Труды Москов. Матем. Общ.
Том 78 (2017), вып. 1

Trans. Moscow Math. Soc.
2017, Pages 75–83
http://dx.doi.org/10.1090/mosc/265
Article electronically published on December 1, 2017

EXAMPLES OF LATTICE-POLARIZED K3 SURFACES WITH AUTOMORPHIC DISCRIMINANT, AND LORENTZIAN KAC–MOODY ALGEBRAS

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Dedicated to É. B. Vinberg on the occasion of his 80th birthday

Abstract. Using our results about Lorentzian Kac–Moody algebras and arithmetic mirror symmetry, we give six series of examples of lattice-polarized K3 surfaces with automorphic discriminant.

1. Introduction

Using results of our recent paper [13] and our previous papers, we construct a series of examples of even hyperbolic lattices S such that S-polarized complex K3 surfaces X have an automorphic discriminant.

We remind the reader that for an S-polarized K3 surface X a primitive embedding $S \subset S_X$ is fixed where S_X is the Picard lattice of X. We say that such X is degenerate (or it belongs to the discriminant) if there exists $\delta \in (S)^{\perp}_{S_X}$ such that $\delta^2 = -2$. By geometry of K3 surfaces, it then follows that X has no polarization h from S. By the Global Torelli Theorem [25] and epimorphicity of the period map for K3 surfaces [17], moduli of such K3 surfaces are covered by the corresponding hermitian symmetric domains, and algebraic functions on moduli are the corresponding automorphic forms on these domains. A holomorphic automorphic form is called *discriminant* if the support of its zero divisor is equal to the preimage of the discriminant of moduli of such K3 surfaces. If a discriminant automorphic form exists, the discriminant is then called *automorphic*.

For example, for $S = \mathbb{Z}h$ of the rank one with $h^2 = n$ where $n \geq 2$ is even (that is, for usual polarized K3 surfaces), it is well known that the discriminant automorphic form exists for $n = 2$. Borcherds constructed the discriminant automorphic form for $n = 2$ explicitly (see [2, pp. 200–201]). It was shown in [24] that for an infinite number of even $n \geq 2$ the discriminant automorphic form does not exist (probably, it was the first result in this direction). Later, Looijenga [18] showed that the discriminant automorphic form does not exist and the discriminant is not automorphic for all $n > 2$.

Here, we find examples of automorphic discriminants for S-polarized K3 surfaces with $\operatorname{rk} S \geq 2$. See some related finiteness results in Ma [19].

In Section 2, we give necessary definitions for S-polarized K3 surfaces and their discriminants and automorphic discriminants.

In Section 3, we prove the main Theorems 3.1 and 3.2, which give six series of even hyperbolic lattices S of $\operatorname{rk} S \geq 2$ such that S-polarized K3 surfaces have an automorphic

2010 *Mathematics Subject Classification.* Primary 14J15, 14J28, 14J33, 14J60, 14J81.

Key words and phrases. K3 surface, Picard lattice, polarization, moduli space, degeneration, discriminant, Lie algebra, Kac–Moody algebra, root system, automorphic form.

The first author was supported by Laboratory of Mirror Symmetry NRU HSE, RF government grant, ag. N 14.641.31.0001.

discriminant. They are given in Tables 1–6. All these examples are related to Lorentzian Kac–Moody algebras constructed in [13], which are hyperbolic automorphic Kac–Moody (Lie super) algebras. The corresponding discriminant automorphic forms are given in [13]. They define such Kac–Moody algebras $\mathfrak{g}$ and give their denominator identities.

It would be interesting to understand the geometric meaning of these automorphic forms and Kac–Moody algebras for the geometry of the corresponding K3 surfaces. For example, we know that if the weight of the discriminant automorphic form is larger than the dimension of the moduli space, then the moduli space is at least uniruled (see Theorem 3.4 in §3).

A preliminary variant of this paper was published as a preprint [14].

2. Lattice-polarized K3 surfaces and their moduli and discriminants

We refer to [21] about lattices. We recall that a lattice M (equivalently, a non-degenerate integral symmetric bilinear form) means that M is a free $\mathbb{Z}$-module M of a finite rank with symmetric $\mathbb{Z}$-bilinear non-degenerate pairing $x \cdot y \in \mathbb{Z}$ for $x, y \in M$. By signature of M, we mean the signature of the corresponding real form $M \otimes \mathbb{R}$ over $\mathbb{R}$ (that is, the numbers $(t_{(+)}, t_{(-)})$ of positive and negative squares respectively). A lattice M of the signature $(1, \operatorname{rk} M - 1)$ is called *hyperbolic.* A lattice M is called *even* if $x^2 = x \cdot x$ is even for any $x \in M$. By $O(M)$, we denote the automorphism group of a lattice M. Each element $\delta \in M$ with $0 \neq \delta^2$ and $\delta^2 | 2(\delta \cdot M)$ (called the *root*) defines the reflection $s_\delta : x \mapsto x - [(2(x \cdot \delta)/\delta^2]\delta$ for $x \in M$. Evidently, $s_\delta \in O(M)$, $s_\delta(\delta) = -\delta$ and s_δ is identical on $\delta^\perp_M$. By $W^{(2)}(M) \subset O(M)$, we denote the subgroup generated by reflections in all elements $\delta \in M$ with $\delta^2 = -2$ (they are all roots).

Let S be a hyperbolic lattice. Let

$$V(S) = \{x \in S \otimes \mathbb{R} \mid x^2 > 0\}$$

be the *cone of* S. It has two connected components $V^+(S)$ and $V^-(S) = -V^+(S)$. We fix one of them, $V^+(S)$, and the corresponding hyperbolic space $\mathcal{L}(S) = V^+(S)/\mathbb{R}_{++}$. Here $\mathbb{R}_{++}$ denotes all positive real numbers, and $\mathbb{R}_+$ denotes all non-negative real numbers. Let $\operatorname{Amp}(S)$, $\operatorname{Amp}(S)/\mathbb{R}_{++}$ be the interior of a fundamental chamber for the reflection group $W^{(2)}(S)$ in $V^+(S)$ and $\mathcal{L}(S)$ respectively. We fix one of them. Thus, we fix the pair $(V^+(S), \operatorname{Amp}(S))$. It is defined uniquely up to the action of $O(S)$. We call the pair the *ample cone of* S. It is equivalent to $\operatorname{Amp}(S)$ or $\operatorname{Amp}(S)/\mathbb{R}_{++}$.

Let X be a Kälerian K3 surface (for example, see [4], [17], [25]–[27] about such surfaces); that is, X is a non-singular compact complex surface with trivial canonical class K_X (equivalently, $0 \neq \omega_X \in H^{2,0}(X) = \Omega^2[X]$ has the zero divisor) and such that the irregularity $q(X)$ is equal to 0 (equivalently, X has no non-zero holomorphic 1-dimensional holomorphic forms). Then $H^{2,0}(X) = \mathbb{C}\omega_X$, and $H^2(X, \mathbb{Z})$ with the intersection pairing is an even unimodular (that is, with the determinant ± 1) lattice L_{K3} of the signature $(3, 19)$. The primitive sublattice

$$S_X = H^2(X, \mathbb{Z}) \cap H^{1,1}(X) = \{x \in H^2(X, \mathbb{Z}) \mid x \cdot \omega_X = 0\} \subset H^2(X, \mathbb{Z})$$

is the *Picard lattice* of X generated by the first Chern classes of all line bundles over X. Here *primitive* means that $H^2(X, \mathbb{Z})/S_X$ has no torsion. By the definition, S_X can be either negative definite or semi-negative definite or hyperbolic lattice. By Kodaira, the last case is exactly the case when X is projective algebraic.

Further, we assume that X is algebraic. We denote by $V^+(S_X) = V(X)$ the half cone of S_X which contains a polarization of X, and by $\operatorname{Amp}(X) \subset V(X)$ the ample cone of X. Then $\operatorname{Amp}(S_X) = \operatorname{Amp}(X)$ gives an ample cone of S_X; see [25].

Further, we fix an even hyperbolic lattice S and its ample cone $\operatorname{Amp}(S)$.

We remind the reader (e.g., see [6], [7], [20]) that a K3 surface X is called *S-polarized* if a primitive embedding $S \subset S_X$ of lattices is fixed such that $\mathrm{Amp}(S) \cap \mathrm{Amp}(X) \neq \emptyset$.

If instead of the last condition only the conditions $\mathrm{Amp}(S) \cap \overline{\mathrm{Amp}(X)} \neq \emptyset$ and $\mathrm{Amp}(S) \cap \mathrm{Amp}(X) = \emptyset$ are satisfied, then we say that X *is a degenerate S-polarized K3 surface; equivalently, X belongs to the discriminant of moduli of S-polarized K3 surfaces.* By geometry of K3 surfaces (see [25]), it happens only if there exists $\delta \in (S)^{\perp}_{S_X}$ such that $\delta^2 = -2$.

By the Global Torelli Theorem for K3 surfaces [25] and epimorphicity of the period map for K3 surfaces [17], for general S-polarized K3 surfaces we have $S_X = S$ and $\mathrm{Amp}(X) = \mathrm{Amp}(S)$, for non-degenerate S-polarized K3 surfaces X the $(S)^{\perp}_{S_X}$ has no elements δ with $\delta^2 = -2$ and $\mathrm{Amp}(X) \cap \mathrm{Amp}(S) \neq \emptyset$, and for degenerate S-polarized K3 surfaces X the $(S)^{\perp}_{S_X}$ has elements δ with $\delta^2 = -2$ and only $\mathrm{Amp}(S) \cap \overline{\mathrm{Amp}(X)} \neq \emptyset$ is valid; equivalently, X *belongs to the discriminant of moduli of S-polarized K3 surfaces.*

For an S-polarized K3 surface X, let us consider periods

$$H^{2,0}(X) = \mathbb{C}\omega_X \subset T_X \otimes \mathbb{C} \subset T \otimes \mathbb{C},$$

where $T_X = (S_X)^{\perp}_{H^2(X,\mathbb{Z})}$ is the transcendental lattice of X and $T = (S)^{\perp}_{H^2(X,\mathbb{Z})}$ is the transcendental lattice of the S-polarization. The periods give a point in the IV type Hermitian symmetric domain

$$\Omega(T) = \{\mathbb{C}\omega \subset T \otimes \mathbb{C} \mid \omega \cdot \omega = 0 \text{ and } \omega \cdot \overline{\omega} > 0\}^+,$$

where $+$ means a choice of one of two connected components. This point belongs to the complement of the discriminant

$$\mathrm{Discr}(T) = \bigcup_{\beta \in T^{(2)}} D_\beta, \tag{2.1}$$

where $D_\beta = \{\mathbb{C}\omega \in \Omega(T) \mid \omega \cdot \beta = 0\}$ is the rational quadratic divisor which is orthogonal to $\beta \in T$ with $\beta^2 < 0$; we recall that $\beta^2 = -2$ for $\beta \in T^{(2)}$. Of course, $D_\beta = D_{-\beta}$, and we identify $\pm\beta$ in this definition. Further,

$$O^+(T) = \{g \in O(T) \mid g(\Omega(T)) = \Omega(T)\}$$

is the group of automorphisms of T which preserve the connected component $\Omega(T)$.

By considering all possible isomorphism classes $T_1, \dots, T_n$ of the transcendental lattice T for all primitive embeddings $S \subset L_{K3}$, we correspond to an S-polarized K3 surface X a point in

$$\mathrm{Mod}(S) = \bigcup_{1 \le k \le n} G_k \backslash (\Omega(T_k) - \mathrm{Discr}(T_k)), \tag{2.2}$$

where $G_k \subset O^+(T_k)$ is an appropriate finite index subgroup. By the Global Torelli Theorem [25] and epimorphicity of the period map [17] for K3 surfaces, each point of $\mathrm{Mod}(S)$ corresponds to some S-polarized K3 surface X.

We recall that a holomorphic function Φ on the affine cone

$$\Omega(T)^\bullet = \{\omega \in T \otimes \mathbb{C} \mid \omega \cdot \omega = 0 \text{ and } \omega \cdot \overline{\omega} > 0\}^+$$

over $\Omega(T)$ is called an *automorphic form on $\Omega(T)$ of a weight $d \in \mathbb{N}$* if Φ is homogeneous of the degree $(-d)$ with respect to the action of $\mathbb{C}^*$, and it is symmetric with respect to a subgroup $H \subset O^+(T)$ of finite index.

Finally, we can give a definition:

Definition 2.1. We fix an even hyperbolic lattice S.

We say that S-polarized K3 surfaces have an **automorphic discriminant** if for each $1 \le k \le n$ in (2.2) there exists a holomorphic automorphic form on $\Omega(T_k)$ such that the support of its zero divisor is equal to $\mathrm{Discr}(T_k)$ in (2.1). Then we call this automorphic form a **discriminant auomorphic form.**

The *stable orthogonal group*

$$\widetilde{O}^+(T) = \{g \in O^+(T) \mid g|_{T^*/T} = \mathrm{id}\}$$

is a subgroup of finite index of $O^+(T)$. For a primitive embedding $S \subset L_{K3}$ and $T = (S)^\perp_{L_{K3}}$, the group $\widetilde{O}^+(T)$ consists of automorphisms from $O^+(T)$ which can be continued to an element of $O(L_{K3})$ identically on S. Thus, we can assume that $\widetilde{O}^+(T_k) \subset G_k$.

3. Lattice-polarized K3 surfaces with automorphic discriminant related to Lorentzian Kac–Moody algebras with Weyl groups of 2-reflections

Below, we use the following notation for lattices. We use $\oplus$ for the orthogonal sum of lattices. By tM, we denote the orthogonal sum of t copies of a lattice M. By A_k, $k \geq 1$, D_m, $m \geq 4$, E_l, $l = 6, 7, 8$, we denote the standard root lattices with Dynkin diagrams $\mathbb{A}_k$, $\mathbb{D}_m$, $\mathbb{E}_l$ respectively and the roots with square (-2). For a lattice M, we denote by $M(t)$ the lattice which is obtained from M by multiplication by $0 \neq t \in \mathbb{Q}$ of the bilinear form of the lattice M if the form of $M(t)$ remains integral. By $\langle A \rangle$, we denote the lattice with the symmetric matrix A. Thus,

$$U = \left\langle \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right\rangle \tag{3.1}$$

is an even unimodular lattice of the signature $(1, 1)$. For example, $L_{K3} \cong 3U \oplus 2E_8$.

We recall that for an integer lattice M we have the canonical embedding $M \subset M^* = \mathrm{Hom}(M, \mathbb{Z})$. It defines a (finite) discriminant group $A_M = M^*/M$. By continuing the symmetric bilinear form of the lattice M to M^*, we obtain a finite symmetric bilinear form b_M on A_M with values in $\mathbb{Q}/\mathbb{Z}$ and a finite quadratic form q_M on A_M with values in $\mathbb{Q}/2\mathbb{Z}$ if M is even. These are called the *discriminant forms of the lattice M.*

If there are no other conditions, by $(M)^\perp_L$ we mean an orthogonal complement to a lattice M in a lattice L for some primitive embedding $M \subset L$. For most cases of Theorems 3.1 and 3.2 below, the orthogonal complement is unique up to isomorphism. For other cases, it does not matter which isomorphism class we shall take.

We have the following six series of examples of even hyperbolic lattices S of $\mathrm{rk}\, S \geq 2$ such that S-polarized K3 surfaces have an automorphic discriminant.

Theorem 3.1. *For hyperbolic lattices S which are given in the last columns of the Tables* 1–6 *below, S-polarized* K3 *surfaces have an automorphic discriminant. We also give the discriminant quadratic form q_S of S in notation of* [5]*. The even hyperbolic lattice S is defined by its rank and q_S uniquely up to isomorphism* (*see proofs below*)*.*

For all these cases, the transcendental lattice $T = (S)^\perp_{L_{K3}}$, where $L_{K3} = 3U \oplus 2E_8$, is unique up to isomorphism, and its isomorphism class is equal to $T = U(m) \oplus S^{mir}$, where the hyperbolic lattice S^{mir} is shown in the first column and m is shown in the second column of the table in the same line as S.

Theorem 3.2. *For all cases of Theorem* 3.1*, the discriminant automorphic form $\Phi(z)$ has the Fourier expansion with integral coefficients at the zero dimensional cusp defined by the decomposition $T = U(m) \oplus S^{mir}$* (*see* [13]), *$z \in S^{mir} \otimes \mathbb{R} + \sqrt{-1}\, V^+(S^{mir})$. The Fourier coefficients define a Lorentzian* (*hyperbolic and automorphic*) *Kac–Moody superalgebra $\mathfrak{g}$, which is graded by the hyperbolic lattice S^{mir}. The $\Phi(z)$ has an infinite product* (*Borcherds*) *expansion which gives multiplicities of roots of this algebra. See* [1], [2], [15], [16].

The divisor of $\Phi(z)$ is the sum of rational quadratic divisors D_α, $\alpha \in T^{(2)}$, with multiplicities one.

The S^{mir}-polarized K3 *surfaces can be considered as mirror symmetric to S-polarized* K3 *surfaces by mirror symmetry considered in* [6], [7], [11], [12]. *They have the remarkable property that there exists $\rho \in S^{mir} \otimes \mathbb{Q}$ such that $\rho \cdot E = 1$ for each irreducible non-singular rational curve $E \subset X$ with $S_X = S^{mir}$ (for $\rho^2 > 0$ and* $\operatorname{rk} S^{mir} = 4$, *such that S^{mir} are in the list of* 14 *lattices which were found by É. B. Vinberg in* [28]*; about other S^{mir} see* [22] *and* [23]*).*

Proof. Theorems 3.1 and 3.2 are mainly reformulations of the results of [13] using the discriminant forms technique for integer lattices which was developed in [21].

Let S be a lattice of one of Tables 1–6. By results of [21], we have $T = (S)^{\perp}_{3U\oplus 2E_8} \cong U(m) \oplus S^{mir}$, where S^{mir} and m are shown in the same line of the table as S. Here, it is important that the discriminant quadratic forms q_T and q_S are related as $q_T \cong -q_S$ since $T \perp S$ in the unimodular lattice $3U \oplus 2E_8$. Vice versa, $(S)^{\perp}_{3U\oplus 2E_8} = T$ and $(T)^{\perp}_{3U\oplus 2E_8} = S$ for some primitive embeddings $S \subset 3U \oplus 2E_8$ and $T \subset 3U \oplus 2E_8$ if signatures of T, S and $3U \oplus 2E_8$ agree and $q_T \cong -q_S$. The signature $(t_{(+)}, t_{(-)})$ together with the discriminant quadratic form q defines the genus of an even lattice. Theorem 1.13.1 in [21] (which uses results by M. Kneser) gives conditions when an even indefinite lattice with the invariants $(t_{(+)}, t_{(-)}, q)$ is unique up to isomorphism.

For all S^{mir} and m which are shown in Tables 1–6, the automorphic form $\Phi(z)$ with the properties mentioned in Theorems 3.1 and 3.2 is constructed in [13]. For lattices of Table 1, it is done in [13, Theorem 4.2 and Proposition 4.1]; of Table 2, in [13, Theorem 4.4]; of Table 3, in [13, Example 6.1 and Theorem 6.1]. The last case $S^{mir} = U \oplus E_8(2)$ and $m = 2$ of this table is related to Enriques surfaces and was considered by Borcherds in [3] and also in [8]. For lattices of Table 4, the automorphic form $\Phi(z)$ is constructed in [13, Theorems 6.2 and 6.3]; of Table 5, in [13, Lemma 6.4]; of Table 6, in [13, Theorem 6.5].

By results of [21] which were mentioned above, we have that $S = (T)^{\perp}_{3U\oplus 2E_8}$ is unique up to isomorphism, and S is shown in the tables.

These considerations give the proof. $\square$

In many cases, existence of the automorphic discriminant tells us that the moduli space of the corresponding S-polarized K3 surfaces has a special geometry. We recall that an algebraic variety V is called *uniruled* if there exists a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow V$ where Y is an algebraic variety with $\dim Y = \dim V - 1$. The following criterion is valid.

Theorem 3.3 (See [9, Theorem 2.1])**.** *Let $\Omega(T)$ be a connected component of the type IV domain associated to a lattice T of signature $(2, n)$ with $n \geq 3$ and let $\Gamma \subset \mathrm{O}^+(T)$ be an arithmetic subgroup of finite index of the orthogonal group. Let $\widetilde{B} = \sum_r D_r$ in $\Omega(T)$ be the divisorial part of the ramification locus of the quotient map $\Omega(T) \to \Gamma\backslash\Omega(T)$. (This means that the reflection s_r or $-s_r$ belongs to Γ.) Assume that a modular form F_k with respect to Γ of weight k with a (finite order) character exists, such that $\{F_k = 0\} = \sum_r m_r D_r$ where the m_r are non-negative integers. Let $m = \max\{m_r\}$ ($m > 0$ by Koecher's principle). If $k > m \cdot n$, then $\Gamma'\backslash\mathcal{D}$ is uniruled for every arithmetic group Γ' containing Γ.*

Using this criterion, we obtain

Theorem 3.4. *The moduli space of S-polarized* K3 *surfaces is at least uniruled if S is any lattice of Table* 1 *and Table* 2, *a lattice from the first five lines of Table* 3 *(till the lattice $\langle 2\rangle \oplus 5A_1$), the first two lines of Table* 4, *and the first two lines of Tables* 5 *and* 6.

Table 1. S-polarized K3 surfaces with automorphic discriminant.

S^{mir}	$T = U(m) \oplus S^{mir}$	weight of $\Phi(z)$	$S = (T)^{\perp}_{3U\oplus 2E_8}$	q_S
$U \oplus A_1$	$m=1$	35	$U \oplus E_8 \oplus E_7$	2_1^{+1}
$U \oplus 2A_1$	$m=1$	34	$U \oplus E_8 \oplus D_6$	2_2^{+2}
$U \oplus A_2$	$m=1$	45	$U \oplus E_8 \oplus E_6$	3^{-1}
$U \oplus 3A_1$	$m=1$	33	$U \oplus E_7 \oplus D_6$	2_3^{+3}
$U \oplus A_3$	$m=1$	54	$U \oplus E_8 \oplus D_5$	4_3^{-1}
$U \oplus 4A_1$	$m=1$	32	$U \oplus D_6 \oplus D_6$	2_4^{+4}
$U \oplus 2A_2$	$m=1$	42	$U \oplus E_6 \oplus E_6$	3^{+2}
$U \oplus A_4$	$m=1$	62	$U \oplus E_8 \oplus A_4$	5^{+1}
$U \oplus D_4$	$m=1$	72	$U \oplus E_8 \oplus D_4$	2_{II}^{-2}
$U \oplus D_4$	$m=2$	40	$U(2) \oplus E_8 \oplus D_4$	2_{II}^{-4}
$U \oplus A_5$	$m=1$	69	$U \oplus E_8 \oplus A_2 \oplus A_1$	$2_{-1}^{+1}, 3^{+1}$
$U \oplus D_5$	$m=1$	88	$U \oplus E_8 \oplus A_3$	4_5^{-1}
$U \oplus 3A_2$	$m=1$	39	$U \oplus E_6 \oplus 2A_2$	3^{-3}
$U \oplus 2A_3$	$m=1$	48	$U \oplus 2D_5$	4_6^{+2}
$U \oplus A_6$	$m=1$	75	$U \oplus E_8 \oplus \left\langle \begin{matrix} -2 & 1 \\ 1 & -4 \end{matrix} \right\rangle$	7^{-1}
$U \oplus D_6$	$m=1$	102	$U \oplus E_8 \oplus 2A_1$	2_{-2}^{+2}
$U \oplus E_6$	$m=1$	120	$U \oplus E_8 \oplus A_2$	3^{+1}
$U \oplus A_7$	$m=1$	80	$U \oplus E_8 \oplus \langle -8 \rangle$	8_{-1}^{+1}
$U \oplus D_7$	$m=1$	114	$U \oplus E_8 \oplus \langle -4 \rangle$	4_{-1}^{+1}
$U \oplus E_7$	$m=1$	165	$U \oplus E_8 \oplus A_1$	2_{-1}^{+1}
$U \oplus 2D_4$	$m=1$	60	$U \oplus 2D_4$	2_{II}^{+4}
$U \oplus D_8$	$m=1$	124	$U \oplus D_8$	2_{II}^{+2}
$U \oplus E_8$	$m=1$	252	$U \oplus E_8$	0
$U(2) \oplus 2D_4$	$m=1$	28	$U(2) \oplus 2D_4$	2_{II}^{+6}
$U \oplus 2E_8$	$m=1$	132	U	0

Table 2. S-polarized K3 surfaces with automorphic discriminant.

S^{mir}	$T = U(m) \oplus S^{mir}$	weight of $\Phi(z)$	$S = (T)^{\perp}_{3U\oplus 2E_8}$	q_S
U	$m=1$	12	$U \oplus E_8 \oplus E_8$	0
$U \oplus A_1(2)$	$m=1$	12	$U \oplus E_8 \oplus D_7$	4_1^{+1}
$U \oplus A_1(3)$	$m=1$	12	$U \oplus E_8 \oplus E_6 \oplus A_1$	$2_{-1}^{+1}, 3^{-1}$
$U \oplus A_1(4)$	$m=1$	12	$U \oplus E_8 \oplus A_7$	8_1^{+1}
$U \oplus 2A_1(2)$	$m=1$	12	$U \oplus D_7 \oplus D_7$	4_2^{+2}
$U \oplus A_2(2)$	$m=1$	12	$U \oplus E_8 \oplus D_4 \oplus A_2$	$2_{II}^{-2}, 3^{+1}$
$U \oplus A_2(3)$	$m=1$	12	$U \oplus E_8 \oplus (A_2(3))^{\perp}_{E_8}$	$3^{-1}, 9^{-1}$
$U \oplus A_3(2)$	$m=1$	12	$U \oplus E_8 \oplus (A_3(2))^{\perp}_{E_8}$	$2_{II}^{-2}, 8_3^{-1}$
$U \oplus D_4(2)$	$m=1$	12	$U \oplus E_8 \oplus D_4(2)$	$2_{II}^{-2}, 4_{II}^{-2}$
$U \oplus E_8(2)$	$m=1$	12	$U \oplus E_8(2)$	2_{II}^{+8}

TABLE 3. S-polarized K3 surfaces with automorphic discriminant.

S^{mir}	$T = U(m) \oplus S^{mir}$	weight of $\Phi(z)$	$S = (T)^{\perp}_{3U\oplus 2E_8}$	q_S
$\langle 2\rangle \oplus A_1$	$m=2$	12	$U(2)\oplus E_8\oplus E_7\oplus A_1$	2_0^{+4}
$\langle 2\rangle \oplus 2A_1$	$m=2$	11	$U(2)\oplus E_7\oplus E_7\oplus A_1$	2_1^{+5}
$\langle 2\rangle \oplus 3A_1$	$m=2$	10	$U(2)\oplus E_7\oplus D_6\oplus A_1$	2_2^{+6}
$\langle 2\rangle \oplus 4A_1$	$m=2$	9	$U(2)\oplus D_6\oplus D_6\oplus A_1$	2_3^{+7}
$\langle 2\rangle \oplus 5A_1$	$m=2$	8	$U\oplus D_6\oplus 6A_1$	2_4^{+8}
$\langle 2\rangle \oplus 6A_1$	$m=2$	7	$U(2)\oplus D_6\oplus 5A_1$	2_5^{+9}
$\langle 2\rangle \oplus 7A_1$	$m=2$	6	$U(2)\oplus D_4\oplus 6A_1$	2_6^{+10}
$\langle 2\rangle \oplus 8A_1$	$m=2$	5	$U(2)\oplus E_8(2)\oplus A_1$	2_7^{+11}
$U\oplus E_8(2)$	$m=2$	4	$U(2)\oplus E_8(2)$	2_{II}^{+10}

TABLE 4. S-polarized K3 surfaces with automorphic discriminant.

S^{mir}	$T = U(m) \oplus S^{mir}$	weight of $\Phi(z)$	$S = (T)^{\perp}_{3U\oplus 2E_8}$	q_S
$U(2)\oplus D_4$	$m=1$	40	$U(2)\oplus E_8\oplus D_4$	2_{II}^{-4}
$U(2)\oplus D_4$	$m=2$	24	$U\oplus 3D_4$	2_{II}^{-6}
$U(4)\oplus D_4$	$m=4$	6	$U(4)\oplus (U(4)\oplus D_4)^{\perp}_{U\oplus 2E_8}$	$2_{II}^{-2}, 4_{II}^{+4}$

TABLE 5. S-polarized K3 surfaces with automorphic discriminant.

S^{mir}	$T = U(m) \oplus S^{mir}$	weight of $\Phi(z)$	$S = (T)^{\perp}_{3U\oplus 2E_8}$	q_S
$U(4)\oplus A_1$	$m=4$	5	$U(4)\oplus (U(4))^{\perp}_{U\oplus E_8}\oplus E_7$	$2_1^{+1}, 4_{II}^{+4}$
$U(4)\oplus 2A_1$	$m=4$	4	$U(4)\oplus (U(4))^{\perp}_{U\oplus E_8}\oplus D_6$	$2_2^{+2}, 4_{II}^{+4}$
$U(4)\oplus 3A_1$	$m=4$	3	$U(4)\oplus (U(4))^{\perp}_{U\oplus E_8}$ $\oplus D_4\oplus A_1$	$2_3^{+3}, 4_{II}^{+4}$
$U(4)\oplus 4A_1$	$m=4$	2	$U(4)\oplus (U(4))^{\perp}_{U\oplus E_8}\oplus 4A_1$	$2_4^{+4}, 4_{II}^{+4}$

TABLE 6. S-polarized K3 surfaces with automorphic discriminant.

S^{mir}	$T = U(m) \oplus S^{mir}$	weight of $\Phi(z)$	$S = (T)^{\perp}_{3U\oplus 2E_8}$	q_S
$U(3)\oplus A_2$	$m=3$	9	$U(3)\oplus (U(3))^{\perp}_{U\oplus E_8}\oplus E_6$	3^{-5}
$U(3)\oplus 2A_2$	$m=3$	6	$U(3)\oplus (U(3))^{\perp}_{U\oplus E_8}\oplus 2A_2$	3^{+6}
$U(3)\oplus 3A_2$	$m=3$	3	$U(3)\oplus (U(3)\oplus A_2)^{\perp}_{U\oplus E_8}\oplus 2A_2$	3^{-7}

Proof. The moduli space of S-polarized K3 surfaces is defined in (2.2). For any lattice S in Tables 1–6, there is only one isomorphism class of the corresponding lattices T; i.e., there is only one term in (2.2). The modular group $G = G_1$ of the moduli space always contains the stable orthogonal group $\widetilde{O}^+(T)$ acting trivially on the discriminant quadratic form of T. The divisor D_r with $r^2 = -2$, $r \in T$, always belongs to the ramification divisor since $s_r \in \widetilde{O}^+(T)$. Remark that $\widetilde{O}^+(T)$ is generated by -2-reflections for the most part of the lattices from Tables 1 and 2 (see [10]). By construction (see [13, §4]), any discriminant automorphic form from Tables 1 and 2 is a modular form with respect to $\widetilde{O}^+(T)$ with the character det, and with the simplest possible divisor $\mathrm{Discr}(T)$ of multiplicity one. The weight of the discriminant automorphic form is shown in the tables. If the dimension n of the moduli space is larger than 2, we apply Theorem 3.3. If $n = 1$ or 2, the corresponding modular varieties are at least unirational.

The construction of the discriminant automorphic forms of Table 3 uses the isomorphism

$$O(U(2) \oplus (\langle 2\rangle \oplus (k+1)\langle -2\rangle)) \cong O(U \oplus (\langle 1\rangle \oplus (k+1)\langle -1\rangle)) \cong O(U \oplus U \oplus D_k)$$

(see [13, Lemma 6.1]). Moreover, reflections with respect to -2-elements of $\langle 2\rangle\oplus (k+1)\langle -2\rangle$ correspond to reflections with respect to -4-roots of $U \oplus D_k$ or -1-roots of $U \oplus D_k^*$. If $k \neq 4$, then all -1-roots of $2U \oplus D_k^*$ belong to a unique $\widetilde{\mathrm{O}}^+(2U \oplus D_k)$-orbit which is equal to the set of -1-elements in $2U \oplus k\langle -1\rangle$. If $k = 4$, then there are three such $\widetilde{\mathrm{O}}^+(2U \oplus D_4)$-orbits, and each of them coincides with the -1-elements in $2U \oplus k\langle -1\rangle$.

The discriminant automorphic forms of Table 3 (see [13, §6]) are modular with respect to the full orthogonal group $O^+(2U \oplus D_k)$ if $k \neq 4$ and with a subgroup $\widetilde{O}^+(2U \oplus D_4)$ containing $\widetilde{O}^+(U(2) \oplus (\langle 2\rangle \oplus (5\langle -2\rangle))$. If $k \leq 5$, then the weight of the discriminant automorphic form is strictly larger than the dimension of the moduli space.

Similar arguments work for the remaining cases of the modular forms constructed in [13, §§6.3–6.5]. □

Remark. In each table 3–6, there exists one discriminant automorphic form with weight which is equal to the dimension of the homogeneous domain. It follows that the Kodaira dimension of a finite quotient of the corresponding moduli space is equal to 0. (See the criterion in [8] and [9, Theorem 1.3].) We hope to consider these cases in detail later.

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Translated by THE AUTHORS

Originally published in Russian