z)\in\mathbb{R}\times[0,\xi];$$

$\varkappa_4$) *there exists a positive number* $\xi_0\in(0,\xi)$ (*see the definition of the number* $\xi>0$ *in Lemma* 3) *such that* $\mu_1(t,\varphi_{\xi_0}(t))\ge\varphi_{\xi_0}(t)$, $\mu_1(t,\xi)\le\varphi_\xi(t)$, $t\in\mathbb{R}$.

Then equation (1) *has a nonnegative nonzero solution in the space* $L_1^0(\mathbb{R})\cap L_\infty(\mathbb{R})$.

Proof. Consider the following iterations:

$$x_{n+1}(t)=\int_{-\infty}^{+\infty}K(t,s)\mu_0(s,x_n(s))\,ds+\mu_1(t,x_n(t)),\quad t\in\mathbb{R},\qquad(41)$$
$$x_0(t)=\xi-f(t),\quad n=0,1,2,\dots,$$

where $f(t)$ is a solution of equation (22) constructed using the successive approximations (24).

We use induction on n to prove that

$$x_n(t)\downarrow\text{ with respect to } n,\quad t\in\mathbb{R}.\qquad(42)$$

By conditions $\varkappa_1$), $\varkappa_3$), $\varkappa_4$), and (32), from (41) we obtain

$$\begin{aligned}x_1(t)&\le\lambda(t)\int_{-\infty}^{+\infty}\mathring{K}(t-s)\mu_0(s,\xi-f(s))\,ds+\mu_1(t,\xi-f(t))\\&\le\alpha\lambda(t)\int_{-\infty}^{+\infty}\mathring{K}(t-s)Q(\xi-f(s))\,ds+\mu_1(t,\xi)\\&\le\lambda(t)\int_{-\infty}^{+\infty}\mathring{K}(t-s)\big(\xi-\widetilde{Q}_\alpha(f(s))\big)\,ds+\varphi_\xi(t)\\&=\xi\lambda(t)-f(t)+\varphi_\xi(t)=\xi-f(t).\end{aligned}$$

Now, assuming that $x_n(t)\le x_{n-1}(t)$ for some $n\in\mathbb{N}$, from (41) in view of condition $\varkappa_1$) we obtain $x_{n+1}(t)\le x_n(t)$. Thus, we have proved (42).

We will now prove that

$$x_n(t)\ge\varphi_{\xi_0}(t),\quad n=0,1,2,\dots,\quad t\in\mathbb{R}.\qquad(43)$$

First we verify that (43) is true when $n=0$. In fact, since the function $\rho(t)=\xi-f(t)$ is a unique nonnegative solution of equation (25) and $\xi_0\in(0,\xi)$, we obtain

$$\xi-f(t)\ge\varphi_\xi(t)\ge\varphi_{\xi_0}(t),\quad t\in\mathbb{R}.$$

Assuming that (43) holds for some $n \in \mathbb{N}$ and taking condition $\varkappa_4$) into account, from (41) we obtain

$$x_{n+1}(t) \geq \mu_1(t, \varphi_{\xi_0}(t)) \geq \varphi_{\xi_0}(t).$$

Consequently, the sequence of functions $\{x_n(t)\}_{n=0}^{\infty}$ has a limit as $n \to \infty$: $\lim_{n\to\infty} x_n(t) = x(t)$, and this limit satisfies equation (1) and the following chain of inequalities:

$$\varphi_{\xi_0}(t) \leq x(t) \leq \xi - f(t), \quad t \in \mathbb{R}. \tag{44}$$

By Lemma 4, from (44) we conclude that $x \in L_1^0(\mathbb{R}) \cap L_\infty(\mathbb{R})$. The theorem is proved. □

The following theorem is proved in a similar way, using Lemma 5.

Theorem 4. *Suppose that conditions* (23) *and* (32) *hold, and the function* $\lambda(t)$ *has the property* $2\lambda - 1 \in L_1^0(\mathbb{R})$. *Suppose that the functions* $\mu_j(t, z)$ *are defined on the set* $\mathbb{R} \times \mathbb{R}$ *and satisfy the following conditions*:

χ_1) $\mu_j(t, z) \uparrow$ *with respect to* z *on the closed interval* $[0, \xi - \theta]$ *for every fixed* $t \in \mathbb{R}$, $j = 0, 1$;

χ_2) *the functions* $\mu_j(t, z)$ $(j = 0, 1)$ *satisfy the Carathéodory condition on the set* $\mathbb{R} \times [0, \xi - \theta]$ *with respect to the argument* z;

χ_3) *there exists a number* $\alpha \in I$ *such that*

$$0 \leq \mu_0(t, z) \leq \widetilde{Q}_\alpha(z + \theta) - 2\theta, \quad t \in \mathbb{R},\ z \in [0, \xi - \theta];$$

χ_4) *there exists a positive number* $\varepsilon \in (0, \theta)$ (*see the definition of the numbers* ξ *and* θ *in Lemma* 3) *such that*

$$\mu_1(t, v_\varepsilon(t)) \geq v_\varepsilon(t), \quad \mu_1(t, \xi - \theta) \leq v_\theta(t), \quad t \in \mathbb{R},$$

where

$$v_\delta(t) \equiv \delta(2\lambda(t) - 1), \quad \delta > 0,\ t \in \mathbb{R}.$$

Then equation (1) *has a nonnegative nonzero solution in the space* $L_1^0(\mathbb{R}) \cap L_\infty(\mathbb{R})$.

To end the paper we present several examples of the functions $\mu_j(t, z)$, $j = 0, 1$.

1. Examples of the functions $\mu_j(t, z)$ for Theorem 2:

$$\mu_0(t, z) = q(t)\frac{z^p}{\eta^{p-1}},$$
$$p > 1, \quad q \in C(\mathbb{R}), \quad 0 \leq q(t) \leq 1, \quad t \in \mathbb{R}, \quad z \in [0, \eta];$$
$$\mu_1(t, z) = \varphi_{\eta_0+\eta_1}(t)\frac{z}{z + \varphi_{\eta_1}(t)},$$
$$\eta_0, \eta_1 > 0, \quad \eta \geq \eta_0 + \eta_1, \quad t \in \mathbb{R}, \quad z \in [0, \eta].$$

2. Examples of the functions $\mu_j(t, z)$ for Theorem 3:

$$\mu_0(t, z) = \Phi(t)\frac{Q^p(z)}{\xi^{p-1}},$$
$$p > 1, \quad \Phi \in C(\mathbb{R}), \quad 0 \leq \Phi(t) \leq \alpha, \quad \alpha \in I, \quad t \in \mathbb{R}, \quad z \in [0, \xi];$$
$$\mu_1(t, z) = \varphi_{\xi_0+\xi_1}(t)\frac{\beta z^2}{(z + \varphi_{\xi_1}(t))^2},$$
$$\beta \geq 1 + \frac{\xi_1}{\xi_0}, \quad \xi \geq \beta(\xi_0 + \xi_1), \quad \xi_0, \xi_1 > 0, \quad t \in \mathbb{R}, \quad z \in [0, \xi].$$

3. Examples of the functions $\mu_j(t,z)$ for Theorem 4:

$$\mu_0(t,z)=\omega(t)\sin\left[\frac{\pi}{2(\xi-\theta)}(\widetilde{Q}_\alpha(z+\theta)-2\theta)\right],$$
$$\omega\in C(\mathbb{R}),\quad 0\le\omega(t)\le\frac{2(\xi-\theta)}{\pi},\quad t\in\mathbb{R};$$
$$\mu_1(t,z)=v_{\varepsilon+\varepsilon_0}(t)\frac{z}{z+v_{\varepsilon_0}(t)},$$
$$\varepsilon,\varepsilon_0>0,\quad \theta\ge\varepsilon+\varepsilon_0,\quad t\in\mathbb{R},\quad z\in[0,\xi-\theta].$$

Remark. Note that in all the examples above, the functions $\mu_j(t,z)$ have the property

$$\mu_j(t,0)\equiv 0,\quad t\in\mathbb{R},\ j=0,1.$$

It follows from Theorems 2–4 proved above that in all the examples given above, in addition to the trivial solution, equation (1) also has a nonzero nonnegative solution.

Note also that in Theorems 2–4 the conditions imposed on the function $\mu_0(t,z)$ immediately imply that

$$\mu_0(t,0)=0,\quad t\in\mathbb{R}. \tag{45}$$

It is possible to give examples of the functions $\mu_1(t,z)$ for Theorems 2–4 such that $\mu_1(t,0)\neq 0$:

$$\mu_1(t,z)=\varphi_{\eta_0}(t)+C_0(t,z)\quad \text{(for Theorem 2)}, \tag{46}$$
$$\mu_1(t,z)=\varphi_{\eta_0}(t)+C_1(t,z)\quad \text{(for Theorem 3)}, \tag{47}$$
$$\mu_1(t,z)=v_{\varepsilon}(t)+C_2(t,z)\quad \text{(for Theorem 4)}, \tag{48}$$

where

A) the functions C_0, C_1, and C_2 satisfy the Carathéodory condition with respect to the argument z on the sets $\mathbb{R}\times[0,\eta]$, $\mathbb{R}\times[0,\xi]$, and $\mathbb{R}\times[0,\xi-\theta]$, respectively;

B) $C_0\uparrow$ with respect to z on the closed interval $[0,\eta]$ for every fixed $t\in\mathbb{R}$,
$C_1\uparrow$ with respect to z on the closed interval $[0,\xi]$ for every fixed $t\in\mathbb{R}$,
$C_2\uparrow$ with respect to z on the closed interval $[0,\xi-\theta]$ for every fixed $t\in\mathbb{R}$;

C) $C_0(t,z)\ge 0$, $C_0(t,\eta)\le\varphi_\eta(t)-\varphi_{\eta_0}(t)$, $t\in\mathbb{R}$, $z\in[0,\eta]$,
$C_1(t,z)\ge 0$, $C_1(t,\xi)\le\varphi_\xi(t)-\varphi_{\xi_0}(t)$, $t\in\mathbb{R}$, $z\in[0,\xi]$,
$C_2(t,z)\ge 0$, $C_2(t,\xi-\theta)\le v_\theta(t)-v_\varepsilon(t)$, $t\in\mathbb{R}$, $z\in[0,\xi-\theta]$.

The following functions can serve as the functions $C_j(t,z)$ $(j=0,1,2)$:

$$C_0(t,z)=\frac{(\varphi_\eta(t)-\varphi_{\eta_0}(t))z^p}{\eta^p},$$
$$C_1(t,z)=\frac{(\varphi_\xi(t)-\varphi_{\xi_0}(t))z^p}{\xi^p},$$
$$C_2(t,z)=\frac{(v_\theta(t)-v_\varepsilon(t))z^p}{(\xi-\theta)^p},\quad p\in\mathbb{N}.$$

The question of the uniqueness of the solutions constructed here in the corresponding spaces will be considered in a separate paper.

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