Труды Москов. Матем. Общ.
Том 75 (2014), вып. 2

Trans. Moscow Math. Soc.
2014, Pages 281–299
S 0077-1554(2014)00234-X
Article electronically published on November 5, 2014

STURM–LIOUVILLE OPERATORS

K. A. MIRZOEV

Abstract. Let $(a,b) \subset \mathbb{R}$ be a finite or infinite interval, let $p_0(x)$, $q_0(x)$, and $p_1(x)$, $x \in (a,b)$, be real-valued measurable functions such that p_0, p_0^{-1}, $p_1^2 p_0^{-1}$, and $q_0^2 p_0^{-1}$ are locally Lebesgue integrable (i.e., lie in the space $L^1_{\mathrm{loc}}(a,b)$), and let $w(x)$, $x \in (a,b)$, be an almost everywhere positive function. This paper gives an introduction to the spectral theory of operators generated in the space $\mathcal{L}^2_w(a,b)$ by formal expressions of the form

$$l[f] := w^{-1}\{-(p_0 f')' + i[(q_0 f)' + q_0 f'] + p_1' f\},$$

where all derivatives are understood in the sense of distributions. The construction described in the paper permits one to give a sound definition of the minimal operator L_0 generated by the expression $l[f]$ in $\mathcal{L}^2_w(a,b)$ and include L_0 in the class of operators generated by symmetric (formally self-adjoint) second-order quasi-differential expressions with locally integrable coefficients. In what follows, we refer to these operators as Sturm–Liouville operators. Thus, the well-developed spectral theory of second-order quasi-differential operators is used to study Sturm–Liouville operators with distributional coefficients. The main aim of the paper is to construct a Titchmarsh–Weyl theory for these operators. Here the problem on the deficiency indices of L_0, i.e., on the conditions on p_0, q_0, and p_1 under which Weyl's limit point or limit circle case is realized, is a key problem. We verify the efficiency of our results for the example of a Hamiltonian with δ-interactions of intensities h_k centered at some points x_k, where

$$l[f] = -f'' + \sum_j h_j \delta(x - x_j) f.$$

1. Introduction

1. Let $I := (a,b) \subset \mathbb{R}$. Let p, q, and w be real-valued functions on I such that $p(x) \neq 0$ and $w(x) > 0$ a.e. for $x \in I$ and p^{-1} $(:= 1/p)$, q, and w are locally integrable (i.e., $p^{-1}, q, w \in L^1_{\mathrm{loc}}(I)$). Finally, let $r \in L^1_{\mathrm{loc}}(I)$ be a complex-valued function on I. We define the quasi-derivative $f^{[1]}$ of a locally absolutely continuous complex-valued function f on I by setting $f^{[1]} := p(f' - rf)$. Assume that $f^{[1]}$ is locally absolutely continuous as well and define the second quasi-derivative $f^{[2]}$ by setting $f^{[2]} := (f^{[1]})' + \bar{r} f^{[1]} - qf$ and the quasi-differential expression $l[f]$ by setting $l[f] := -w^{-1} f^{[2]}$. Thus,

$$l[f](x) = w^{-1}(x)\bigl\{-[p(f' - rf)]' - \bar{r}p(f' - rf) + qf\bigr\}(x), \tag{1}$$

the domain Δ of $l[f]$ is the set of all complex-valued functions f such that f and $f^{[1]}$ are locally absolutely continuous on I, and $l[f] (\in L^1_{\mathrm{loc}}(I))$ is computed by formula (1) for $f \in \Delta$.

2010 *Mathematics Subject Classification.* Primary 34B24; Secondary 34B20, 34B40.

Key words and phrases. Second-order quasi-differential operator, minimal operator, maximal operator, Sturm–Liouville theory, deficiency index, limit point–limit circle, linear differential equation with distributional coefficients, finite-difference equation, Jacobi matrix.

Supported by RFBR Grant No. 14-01-00349.

Green's formula

$$\int_\alpha^\beta \{\bar{g}l[f] - f\overline{l[g]}\}w = [fg](\beta) - [fg](\alpha), \qquad a < \alpha \le \beta < b,$$

holds for all $f, g \in \Delta$, where the sesquilinear form $[fg]$ is given by

$$[fg](x) := f(x)\overline{g^{[1]}(x)} - f^{[1]}(x)\overline{g(x)}, \qquad x \in I,$$

and is antisymmetric,

$$\overline{[fg]}(x) = -[gf](x).$$

The study of spectral properties of the operators generated by the expression $l[f]$ in the Hilbert space $\mathcal{L}_w^2(I)$ of complex-valued measurable functions f such that $|f|^2w$ is integrable on I can be reduced to studying the quasi-differential equation

$$l[f] = \lambda f + h \tag{2}$$

with a parameter $\lambda \in C$ and a given function h and the corresponding homogeneous equation

$$l[f] = \lambda f. \tag{3}$$

Equation (2) is obviously equivalent to the system of first-order differential equations

$$\widehat{f}\,' = \mathcal{F}\widehat{f} + \widehat{h},$$

where

$$\mathcal{F} = \begin{pmatrix} r & p^{-1} \\ q - \lambda w & -\bar{r} \end{pmatrix}, \qquad \widehat{f} = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}, \qquad \widehat{h} = \begin{pmatrix} 0 \\ -wh \end{pmatrix}.$$

One can use this reduction and the standard existence and uniqueness theorems for the Cauchy problem for a system of linear first-order differential equations to prove the corresponding theorem for (2) (see [11, Section V.16.1, Theorem 1], [38, Section 1.2, Theorem 1.2.1], and [28]). Thus, the following theorem holds.

Theorem 1. *Assume that the functions p, q, r, and w satisfy the conditions listed above and $wh \in L^1_{\mathrm{loc}}(I)$. Then for any $t \in I$ and $\alpha, \beta \in \mathbb{C}$ there exists a unique function $f\colon I \times \mathbb{C} \to \mathbb{C}$ such that the functions $f(\cdot, \lambda)$ and $f^{[1]}(\cdot, \lambda)$ are locally absolutely continuous on I, the function $f(\cdot, \lambda)$ satisfies* (2) *almost everywhere, $f(x, \lambda)|_{x=t} = \alpha$, and $f^{[1]}(x, \lambda)|_{x=t} = \beta$.*

It is also well known that the converse of Theorem 1 is true (see [28] and[38, Section 1.2, Theorem 1.2.3]).

One sometimes uses the terminology adopted in the theory of linear quasi-differential operators of arbitrary order [19, 27] and says that the quasi-derivatives $f^{[0]}(:= f)$, $f^{[1]}$, and $f^{[2]}$ are generated by the matrix

$$F = \begin{pmatrix} r & p^{-1} \\ q & -\bar{r} \end{pmatrix}.$$

Recall that the expression $l[f]$ is said to be regular if the interval (a, b) is finite and the entries of F, as well as the function w, are integrable on the entire interval $[a, b]$. Otherwise, $l[f]$ is said to be singular. Furthermore, the left endpoint a of the interval (a, b) is said to be regular if $a > -\infty$ and the entries of F, as well as w, are integrable on each interval $[a, \beta] \subset [a, b)$, $\beta < b$. Otherwise, the endpoint a is said to be singular. The notions of regularity and singularity for the other endpoint, b, are defined in a similar way.

Theorem 1 obviously remains true if t is a regular endpoint of I.

If $r = 0$ on I, then (3) becomes

$$-(pf')' + qf = \lambda wf \tag{4}$$

with the quasi-derivatives $f^{[0]} = f$, $f^{[1]} = pf'$, and $f^{[2]} = (pf')' - qf$.

Spectral analysis of ordinary differential operators goes back to Sturm and Liouville's work in the early 1830s, especially to their legendary paper [35], where the eigenvalue problem for an equation of the form (4) with separated boundary conditions at the endpoints of an interval $I = [a, b]$ of the real line was studied for the case in which $q, w \in C(I)$, $p \in C^1(I)$, and $p, q, w > 0$ on I. At the beginning of the 20th century, Weyl published several papers dealing with (4), where it was assumed that $p, q \in C[0, +\infty)$, $p > 0$, and $w = 1$ on $[0, +\infty)$; e.g., see [36]. (It was in these papers that Weyl had already used the quasi-derivative $f^{[1]} = pf'$.) In 1912, Dixon [21] was the first to replace the condition of continuity of the coefficients in (4) with the condition $p^{-1}, q, w \in L^1[a, b]$, where $p, w > 0$ a.e. on $[a, b]$. The subsequent development timeline of and the state of the art in the theory of operators generated by the expression $l[f] = -(pf')' + qf$ in the space $\mathcal{L}^2_w(I)$ are presented in much detail in the proceedings [18] of a conference dedicated to Sturm's 200th birthday and in Zettl's book [38]. Unfortunately, the early 1950s work of Soviet mathematicians is underrepresented in the literature (in particular, in these two books). In this connection, we note only that Levitan's book [7] dealing with spectral expansions for operators of the form $l[f] = -f'' + qf$ has played an extremely important role in the development of mathematical analysis in the USSR.

2. Let p_0, q_0, and p_1 be real-valued Lebesgue measurable functions such that p_0^{-1}, $p_1^2 p_0^{-1}$, and $q_0^2 p_0^{-1}$ are locally Lebesgue integrable (i.e., $p_0^{-1}, p_1^2 p_0^{-1}, q_0^2 p_0^{-1} \in L^1_{\mathrm{loc}}(I)$). Set $\varphi := p_1 + iq_0$. Consider the matrix

$$F = \frac{1}{p_0} \begin{pmatrix} \varphi & 1 \\ -|\varphi|^2 & -\overline{\varphi} \end{pmatrix}.$$

The Cauchy–Schwarz inequality readily shows that

$$\varphi p_0^{-1} \in L^1_{\mathrm{loc}}(I).$$

Indeed,

$$\int_\alpha^\beta |\varphi| \cdot |p_0^{-1}| = \int_\alpha^\beta \frac{1}{\sqrt{|p_0|}} \cdot \frac{|\varphi|}{\sqrt{|p_0|}} \le \left(\int_\alpha^\beta \frac{1}{|p_0|} \right)^{1/2} \cdot \left(\int_\alpha^\beta \frac{|\varphi|^2}{|p_0|} \right)^{1/2} < +\infty$$

for any $[\alpha, \beta] \subset I$. Thus, all entries of F lie in $L^1_{\mathrm{loc}}(I)$.

We use the matrix F to define the quasi-derivatives $f^{[0]}$, $f^{[1]}$, and $f^{[2]}$ by setting, as usual,

$$f^{[0]} = f, \qquad f^{[1]} = p_0 f' - \varphi f, \qquad f^{[2]} = (f^{[1]})' + \frac{\overline{\varphi}}{p_0} f^{[1]} + \frac{|\varphi|^2}{p_0} f.$$

Next, we apply Theorem 1 and the subsequent remark (see Section 1.1) to conclude that the existence and uniqueness theorem holds for the Cauchy problem with data at an arbitrary point of I for the equation

$$-f^{[2]} = \lambda w f + wh \tag{5}$$

and that the conditions $p_0^{-1}, p_1^2 p_0^{-1}, q_0^2 p_0^{-1}, w, wh \in L^1_{\mathrm{loc}}(I)$ are the most general conditions guaranteeing that this is the case.

One can treat the entries of F in (5) as regular distributions and the prime $'$ as the generalized derivative (i.e., the derivative in the sense of distributions.) Since

$$q_0 f' = \frac{q_0}{p_0}(p_0 f') = \frac{q_0}{p_0} f^{[1]} + \frac{q_0 \varphi}{p_0} f,$$

$f, f^{[1]} \in AC_{\mathrm{loc}}(I)$, and the coefficients $q_0 p_0^{-1}$ and $q_0\varphi p_0^{-1}$ lie in $L^1_{\mathrm{loc}}(I)$ (see above), we conclude that $q_0 f'$ is a regular distribution. Now assume that p_0 lies in $L^1_{\mathrm{loc}}(I)$ as well. Then $\varphi \in L^1_{\mathrm{loc}}(I)$. Indeed,

$$\int_\alpha^\beta |\varphi| = \int_\alpha^\beta \sqrt{|p_0|} \cdot \frac{|\varphi|}{\sqrt{|p_0|}} \le \left(\int_\alpha^\beta |p_0|\right)^{1/2} \cdot \left(\int_\alpha^\beta \frac{|\varphi|^2}{|p_0|}\right)^{1/2} < +\infty$$

for any $[\alpha, \beta] \subset I$.

We rewrite the formula for $f^{[1]}$ as

$$p_0 f' = f^{[1]} + \varphi f$$

and find that $p_0 f'$ is a regular distribution as well.

On the other hand, if we define the product of the distribution a', where $a(x)$ is an entry of F, by a function f locally absolutely continuous on I by setting, as usual,

$$(a' f)(\phi) = -\int_a^b \overline{a}(\overline{f}\phi)'$$

for an arbitrary infinitely differentiable complex-valued function ϕ compactly supported on I, then it is easily seen that

$$(af)' = a'f + af'$$

in the sense of distributions.

This argument shows that one can multiply out completely in the expression for $f^{[2]}$, thus obtaining

$$f^{[2]} = (p_0 f')' - i(2q_0 f' + q_0' f) - p_1' f,$$

or

$$f^{[2]} = (p_0 f')' - i((q_0 f)' + q_0 f') - p_1' f.$$

We point out that the monomials $(p_0 f')'$, $(q_0 f)'$, and $p_1' f$ in this formula are singular distributions, the first two of them being the generalized derivatives of regular functions, while $q_0 f'$ and $f^{[2]}$ are regular distributions.

Thus, the expression $l[f]$ (see (1)) becomes

$$l[f] = w^{-1}\{-(p_0 f')' + i((q_0 f)' + q_0 f') + p_1' f\}$$

in terms of distributions.

Note that if the condition $p_0 \in L^1_{\mathrm{loc}}(I)$ is not satisfied, then one apparently cannot multiply out in the expression for $f^{[2]}$ without, say, ascribing a meaning to the expression p_1'. In such cases, we do not multiply out in (5).

Note some special cases of the expression $l[f]$.

If $q_0(x) = 0$, $w(x) = 1$ on I, and $p_0, p_0^{-1}, p_1^2 p_0^{-1} \in L^1_{\mathrm{loc}}(I)$, then the matrix F, the quasi-derivatives $f^{[0]}$, $f^{[1]}$, and $f^{[2]}$, and the quasi-differential expression $l[f]$ acquire the form

$$F = \frac{1}{p_0}\begin{pmatrix} p_1 & 1 \\ -p_1^2 & -p_1 \end{pmatrix},$$

$$f^{[0]} = f, \qquad f^{[1]} = p_0 f' - p_1 f, \qquad f^{[2]} = (f^{[1]})' + \frac{p_1}{p_0} f^{[1]} + \frac{p_1^2}{p_0} f$$

(6) $$l[f] = -(p_0 f')' + p_1' f.$$

In particular, if $p_0(x) = 1$ and $p_1(x) = \sigma(x)$, where $\sigma^2(x) \in L^1_{\mathrm{loc}}(I)$, then F, $f^{[0]}$, $f^{[1]}$, $f^{[2]}$, and $l[f]$ have the form

$$F = \begin{pmatrix} \sigma & 1 \\ -\sigma^2 & -\sigma \end{pmatrix}, \qquad f^{[0]} = f, \quad , f^{[1]} = f' - \sigma f, \quad f^{[2]} = (f^{[1]})' + \sigma f^{[1]} + \sigma^2 f$$

$$l[f] = -f'' + \sigma' f. \tag{7}$$

Moreover, if $\sigma(x)$ is a step function with jumps h_j at some points $x_j \in I$, then

$$l[f] = -f'' + \sum_j h_j \delta(x - x_j) f, \tag{8}$$

where $\delta(x)$ is the Dirac delta function.

Thus, the theory of a Hamiltonian with δ-interactions of intensities h_k centered at the points x_k is covered by the theory of operators generated by second-order quasi-differential expressions.

The operators generated by expressions of the form (7) were thoroughly studied in [14] and [15], mostly in the case of a finite interval. Later, the Titchmarsh–Weyl theory was constructed in [24] for an expression of the form (1) with real coefficients, and hence, in particular, some results in [14] and [15] were generalized. Note also the related earlier papers [29] and [37]. In [31] the authors study a differential expression of the form (8) on a finite interval.

Expressions of the form (8) are physically important and hence have long been a focus of attention for numerous mathematicians (see [5, 17, 20, 32, 34]).

The book [13, Chapter 7] studies the operators generated on an interval by expressions of the form (6) and the Dirichlet boundary conditions for the case in which p_0 and p_1 are functions of bounded variation.

2. The operators L_1 and L_0. Deficiency indices and a description of self-adjoint extensions of L_0

1. Let p, q, r, and w satisfy the conditions listed at the beginning of Section 1. The expression $l[f]$ generates an operator L_1 with domain $\mathcal{D}(L_1)$ in $\mathcal{L}^2_w(I)$ by the formula $L_1 f = l[f]$, $f \in \mathcal{D}(L_1)$, where

$$\mathcal{D}(L_1) := \{f \in \mathcal{L}^2_w(I) \mid f \in \Delta,\, l[f] \in \mathcal{L}^2_w(I)\}.$$

We write out Green's identity in the form

$$\int_\alpha^\beta [(\overline{g} L_1 f - f \overline{L_1 g}) w] = [fg](\beta) - [fg](\alpha)$$

and note that if $f, g \in \mathcal{D}(L_1)$, then there exist limits

$$\lim_{x \to a+0} [fg](x) (=: [fg](a)), \qquad \lim_{x \to b-0} [fg](x) (=: [fg](b))$$

and one has

$$(L_1 f, g) - (f, L_1 g) = [fg](b) - [fg](a),$$

where $(\cdot, \cdot)$ is the inner product on $\mathcal{L}^2_w(I)$.

The operator L_1 is called the maximal operator generated by the quasi-differential expression l in the space $\mathcal{L}^2_w(I)$.

Next, let $\mathcal{D}_0(l)$ be the set of all compactly supported functions in $\mathcal{D}(L_1)$, and let

$$\mathcal{D}(L_0) := \{f \in \mathcal{D}(L_1) \mid [fg](b) = [fg](a)\, \forall g \in \mathcal{D}(L_1)\}.$$

Clearly, $\mathcal{D}(L_1) \cap \mathcal{D}_0(l) \subset \mathcal{D}(L_0)$. Furthermore, the set $\mathcal{D}(L_1) \cap \mathcal{D}_0(l)$ is everywhere dense in $\mathcal{L}^2_w(I)$ [27, Appendix A, Theorem 1]. Consequently, the formula

$$L_0 f = l[f], \qquad f \in \mathcal{D}(L_0),$$

defines a closed symmetric operator with dense domain $\mathcal{D}(L_0)$. Note that $\mathcal{D}(L_0)$ can also be characterized as

$$\mathcal{D}(L_0) = \{f \in \mathcal{D}(L_1) \mid [fg](b) = [fg](a) = 0 \,\forall g \in \mathcal{D}(L_1)\}.$$

This operator is called the minimal operator generated by l.

It is well known that

$$L_0^* = L_1, \quad L_1^* = L_0.$$

Now we define the deficiency subspaces

$$D^{\pm} := \{f \in \mathcal{D}(L_0^*) \mid L_0^* f = \pm i f\} = \{f \in \Delta \cap \mathcal{L}_w^2(I) \mid l[f] = \pm i f\}$$

and deficiency indices $d^{\pm} := \dim D^{\pm}$ of the operator L_0.

It is well known that the deficiency subspaces and the domains of L_1 and L_0 are related by

$$\mathcal{D}(L_1) = \mathcal{D}(L_0) \oplus D^+ \oplus D^- \tag{9}$$

and that the deficiency indices $d^{\pm}$ satisfy the inequality $0 \le d^{\pm} \le 2$.

Note that if the expression l is regular, then, obviously, $d^+ = d^- = 2$. Moreover, the problem on the deficiency indices of L_0 on an interval I with two singular endpoints can be reduced to the same problem for the minimal operators generated by l on the intervals $(a, c]$ and $[c, b)$, where $c \in (a, b)$ is arbitrary. Let a be a regular and b a singular endpoint of I. Then the deficiency indices d^+ and d^- of L_0 coincide ($d^+ = d^- =: d$) and can take the values $d = 1$ or $d = 2$. Following Weyl, one speaks of the case of a limit point (respectively, limit circle) for l at b if $d = 1$ (respectively, $d = 2$). The following theorem holds.

Theorem 2. *Let the functions p, q, r, and w satisfy the conditions listed at the beginning of Section* 1. *If all solutions of* (3) *lie in $\mathcal{L}_w^2(I)$ for some $\lambda_0 \in C$, then all solutions of this equation lie in $\mathcal{L}_w^2(I)$ for every $\lambda \in C$.*

This theorem obviously implies that if one of the deficiency indices is 2, then so is the other.

Let us state one more theorem pertaining to the case of $d^+ = d^- = 2$.

Theorem 3. *A necessary and sufficient condition for all solutions of* (3) *to lie in $\mathcal{L}_w^2(I)$ for each $\lambda \in C$ is that, for every sequence $(\alpha_k, \beta_k) \subset (a, b)$, $k = 1, 2, \dots,$ of pairwise disjoint intervals,*

$$\sum_{k=1}^{+\infty} \left(\int_{\alpha_k}^{\beta_k} w(x)\, dx \int_{\alpha_k}^{x} |K(x,t)|^2 w(t)\, dt \right)^{1/2} < +\infty,$$

where $K(x,t)$ is the Cauchy function of the equation $l[f] = 0$, i.e., the solution of this equation with respect to the variable x with the initial conditions $K(x,t)|_{x=t} = 0$ and $K^{[1]}(x,t)|_{x=t} = 1$.

Weyl's paper [36], published in 1910, was the first to introduce the limit point–limit circle classification of singular second-order differential operators generated by expressions of the form $-(pf')' + qf$, where p and q are continuous functions, and the first test for the limit point case for the corresponding operators. The problem of finding necessary and sufficient conditions on the coefficients of a second-order differential expression guaranteeing, say, that the limit circle case occurs was not posed in [36]. However, Everitt later stated this problem and indicated its importance in the survey paper [25], and Zettl included it in a list of unsolved problems in his 2005 book [38]. Theorem 1 in [9] (see also [10] and references therein), of which Theorem 3 is a special case, solves a similar problem stated for arbitrary differential operators of arbitrary order and for the spaces $\mathcal{L}_w^p$ (see

[10] and references therein), where $\mathcal{L}_w^p(I)$, $1 \le p < +\infty$, is the space of all measurable functions f such that $|f|^p w$ is integrable on I. We return to this problem at the end of Section 3.1.

When proving Theorem 8 in Section 3, we use the following restatement of Theorem 3.

Theorem 4. *Let a be a regular endpoint of an interval I, and let b be a singular endpoint of I. A necessary and sufficient condition for the limit point case to occur for the expression l at b is that, for some sequence $(\alpha_k, \beta_k) \subset (a, b)$, $k = 1, 2, \dots$, of pairwise disjoint intervals,*

$$\sum_{k=1}^{+\infty} \Big(\int_{\alpha_k}^{\beta_k} w(x)\, dx \int_{\alpha_k}^{x} |K(x,t)|^2 w(t)\, dt \Big)^{1/2} = \infty. \tag{10}$$

To make the picture complete, we conclude the subsection with the proof of this theorem.

Proof. Let $t_0 \in I$, and let u and v form a fundamental solution system of the equation $l[f] = 0$ with the initial conditions $u(t_0) = v^{[1]}(t_0) = 1$ and $u^{[1]}(t_0) = v(t_0) = 0$. First, let us show that the functions $K(x,t)$, $u(x)$, and $v(x)$ are related by the formula

$$K(x,t) = \bar{u}(t)v(x) - \bar{v}(t)u(x). \tag{11}$$

Indeed, all three functions are solutions of the same equation. Hence $K(x,t) = c_1(t)u(x) + c_2(t)v(x)$. Green's formula obviously implies that if functions f and g are solutions of the equation $l[f] = 0$, then the form $[fg](x)$ is independent of x. Consequently,

$$[K(\cdot,t)v(\cdot)](x)\big|_{x=t_0} = c_1(t), \qquad [K(\cdot,t)v(\cdot)](x)\big|_{x=t} = -\bar{v}(t).$$

Thus, $c_1(t) = -\bar{v}(t)$. In a similar way, one proves that $c_2(t) = \bar{u}(t)$.

Sufficiency. Formula (11) shows that the function $K(x,t)$ with given x and t is the inner product of the vectors $(\bar{u}(t), \bar{v}(t))$ and $(v(x), -u(x))$. Hence it follows from the Cauchy–Schwarz inequality that

$$|K(x,t)|^2 \le (|u(x)|^2 + |v(x)|^2)(|u(t)|^2 + |v(t)|^2).$$

Let $[\alpha, \beta] \subset I$. We first multiply the last inequality by $w(t)$ and integrate with respect to t from α to x and then multiply the resulting inequality by $w(x)$ and integrate with respect to x from α to β; elementary computations show that

$$\int_\alpha^\beta (|u(x)|^2 + |v(x)|^2) w(x)\, dx \ge \sqrt{2} \Big(\int_\alpha^\beta w(x)\, dx \int_\alpha^x |K(x,t)|^2 w(t)\, dt \Big)^{1/2}. \tag{12}$$

Now let $(\alpha_k, \beta_k) \subset I$ be a sequence of intervals satisfying condition (10). We apply inequality (12) with $\alpha = \alpha_k$ and $\beta_k = \beta$ to each term on the right-hand side in the inequality

$$\int_a^b (|u|^2 + |v|^2) w \ge \sum_{k=1}^{\infty} \int_{\alpha_k}^{\beta_k} (|u|^2 + |v|^2) w$$

and use (10) to obtain

$$\int_a^b (|u|^2 + |v|^2) w = +\infty. \tag{13}$$

Thus, for some $\lambda_0 \in C$ (namely, for $\lambda_0 = 0$), not all solutions of (3) lie in $\mathcal{L}_w^2(I)$. Consequently, the limit circle case does not hold for l at b by Theorem 2. We have proved the sufficiency.

Necessity. Assume that the limit point case holds for l; i.e., (13) holds. Let us show then that

$$\int_{\alpha}^{\beta} w(x)\,dx \int_{\alpha}^{x} |K(x,t)|^2 w(t)\,dt = +\infty \tag{14}$$

for every $\alpha \in I$. Assume the contrary:

$$J := \int_{\alpha_0}^{\beta} w(x)\,dx \int_{\alpha_0}^{x} |K(x,t)|^2 w(t)\,dt < +\infty$$

for some $\alpha_0 \in I$. We use (11) and note then that

$$\int_{\alpha_0}^{\beta} w(x)\,dx \int_{\alpha_0}^{\beta} |K(x,t)|^2 w(t)\,dt = 2J < +\infty.$$

It follows by Fubini's theorem that $K(\cdot,t) \in \mathcal{L}^2_w(I)$ for almost all $t \geq \alpha_0$. On the other hand, the functions u and v are linearly independent and continuous. Hence, first, there exist numbers t_1^0 and t_2^0 ($> \alpha_0$) such that

$$\begin{vmatrix} u(t_1) & v(t_1) \\ u(t_2) & v(t_2) \end{vmatrix} \neq 0$$

for $t_1 = t_1^0$ and $t_2 = t_2^0$; second, there exists a $\delta > 0$ such that this inequality holds for all $t_1 \in (t_1^0 - \delta, t_1^0 + \delta)$ and $t_2 \in (t_2^0 - \delta, t_2^0 + \delta)$. Thus, there exist numbers t_1^1 and t_2^1 in the respective intervals such that $K(\cdot, t_i^1) \in \mathcal{L}^2_w(I)$ for $i = 1, 2$. Moreover, it is obvious that the functions $K(\cdot, t_1^1)$ and $K(\cdot, t_2^1)$ are linearly independent. Consequently, all solutions of the equation $l[f] = 0$ lie in $\mathcal{L}^2_w(I)$, which contradicts condition (13). The proof of (14) is complete. It remains to note that if (14) holds, then there exists a sequence of pairwise disjoint intervals $(\alpha_k, \beta_k) \subset I$, $k = 1, 2, \ldots$, such that, say,

$$\int_{\alpha_k}^{\beta_k} w(x)\,dx \int_{\alpha_k}^{x} |K(x,t)|^2 w(t)\,dt \geq 1,$$

and (10) obviously holds for this sequence. The proof of Theorem 4 is complete. □

2. It is well known that the operator L_0 admits self-adjoint extensions if and only if $d^+ = d^- = d$. If $d = 0$, then it follows from (9) that $\mathcal{D}(L_1) = \mathcal{D}(L_0)$, i.e., that the operator L_1 (and hence L_0) is self-adjoint. This case obviously holds if and only if

$$[fg](a) = [fg](b) = 0$$

for any $f, g \in \mathcal{D}(L_1)$.

Now let $d = 1$. Assume that either a is a regular endpoint or the limit circle case holds for it and that the limit point case holds for b. Then the domain $\mathcal{D}(L)$ of every self-adjoint extension L of L_0 is given by

$$\mathcal{D}(L) = \{y \in \mathcal{D}(L_1) \colon [yw](a) = 0\},$$

where w is a function such that $w \in \mathcal{D}(L_1) \setminus \mathcal{D}(L_0)$ and $[ww](a) = 0$. The converse is also true.

The following theorem gives a description of all self-adjoint extensions of L_0 for $d = 2$.

Theorem 5. *Let $d^+ = d^- = 2$. The domain $\mathcal{D}(L)$ of every self-adjoint extension L of L_0 is the set of functions $y \in \mathcal{D}(L_1)$ satisfying the conditions*

$$[yw_k](b) - [yw_k](a) = 0, \qquad k = 1, 2, \tag{15}$$

where $w_1, w_2 \in \mathcal{D}(L_1)$ are some functions linearly independent modulo $\mathcal{D}(L_0)$ such that

$$[w_j w_k](b) - [w_j w_k](a) = 0, \qquad j, k = 1, 2. \tag{16}$$

Conversely, for arbitrary functions $w_1, w_2 \in \mathcal{D}(L_1)$ linearly independent modulo $\mathcal{D}(L_0)$ and satisfying condition (16)*, the set of functions $y(x) \in \mathcal{D}(L_1)$ satisfying condition* (15) *is the domain of a self-adjoint extension L of L_0.*

In some cases, one can represent conditions (15) and (16) in a more explicit form. In particular, the following theorem holds.

Theorem 6. *Assume that the expression l is regular on the interval I. Every self-adjoint extension L of L_0 is determined by linearly independent boundary conditions of the form*

$$\alpha_{i1}y(a) + \alpha_{i2}y^{[1]}(a) + \beta_{i1}y(b) + \beta_{i2}y^{[1]}(b) = 0, \qquad i = 1, 2, \tag{17}$$

and moreover,

$$AJA^* = BJB^*, \tag{18}$$

where

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Conversely, any linearly independent boundary conditions (17) *satisfying relations* (18) *define some self-adjoint extension L of L_0.*

If the point a of the interval I is a regular endpoint for l and $d = 1$, then every self-adjoint extension L of L_0 is determined by a boundary condition of the form

$$\alpha y(a) + \beta y^{[1]}(a) = 0,$$

where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 \neq 0$ and $\alpha\bar{\beta} = \bar{\alpha}\beta$. The converse is also true.

Now let us state a theorem describing the case in which a is a regular endpoint, b is a singular endpoint, and $d = 2$.

Let u and v be the fundamental solution system of the equation $l[f] = 0$ with the conditions $u(a) = v^{[1]}(a) = 1$ and $u^{[1]}(a) = v(a) = 0$ (see Theorem 1), and let $(Sf)(x)$, $f \in \mathcal{D}(L_1)$, be the linear transformation

$$(Sf)(x) = \begin{pmatrix} (Sf)_1(x) \\ (Sf)_2(x) \end{pmatrix} = \begin{pmatrix} u & v \\ u^{[1]} & v^{[1]} \end{pmatrix}^{-1} \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}(x).$$

We see from this definition of S that

$$(Sf)_1(x) = W\{u, f\}(x), \qquad (Sf)_2(x) = W\{f, v\}(x),$$

where $W\{f_1, f_2\}$ is the Wronskian of the system of functions f_1, f_2 (with the derivatives everywhere replaced with first-order quasi-derivatives). It follows from the last formula and the definition of quasi-derivative that

$$(Sf)_1(x) = (Sf)_1(a) - \int_a^x f(t)\overline{v(t)}\,dt, \qquad (Sf)_2(x) = (Sf)_2(a) + \int_a^x f(t)\overline{u(t)}\,dt.$$

Consequently, there exists a limit $\lim_{x\to b-0}(Sf)(x) =: (Sf)(b)$.

Theorem 7. *Let a be a regular endpoint of I for the expression l, let b be a singular endpoint of I, and let $d = 2$. Then every self-adjoint extension L of the operator L_0 is determined by linearly independent boundary conditions of the form*

$$\alpha_{i1}y(a) + \alpha_{i2}y^{[1]}(a) + \beta_{i1}(Sf)_1(b) + \beta_{i2}(Sf)_2(b) = 0, \qquad i = 1, 2,$$

where the complex numbers α_{ik} and β_{ik} satisfy conditions (17)*. The converse is also true.*

Clearly, a similar theorem holds for the case in which both endpoints of I are singular and $d = 2$.

Interestingly, self-adjoint extensions L_0 can be defined under the assumptions of Theorem 7 not only by analogs of Sturm type boundary conditions (separated boundary conditions) but also by other types of boundary conditions (e.g., analogs of periodic boundary conditions).

Theorems 6 and 7 provide broad possibilities for studying the spectrum and, more generally, the Green function of the corresponding boundary value problems. In particular, theorems on uniform resolvent convergence apparently hold in this context. However, a detailed exposition of these results is beyond the framework of the present paper. We note only that theorems on uniform resolvent convergence under the assumptions of Theorem 6 were given in [14] and [15] for differential expressions $l[f]$ of the form (7) and later in [31] for $l[f]$ of the form (6).

Krein's paper [6] deals with spectral analysis of self-adjoint extensions of the operator generated in $\mathcal{L}^2_\rho[0,+\infty)$ by the expression $\frac{1}{\rho}(-y'' + q(x)y)$ for the case in which the deficiency indices of L_0 are $(2,2)$ and the self-adjoint extension is defined by Sturm type boundary conditions. Thus, the starting point of [6] is the assertion of Theorem 7 for the case in which the first boundary condition is posed at zero and the second, at infinity; moreover, all possible conditions of this kind are considered. Fulton's paper [30] was the first to introduce the transformation S for the case in which $\rho = 1$ and all other assumptions in [6] are satisfied; this transformation was used to describe all self-adjoint extension of L_0 with separated boundary conditions and write out the Titchmarsh–Weyl $m(\lambda)$-function for the corresponding operator.

Theorems 5 and 6 and the subsequent remarks can be found in [11, Section 18.1–18.3] and, in a slightly different form, in [1, Supplement 2, Sections 125 and 127] for quasi-differential operators of arbitrary even order with real coefficients and with quasi-derivatives defined in a special way. However, these books present a general theory of self-adjoint extensions of symmetric operators with finite deficiency indices in abstract Hilbert spaces (see [11, Section V.14] and [1, Chapter VIII]), which permits one to construct the corresponding theory for quasi-differential expressions associated with matrices F of general form (see [27]). Most of [27] deals with the theory outlined here and uses symplectic algebra techniques to this end.

Now let L be an arbitrary self-adjoint extension of L_0. Clearly, Krein's method of directing functionals (see [11, Section 6.14] and [1]) can be used to prove the theorem on eigenfunction expansions (the spectral expansion) for the self-adjoint operator L and in particular obtain inversion formulas.

A modern version of the proof of the Titchmarsh–Weyl theorem on eigenfunction expansions for an operator L with separated boundary conditions can be found in a joint paper by Bennewitz and Everitt in [18, pp. 137–171]. Although this paper deals with an equation of the form (4), one can see from the joint paper [19] by the same authors that everything remains true for an equation of the form (3).

Note also that Levitan viewed the spectral expansion theorem for an ordinary symmetric second-order differential equation with real coefficients and with one singular endpoint as an elementary theorem of calculus. In particular, he had in mind his proof published in [7]. In [8], he extended this proof to symmetric differential equations of higher order with real coefficients. Apparently, his proof also remains valid for equations of the form (3).

3. Limit point and limit circle theorems

1. In this section, we assume that $I = [0, +\infty)$ and p_0, q_0, and p_1 satisfy the conditions on I listed at the beginning of Section 1.2. Let $(\alpha_n, \beta_n) \subset I$, $n = 1, 2, \dots$, be a sequence of pairwise disjoint intervals, and let $K(x,t)$ be the Cauchy function of the equation $l[f] = 0$. It follows from Theorem 1 that, for given n, the function $K(x,t)$ is uniquely determined in the square $\{(x,t) \mid x, t \in [\alpha_n, \beta_n]\}$ by the values of the functions $p_0(x)$, $q_0(x)$, and $p_1(x)$ for $x \in [\alpha_n, \beta_n]$. Thus, if condition (10) is satisfied, then, regardless of the values of these functions outside the set $\bigcup_{n=1}^{+\infty}[\alpha_n, \beta_n]$, the limit point case holds for the expression l; it is only required that p_0^{-1}, $p_1^2 p_0^{-1}$, $q_0^2 p_0^{-1} \in L^1_{\mathrm{loc}}(I)$. Using this fact, let us prove the following theorem.

Theorem 8. *Suppose that $p_0^{-1}, p_1^2 p_0^{-1}, q_0^2 p_0^{-1}, w \in L^1_{\mathrm{loc}}(I)$, $w(x) > 0$ a.e. on I, and (α_n, β_n), $n = 1, 2, \dots$, is a sequence of pairwise disjoint intervals such that $p_0(x) = 1$, $q_0(x) = 0$, $w(x) = 1$, and $p_1(x) = \sigma(x)$ for $x \in [\alpha_n, \beta_n]$, $n = 1, 2, \dots$, and the function $\sigma(x)$ is absolutely continuous on each of the intervals $[\alpha_n, \beta_n]$. In addition, assume that one of the following four conditions is satisfied*:

(a) *$\sigma'(x) = -k_n$ a.e. for $x \in [\alpha_n, \beta_n]$, where the $k_n > 0$ are constants such that*

$$\sum_{n=1}^{+\infty} \frac{1}{k_n}\{\gamma_n^2 - \sin^2 \gamma_n\}^{1/2} = +\infty;$$

here $\gamma_n = \sqrt{k_n}(\beta_n - \alpha_n)$.

(b) *The sequences k_n and γ_n are the same as in* (a), *$\sigma'(x) \geq k_n$ a.e. for $x \in [\alpha_n, \beta_n]$, and*

$$\sum_{n=1}^{+\infty} \frac{1}{k_n}\{\sinh^2 \gamma_n - \gamma_n^2\}^{1/2} = +\infty.$$

(c) *One has*

$$\int_{\alpha_n}^{\beta_n} (\beta_n - x)(x - \alpha_n)\sigma'_-(x)\,dx \leq \nu(\beta_n - \alpha_n),$$

$$\sum_{n=1}^{+\infty}\left((\beta_n - \alpha_n)\int_{\alpha_n}^{\beta_n} [(\beta_n - x)(x - \alpha_n)]^2 \sigma'_+(x)\,dx\right)^{1/2} = +\infty,$$

where $\sigma'_-(x) = -\min\{\sigma'(x), 0\}$, $\sigma'_+(x) = \sigma'(x) + \sigma'_-(x)$, and $\nu \in [0,1)$ is a constant. Q1

(d) *One has*

$$\sum_{n=1}^{+\infty} \frac{(\beta_n - \alpha_n)^2}{1 + \left((\beta_n - \alpha_n)\int_{\alpha_n}^{\beta_n} \sigma'_-(x)\,dx\right)^{1/2}} = +\infty.$$

Then the limit point case holds for the expression l in the space $\mathcal{L}^2_w(I)$.

Proof. Assume that $0 \leq \alpha < \beta < +\infty$, the expression $l[f]$ can be represented in the form (7) for $x \in (\alpha, \beta)$, and the function $\sigma(x)$ is absolutely continuous on the interval $[\alpha, \beta]$.

If $\sigma'(x) = -k$ $(k > 0)$ for $x \in [\alpha, \beta]$, then

$$K(x,t) = \frac{\sin\sqrt{k}(x - t)}{\sqrt{k}} \qquad \text{for } \alpha \leq t \leq x \leq \beta,$$

and hence

$$\int_\alpha^\beta dx \int_\alpha^x K^2(x,t)\,dt = \frac{1}{4k^2}[k(\beta - \alpha)^2 - \sin^2\sqrt{k}(\beta - \alpha)].$$

Now we set $[\alpha, \beta] = [\alpha_n, \beta_n]$ and $k = k_n$ in this equation and apply Theorem 4 to find that the assertion of Theorem 8 holds under condition (a).

The Cauchy function $K(x,t)$ satisfies the differential equation

$$f'' + \sigma'_-(x)f = \sigma'_+(x)f \tag{19}$$

with respect to $x(\in(\alpha,\beta))$ and the initial conditions

$$K(x,t)|_{x=t} = 0 \quad \text{and} \quad K'_x(x,t)|_{x=t} = 1, t \in [\alpha,\beta].$$

Consequently, this function satisfies the Volterra integral equation

$$K(x,t) = u(x,t) + \int_t^x u(x,\tau)\sigma'_+(\tau)K(\tau,t)\,d\tau,$$

where $u(x,t)$ is the Cauchy function of the equation $f'' = -\sigma'_-(x)f$.

We find by the successive approximation method that $u(x,t)$ can be represented in the form

$$u(x,t) = x - t - \int_t^x (x-\tau)\sigma'_-(\tau)(\tau-t)\,d\tau + \sum_{m=1}^{+\infty}\int_t^x [u_{2m}(x,\tau) - u_{2m+1}(x,\tau)](\tau-t)\,d\tau, \tag{20}$$

where

$$u_1(x,\tau) = (x-\tau)\sigma'_-(\tau),$$
$$u_{m+1}(x,\tau) = \int_\tau^x u_1(x,\xi)u_m(\xi,\tau)\,d\xi, \qquad m = 1,2,\dots,$$

and $K(x,t)$ can be represented in the form

$$K(x,t) = u(x,t) + \sum_{m=1}^{+\infty}\int_t^x K_m(x,\tau)u(\tau,t)\,d\tau, \tag{21}$$

where

$$K_1(x,\tau) = u(x,\tau)\sigma'_+(\tau),$$
$$K_{m+1}(x,\tau) = \int_\tau^x K_1(x,\xi)K_m(\xi,\tau)\,d\xi, \qquad m = 1,2,\dots$$

Moreover, if $\sigma'_-(x) = 0$ and $\sigma'_+ \geq k$ $(k > 0)$ for $x \in [\alpha,\beta]$, then $u(x,t) = x - t$, and it readily follows from (21) that

$$K(x,t) \geq \frac{1}{\sqrt{k}}\sinh\sqrt{k}(x-t), \qquad \alpha \leq t \leq x \leq \beta.$$

Thus, in this case we have

$$\int_\alpha^\beta dx \int_\alpha^x K^2(x,t)\,dt \geq \frac{1}{4k^2}[\sinh^2\sqrt{k}(\beta-\alpha) - k(\beta-\alpha)^2].$$

Now we set $[\alpha,\beta] = [\alpha_n,\beta_n]$ and $k = k_n$ in this inequality and apply Theorem 4, thus proving the assertion of Theorem 8 under condition (b).

Now assume that

$$\int_\alpha^\beta (\beta-\tau)(\tau-\alpha)\sigma'_-(\tau)\,d\tau \leq \nu(\beta-\alpha), \tag{22}$$

Q1 where $\nu \in (0,1]$ is some constant. The function

$$f(x,t) = \frac{1}{x-t}\int_t^x (x-\tau)(\tau-t)\sigma'_-(\tau)\,d\tau,$$

$\alpha \le t < x \le \beta$, increases in x on the set $(t, \beta]$ and decreases in t on the set $[\alpha, x)$. Hence inequality (22) implies the inequality

$$\int_t^x (x-\tau)(\tau-t)\sigma'_-(\tau)\,d\tau \le \nu(x-t), \qquad \alpha \le t \le x \le \beta. \tag{23}$$

The last inequality, together with the definition of the functions $u_m(x,t)$, implies that

$$u_m(x,t) - u_{m+1}(x,t) \ge (1-\nu)u_m(x,t), \qquad m = 1, 2, \dots \tag{24}$$

We take into account inequality (23) and the inequality $u_{2m}(x,t) - u_{2m+1}(x,t) \ge 0$, $\alpha \le t \le x \le \beta$, which follows from (24), and obtain $u(x,t) \ge (1-\nu)(x-t)$ for $\alpha \le t \le x \le \beta$ and $0 < \nu < 1$. Now it follows from (21) that

$$K(x,t) \ge (1-\nu)(x-t) + (1-\nu)^2 \int_t^x (x-\tau)(\tau-t)\sigma'_+(\tau)\,d\tau.$$

Thus, if α and β are numbers such that condition (22) is satisfied for $0 < \nu < 1$, then

$$K^2(x,t) \ge 2(1-\nu)^3(x-t) \int_t^x (x-\tau)(\tau-t)\sigma'_+(\tau)\,d\tau$$

for $\alpha \le t \le x \le \beta$, Consequently,

$$\begin{aligned}\int_\alpha^\beta dx \int_\alpha^x K^2(x,t)\,dt \ge 2(1-\nu)^3 \int_\alpha^\beta dx \int_\alpha^x (x-t)\left(\int_t^x (x-\tau)(\tau-t)\sigma'_+(\tau)\,d\tau\right)dt \\ = \frac{(1-\nu)^3}{3}(\beta-\alpha)\int_\alpha^\beta [(\beta-x)(x-\alpha)]^2\sigma'_+(x)\,dx.\end{aligned}$$

To prove the assertion of Theorem 8 in case (c), it remains to set $[\alpha, \beta] = [\alpha_n, \beta_n]$ and apply Theorem 4.

Let us define a partition of the interval $[\alpha, \beta]$ by points $\alpha =: a_0 < a_1 < \ldots < a_n := \beta$ as follows. Note that the function $f(x, a_0)$ increases for $x > a_0$ and $f(x, a_0) \to 0$ as $x \to a_0+$; hence either $f(x, a_0) < 1$ for all $x \in (\alpha, \beta]$, or the equation $f(x, a_0) = 1$ has a solution on $(\alpha, \beta]$. In the former case, we set $a_1 := \beta$ and complete the partition. In the latter case, we set $a_1 := \inf\{x \mid x > a_0,\ f(x, a_0) = 1\}$. If it turns out that $a_1 = \beta$, we also complete the partition; otherwise, we define a_2 by replacing a_0 with a_1 in the preceding argument and continue the procedure. Now let us show that the partition of $[\alpha, \beta]$ thus constructed contains finitely many points and estimate their number from above. Let $N \ge 2$ be a positive integer, and assume that the number of points in the partition is greater than N. Then

$$f(a_m, a_{m-1}) \le 1, \qquad m = 1, 2, \dots, N-1.$$

It readily follows from this inequality that

$$\frac{1}{a_m - a_{m-1}} \le \frac{1}{4}\int_{a_{m-1}}^{a_m} \sigma'_-(\tau)\,d\tau, \qquad m = 1, 2, \dots, N-1,$$

and we obtain, by summing with respect to m,

$$\sum_{m=1}^{N-1} \frac{1}{a_m - a_{m-1}} \le \frac{1}{4}\int_\alpha^\beta \sigma'_-(\tau)\,d\tau.$$

Next, we apply the arithmetic–harmonic mean inequality and find after some elementary transformations that

$$N \le \frac{1}{2}\left((\beta-\alpha)\int_\alpha^\beta \sigma'_-(\tau)\,d\tau\right)^{1/2} + 1.$$

Now note that, by construction, inequality (22) becomes an equality with $\nu = 1$ for $\alpha = a_{m-1}$ and $\beta = a_m$, and hence inequality (23) holds for these α, β, and ν. Thus, by formulas (20) and (21),

$$K(x,t) \geq x - t - \int_t^x (x-\tau)\sigma'_-(\tau)(\tau - t)\,d\tau, \qquad a_{m-1} \leq t \leq x \leq a_m. \tag{25}$$

Now let $[\alpha, \beta] = [\alpha_n, \beta_n]$, $n = 1, 2, \ldots$ We divide the interval $[\alpha_n, \beta_n]$ into N_n parts by points

$$\alpha_n =: a_n^{(0)} < a_n^{(1)} < \ldots < a_n^{(N_n)} := \beta_n$$

using the method indicated above. Inequality (25) holds on each $[a_n^{(m-1)}, a_n^{(m)}]$ with $a_m = a_n^{(m)}$, $m = 1, 2, \ldots, N_n$, and $a_n = a_n^{(0)}$. Next, we use the Cauchy–Schwarz inequality, then estimate the resulting integrals with the use of inequality (25), and finally sum with respect to m, thus obtaining

$$\sum_{m=1}^{N_n} \left(\int_{a_n^{(m-1)}}^{a_n^{(m)}} dx \int_{a_n^{(m-1)}}^{x} K^2(x,t)\,dt \right)^{1/2} \geq c \sum_{m=1}^{N_n} (a_n^{(m)} - a_n^{(m-1)})^2 \geq \frac{c(\beta_n - \alpha_n)^2}{N_n},$$

where $c = 5\sqrt{2}/48$. It remains to take into account the upper bound for N_n, sum these inequalities with respect to n, and apply Theorem 4. Thus, the assertion of Theorem 8 holds under condition (d). The proof of Theorem 8 is complete. □

It is easily seen that the upper bound for N is an estimate for the number of zeros of a nonzero solution of (19) on the interval $[\alpha, \beta]$, and inequality (22) with $\nu = 1$ is a condition under which an arbitrary nonzero real solution of the same equation has at most one zero on $[\alpha, \beta]$. Thus, we have obtained yet another proof of Theorem 5.1 and inequality (5.9) in [16, Section 11.5]. Moreover, the number N also satisfies the estimate

$$N \leq \int_\alpha^\beta (x - \alpha)\sigma'_-(x)\,dx + 1$$

(see inequality (5.11) in [16, Section 11.5]). One can use this inequality to show that Theorem 8 remains valid if condition (d) is replaced by the condition

$$\sum_{n=1}^{+\infty} \frac{(\beta_n - \alpha_n)^2}{1 + \int_{\alpha_n}^{\beta_n} (x - \alpha_n)\sigma'_-(x)\,dx} = +\infty. \tag{26}$$

Needless to say, there are quite a few well-known sufficient conditions for the limit point and limit circle cases to hold provided that $l[f]$ has the form (6),

$$l[f] = -(pf')' + qf, \qquad p(:= p_0),\ q(:= p_1') \in L^1_{\mathrm{loc}}(I)$$

(e.g., see [26] and references therein). Basic information about Weyl's limit point–limit circle alternative for expressions $l[f]$ of this form can be found in [38, Section 7.4]. The same book [38, Section 7.5] provides quite an impressive list of references on the topic. In what follows, we present a brief analysis of how the assertions in Theorem 8 are related to the result obtained earlier by other authors for the case of $l[f] = -f'' + qf$, where $q(:= \sigma') \in L^1_{\mathrm{loc}}(I)$ (see (6)).

Examples of step function potentials q such that the limit point or limit circle case is realized for l were used earlier in various issues of quantum physics and differential equations (see [23]). I do not know any other conditions of the form (a). Neither am I aware of any papers by other authors where conditions of the form (c) have been obtained for which the limit point case is realized.

Ismagilov [3] showed, as early as in 1963, that if

$$q(x) \geq k_n > 0 \quad \text{for } x \in (\alpha_n, \beta_n) \qquad \text{and} \quad \sum_{n=1}^{+\infty} \sqrt{k_n}(\beta_n - \alpha_n)^3 = +\infty,$$

then the limit point case holds for l. Later, this theorem was generalized in [12], [26], and [33]. The results obtained in these papers are stated in slightly different terms than in Theorem 8(b), but are close to these.

To compare earlier obtained results with condition (d), we state the following theorem.

Theorem 9. *The limit point case is realized for l if $q \in L^1_{\mathrm{loc}}(I)$ and there exists a sequence of pairwise disjoint intervals $[\alpha_n, \beta_n]$, $n = 1, 2, \dots$, such that one of the following two conditions is satisfied*:

1. *$q(x) \geq -\gamma_n$ for $x \in [\alpha_n, \beta_n]$, where $\gamma_n > 0$ and*

$$\sum_{n=1}^{+\infty} \frac{1}{(\beta_n - \alpha_n)^{-2} + \gamma_n} = +\infty.$$

2. *There exists a sequence of positive numbers ν_n and constants k and K such that*

$$\sum_{n=1}^{+\infty} \nu_n^{-1} = +\infty, \qquad (\beta_n - \alpha_n)^2 \nu_n \geq k, \qquad \int_{\alpha_n}^{\beta_n} q_-(x)\, dx \leq K(\beta_n - \alpha_n)^3 \nu_n^2.$$

The first part of Theorem 9 is the case of $n = 1$ of a theorem in Ismagilov's paper [2], where it is stated for the expression $(-1)^n f^{(2n)} + q(x)f$, and the second part was proved by Eastham [22]. One can readily show that condition 1 in Theorem 9 implies that conditions (26) are satisfied, while condition 2 and condition (d) in Theorem 8 are equivalent. Thus, condition (d) in Theorem 8 and its modification (26) are none other than Theorem 9, and the proof given here is new.

We have already noted that, as was indicated in the survey paper [25] and in [38, Chapter 17, Open problems II], the following problem is a key classification problem for the expression $l[f] = -(pf')' + qf$ in the space $\mathcal{L}^2_w(I)$:

Find necessary and sufficient conditions on the coefficients p, q, and w for the limit point or limit circle case to be realized for the expression l.

Theorem 8 and its proof show that Theorem 4 provides quite an efficient solution of this problem.

2. Here we still assume that $I = [0, +\infty)$ and $w(x) = 1$. Along with the quasi-differential expression $l[f]$ of the form (7) generated by the matrix F, consider the expression $s[f]$ generated by the matrix

$$G = \begin{pmatrix} Q & 1 \\ -Q^2 & -Q \end{pmatrix},$$

where Q, as well as σ, is a real-valued function on I and lies in $L^1_{\mathrm{loc}}(I)$. By S_0 we denote the minimal operator generated by $s[f]$ in the space $\mathcal{L}^2(I)$. Next, let u and v be a fundamental solution system of the equation

$$-(f' - Qf)' - Q(f' - Qf) - Q^2 f = 0.$$

Theorem 10. *Let $\sigma(x), Q(x) \in L^2_{\mathrm{loc}}(I)$ be real-valued functions such that, for some $a > 0$,*

$$\text{(i)} \quad \int_a^{+\infty} (\sigma - Q)^2 \varphi^2 < +\infty, \qquad \text{(ii)} \quad \int_a^{+\infty} |\sigma - Q| \cdot |\varphi\varphi'| < +\infty,$$

where $\varphi = u$ or $\varphi = v$, and

$$\text{(iii)} \qquad \int_a^{+\infty} |\sigma - Q| \cdot |(uv)'| < +\infty.$$

Then the deficiency indices of the operators L_0 and S_0 are the same.

It is well known that if $Q(x)$ is sufficiently smooth and satisfies certain conditions (see Theorem 11 below), then u and v form a fundamental solution system of the equation

$$-f'' + Q'f = 0$$

and satisfy the inequalities (the Liouville–Green estimates)

$$|u|, |v| \le c(-Q')^{-1/4}, \qquad |u'|, |v'| \le c(-Q')^{1/4},$$

where c is some constant. Using this fact and Theorem 10, one can prove the following theorem.

Theorem 11. *Assume that $Q(x) \in L^1_{\mathrm{loc}}(I)$ has locally absolutely continuous second derivative on $(a, +\infty)$ for some $a > 0$, $Q'(x) < 0$ for $x \in (a, +\infty)$, and the conditions*

$$\int_a^{+\infty} \sqrt{-Q'} = +\infty, \qquad \int_a^{+\infty} (-Q')^{-1/4} \cdot |[(-Q')^{-1/4}]''| < +\infty$$

are satisfied. Moreover, assume that $\sigma(x)$ and $Q(x)$ satisfy

$$\int_a^{+\infty} \frac{(\sigma - Q)^2}{\sqrt{-Q'}} < +\infty, \qquad \int_a^{+\infty} |\sigma - Q| < +\infty.$$

Then the assertion of Theorem 10 holds.

We do not dwell upon generalizations and applications of Theorems 10 and 11, in particular those to differential expressions of the form (8), and refer the reader to Konechnaya's paper [4].

3. Let x_n, $n = 1, 2, \ldots$, be an increasing sequence of positive numbers such that $x_0 = 0$ and $\lim_{n\to+\infty} x_n = +\infty$. In what follows, we assume that the expression $l[f]$ has the form (7) and $\sigma'(x)$ satisfies

$$\sigma'(x) = q(x) + \sum_{k=0}^{+\infty} h_k \delta(x - x_k), \tag{27}$$

where $q(x) \in L^1_{\mathrm{loc}}(I)$, $h_0 = 0$, and the h_k, $k = 1, 2, \ldots$, are some real constants. If the function $q(x)$ satisfies some of conditions (a)–(d) in Theorem 8, say, on the intervals

$$[\alpha_k, \beta_k] = \left[\frac{2x_k + x_{k+1}}{3}, \frac{x_k + 2x_{k+1}}{3}\right],$$

then the limit point case obviously holds for the expression $l[f]$. Thus, Theorem 8 provides various examples of the limit point case for $l[f]$ regardless of the sequences h_k and x_k. In particular, if $q(x) = 0$ for $x \in I$, then $\sigma'_-(x) = 0$ on the above-mentioned sequence of intervals $[\alpha_k, \beta_k]$; by Theorem 8(d), the limit point case holds for $l[f]$ whenever

$$\sum_{k=0}^{+\infty} (x_{k+1} - x_k)^2 = +\infty.$$

In our situation, the expression $l[f]$ of the form (8) is considered, and the conditions given above ensure that the limit point case is realized for this expression. As we have already noted, the expression $l[f]$ of the form (8) is discussed in numerous papers. In particular, it was shown in [20] that the deficiency indices of the operator L_0 generated by this expression in $\mathcal{L}^2(I)$ do not exceed 2 and that Weyl's alternative holds. The fact that the theory of operators generated by quasi-differential expressions with locally integrable coefficients covers the operators generated by expressions $l[f]$ of the form (8) implies these assertions automatically. The paper by Christ and Stolz [34] was the first to provide an example in which the deficiency indices of this operator are $(2, 2)$. Later, quite a few various examples of this type were constructed in [4] and [5]. The state of the art in spectral analysis of operators generated by expressions of the form (8) is presented in [5] and [32]. The paper [17] deals with spectral analysis of lower semibounded Sturm–Liouville operators with potentials of the form (27). The papers [5] and [32] introduce a Jacobi matrix J and establish relationships between the spectral characteristics of operators generated in $\mathcal{L}^2(I)$ by expressions $l[f]$ of the form (8) and the boundary condition $f'(0) = 0$ and the difference operators generated by J in the space l^2.

It is obviously advantageous to include the operators generated by expressions of the form (8) in the class of operators generated by quasi-differential expressions with locally integrable coefficients. To illustrate this, let us state one result due to Malamud and Kostenko [5, 32] and give a simple proof.

Theorem 12. *The minimal closed symmetric operator L_0 generated by $l[f]$ (see (8)) in $\mathcal{L}^2(I)$ has deficiency indices $(2, 2)$ if and only if all solutions of the equation*

$$\frac{Z_{k+1}}{r_{k+1}r_{k+2}d_{k+1}} - \frac{1}{r_{k+1}^2}\left(h_k + \frac{1}{d_k} + \frac{1}{d_{k+1}}\right)Z_k + \frac{Z_{k-1}}{r_k r_{k+1} d_k} = 0, \qquad k = 1, 2, \dots, \tag{28}$$

belong to the space l^2, where $d_k = x_k - x_{k-1}$ and $r_{k+1} = \sqrt{d_k + d_{k-1}}$.

Proof. Let f be an arbitrary solution of the equation $l[f] = 0$. Then

$$f = X_k + Y_k(x - x_{k-1}), \qquad k = 1, 2, \dots,$$

for $x_{k-1} < x < x_k$, where X_k and Y_k are some constants. The functions f and $f^{[1]}$ are locally absolutely continuous on I by Theorem 1. Hence, in particular, $f(x_k-) = f(x_k+)$ and $f'(x_k+) - f'(x_k-) = h_k f(x_k)$; i.e.,

$$\begin{cases} X_k + Y_k d_k = X_{k+1}, \\ Y_{k+1} - Y_k = h_k X_{k+1}, \quad k = 1, 2, \dots \end{cases}$$

We find from this system that the coefficients X_k and Y_k are related by the formula

$$Y_k = \frac{X_{k+1} - X_k}{d_k},$$

and moreover,

$$\frac{X_{k+2}}{d_{k+1}} - \left(h_k + \frac{1}{d_k} + \frac{1}{d_{k+1}}\right)X_{k+1} + \frac{X_k}{d_k} = 0, \qquad k = 1, 2, \dots \tag{29}$$

On the other hand, one can readily establish that

$$\int_0^{+\infty} f^2(x)\,dx = \sum_{k=1}^{+\infty}\int_{x_{k-1}}^{x_k} f^2(x)\,dx = \frac{1}{3}\sum_{k=1}^{+\infty} d_k(X_k^2 + X_k X_{k+1} + X_{k+1}^2),$$

where the quadratic form in parentheses on the right-hand side satisfies the two-sided inequality

$$\frac{1}{2}(X_k^2 + X_{k+1}^2) \le X_k^2 + X_k X_{k+1} + X_{k+1}^2 \le \frac{3}{2}(X_k^2 + X_{k+1}^2).$$

Hence $f \in \mathcal{L}^2(I)$ if and only if the series

$$\sum_{k=1}^{+\infty} d_k(X_k^2 + X_{k+1}^2) = d_1 X_1^2 + \sum_{k=1}^{+\infty}(d_k + d_{k+1})X_{k+1}^2,$$

where X_k satisfies the finite-difference equation (29), converges. By making the change of variables

$$Z_k = \sqrt{d_k + d_{k+1}} X_{k+1} = r_{k+1} X_{k+1}, \qquad k = 1, 2, \ldots,$$

in this equation and by multiplying the resulting equation by $1/r_{k+1}$, we find that Z_k satisfies (28) and thus complete the proof of Theorem 12. $\square$

References

[1] N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, vol. 2, Vyshcha shkola, Kharkov, 1978; English transl., Pitman, Boston, MA–London, 1981. MR509335 (82g:47001b)

[2] R. S. Ismagilov, *Conditions for self-adjointness of differential operators of higher order*, Dokl. Akad. Nauk SSSR **142** (1962), no. 6, 1239–1242; English transl., Sov. Math. Dokl. **3** (1962), 279–283. MR0131594 (24:A1443)

[3] R. S. Ismagilov, *On the self-adjointness of the Sturm–Liouville operator*, Uspekhi Mat. Nauk **18** (1963), no. 5 (113), 161–166. (Russian) MR0155037 (27:4979)

[4] N. N. Konechnaya, *Asymptotic integration of symmetric second-order quasidifferential equations*, Mat. Zametki **90** (2011), no. 6, 875–884; English transl., Math. Notes **90** (2011), no. 5–6, 850–858. MR2962962

[5] A. S. Kostenko and M. M. Malamud, *On the one-dimensional Schrödinger operator with δ-interactions*, Funktsional. Anal. i Prilozhen. **44** (2010), no. 2, 87–91; English transl., Funct. Anal. Appl. **44** (2010), no. 2, 151–155. MR2681961 (2011j:47135)

[6] M. G. Krein, *On the indeterminate case of the Sturm–Liouville boundary problem in the interval* $(0, \infty)$, Izv. Akad. Nauk SSSR Ser. Mat. **16** (1952) no. 4, 293–324. (Russian) MR0052004 (14:558g)

[7] B. M. Levitan, *Expansions in eigenfunctions of second-order differential equations*, Gos. Izdat. Tekhn.-Teor. Lit., Moscow–Leningrad, 1950. (Russian) MR0036918 (12:183e)

[8] B. M. Levitan, *Proof of the theorem on the expansion in eigenfunctions of self-adjoint differential equations*, Dokl. Akad. Nauk SSSR **73** (1950), 651–654. (Russian) MR0039152 (12:502e)

[9] K. A. Mirzoev, *The Cauchy function and $\mathcal{L}^p_w$-properties of solutions of quasidifferential equations*, Uspekhi Mat. Nauk **46** (1991), no. 4, 161–162; English transl., Russ. Math. Surveys **46** (1991), no. 4, 190–191. MR1138966 (93d:34012)

[10] K. A. Mirzoev, *Analogs of limit-point theorems*, Mat. Zametki **57** (1995), no. 3, 394–414; English transl., Math. Notes **57** (1995), no. 3, 275–287. MR1346439 (97c:34045)

[11] V. A. Naimark, *Linear differential operators*, 2d ed., Nauka, Moscow, 1969; English transl. of the 1st ed., Frederick Ungar, New York, 1967 (part 1) and 1968 (part 2). MR0353061 (50:5547)

[12] Yu. B. Orochko, *Sufficient conditions for the self-adjointness of the Sturm–Liouville operator*, Mat. Zametki **15** (1974), no. 2, 271–280; English transl., Math. Notes, **15** (1974), no. 2, 155–160. MR0375002 (51:11198)

[13] Yu. V. Pokornyi et al., *Differential equations on geometric graphs*, Fizmatlit, Moscow, 2004. (Russian)

[14] A. M. Savchuk and A. A. Shkalikov, *Sturm–Liouville operators with singular potentials*, Mat. Zametki **66** (1999), no. 6, 897–912; English transl., Math. Notes **66** (1999), no. 6, 741–753. MR1756602 (2000m:34061)

[15] A. M. Savchuk and A. A. Shkalikov, *Sturm–Liouville operators with distribution potentials*, Trudy Moskov. Matem. Obshch. **64** (2003), 159–212; English transl., Trans. Moscow Math. Soc. **64** (2003), 143–192 MR2030189 (2004j:34198)

[16] P. Hartman, Ordinary differential equations, Wiley, New York–London–Sydney, 1964. MR0171038 (30:1270)

[17] S. Albeverio, A. Kostenko, and M. Malamud, *Spectral theory of semibounded Sturm–Liouville operators with local interactions on a discrete set*, J. Math. Phys. **51** (2010), no. 10, 102102. MR2761288 (2012a:47105)

[18] W. O. Amrein, A. M. Hinz, and D. B. Pearson, *Sturm–Liouville theory. Past and present*, Birkhäuser, Basel, 2005. MR2132131 (2006a:34003)

[19] C. Bennewitz and W. N. Everitt, *Some remarks on the Titchmarsh–Weyl m-coefficient*, Proc. of the Pleijel Conference, University of Uppsala, Amer. Math. Soc., Providence, RI, 1980, pp. 49–108.

[20] D. Buschmann, G. Stolz, and J. Weidmann, *One-dimensional Schrödinger operators with local point interactions*, J. Reine Angew. Math. **467** (1995), 169–186. MR1355927 (97c:47055)

[21] A. C. Dixon, *On the series of Sturm and Liouville, as derived from a pair of fundamental integral equations instead of a differential equation*, Philos. Trans. Roy. Soc. London. Ser. A **211** (1912), 411–432.

[22] M. S. P. Eastham, *On a limit-point method of Hartman*, Bull. London Math. Soc. **4** (1972), 340–344. MR0316801 (47:5349)

[23] M. S. P. Eastham and M. L. Thompson, *On the limit-point, limit-circle classification of second-order ordinary differential equations*, Quart. J. Math. Oxford Ser. (2) **24** (1973), 531–535. MR0417481 (54:5531)

[24] J. Eckhardt, F. Gesztesy, R. Nichols, and G. Teschl, *Weyl–Titchmarsh theory for Sturm–Liouville operators with distributional potentials*, arXiv:1208.4677v3[math.SP]. MR3046408

[25] W. N. Everitt, *On the deficiency index problem for ordinary differential operators 1910–1976*, Differential Equations, Proc. Internat. Conf. Uppsala, 1977, pp. 62–81. MR0477247 (57:16788)

[26] W. N. Everitt, I. W. Knowles, and T. T. Read, *Limit-point and limit-circle criteria for Sturm–Liouville equations with intermittenly negative principal coefficients*, Proc. Roy. Soc. Edinburgh. Sect. A **103** (1986), 215–228. MR866835 (87k:34028)

[27] W. N. Everitt and L. Marcus, *Boundary value problems and symplectic algebra for ordinary differential and quasi-differential operators*, Mathematical Surveys and Monographs, vol. 61, Amer. Math. Soc., Providence, RI, 1999. MR1647856 (2000c:34030)

[28] W. N. Everitt and D. Race, *On necessary and sufficient conditions for the existence of Carathéodory solutions of ordinary differential equations*, Quaestiones Math. **2** (1977/78), no. 4, 507–512. MR0477222 (57:16763)

[29] W. N. Everitt and D. Race, *The regular representation of singular second-order differential expressions using quasi-derivatives*, Proc. London Math. Soc. (3) **65** (1992), no. 2, 383–404. MR1168193 (93m:34140)

[30] Ch. T. Fulton, *Parametrizations of Titchmarsh's $m(\lambda)$-functions in the limit circle case*, Trans. Amer. Math. Soc. **229** (1977), 51–63. MR0450657 (56:8950)

[31] A. Goriunov and V. Mikhailets, *Regularization of singular Sturm–Liouville equations*, Methods Funct. Anal. Topology **16** (2010), no. 2, 120–130. MR2667807 (2011e:34060)

[32] A. S. Kostenko and M. M. Malamud, *1-D Schrödinger operators with local point interactions on a discrete set*, J. Differential Equations **249** (2010), no. 2, 253–304. MR2644117 (2011g:34206)

[33] T. T. Read, *A limit-point criterion for expressions with intermittenly positive coefficients*, J. London Math. Soc. (2) 15 (1977), no. 2, 271–276. MR0437844 (55:10765)

[34] C. S. Christ and G. Stolz, *Spectral theory of one-dimensional Schrödinger operators with point interactions*, J. Math. Anal. Appl. **184** (1994), no. 3, 491–516. MR1281525 (95k:47072)

[35] Ch. Sturm and J. Liouville, *Extrait d'un Mémoire sur le developpement des fonctions en séries dont les différents termes sont assujettis à satisfaire à une même équation différentielle linéaire, contenant un paramètre variable*, J. Math. Pures Appl. **II** (1837), no. 2, 220–223.

[36] H. Weyl, *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen entwicklungen willkürlicher Funktionen*, Math. Ann. **68** (1910), 220–269. MR1511560

[37] R. E. White, *Weak solutions of $(p(x)u'(x))' + g(x)u'(x) + qu(x) = f$ with $q, f \in H_{-1}[a,b]$, $0 < p(x) \in L_\infty[a,b]$, $g(x) \in L_\infty[a,b]$ and $u \in H_1[a,b]$*, SIAM J. Math. Anal. **10** (1979), no. 6, 1313–1325. MR547816 (80k:34022)

[38] A. Zettl, *Sturm–Liouville theory*, Math. Surveys and Monographs, vol. 121, Amer. Math. Soc., Providence, RI, 2005. MR2170950 (2007a:34005)

Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia
E-mail address: `mirzoev.karahan@mail.ru`

Translated by V. E. NAZAIKINSKII

Originally published in Russian