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RIESZ BASIS PROPERTY OF HILL OPERATORS WITH POTENTIALS IN WEIGHTED SPACES

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Dedicated to the memory of Boris Moiseevich Levitan on the occasion of the 100th anniversary of his birthday

Abstract. Consider the Hill operator $L(v) = -d^2/dx^2 + v(x)$ on $[0, \pi]$ with Dirichlet, periodic or antiperiodic boundary conditions; then for large enough n close to n^2 there are one Dirichlet eigenvalue μ_n and two periodic (if n is even) or antiperiodic (if n is odd) eigenvalues λ_n^-, λ_n^+ (counted with multiplicity).

We describe classes of complex potentials $v(x) = \sum_{2\mathbb{Z}} V(k)e^{ikx}$ in weighted spaces (defined in terms of the Fourier coefficients of v) such that the periodic (or antiperiodic) root function system of $L(v)$ contains a Riesz basis if and only if

$$V(-2n) \asymp V(2n) \quad \text{as } n \in 2\mathbb{N} \text{ (or } n \in 1 + 2\mathbb{N}), \quad n \to \infty.$$

For such potentials we prove that $\lambda_n^+ - \lambda_n^- \sim \pm 2\sqrt{V(-2n)V(2n)}$ and

$$\mu_n - \frac{1}{2}(\lambda_n^+ + \lambda_n^-) \sim -\frac{1}{2}(V(-2n) + V(2n)).$$

§ 1. Introduction

The theory of self-adjoint ordinary differential operators (o.d.o.) is well-developed, and the spectral decompositions play a central role in it [24, 29, 26].

Convergence of the spectral decompositions of non-self-adjoint o.d.o., considered on a finite interval I and subject to strictly regular boundary conditions (see [29, § 4.8]), has been understood completely in the early 1960's [27, 23, 17]. In this case, we not only have convergence, but the system of eigenfunctions (SEF) is a Riesz basis in $L^2(I)$. However, in the case of regular but not strictly regular boundary conditions—even in the case of periodic or antiperiodic boundary conditions—complete understanding appeared only in the 2000s as a result of the interaction of two lines of research.

One stems from a question raised by A. Shkalikov in 1996/1997 in the Kostyuchenko–Shkalikov seminar on Spectral Analysis at Moscow State University. Shkalikov formulated the following assertion and sketched an approach to its proof.

Consider the Hill operator

$$Ly = -y'' + q(x)y, \quad 0 \le x \le \pi,$$

with a smooth potential q such that for some $s \ge 0$

$$q^{(k)}(0) = q^{(k)}(\pi), \quad 0 \le k \le s - 1,$$

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and

$$q^{(s)}(0) - q^{(s)}(\pi) \neq 0.$$

Then the system of normalized periodic (or antiperiodic) root functions of the operator $L = L(q)$ is a Riesz basis in $L^2([0,\pi])$.

In the framework of the scheme suggested by Shkalikov, this claim was proved in the case $q \in C^4([0,\pi])$, $s = 0$, by Kerimov and Mamedov [22]. Further results of Dernek–Veliev [3], Makin [25] and Veliev-Shkalikov [32] confirmed the general case $s \geq 0$. Moreover, Makin [25] considered potentials $q(x) = \sum_{k\in 2\mathbb{Z}} q_k e^{ikx}$ such that

$$q \in W_1^s([0,\pi]), \quad q^{(p)}(0) = q^{(p)}(\pi), \quad 0 \leq p \leq s-1, \tag{1.1}$$

and

$$\exists c > 0\colon |q_{\pm 2n}| > cn^{-s-1} \quad \forall n \gg 1, \tag{1.2}$$

and proved that *the periodic (antiperiodic) SEF is a Riesz basis if and only if there are constants $C > c > 0$ such that*

$$cq_{-2n} \leq q_{2n} \leq Cq_{-2n} \quad \forall \text{even (odd) } n \gg 1. \tag{1.3}$$

He used this result to construct examples of potentials for which the periodic SEF is not a Riesz basis. Veliev and Shkalikov [32] extended the results of Makin by providing more general conditions for the existence of Riesz bases.

Notice, however, that the above results were obtained for potentials of finite smoothness, i. e., in the framework of Sobolev spaces W_1^m, where m is a positive integer.

Another line of research comes from the papers [20, 21, 4, 5, 6]. The goal of the papers was the analysis of spectral gaps $\gamma_n = \lambda_n^+ - \lambda_n^-$ and deviations $\delta_n = \mu_n - \frac{1}{2}(\lambda_n^+ + \lambda_n^-)$, but the analytical methods developed in these papers allowed us to understand the structures responsible for the Riesz basis property of SEF in the case of periodic or antiperiodic boundary conditions. Already in [28] it had been announced that the authors had constructed examples of 1D Dirac operators such that their periodic or antiperiodic SEF is not a Riesz basis. With all details these constructions were presented, both for Hill and 1D Dirac operators in [7, Section 5.2], in particular Theorem 71.

Recently, the same approach has led to general necessary and sufficient conditions for the existence of Riesz bases consisting of periodic (or antiperiodic) root functions [11, 12, 13].

This note gives a further development of those results in the framework of the approach in [7, Section 5.2]. (See Theorems 4.2 and 4.3 below.) We work in weighted spaces of potentials, which allows us to consider potentials of arbitrary smoothness (including singular potentials or potentials of smoothness beyond C^∞, say in the Carlemann–Gevrey classes).

We describe classes of complex potentials $v(x) = \sum_{2\mathbb{Z}} V(k)e^{ikx}$ (in weighted spaces defined in terms of the Fourier coefficients $V(k)$ of v) such that the periodic or antiperiodic root function system of the Hill operator $L(v)$ contains a Riesz basis if and only if

$$V(-2n) \asymp V(2n) \quad \text{as } n \in 2\mathbb{N} \text{ (or } n \in 1 + 2\mathbb{N}), \quad n \to \infty.$$

For such potentials we prove that

$$\lambda_n^+ - \lambda_n^- \sim \pm 2\sqrt{V(-2n)V(2n)}$$

and

$$\mu_n - \frac{1}{2}(\lambda_n^+ + \lambda_n^-) \sim -\frac{1}{2}(V(-2n) + V(2n)).$$

Moreover, we give several examples (Section 5) to illustrate our main statements, where we overcome additional difficulties when verifying the general conditions. (For

asymptotics of crucial sequences $\beta_n^{\pm}$, see Propositions 5.2, 5.3 and other claims in Section 5.)

§ 2. Preliminaries

Let $L = L(v)$ be the Hill operator

$$Ly = -y'' + v(x)y, \tag{2.1}$$

with a complex valued potential $v \in L^2([0,\pi]$ or more generally, with a singular complex valued potential $v \in H^{-1}_{\mathrm{per}}(\mathbb{R})$. For potentials $v \in L^2([0,\pi]$ we consider $L(v)$ on the interval $[0,\pi])$ with Dirichlet (Dir), periodic (Per^+) and antiperiodic (Per^-) boundary conditions (bc):

$$\mathrm{Dir}: \quad y(0) = 0, \qquad y(\pi) = 0, \tag{2.2}$$

$$\mathrm{Per}^{\pm}: \quad y(\pi) = \pm y(0), \quad y'(\pi) = \pm y'(0). \tag{2.3}$$

Singular π-periodic potentials $v \in H^{-1}_{\mathrm{per}}(\mathbb{R})$ have the form $v = C + Q'$, where C is a constant and Q is a π-periodic function such that $Q \in L^2_{\mathrm{loc}}(\mathbb{R})$. Since adding constant results in a shift of the spectra, we may consider without loss of generality only π-periodic potentials of the form

$$v(x) = Q'(x), \quad Q \in L^2_{\mathrm{loc}}(\mathbb{R}), \quad Q(x+\pi) = Q(x). \tag{2.4}$$

Let us notice that if v is a π-periodic function with $v \in L^1_{\mathrm{loc}}(\mathbb{R})$, then it has the form (2.4) if, and only if,

$$\int_0^{\pi} v(t)\,dt = 0, \tag{2.5}$$

because the latter condition implies that

$$Q(x) = \int_0^x v(t)\,dt$$

is a π-periodic function.

In the case of potentials $v \in H^{-1}_{\mathrm{per}}(\mathbb{R})$ the classical periodic and antiperiodic boundary conditions (2.3) are replaced by

$$y(\pi) = \pm y(0), \quad y^{[1]}(\pi) = \pm y^{[1]}(0), \tag{2.6}$$

where

$$y^{[1]}(x) := y'(x) - Q(x)y(x)$$

is the *quasi-derivative* of y. We refer to [31, 19, 9, 10] for basics and details about Hill–Schrödingier operators with singular potentials of the form (2.4). The Fourier method for such operators is developed in [9]. We recall that the Fourier coefficients of v with respect to the orthonormal system $(e^{ikx})_{k\in 2\mathbb{Z}}$ are defined by

$$V(k) = ikq(k), \quad \text{where } q(k) = \frac{1}{\pi}\int_0^{\pi} Q(x)e^{ikx}\,dx, \quad k \in 2\mathbb{Z}. \tag{2.7}$$

It is known (see [26, 7] for L^2-potentials or [9, 10] for H^{-1}_{per}-potentials) that the following holds.

Lemma 2.1. *Let v be a potential of the form (2.4). Then the periodic, antiperiodic and Dirichlet spectra of the operator $L(v)$ are discrete. Moreover, there is an integer $N_* = N_*(v)$ such that for each $n > N_*$ the disc*

$$D_n = \{\lambda \in \mathbb{C}\colon |\lambda - n^2| < n/4\} \tag{2.8}$$

contains one simple Dirichlet eigenvalue and two periodic (if n is even) or antiperiodic (if n is odd) eigenvalues λ_n^-, λ_n^+ (counted with multiplicity). There are at most finitely

many periodic, antiperiodic and Dirichlet eigenvalues outside the union $\bigcup_{n\geq N_*} D_n$*, and those eigenvalues are situated in the half-plane* $\Re z < (N_* + 1/2)^2$.

The smoothness of potentials v can be characterized in terms of the decay rate of the *spectral gaps* $\gamma_n = \lambda_n^+ - \lambda_n^-$ and *deviations* $\delta_n = \mu_n - \lambda_n^+$ (see [7] and the bibliography therein for Hill operators with L^2-potentials and [10] for Hill operators with singular potentials). The proofs of these results use essentially the following statement (see [7, Section 2.2] for Hill operators with L^2-potentials and [10, Lemma 6] for Hill– Schrödinger operators with H^{-1}_{per}-potentials).

Lemma 2.2. *There are functionals* $\alpha_n(v;z)$ *and* $\beta_n^\pm(v;z)$ *defined for large enough* $n \in \mathbb{N}$ *and* $|z| < n$ *such that* $\lambda = n^2 + z$ *is a periodic (for even* n*) or antiperiodic (for odd* n*) eigenvalue of* L *if and only if* z *is an eigenvalue of the matrix*

$$\begin{bmatrix} \alpha_n(v;z) & \beta_n^-(v;z) \\ \beta_n^+(v;z) & \alpha_n(v;z) \end{bmatrix}. \tag{2.9}$$

Moreover, $\alpha_n(z;v)$ *and* $\beta_n^\pm(z;v)$ *depend analytically on* v *and* z*, and* $z_n^\pm = \lambda_n^\pm - n^2$ *are the only solutions of the basic equation*

$$(z - \alpha_n(v;z))^2 = \beta_n^-(v;z)\beta_n^+(v;z) \tag{2.10}$$

in the disc $|z| < n/4$.

The functionals $\alpha_n(v;z)$ and $\beta_n^\pm(v;z)$ are well defined for large enough n by the following expressions in terms of the Fourier coefficients of the potential (see (2.16)–(2.33) in [7] for Hill operators with L^2-potentials and (3.23)–(3.30) in [10] for Hill operators with H^{-1}_{per}-potentials).

$$\alpha_k = \sum_{k=1}^{\infty} S_k^{11}, \quad \beta_n^- = V(-2n) + \sum_{k=1}^{\infty} S_k^{12}, \quad \beta_n^+ = V(2n) + \sum_{k=1}^{\infty} S_k^{21}, \tag{2.11}$$

where for $k = 1, 2, \dots$

$$S_k^{11} = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(-n-j_1)V(j_1-j_2)\dots V(j_{k-1}-j_k)V(j_k+n)}{(n^2-j_1^2+z)\dots(n^2-j_k^2+z)}, \tag{2.12}$$

and

$$S_k^{12} = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(-n-j_1)V(j_1-j_2)\dots V(j_{k-1}-j_k)V(j_k-n)}{(n^2-j_1^2+z)\dots(n^2-j_k^2+z)}, \tag{2.13}$$

$$S_k^{21} = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(n-j_1)V(j_1-j_2)\dots V(j_{k-1}-j_k)V(j_k+n)}{(n^2-j_1^2+z)\dots(n^2-j_k^2+z)}. \tag{2.14}$$

In the sequel, we suppress the dependence on v in the notations and write only $\beta_n^\pm(z), \alpha_n(z)$.

Lemma 2.3. *If* v *is a singular potential of the form* (2.4)*, and* $\lambda_n^\pm$ *are the corresponding periodic or antiperiodic eigenvalues in the disc* D_n*, then*

$$|\lambda_n^\pm - n^2| = o(n), \quad n \to \infty. \tag{2.15}$$

Proof. In view of [10, (4.32)],

$$|\alpha_n(z)| \leq n\varepsilon_n, \quad |\beta_n^\pm(z)| \leq n\varepsilon_n + |V(\pm 2n)| \quad \text{for } |z| \leq \frac{n}{2},$$

where $V(k)$ are given by (2.7) and

$$\varepsilon_n := C_1 \Big(\sum_{|k|\geq\sqrt{n}} |q(k)|^2 \Big)^{1/2} + \frac{C_2}{\sqrt{n}} \tag{2.16}$$

with some constants C_1, C_2. Therefore, $\varepsilon_n \to 0$ as $n \to \infty$. On the other hand, by (2.7) we have $V(\pm 2n) = \pm 2inq(\pm 2n)$ with $q(\pm 2n) \to 0$. Since $z_n^{\pm} = \lambda_n^{\pm} - n^2$ are roots of (2.10), it follows that

$$\frac{|z_n^{\pm}|}{n} \le \varepsilon_n + \sqrt{(\varepsilon_n + 2|q(-2n)|)(\varepsilon_n + 2|q(2n)|)} \to 0. \tag{2.17}$$

Thus, (2.15) holds. □

Remark. The estimate in (2.15) could be improved if $v(x) = Q'(x)$ with $Q \in H^{\alpha}$, $0 < \alpha < 1$. Then one can show that

$$|\lambda_n^{\pm} - n^2| = o(n^{1-\alpha}), \quad n \to \infty. \tag{2.18}$$

The asymptotic behavior of $\beta_n^{\pm}(z)$ (or γ_n and δ_n) also plays a crucial role in studying the Riesz basis property of the system of root functions of the operators $L_{\mathrm{Per}^{\pm}}$. In [7, Section 5.2], it is shown (for potentials $v \in L^2([0,\pi])$) that if the ratio $\beta_n^+(z_n^*)/\beta_n^-(z_n^*)$ is not separated from 0 or ∞, then the system of root functions of $L_{\mathrm{Per}^{\pm}}$ does not contain a Riesz basis (see Theorem 71 in [7] and its proof therein). Theorem 1 in [12] (or Theorem 2 in [11]) gives, for wide classes of L^2-potentials, the following criteria for Riesz basis property.

Criterion 2.1. *Consider the Hill operator with $v \in L^2([0,\pi])$. If*

$$\beta_n^+(0) \ne 0, \quad \beta_n^-(0) \ne 0 \tag{2.19}$$

and

$$\exists c \ge 1\colon\ c^{-1}|\beta_n^{\pm}(0)| \le |\beta_n^{\pm}(z)| \le c|\beta_n^{\pm}(0)|, \quad |z| \le 1, \tag{2.20}$$

for all sufficiently large even n (if $\mathrm{bc} = \mathrm{Per}^+$) or odd n (if $\mathrm{bc} = \mathrm{Per}^-$), then

(a) *There is $N = N(v)$ such that for $n > N$ the operator $L_{\mathrm{Per}^{\pm}}(v)$ has exactly two simple periodic (for even n) or antiperiodic (for odd n) eigenvalues in the disc $\{z\colon |z - n^2| < 1\}$.*

(b) *The system of root functions of $L_{\mathrm{Per}^+}(v)$ or $L_{\mathrm{Per}^-}(v)$ contains a Riesz basis in $L^2([0,\pi])$ if and only if, respectively,*

$$\limsup_{n \in 2\mathbb{N}} t_n(0) < \infty \quad \text{or} \quad \limsup_{n \in 1+2\mathbb{N}} t_n(0) < \infty, \tag{2.21}$$

where

$$t_n(z) = \max\left\{\frac{|\beta_n^-(z)|}{|\beta_n^+(z)|}, \frac{|\beta_n^+(z)|}{|\beta_n^-(z)|}\right\}. \tag{2.22}$$

In general form, i. e., without the restrictions (2.19) and (2.20), this criterion is given in [14] in the context of 1D Dirac operators, but in the case of Hill operators the formulation and the proof are the same (see Proposition 19 in [13]). Moreover, the same argument gives the following more general statement.

Criterion 2.2. *Let $\Gamma^+ = 2\mathbb{N}$, $\Gamma^- = 2\mathbb{N} - 1$ in the case of Hill operators, and $\Gamma^+ = 2\mathbb{Z}$, $\Gamma^- = 2\mathbb{Z} - 1$ in the case of one dimensional Dirac operators. There exists $N_* = N_*(v)$ such that for $|n| > N_*$ the operator $L = L_{\mathrm{Per}^{\pm}}(v)$ has in the disc $D_n = \{z\colon |z - n^2| < n/4\}$ (respectively $D_n = \{z\colon |z - n| < 1/2\}$) exactly two periodic (for $n \in \Gamma^+$) or antiperiodic (for $n \in \Gamma^-$) eigenvalues, counted with algebraic multiplicity. Let*

$$\mathcal{M}^{\pm} = \{n \in \Gamma^{\pm}\colon\ |n| \ge N_*,\ \lambda_n^- \ne \lambda_n^+\},$$

and let $\{u_{2n-1}, u_{2n}\}$ be a pair of normalized eigenfunctions associated, respectively, with the eigenvalues λ_n^- and λ_n^+, $n \in \mathcal{M}^{\pm}$.

(a) *If* $\Delta \subset \Gamma^{\pm}$, *then the system* $\{u_{2n-1}, u_{2n},\ n \in \Delta \cap \mathcal{M}^{\pm}\}$ *is a (Riesz) basis in its closed linear span if and only if*

$$\limsup_{n\in\Delta\cap\mathcal{M}^{\pm}} t_n(z_n^*) < \infty, \tag{2.23}$$

where $z_n^* = \frac{1}{2}(\lambda_n^- + \lambda_n^+) - \lambda_n^0$ *with* $\lambda_n^0 = n^2$ *for Hill operators and* $\lambda_n^0 = n$ *for Dirac operators.*

(b) *The system of root functions of* L *contains a Riesz basis if and only if (*2.23*) holds for* $\Delta = \Gamma^{\pm}$.

Another interesting abstract criterion of basisness is the following.

Criterion 2.3. *The system of root functions of the operator* $L_{\mathrm{Per}^{\pm}}(v)$ *contains a Riesz basis in* $L^2([0,\pi])$ *if only if*

$$\limsup_{n\in\mathcal{M}^{\pm}} \frac{|\lambda_n^+ - \mu_n|}{|\lambda_n^+ - \lambda_n^-|} < \infty. \tag{2.24}$$

This criterion was given (with completely different proofs) in [18] for Hill operators with L^2-potentials and in [13] for Hill operators with H^{-1}_{per}-potentials and for one-dimensional Dirac operators with L^2-potentials as well.

Recently we have obtained in [16] asymptotic formulas for spectral gaps γ_n and deviations $\delta_n = \mu_n - \lambda_n^+$ under the assumptions (2.19) and (2.20). The following holds.

Proposition 2.1. *Assume that there is an infinite set* $\Delta \subset \mathbb{N}$ *such that (*2.19*) and (*2.20*) hold. Then there exist branches* $\sqrt{\beta_n^-(z)}$ *and* $\sqrt{\beta_n^+(z)}$ *such that*

$$\gamma_n \sim 2\sqrt{\beta_n^-(z_n^*)}\sqrt{\beta_n^+(z_n^*)}, \quad n \in \Delta. \tag{2.25}$$

Moreover,

(a) *If* -1 *is not a cluster point of the sequence* $\big(\sqrt{\beta_n^-(z_n^*)}/\sqrt{\beta_n^+(z_n^*)}\big)_{n\in\Delta}$, *then*

$$\mu_n - \lambda_n^+ \sim -\frac{1}{2}\big(\sqrt{\beta_n^+(z_n^*)} + \sqrt{\beta_n^-(z_n^*)}\big)^2, \quad n \in \Delta. \tag{2.26}$$

(b) *If* 1 *is not a cluster point of the sequence* $\big(\sqrt{\beta_n^-(z_n^*)}/\sqrt{\beta_n^+(z_n^*)}\big)_{n\in\Delta}$, *then*

$$\mu_n - \lambda_n^- \sim -\frac{1}{2}\big(\sqrt{\beta_n^+(z_n^*)} - \sqrt{\beta_n^-(z_n^*)}\big)^2, \quad n \in \Delta. \tag{2.27}$$

(c) *If* -1 *is not a cluster point of the sequence* $\big(\beta_n^-(z_n^*)/\beta_n^+(z_n^*)\big)_{n\in\Delta}$, *then in the Hill case*

$$\mu_n - \frac{1}{2}(\lambda_n^- + \lambda_n^+) \sim -\frac{1}{2}\big(\beta_n^+(z_n^*) + \beta_n^-(z_n^*)\big), \quad n \in \Delta. \tag{2.28}$$

Here and thereafter, we write for two sequences (a_n) and (b_n) that $a_n \sim b_n$ as $n \to \infty$ if $a_n/b_n \to 1$ as $n \to \infty$. We write $a_n \asymp b_n$ if there are constants $C > c > 0$ such that $ca_n \le b_n \le Ca_n$ for large enough n. In this paper we study the class of Hill potentials v with the property that the main term in the asymptotics of $\beta_n^{\pm}$ equals the Fourier coefficient $V(\pm 2n)$. In the context of Sobolev spaces, a natural example of such potentials is given by the following assertion (compare to (1.1) and (1.2); see also [32]).

Lemma 2.4. *Suppose* $v(x)$, $0 \le x \le \pi$, *is* m *times differentiable and the function* $v^{(m)}(x)$, *is absolutely continuous. If the conditions*

(a) $v^{(s)}(\pi) = v^{(s)}(0)$ *for* $s = 0, \dots, m-1$ *(if* $m > 0$*);*

(b) $v^{(m)}(\pi) \neq v^{(m)}(0)$

hold, then we have

$$\beta_n^-(z) \sim V(-2n) \sim \frac{1}{(-2in)^{m+1}}\big(v^{(m)}(0) - v^{(m)}(\pi)\big), \quad |z| \le n, \tag{2.29}$$

$$\beta_n^+(z) \sim V(2n) \sim \frac{1}{(2in)^{m+1}}\big(v^{(m)}(0) - v^{(m)}(\pi)\big), \qquad |z| \le n. \tag{2.30}$$

In Section 3 we introduce weighted spaces of Hill potentials (in terms of their Fourier coefficients) and consider general classes of potentials such that $\beta_n^\pm \sim V(\pm 2n)$; see Theorem 4.1. Lemma 2.4 is a partial case of that theorem, which corresponds to the weight $\Omega(k) = k^m$.

§ 3. Weights and weighted spaces

Since we study the Hill operator on $[0, \pi]$, our basic index set is $2\mathbb{Z}$. A sequence of *positive* numbers $\Omega = (\Omega(k))_{k\in 2\mathbb{Z}}$ is called *weight*, or *weight sequence*. We consider only even weights, i. e.,

$$\Omega(-k) = \Omega(k), \quad k \in 2\mathbb{Z}, \tag{3.1}$$

such that

$$\Omega(0) = 1, \quad \Omega(k) \le \Omega(m) \quad \text{for } m \ge k \ge 0. \tag{3.2}$$

For every weight Ω we consider the corresponding ℓ^∞-type weighted space of Hill potentials

$$W_\infty(\Omega) = \Big\{ v(x) = \sum_{k\in 2\mathbb{Z}} V(k) e^{ikx} \colon \|v\|_\Omega = \sup_{k\in 2\mathbb{Z}} |V(k)|\Omega(k) < \infty \Big\}. \tag{3.3}$$

We say that two weights Ω_1 and Ω_2 are *equivalent* if

$$\exists C \ge 1\colon\ C^{-1}\Omega_1(k) \le \Omega_2(k) \le C\Omega_1(k), \quad k \in 2\mathbb{Z}. \tag{3.4}$$

Obviously, equivalent weights generate one and the same weighted space.

A weight Ω is called *submultiplicative* if

$$\Omega(k+m) \le \Omega(k)\Omega(m), \quad k, m \in 2\mathbb{Z}. \tag{3.5}$$

Of course, if Ω_1 and Ω_2 are equivalent weights and one of them is submultiplicative, then the other one satisfies

$$\Omega(k+m) \le C\Omega(k)\Omega(m), \quad k, m \in 2\mathbb{Z} \tag{3.6}$$

for some constant $C > 0$. Obviously, if Ω satisfies (3.6), then $\widetilde{\Omega} = C\Omega$ satisfies (3.5). Moreover, it is easy to see that if (3.6) holds for $|k|, |m| \ge k_0$, then it holds for all $k, m \in 2\mathbb{Z}$, maybe with another constant C. In the sequel we call a weight *almost submultiplicative* if it satisfies (3.6).

A weight ω is called *slowly increasing* if

$$A := \sup_{k\in 2\mathbb{N}} \frac{\omega(2k)}{\omega(k)} < \infty. \tag{3.7}$$

Every slowly increasing weight is almost submultiplicative. Indeed, if $0 < k \le m$ then from (3.7) and (3.2) it follows that

$$\omega(m+k) \le \omega(2m) \le A\omega(m) \le A\omega(m)\omega(k), \tag{3.8}$$

so (3.6) holds with $C = A$.

If $\sup \Omega(2k)/\Omega(k) = \infty$ (i. e., if Ω is not slowly increasing), then Ω is called *rapidly increasing* weight. A rapidly increasing submultiplicative weight Ω is growing at most exponentially because

$$\Omega(k) \le (\Omega(2))^{k/2} = e^{ak}, \quad a = \frac{1}{2}\log \Omega(2).$$

Each weight may be written in the form

$$\Omega(k) = \exp(h(|k|)), \quad \text{where } h(k) = \log \Omega(k),\ h(0) = 0. \tag{3.9}$$

Then Ω is submultiplicative if and only if h is subadditive, i. e.,

$$h(k+m) \le h(k) + h(m) \quad \forall k, m \in 2\mathbb{N}. \tag{3.10}$$

It is well known (e. g., see [30, Problem 98]) that if $(h(k))$ is a subadditive sequence, then the limit

$$\ell = \lim_{k\to\infty} \frac{h(k)}{k} \tag{3.11}$$

exists. A submultiplicative weight Ω of the form (3.9) is called *subexponential* if $\ell = 0$ and *exponential* if $\ell > 0$.

Lemma 3.1. *Let Ω be a weight of the form (3.9). If the corresponding sequence $(h(k))_{k\in 2\mathbb{Z}_+}$ is concave, i. e.,*

$$h(k+4) - h(k+2) \le h(k+2) - h(k) \quad \text{for } k \ge 0, \tag{3.12}$$

then Ω is submultiplicative.

Proof. Fix $k, m \in \mathbb{N}$. By (3.12), we have

$$\begin{aligned} h(2k+2m) - h(2k) &= \sum_{i=1}^{m} [h(2k+2j) - h(2k+2j-2)] \\ &\le \sum_{j=1}^{m} [h(2j) - h(2j-2)] = h(2m). \end{aligned}$$

Thus (3.10) holds, i. e., the weight $\Omega(k) = \exp(h(|k|))$ is submultiplicative. □

Typical examples of submultiplicative weights are

$$\omega_a(0) = 1, \quad \omega_a(k) = |k|^a \quad \text{for } k \ne 0,\ a > 0 \tag{3.13}$$

(known as the Sobolev weights), and

$$\Omega_{c,\gamma}(k) = \exp(c|k|^\gamma), \quad c > 0,\ \gamma \in (0,1) \tag{3.14}$$

(known as the Gevrey weights). The corresponding functions h are concave.

Further we need the following technical assertion.

Lemma 3.2. *For every $c > 0$, $\gamma \in (0,1)$ and $a > 0$ the weight $\Omega = (\Omega(k))_{k\in 2\mathbb{Z}}$ defined by*

$$\Omega(0) = 1, \quad \Omega(k) = \exp(c|k|^\gamma)|k|^{-a} \quad \text{for } k \ne 0, \tag{3.15}$$

is almost submultiplicative. Moreover, if $a \le c\gamma(1-\gamma)2^\gamma$, then the weight Ω is submultiplicative.

Proof. We have $\Omega(k) = e^{h(|k|)}$, where

$$h(0) = 0, \quad h(x) = cx^\gamma - a\log x \quad \text{for } x > 0.$$

For large enough x the function h is concave. Indeed,

$$h''(x) = \frac{1}{x^2}(c\gamma(\gamma-1)x^\gamma + a) < 0 \quad \text{for } x > x_0,$$

where $x_0 = \left(\frac{a}{c\gamma(1-\gamma)}\right)^{1/\gamma}$. Set

$$h_1(x) = \begin{cases} d + h(x) & \text{for } x > x_0, \\ (d + h(x_0))\dfrac{x}{x_0} & \text{for } 0 \le x \le x_0, \end{cases}$$

where the constant $d > 0$ is chosen so large that

$$\frac{d + h(x_0)}{x_0} \ge h'(x_0) \quad \text{and} \quad h(x) \le (d + h(x_0))\frac{x}{x_0} \quad \text{for } 2 \le x \le x_0.$$

Then h_1 is a concave function on $[0,\infty)$ with $h_1(0) = 0$, so by Lemma 3.1 the weight $\Omega_1(k) = e^{h_1(|k|)}$ is submultiplicative. Since

$$h(x) \le h_1(x) \le h(x) + d + h(x_0) \quad \text{for } x \ge 2,$$

the weights Ω and Ω_1 are equivalent.

If $a \le c\gamma(1-\gamma)2^\gamma$, then $x_0 = \left(\frac{a}{c\gamma(1-\gamma)}\right)^{1/\gamma} \le 2$, so it follows that the function h is concave for $x \ge 2$. Thus, (3.12) holds for $k \ge 2$.

If $k = 0$, then (3.12) reduces to $h(4) \le 2h(2)$, i. e.,

$$c4^\gamma - a\log 4 \le 2(c2^\gamma - a\log 2) = c2^{1+\gamma} - a\log 4.$$

Since $4^\gamma = 2^{2\gamma} < 2^{1+\gamma}$, (3.12) holds for $k = 0$ as well, so by Lemma 3.1 it follows that in this case the weight Ω is submultiplicative. □

§ 4. Main results

Theorem 4.1. *Suppose $\Omega = (\Omega(k))_{k\in 2\mathbb{Z}}$ is a weight of the form*

$$\Omega(k) = \omega(k)\cdot\widetilde{\Omega}(k), \tag{4.1}$$

where $\widetilde{\Omega}$ is an almost submultiplicative weight and ω is a slowly increasing weight with

$$M := \sum_{k\ne 0}\frac{1}{|k|\omega(k)} < \infty. \tag{4.2}$$

Let $v \in W_\infty(\Omega)$, and let $(V(k))_{k\in 2\mathbb{Z}}$ be its Fourier coefficients.

(a) *If $\Delta \subset \mathbb{N}$ is an infinite set such that*

$$|V(\pm 2n)|\, n\Omega(2n) \to \infty \quad \text{as} \quad n \in \Delta,\ n \to \infty, \tag{4.3}$$

then

$$\beta_n^\pm(v,z) \sim V(\pm 2n) \quad \text{as} \quad |z| \le \frac{n}{2},\ n \in \Delta,\ n \to \infty. \tag{4.4}$$

(b) *If*

$$\lim_{|k|\to\infty} |V(k)|\Omega(k) = 0 \tag{4.5}$$

and $\Delta \subset \mathbb{N}$ is an infinite set such that

$$\exists c > 0\colon\ |V(\pm 2n)| n\Omega(2n) \ge c \quad \text{for } n \in \Delta, \tag{4.6}$$

then (4.4) holds.

Proof. We prove (4.4) for β_n^+ only; the proof for β_n^- is the same.

In view of (2.11),

$$|\beta_n^+(z) - V(2n)| \le \sum_{k=1}^{\infty} |S_k^{21}(z)|, \tag{4.7}$$

where S_k^{21} are defined by (2.14).

Set

$$r(k) = |V(k)|\Omega(k), \quad R_m = \sup_{|k|\ge m} r(k). \tag{4.8}$$

By $v \in W_\infty(\Omega)$, we have that $r(k) \le \|v\|_\Omega$, so $R_m \le \|v\|_\Omega$. Moreover, since $\widetilde{\Omega}$ satisfies (3.6) with some constant C and ω satisfies (3.8) with a constant A, it follows that

$$\begin{aligned} &\big|V(n-j_1)V(j_1-j_2)\dots V(j_{k-1}-j_k)V(j_k+n)\big|\Omega(2n) \\ &\qquad \le (AC)^k r(n-j_1)r(j_1-j_2)\dots r(j_{k-1}-j_k)r(j_k+n). \end{aligned} \tag{4.9}$$

First we estimate S_1^{21}. By (2.14), (4.1) and (4.8),

$$|S_1^{21}(z)|\Omega(2n) \le \sum_{j\in n+2\mathbb{Z}\setminus\{\pm n\}} \frac{Cr(n-j)r(j+n)}{|n^2-j^2+z|} \cdot \frac{\omega(2n)}{\omega(n-j)\omega(j+n)}.$$

It is easy to see that

$$|n^2-j^2+z| \ge |n^2-j^2| - |z| \ge \frac{1}{2}|n^2-j^2| \quad \text{if} \quad |z| \le \frac{n}{2}. \tag{4.10}$$

Therefore, for $|z| \le n/2$ we have

$$|S_1^{21}(z)|\Omega(2n) \le \sigma_1 + \sigma_2,$$

with

$$\begin{aligned} \sigma_1 &= \sum_{j<0,\, j\ne -n} \frac{2Cr(n-j)r(j+n)}{|n-j|\cdot|n+j|} \cdot \frac{\omega(2n)}{\omega(n-j)\omega(j+n)} \\ &\le \sum_{j<0,\, j\ne -n} \frac{2CR_n\|v\|_\Omega}{n|n+j|\omega(n+j)} \cdot \frac{\omega(2n)}{\omega(n)} \le \quad \text{(by (4.8) and } n-j \ge n) \\ &\le 2AC\|v\|_\Omega R_n \frac{1}{n} \sum_{j\ne -n} \frac{1}{|n+j|\omega(n+j)} \le 2ACM\|v\|_\Omega R_n \frac{1}{n} \end{aligned}$$

by (3.7) and (4.2), and similarly,

$$\sigma_2 = \sum_{j>0,\, j\ne n} \frac{2Cr(n-j)r(j+n)}{|n-j|\cdot|n+j|} \cdot \frac{\omega(2n)}{\omega(n-j)\omega(j+n)} \le 2ACM\|v\|_\Omega R_n \frac{1}{n}.$$

Since $R_n \le \|v\|_\Omega$, it follows that

$$|S_1^{21}(z)| = O\Big(\frac{1}{n\Omega(2n)}\Big), \quad |z| \le \frac{n}{2}. \tag{4.11}$$

If (4.5) holds, then $R_n \to 0$, so we obtain that

$$|S_1^{21}(z)| = o\Big(\frac{1}{n\Omega(2n)}\Big), \quad |z| \le \frac{n}{2}. \tag{4.12}$$

Next we estimate S_k^{21} for $k = 2, 3, \dots$ In view of (2.14), (4.1) and (4.8)–(4.10), we have

$$|S_k^{21}(z)|\Omega(2n) \le \sum_{j_1,\dots,j_k \ne \pm n} 2^k (AC)^k \frac{r(n-j_1)r(j_1-j_2)\dots r(j_k+n)}{|n^2-j_1^2|\cdot|n^2-j_2^2|\dots|n^2-j_k^2|}$$
$$\le \|v\|_\Omega^{k+1}(2AC)^k \Big(\sum_{j\ne\pm n} \frac{1}{|n^2-j^2|}\Big)^k.$$

Since

$$\sum_{j\ne\pm n} \frac{1}{|n^2-j^2|} \le \frac{2\log(6n)}{n},$$

it follows that

$$|S_k^{21}(z)|\Omega(2n) \le \|v\|_\Omega^{k+1}(2AC)^k \Big(\frac{2\log(6n)}{n}\Big)^k, \quad |z| \le \frac{n}{2}. \tag{4.13}$$

Now, if n is so large that

$$\frac{4\|v\|_\Omega AC\log(6n)}{n} < \frac{1}{2},$$

we obtain that

$$\sum_{k=2}^\infty |S_k^{21}| \le \|v\|_\Omega \Big(\frac{4\|v\|_\Omega \log(6n)}{n}\Big)^2 \frac{1}{\Omega(2n)} = O\Big(\frac{(\log n)^2}{n^2\Omega(2n)}\Big). \tag{4.14}$$

Thus, if (4.3) holds, then (4.7), (4.11) and (4.14) imply (4.4).

Moreover, in the case when (4.5) holds, (4.7), (4.6), (4.12) and (4.14) prove (4.4) for β_n^+. □

Corollary 4.1. *Lemma* 2.4 *holds.*

Proof. Indeed, integration by parts and the Riemann–Lebesgue Lemma show that

$$V(k) \sim \frac{1/\pi}{(ik)^{m+1}}(v^{(m)}(0) - v^{(m)}(\pi)).$$

Consider the weight Ω defined by

$$\Omega(0) = 1, \quad \Omega(k) = |k|^{m+1} \quad \text{for } k \ne 0.$$

Then $v \in W_\infty(\Omega)$ and $|V(\pm 2n)|n\Omega(2n) \to \infty$.

We can apply Theorem 4.1, since the weight Ω satisfies (4.1) with $\omega(k) = |k|$, $\widetilde{\Omega}(k) = |k|^m$. Hence, Lemma 2.4 follows from (4.4). □

In view of Lemma 3.2 one can apply Theorem 4.1 to weights Ω of the form

$$\Omega(k) = |k|^a e^{c|k|^\gamma} \quad \text{for } k \ne 0,\ a \in \mathbb{R},\ c > 0,\ \gamma \in (0,1). \tag{4.15}$$

But it is impossible to apply Theorem 4.1 if the weight Ω is growing so slowly that

$$\sum \frac{1}{|k|\Omega(k)} = \infty.$$

For example, this is the case if we consider the weight Ω given by

$$\Omega(0) = 1, \quad \Omega(k) = \log(e|k|) \quad \text{for } k \ne 0. \tag{4.16}$$

For such weights, the next theorem gives conditions which guarantee that $V(\pm 2n)$ is the main term in the asymptotics of $\beta_n^\pm$.

Theorem 4.2. *Suppose $\Omega = (\Omega(k))_{k\in 2\mathbb{Z}}$ is an almost submultiplicative weight. Let $v \in W_\infty(\Omega)$, and let $(V(k))_{k\in 2\mathbb{Z}}$ be its Fourier coefficients.*

(a) *If $\Delta \subset \mathbb{N}$ is an infinite set such that*

$$|V(\pm 2n)|\frac{n}{\log n}\Omega(2n) \to \infty \quad \text{as} \quad n \in \Delta, \quad n \to \infty, \tag{4.17}$$

then

$$\beta_n^{\pm}(v,z) \sim V(\pm 2n) \quad \text{as} \quad |z| \le \frac{n}{2},\ n \in \Delta,\ n \to \infty. \tag{4.18}$$

(b) *If*

$$\lim_{|k|\to\infty} |V(k)|\Omega(k) = 0 \tag{4.19}$$

and $\Delta \subset \mathbb{N}$ is an infinite set such that

$$\exists c > 0\colon\ |V(\pm 2n)|\frac{n}{\log n}\Omega(2n) \ge c \quad \text{for } n \in \Delta, \tag{4.20}$$

then (4.18) holds.

Proof. As in the proof of Theorem 4.1, we consider β_n^+ only and use the notations (4.7) and (4.8).

Since the weight Ω is almost submultiplicative, we have

$$\begin{aligned} &\big|V(n-j_1)V(j_1-j_2)\dots V(j_{k-1}-j_k)V(j_k+n)\big|\Omega(2n) \\ &\qquad \le C^k r(n-j_1)r(j_1-j_2)\dots r(j_{k-1}-j_k)r(j_k+n) \end{aligned} \tag{4.21}$$

with some constant $C \ge 1$.

As in the proof of Theorem 4.1 we obtain

$$\sum_{k=2}^{\infty} |S_k^{21}(z)| = O\Big(\frac{(\log n)^2}{n^2\Omega(2n)}\Big), \quad |z| \le \frac{n}{2}. \tag{4.22}$$

Next we estimate S_1^{21}. Since

$$r(n-j)r(n+j) \le R_n\|v\|_\Omega,$$

by (2.14), (4.8) and (4.21) we obtain

$$\begin{aligned} |S_1^{21}(z)|\Omega(2n) &\le \sum_{j\in(n+2\mathbb{Z})\setminus\{\pm n\}} \frac{2Cr(n-j)r(j+n)}{|n^2-j^2|} \\ &\le 2CR_n\|v\|_\Omega \sum_{j\ne\pm n} \frac{1}{|n^2-j^2|} \le 4CR_n\|v\|_\Omega \frac{\log(6n)}{n}. \end{aligned}$$

Since $R_n \le \|v\|_\Omega$, it follows that

$$|S_1^{21}(z)| = O\Big(\frac{\log n}{n\Omega(2n)}\Big), \quad |z| \le \frac{n}{2}. \tag{4.23}$$

If (4.19) holds, then $R_n \to 0$, so we obtain that

$$|S_1^{21}(z)| = o\Big(\frac{\log n}{n\Omega(2n)}\Big), \quad |z| \le \frac{n}{2}. \tag{4.24}$$

Finally, if (4.17) holds, then (4.22) and (4.23) imply (4.18) for β_n^+. Moreover, if (4.19) holds, then (4.20), (4.22) and (4.24) prove (4.18) for β_n^+. □

Theorem 4.3. *Let $L = L(v)$ be the Hill operator with a potential v that satisfies (with some infinite set of indices $\Delta \subset 2\mathbb{N}$ or $\Delta \subset 2\mathbb{N}+1$) the assumptions of either part* (a) *or part* (b) *of Theorem* 4.1*, or either part* (a) *or part* (b) *of Theorem* 4.2*. Then there are square roots $\sqrt{V(-2n)}$ and $\sqrt{V(2n)}$ such that:*

$$\text{(a)} \qquad \lambda_n^+ - \lambda_n^- \sim 2\sqrt{V(-2n)}\sqrt{V(2n)} \quad \text{as} \ \ n \in \Delta,\ n \to \infty. \tag{4.25}$$

(b) *If -1 is not a cluster point of the sequence $\big(\sqrt{V(-2n)}/\sqrt{V(2n)}\big)_{n\in\Delta}$, then*

$$\mu_n - \lambda_n^+ \sim -\frac{1}{2}\big(\sqrt{V(-2n)} + \sqrt{V(2n)}\big)^2 \quad \text{as} \ \ n \in \Delta,\ n \to \infty. \tag{4.26}$$

(c) *If 1 is not a cluster point of the sequence $\big(\sqrt{V(-2n)}/\sqrt{V(2n)}\big)_{n\in\Delta}$, then*

$$\mu_n - \lambda_n^- \sim -\frac{1}{2}\big(\sqrt{V(-2n)} - \sqrt{V(2n)}\big)^2 \quad \text{as} \ \ n \in \Delta,\ n \to \infty. \tag{4.27}$$

(d) *If -1 is not a cluster point of $(V(-2n)/V(2n))_{n\in\Delta}$, then*

$$\mu_n - \frac{1}{2}(\lambda_n^+ + \lambda_n^-) \sim -\frac{1}{2}(V(-2n) + V(2n)) \quad \text{as} \ \ n \in \Delta,\ n \to \infty. \tag{4.28}$$

(e) *Moreover, if u_n^-, u_n^+ are normalized eigenvectors corresponding to the eigenvalues λ_n^-, λ_n^+, then the system $\{u_n^\pm,\ n \in \Delta\}$ is a Riesz basis in its closed linear span if and only if*

$$V(-2n) \asymp V(2n) \quad \text{as} \ \ n \in \Delta,\ n \to \infty. \tag{4.29}$$

Notice that part (e) of Theorem 4.3 generalizes the results of Makin [25] to much wider classes of potentials that include both singular potentials and potentials in Carlemann–Gevrey classes far beyond the Sobolev spaces.

§ 5. Examples

1. Further we say that a Hill operator $L(v)$ has the periodic (or antiperiodic) Riesz basis property (RBP) if the periodic (or antiperiodic) root function system of $L(v)$ contains Riesz bases. In view of Theorems 4.1, 4.2 and 4.3 it is easy to give nontrivial examples of potentials v such that the operator $L(v)$ has or has not the periodic and/or antiperiodic Riesz basis property.

Indeed, the relation $v \in W_\infty(\Omega)$ means that the Fourier coefficients $(V(k))_{k\in 2\mathbb{Z}}$ of a potential v have the form

$$V(k) = \frac{\eta(k)}{\Omega(k)} \quad \text{with } (\eta(k)) \in \ell^\infty(2\mathbb{Z}). \tag{5.1}$$

Conversely, we may determine a potential v by defining its Fourier coefficients by (5.1).

Theorems 4.1 and 4.3 immediately imply the following.

Proposition 5.1. *Let Ω be a weight that satisfies the conditions* (4.1) *and* (4.2)*. Choose a bounded scalar sequence $(\eta(k))_{k\in 2\mathbb{Z}}$ so that*

(i) $n \cdot \eta(\pm 2n) \to \infty$ *as* $n \in \mathbb{N},\ n \to \infty$*;*

(ii) $\eta(-2n) \asymp \eta(2n)$ *or* (ii*)$\eta(-2n) \not\asymp \eta(2n)$ *as* $n \in 2\mathbb{N}$*;*

(iii) $\eta(-2n) \asymp \eta(2n)$ *or* (iii*)$\eta(-2n) \not\asymp \eta(2n)$ *as* $n \in 1 + 2\mathbb{N}$.

Then the operator $L(v)$ with a potential v given by (5.1) *has/(has not) the periodic RBP if respectively* (ii)/(ii*) *holds, and has/(has not) the antiperiodic RBP if respectively* (iii)/(iii*) *holds.*

To be more specific, let us consider the following example, where conditions (ii) and (iii*) hold.

Example 5.1. Let v be defined by (5.1) with a sequence $(\eta(k))$ given by $\eta(0)=0$ and

$$\eta(2n)=\frac{\log n}{n} \quad \text{for} \quad n\in\mathbb{N}, \quad \eta(-2n)=\begin{cases} \dfrac{\log n}{n} & \text{for } n\in 2\mathbb{N}, \\ \dfrac{(\log n)^2}{n} & \text{for } n\in 1+2\mathbb{N}. \end{cases} \tag{5.2}$$

Then the operator $L(v)$ has the periodic Riesz basis property and fails the antiperiodic Riesz basis property.

Notice that we cannot apply Theorem 4.2 to the case given by Example 5.1. But on the other hand Theorem 4.2 works for a wider class of potentials as the following example shows.

Example 5.2. Consider the potential

$$\Omega=(\Omega(k)), \quad \Omega(k)=1 \quad \forall k\in 2\mathbb{Z}.$$

Let v be the potential defined formally by its Fourier coefficients $(V(k))$ given by $V(0)=0$ and

$$V(2n)=1 \quad \text{for} \quad n\in\mathbb{N}, \quad V(-2n)=\begin{cases} \dfrac{1}{\sqrt{n}} & \text{for } n\in 2\mathbb{N}, \\ 1 & \text{for } n\in 1+2\mathbb{N}. \end{cases} \tag{5.3}$$

Then, by Theorems 4.2 and 4.3, the operator $L(v)$ has the antiperiodic Riesz basis property and fails the periodic Riesz basis property.

Of course, one can easily modify the above example and get a potential v such that $L(v)$ has (or fails) the periodic RBP and fails the antiperiodic RBP.

2. In Section 2 we consider classes of potentials v such that $\beta_n^{\pm}(z)\sim V(\pm 2n)$, where $(V(k))$ are the Fourier coefficients of v. By (2.11),

$$\beta_n^-(z)=V(-2n)+\sum_{k=1}^{\infty} S_k^{12}(n,z), \quad \beta_n^+(z)=V(2n)+\sum_{k=1}^{\infty} S_k^{21}(n,z), \tag{5.4}$$

where S_k^{12} and S_k^{21} are given by (2.13) and (2.14). Of course, for a generic potential v it is not true that the first term $V(\pm 2n)$ of the series defining $\beta_n^{\pm}(z)$ dominates the sum of all others and determines the asymptotics.

Moreover, let v be a trigonometric polynomial, say

$$v(x)=\sum_{|k|\le M} V(k)e^{ikx}.$$

Every term of the sum S_k^{21} (or S_k^{12}) given by (2.13) or (2.14) is a fraction whose numerator has the form

$$V(\pm n-j_1)V(j_1-j_2)\dots V(j_{k-1}-j_k)V(j_k\pm n).$$

Notice that

$$(\pm n-j_1)+(j_1-j_2)+\ldots+(j_{k-1}-j_k)+(j_k\pm n)=\pm 2n.$$

Therefore, if $(k+1)M<2n$, then the absolute value of one of these numbers will be strictly greater than M, so the corresponding Fourier coefficient will be zero. Thus, whenever $(k+1)M<2n$ we have $S_k^{21}=0$ and $S_k^{12}=0$. In other words, if v is a trigonometric polynomial, then no fixed partial sum of the series in (5.4) gives the asymptotics of $\beta_n^{\pm}(z)$. We refer to [11, 12, 15, 16] for results about the asymptotics of $\beta_n^{\pm}$, $\gamma_n=\lambda_n^+-\lambda_n^-$, $\delta_n=\mu_n-\lambda_n^+$ and Riesz basis property of root function systems in the case of potentials that are trigonometric polynomials. See also [2, 1] and the bibliography therein.

The situation is similar in the case of potentials whose Fourier coefficients (by absolute value) decay superexponentially, i. e.,

$$\exists \gamma > 1c\colon\ |V(k)| \le e^{-|k|^\gamma}, \quad n \ge N_*.$$

In [5], it is shown that no fixed partial sum of the series in (5.4) gives the asymptotics of $\beta_n^\pm(z)$.

In the context of Sobolev spaces W_1^m Shkalikov and Veliev [32, Theorems 2–4] gave conditions on v for existence (or nonexistence) of Riesz bases (consisting of periodic or antiperiodic root functions) in terms of partial sums

$$\Sigma_m^-(n,z) = V(-2n) + \sum_{k=1}^m S_k^{12}(n,z), \quad \Sigma_m^+(n,z) = V(2n) + \sum_{k=1}^m S_k^{21}(n,z). \tag{5.5}$$

In fact the assumptions of Theorem 2 in [32] say that the partial sums Σ_m^- and Σ_m^+ give the main terms in the asymptotics of β_n^- and β_n^+ respectively.

The following statement generalizes part (a) of Theorem 4.2. Moreover, in view of Criterion 2.1 it could be considered as a generalization of the results of Shkalikov and Veliev [32].

Theorem 5.1. *Suppose $\Omega = (\Omega(k))_{k\in 2\mathbb{Z}}$ is an almost submultiplicative weight. Let $v \in W_\infty(\Omega)$, and let $(V(k))_{k\in 2\mathbb{Z}}$ be the Fourier coefficients of v.*

If $\Delta \subset \mathbb{N}$ is an infinite set such that

$$|\Sigma_m^\pm(n,z_n^*)|\Omega(2n)\Big(\frac{n}{\log n}\Big)^{m+1} \to \infty \quad \text{as} \ \ n \in \Delta,\ |n| \to \infty, \tag{5.6}$$

then

$$\beta_n^\pm(v,z_n^*) \sim \Sigma_m^\pm(n,z_n^*) \quad \text{as} \ \ n \in \Delta,\ n \to \infty. \tag{5.7}$$

Proof. As in the proof of Theorem 4.1, we have

$$|S_k^{21}(z)|\Omega(2n) \le \|v\|_\Omega^{k+1}(2C)^k\Big(\frac{2\log(6n)}{n}\Big)^k, \quad |z| \le \frac{n}{2}, \tag{5.8}$$

which leads to

$$\sum_{k=m+1}^\infty |S_k^{21}(z)| = O\Big(\frac{(\log n)^2}{n^2\Omega(2n)}\Big), \quad |z| \le \frac{n}{2}. \tag{5.9}$$

Now, (5.4), (5.9) and (5.6) imply (5.7). □

3. Next we give examples where the asymptotics of $\beta_n^\pm$ is determined by S_1^{12} and S_1^{21} but not by $V(\pm 2n)$.

Proposition 5.2. *Let $\Omega = (\Omega(k))_{k\in 2\mathbb{Z}}$ be an almost submultiplicative weight that satisfies the conditions of Theorem 4.1, and let v be the potential with Fourier coefficients $(V(k))_{k\in 2\mathbb{Z}}$ defined by*

$$V(\pm 2) = \pm\frac{1}{\Omega(2)}, \tag{5.10}$$

$$V(\pm 4p) = \pm\frac{1}{\Omega(4p)}, \quad p \in \mathbb{N}, \tag{5.11}$$

$$V(4p+2) = \frac{\xi_p}{p\Omega(4p+2)}, \quad \xi_p \ge 0,\ p \in \mathbb{N}, \tag{5.12}$$

$$V(-4p-2) = -\frac{\eta_p}{p\Omega(4p+2)}, \quad \eta_p \ge 0,\ p \in \mathbb{N}. \tag{5.13}$$

If

$$\xi_p \to 0 \quad \textit{and} \quad \eta_p \to 0, \tag{5.14}$$

then

$$|\beta^{\pm}_{2p+1}(z^*_{2p+1}| \asymp \frac{1}{p\Omega(4p)}, \tag{5.15}$$

and there is a Riesz basis in $L^2([0,\pi])$ *which consists of antiperiodic root functions.*

Proof. In view of Criterion 2.1, (5.15) implies that the system of antiperiodic root functions contains Riesz bases. Therefore, we need to prove (5.15) only.

By (2.11) we have

$$\big|\beta^{+}_{2p+1}(z^*_{2p+1}) - S^{21}_1(2p+1, z^*_{2p+1})\big| \le |V(2n)| + \sum_{k=2}^{\infty} \big|S^{21}_k(2p+1, z^*_{2p+1})\big|.$$

From (5.11)–(5.14) it follows that $v \in W_\infty(\Omega)$, so (4.14) holds since its proof uses only that $v \in W_\infty(\Omega)$. Therefore, in view of (5.12) we obtain that

$$\big|\beta^{+}_{2p+1}(z^*_{2p+1}) - S^{21}_1(2p+1, z^*_{2p+1})\big| = o\Big(\frac{1}{p\Omega(4p)}\Big). \tag{5.16}$$

Next we estimate $S^{21}_1(2p+1, z^*_{2p+1})$. Consider $S^{21}_1(n,0)$ with $n = 2p+1$. It is easy to see that

$$S^{21}_1(n,0) = \sum_{j\ne n} \frac{V(n-j)V(j+n)}{n^2-j^2}$$

is a sum of positive terms. Indeed, if $-n < j < n$, then $n^2 - j^2 > 0$ and $V(n \pm j) > 0$ due to (5.10)–(5.13), so the corresponding term is positive. If $j > n$ or $j < -n$, then $n^2 - j^2 < 0$ and either $V(n-j) < 0$, $V(n+j) > 0$ or $V(n-j) > 0$, $V(n+j) < 0$ so again the corresponding term is positive.

Therefore, we have

$$S^{21}_1(2p+1, 0) > \frac{V(4p)V(2)}{8p} = \frac{1}{8p\Omega(4p)\Omega(2)}, \tag{5.17}$$

where the expression on the right is the term of $S^{21}_1(n,0)$ associated with $j = 2p-1$. Next we show that

$$A := \big|S^{21}_1(2p+1, z^*_{2p+1}) - S^{21}_1(2p+1, 0)\big| = o\Big(\frac{1}{p\Omega(4p)}\Big). \tag{5.18}$$

We have, with $n = 2p+1$ and $z = z^*_{2p+1}$,

$$A \le \sum_{j\ne\pm n} \Big|\frac{V(n-j)V(j+n)}{n^2-j^2+z} - \frac{V(n-j)V(j+n)}{n^2-j^2}\Big| = \sum_{j\ne\pm n} \Big|\frac{V(n-j)V(j+n)z}{(n^2-j^2+z)(n^2-j^2)}\Big|.$$

Since $v \in W_\infty(\Omega)$ and the weight Ω is almost submultiplicative, we have

$$|V(n-j)V(j+n)| \le \frac{\|v\|^2_\Omega}{\Omega(n-j)\Omega(n+j)} \le \frac{C\|v\|^2_\Omega}{\Omega(2n)}.$$

Therefore, from (4.10) and the elementary estimate

$$\sum_{j\ne\pm n} \frac{1}{(n^2-j^2)^2} \le \frac{4}{n^2}$$

it follows that

$$A \le \frac{C\|v\|^2_\Omega |z|}{\Omega(2n)} \cdot \sum_{j\ne\pm n} \frac{2}{(n^2-j^2)^2} \le \frac{8C\|v\|^2_\Omega |z|}{n^2\Omega(2n)}.$$

On the other hand, by (2.15) we have

$$\frac{|z_n^*|}{n} \to 0 \quad \text{as} \quad n \to \infty$$

even in the case $v \in H^{-1}_{\text{per}}$. So, with $n = 2p+1$ and $z = z^*_{2p+1}$ it follows that (5.18) holds. Now (4.11), (5.17) and (5.18) imply

$$\left|S_1^{21}(2p+1, z^*_{2p+1})\right| \asymp \frac{1}{p\Omega(4p)}. \tag{5.19}$$

Now (5.16) and (5.19) imply (5.15) for β_n^+.

The proof of (5.15) for β_n^- is similar. By (2.11) we have

$$\left|\beta^-_{2p+1}(z^*_{2p+1}) - S_1^{12}(2p+1, z^*_{2p+1})\right| \le |V(-2n)| + \sum_{k=2}^{\infty} \left|S_k^{12}(2p+1, z^*_{2p+1})\right|.$$

One can use the above argument to prove that

$$\left|S_1^{12}(2p+1, z^*_{2p+1})\right| \asymp \frac{1}{p\Omega(4p)}. \tag{5.20}$$

Also, the same argument that proves (4.14) shows that

$$\sum_{k=2}^{\infty} |S_k^{12}(z)| = O\Big(\frac{(\log n)^2}{n^2\Omega(2n)}\Big), \quad |z| \le \frac{n}{2}. \tag{5.21}$$

Now (5.13), (5.14), (5.20) and (5.21) imply (5.15) for β_n^-. □

Next we modify the construction in Proposition 5.2 in order to give examples of potentials without Riesz basis property.

Proposition 5.3. *Let $\Omega = (\Omega(k))_{k\in 2\mathbb{Z}}$ be an almost submultiplicative weight that satisfies the conditions of Theorem 4.1, and let v be the potential with Fourier coefficients $(V(k))_{k\in 2\mathbb{Z}}$ defined by*

$$V(\pm 2) = \pm\frac{1}{\Omega(2)}, \tag{5.22}$$

$$V(4p) = \frac{1}{\log(4p)\Omega(4p)}, \quad p \in \mathbb{N}, \tag{5.23}$$

$$V(-4p) = -\frac{1}{\Omega(4p)}, \quad p \in \mathbb{N}, \tag{5.24}$$

$$V(4p+2) = \frac{\xi_p}{p\log(4p)\Omega(4p+2)}, \quad \xi_p \ge 0,\ p \in \mathbb{N}, \tag{5.25}$$

$$V(-4p-2) = -\frac{\eta_p}{p\Omega(4p+2)}, \quad \eta_p \ge 0,\ p \in \mathbb{N}. \tag{5.26}$$

If

$$\xi_p \to 0 \quad \textit{and} \quad \eta_p \to 0, \tag{5.27}$$

then

$$\beta^-_{2p+1}(z^*_{2p+1}) \asymp \frac{1}{p\Omega(4p)} \tag{5.28}$$

and

$$\beta^+_{2p+1}(z^*_{2p+1}) \asymp \frac{1}{p\log(4p)\Omega(4p)}. \tag{5.29}$$

Moreover, there is no Riesz basis in $L^2([0,\pi])$ which consists of antiperiodic root functions.

Proof. In view of Criterion 2.1, (5.28) and (5.29) imply that the system of antiperiodic root functions does not contain Riesz bases. On the other hand, following the proof of Proposition 5.2 one can see that (5.28) holds. Therefore, we need to prove (5.29) only.

By (2.11) we have

$$\left|\beta^+_{2p+1}(z^*_{2p+1}) - S^{21}_1(2p+1, z^*_{2p+1})\right| \le |V(2n)| + \sum_{k=2}^{\infty} \left|S^{21}_k(2p+1, z^*_{2p+1})\right|.$$

Therefore, from (5.25) and (5.27) and (4.14) it follows that

$$\left|\beta^+_{2p+1}(z^*_{2p+1}) - S^{21}_1(2p+1, z^*_{2p+1})\right| = o\Big(\frac{1}{p\log(4p)\Omega(4p)}\Big). \tag{5.30}$$

As in the proof of Proposition 5.2 one can show that

$$S^{21}_1(2p+1, 0) > \frac{1}{8p\log(4p)\Omega(4p)\Omega(2)}$$

and

$$\left|S^{21}_1(2p+1, z^*_{2p+1}) - S^{21}_1(2p+1, 0)\right| = o\Big(\frac{1}{p\log(4p)\Omega(4p)}\Big).$$

Therefore, it remains to show that

$$\left|S^{21}_1(2p+1, z^*_{2p+1})\right| = O\Big(\frac{1}{p\log(4p)\Omega(4p)}\Big). \tag{5.31}$$

By (4.10) we have, with $|z| \le n/2$,

$$|S^{21}_1(2p+1, z)| \le \sum_{j\ne\pm(2p+1)} \frac{2|V(2p+1-j)V(j+2p+1)|}{|(2p+1)^2 - j^2|} = \sigma_1 + \sigma_2 + \sigma_3,$$

where σ_1, σ_2 and σ_3 are the partial sums of the above sum, respectively over $\{j < -2p-1\}$, $\{|j| < 2p+1\}$ and $\{j > 2p+1\}$. First we estimate σ_2. Consider the potentials $\widetilde{v}$ defined by its Fourier coefficients

$$\widetilde{V}(k) = \begin{cases} V(k) & \text{if } k > 0, \\ 0 & \text{if } k \le 0. \end{cases}$$

From (5.23) and (5.25) it follows that $\widetilde{v} \in W_\infty(\widetilde{\Omega})$, where $\widetilde{\Omega}(k) = \Omega(k)\log k$. The weight $\widetilde{\Omega}$ satisfies the assumptions of Theorem 4.1. Therefore, by (4.11) we have

$$\sigma_2 = S^{21}_1(\widetilde{v}; 2p+1, z^*_{2p+1}) = O\Big(\frac{1}{p\log(4p)\Omega(4p)}\Big).$$

The change of variable $j \to -j$ shows that $\sigma_1 = \sigma_3$. Next we estimate

$$\sigma_3 = \sum_{s=1}^{\infty} \frac{|V(-2s)|V(4p+2+2s)}{s(4p+2+2s)} = \sigma_{3,1} + \sigma_{3,2},$$

where $\sigma_{3,1}$ and $\sigma_{3,2}$ are respectively the parts of the above sum over odd s and even s. By (5.22)–(5.26), we have

$$\begin{aligned}\sigma_{3,1} &= \frac{|V(-2)|V(4p+4)}{4p+4} + \sum_{k=1}^{\infty} \frac{|V(-4k-2)|V(4p+4k+4)}{(2k+1)(4p+4k+4)} \\ &\le \frac{1}{p\log(4p)\Omega(4p)}\Big(1 + \sum_{k=1}^{\infty} \frac{\text{const}}{k\Omega(4k+2)}\Big) = O\Big(\frac{1}{p\log(4p)\Omega(4p)}\Big).\end{aligned}$$

Similarly, we obtain

$$\sigma_{3,2} = \sum_{k=1}^{\infty} \frac{|V(-4k)|V(4p+4k+2)}{2k(4p+4k+2)}$$
$$\leq \Big(\sum_{k=1}^{\infty} \frac{\text{const}}{k\Omega(4k)}\Big) \frac{1}{p\log(4p)\Omega(4p)} = O\Big(\frac{1}{p\log(4p)\Omega(4p)}\Big).$$

Thus (5.30) holds, which completes the proof of (5.29). □

4. The weighted spaces $W_\infty(\Omega)$ provide a suitable framework when we study L^1-potentials or even potentials that are finite measures. The next theorem extends the results of Theorem 4.2 to a wider class of singular potentials.

Theorem 5.2. *Let $\Omega = (\Omega(k))_{k\in 2\mathbb{Z}}$ be an almost submultiplicative weight, and let v be the potential defined by its Fourier coefficients $(V(k))_{k\in 2\mathbb{Z}}$ given by*

$$V(k) = |k|^\alpha q(k), \quad \alpha \in \Big(0, \frac{1}{2}\Big),\ q = (q(k)) \in \ell^\infty(\Omega). \tag{5.32}$$

(a) *If $\Delta \subset \mathbb{N}$ is an infinite set such that*

$$|V(\pm 2n)|n^{1-2\alpha}\Omega(2n) \to \infty \quad \text{as} \ \ n \in \Delta,\ n \to \infty, \tag{5.33}$$

then

$$\beta_n^\pm(v,z) \sim V(\pm 2n) \quad \text{as} \ \ |z| \leq \frac{n}{2},\ n \in \Delta,\ n \to \infty. \tag{5.34}$$

(b) *If*

$$\lim_{|k|\to\infty} |q(k)|\Omega(k) = 0 \tag{5.35}$$

and $\Delta \subset \mathbb{N}$ is an infinite set such that

$$\exists c > 0\colon\ |V(\pm 2n)|n^{1-2\alpha}\Omega(2n) \geq c \quad \text{for } n \in \Delta, \tag{5.36}$$

then (5.34) holds.

Proof. We prove (5.34) for β_n^+ only since the proof is the same for β_n^-. The following formula (which one can easily verify) will be used:

$$\sum_{j\neq\pm n} \frac{1}{|n^2-j^2|^\beta} \asymp n^{1-2\beta} \quad \text{if} \ \ \frac{1}{2} < \beta < 1. \tag{5.37}$$

By (2.11) we have

$$|\beta_n^+(z) - V(2n)|\Omega(2n) \leq \sum_{k=1}^{\infty} |S_k^{21}(n,z)|\Omega(2n). \tag{5.38}$$

Next we estimate the sum on the right.

Let $r(k) = |k|^\alpha q(k)$, $R_m = \sup\{r(k)\colon |k| \geq m\}$. In view of (3.6) and (5.32), we have

$$|V(n-j)V(n+j)|\Omega(2n) \leq C|n^2-j^2|^\alpha\, |r(n-j)r(n+j)| \leq CR_n\|q\|_\Omega.$$

Therefore, from (2.14) and (4.10) it follows that

$$|S_1^{21}(n,z)|\Omega(2n) \leq 2C \sum_{j\neq\pm n} \frac{R_n\|q\|_\Omega}{|n^2-j^2|^{1-\alpha}},$$

which implies, in view of (5.37),

$$|S_1^{21}(n,z)| = O(n^{-1+2\alpha}) \quad \text{as} \ \ |z| \leq \frac{n}{2},\ n \to \infty. \tag{5.39}$$

If (5.35) holds, then $R_n \to 0$, so in that case we obtain

$$|S_1^{21}(n,z)| = o(n^{-1+2\alpha}) \quad \text{as} \quad |z| \le \frac{n}{2},\ n \to \infty. \tag{5.40}$$

Next we estimate $|S_k^{21}(n,z)| \cdot \Omega(2n)$ for $k \ge 2$. If $j_1, \dots, j_k \in (n + 2\mathbb{Z}) \setminus \{\pm n\}$, then $|n \pm j_s| \ge 2$, $1 \le s \le k$, so we have

$$\begin{aligned}
&\frac{|n-j_1| \cdot |j_1-j_2| \dots |j_{k-1}-j_k| \cdot |j_k+n|}{|n^2-j_1^2| \cdot |n^2-j_2^2| \dots |n^2-j_k^2|} \\
&= \frac{|j_1-j_2|}{|n+j_1| \cdot |n-j_2|} \cdot \frac{|j_2-j_3|}{|n+j_2| \cdot |n-j_3|} \\
&\dots \frac{|j_{k-1}-j_k|}{|n+j_{k-1}| \cdot |n-j_k|} \\
&= \Big|\frac{1}{n+j_1} + \frac{1}{n-j_2}\Big| \cdot \Big|\frac{1}{n+j_2} + \frac{1}{n-j_3}\Big| \\
&\dots \Big|\frac{1}{n+j_{k-1}} + \frac{1}{n-j_k}\Big| \le 1.
\end{aligned}$$

On the other hand, the weight Ω is almost submultiplicative, so we have

$$\Omega(2n) \le C^k \Omega(n-j_1)\Omega(j_1-j_2) \dots \Omega(j_{k-1}-j_k)\Omega(j_k+n).$$

Therefore by (2.14), (4.10), (5.32), the above inequalities and (5.37), we obtain that

$$\begin{aligned}
|S_k^{21}(n,z)|\Omega(2n) &\le (2C)^k \|q\|_\Omega^{k+1} \sum_{j_1,\dots,j_k \ne \pm n} \frac{1}{|n^2-j_1^2|^{1-\alpha} \dots |n^2-j_k^2|^{1-\alpha}} \\
&= (2C)^k \|q\|_\Omega^{k+1} \Big(\sum_{j \ne \pm n} \frac{1}{|n^2-j^2|^{1-\alpha}} \Big)^k \le \widetilde{C} \Big(\frac{2C\|q\|_\Omega}{n^{1-2\alpha}} \Big)^k,
\end{aligned}$$

where the constant $\widetilde{C}$ does not depend on k. Now it follows that

$$\sum_{k=2}^{\infty} |S_k^{21}(n,z)|\Omega(2n) = O(n^{2(2\alpha-1)}) \quad \text{as} \quad |z| \le \frac{n}{2},\ n \to \infty. \tag{5.41}$$

By (5.38), (5.39) and (5.41) we obtain

$$|\beta_n^+(z) - V(2n)|\Omega(2n) = O(n^{-1+2\alpha}) \quad \text{as} \quad |z| \le \frac{n}{2},\ n \to \infty,$$

so (5.33) implies (5.34).

Moreover, if (5.35) holds, then (5.38), (5.40) and (5.41) imply that

$$|\beta_n^+(z) - V(2n)|\Omega(2n) = o(n^{-1+2\alpha}) \quad \text{as} \quad |z| \le \frac{n}{2},\ n \to \infty,$$

so if (5.36) holds then (5.34) holds also. This completes the proof. □

5. In this paper, we consider only weighted spaces of ℓ^∞-type. This approach is good in the case of smooth potentials or even for some classes of singular potentials. But in the case of singular potentials $v \in H^{-1}(\mathbb{R})$ (see (2.4)) it is "natural" to work with ℓ^2-weighted spaces in order to obtain results similar to Theorem 5.1 for the whole class of such potentials. We are going to present such results in another paper.

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