

MATRIX DIVISORS ON RIEMANN SURFACES AND LAX OPERATOR ALGEBRAS

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Dedicated to E. B. Vinberg on the occasion of his 80th birthday

ABSTRACT. Tyurin parametrization of framed vector bundles is extended to the matrix divisors with an arbitrary semi-simple structure group. The considerations are based on the recently obtained description of Lax operator algebras and finite-dimensional integrable systems in terms of \mathbb{Z} -gradings of semi-simple Lie algebras.

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1. INTRODUCTION

Matrix divisors are introduced in the work by A. Weil [30], which is considered as a starting point of the theory of holomorphic vector bundles on Riemann surfaces. The classification of the holomorphic vector bundles on Riemann surfaces by A. N. Tyurin [24, 25, 26] based on matrix divisors, the well-known Narasimhan–Seshadri description of stable vector bundles [14], and subsequent description of the moduli space of vector bundles with the parabolic structure [15, 13] go back to [30]. In the theory of holomorphic vector bundles the matrix divisors play the role similar to the role of usual divisors in the theory of line bundles.

The matrix divisor approach to classification of holomorphic vector bundles provides invariants not only of stable bundles but also of families of smaller dimensions. Moreover, it provides explicit coordinates, invented in [25], in an open subset of the moduli space of stable vector bundles. In [9], these coordinates were given the name of *Tyurin parameters* and successfully applied in integration of soliton equations.

To be more specific, assume that a holomorphic rank n vector bundle has the n -dimensional space of holomorphic sections. Then any base of the space of the holomorphic sections is called *framing*, and the bundle with a given framing is called a *framed bundle*. The classification of the framed holomorphic vector bundles is one of the main results of [24, 25, 26]. In particular, it follows from [25, 26] that the moduli space of stable framed rank n holomorphic vector bundles of degree 0 is a quasiprojective variety of dimension

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$(n^2 - 1)(g - 1)$ where $g \geq 2$ is the genus of the Riemann surface, and if in the same set-up we consider the bundles of degree ng , then the dimension of the corresponding quasiprojective variety is equal to $n^2(g - 1) + 1$. It has also been shown by Tyurin that the bundles which do not possess any natural framing depend on a smaller number of parameters.

In the present paper, we address the problem of classifying the matrix divisors. It is a straightforward generalization of the problem of classifying the framed vector bundles. Indeed, let $\psi_1^U, \dots, \psi_n^U$ be the elements of a framing represented in local coordinates (i.e., the local meromorphic vector-functions defined at the local coordinate set denoted by U). Then the collection of matrices Ψ^U formed by them at every U form a matrix divisor.

We would like to draw attention to one more relationship between matrix divisors and the theory of integrable systems, namely, to the relationship with Lax operator algebras. Those came into existence due to the theory by Krichever [10] of integrable systems with the spectral parameter on a Riemann surface. Originally, this theory had been motivated in part by the Tyurin parametrization of framed vector bundles. Later (see [19, 23]) it was developed in the different and more general set-up related to \mathbb{Z} -gradings of the semi-simple Lie algebras. The main purpose of the present work is to develop the corresponding set-up in the theory of matrix divisors. The result we have obtained in this way can be briefly formulated as follows.

Theorem. *The moduli space \mathcal{M} of matrix divisors with certain discrete invariants and fixed support is a homogeneous space. For its tangent space at the unit we have*

$$(1.1) \quad T_e \mathcal{M} \cong \mathcal{M}^{\mathcal{L}} / \mathcal{L},$$

where \mathcal{L} is the Lax operator algebra essentially defined by the same invariants and $\mathcal{M}^{\mathcal{L}}$ is the corresponding space of M -operators.

This result goes back to [10]. We refer to Section 4, in particular to Theorem 4.1, for the details, notation, and a more precise statement.

We were not able to find any reference for the matrix divisors of G -bundles for G a complex semi-simple group.¹ It is one of the purposes of the present work, closely related to the main purpose, to propose a treatment of such matrix divisors. To do that, we use the Chevalley groups over the field of Laurent series and the ring of Taylor series. It is a very adequate set-up for matrix divisors in our opinion, because a Chevalley group is defined by a (complex, semi-simple) Lie algebra and its faithful representation is given by a highest weight. Such data contain information both on the group structure and on the fibre of the bundle. Moreover, the Cartan decomposition of Chevalley groups in its general form provides a convenient description of the canonical form of a matrix divisor (Theorems 3.2, 3.3 below). For an arbitrary Chevalley group G over the field of Laurent series the Cartan decomposition states that $G = KA^+K$, where K is the same group considered over the ring of Taylor series, and A^+ is a chamber in the maximal torus. In [25] the same role is played by Lemma 1.2.1. However, the last claims a stronger statement; namely it specifies the form of the K -component of the decomposition in the following quite beautiful way: let k be the K -component at a certain point of the divisor support, let $\text{diag}(z^{d_1}, \dots, z^{d_n}) \in A^+$ be the toric component, and let $d_1 \leq \dots \leq d_n$, E_{ij} be the matrix units. Then

$$k = E + \sum_{i < j} a_{ij}(z) E_{ij}, \quad a_{ij}(z) \in \mathbb{C}[[z]] / z^{d_j - d_i} \mathbb{C}[[z]],$$

where $\mathbb{C}[[z]]$ is the ring of Taylor series. Since we are not able to follow all the arguments by A. N. Tyurin in the course of deriving that expression, we reinterpret it, generalize it to

¹In this regard we mention the work [2] addressing a closely related classification problem.

the case of an arbitrary reduced root system R , and thus obtain the following description of the tangent space to the moduli space of matrix divisors (see Theorem 3.10 below for the more precise statement).

Theorem. *The tangent space to \mathcal{M} at the unit consists of elements of the form*

$$\bigoplus_{\gamma \in \Gamma} \sum_{\alpha \in R^+} a_\alpha^\gamma(z) x_\alpha, \quad a_\alpha^\gamma(z) \in \mathbb{C}[[z]]/z^{\alpha(h_\gamma)} \mathbb{C}[[z]],$$

where Γ is the divisor support, h_γ comes from the maximal torus component of the Cartan decomposition at γ , and x_α is the root vector of the root α .

Finally the obtained expression turns out to be an important argument for establishing the above relationship (given by (1.1)) between matrix divisors and Lax operator algebras.

In the present paper we assume G to be semi-simple, which corresponds to the case of topologically trivial holomorphic vector bundles. To include the topologically non-trivial bundles we would need to consider the conformal extensions of semi-simple groups [12] instead. Let G be a complex semi-simple group with the finite center \mathcal{Z} equal to a direct sum of r cyclic components. By *conformal extension* of G we mean $G_c = G \times_{\mathcal{Z}} (\mathbb{C}^*)^r$. For example, $GL(n, \mathbb{C})$ is a conformal extension of $SL(n, \mathbb{C})$. We do not focus on this easy modification here.

The plan of the present paper is as follows. In Section 2 we give the preliminaries on matrix divisors in terms of Chevalley groups and a description of the sheaf of sections of a matrix divisor in terms of certain flag configurations. In Section 3 we define the moduli space of matrix divisors as a certain coset and prove Theorem 3.10, giving a description of its tangent space at the unit in terms of the root system of the group and the weight lattice of the underlying module. In Section 4 we give preliminaries on Lax operator algebras (see [23] for the details) and then complete the interpretation of the moduli space from the point of view of integrable systems identifying the tangent space at the unit with the coset of the space of M -operators by the space of L -operators in the spirit of [10], relying on the results of [23].

2. MATRIX DIVISORS AND FLAG CONFIGURATIONS

Let G denote a Chevalley group given by a semi-simple complex Lie algebra \mathfrak{g} , a faithful \mathfrak{g} -module V with dominant highest weight, and a field k . We recall that G is the group of automorphisms of the k -space $V^k = V \otimes_{\mathbb{Q}} k$ generated by the 1-parameter subgroups of automorphisms of the form $\exp tX_\alpha = \sum_{n=0}^{\infty} t^n X_\alpha^n / n!$ (the sums being actually finite on V^k) where X_α is the representation operator of the root vector of the root α .

Let Σ be a compact Riemann surface with a given complex structure.

The following system of definitions reproduces the corresponding definitions given in [25] for $GL(n)$.

Definition 2.1. Assume each point of Σ to be assigned with a germ of meromorphic G -valued functions holomorphic except at a finite set $\Gamma \subset \Sigma$. Such correspondence is called a *distribution* with the support Γ .

Definition 2.2. Two distributions A_x and B_x , $x \in \Sigma$, are *equivalent* if there exists a third distribution C_x holomorphic for every $x \in \Sigma$ and such that $C_x A_x = B_x$. Any class of equivalent distributions is called *matrix divisor*.

That C_x is holomorphic and G -valued implies in particular that it is holomorphically invertible.

We delay the discussion of equivalent matrix divisors until Section 3 (Definition 3.6 and below).

Definition 2.3. Given a matrix divisor Ψ , by its *local section* (or just *section*) we mean a meromorphic V -valued function f on an open subset $U \subset \Sigma$ such that f is holomorphic on $U \setminus \Gamma$ and $\Psi_\gamma f$ is holomorphic in the neighborhood of any $\gamma \in \Gamma \cap U$.

We denote the sheaf of sections by $\Gamma_V(\Psi)$. It has a simple description in terms of flag configurations related to the divisor.

Given a matrix divisor Ψ we assign a flag in V to every point in its support. Thus the divisor turns out to be assigned with the system of flags which we call a *flag configuration* (this term assumes Γ to be fixed).

Let f be a meromorphic V -valued function on $U \subset \Sigma$. In order for f to be a section it is required that

$$(2.1) \quad s = \Psi f$$

be holomorphic at every $\gamma \in \Gamma \cap U$ where $\Gamma = \text{support } \Psi$. Assume Ψ to have an expansion of the form

$$\Psi = \sum_{i=-m}^{\infty} \Psi_i z^i$$

at γ and f to have an expansion of the form

$$f = \sum_{i=-k}^{\infty} f_i z^i$$

there. We take $-k = \text{ord}_\gamma \Psi_\gamma^{-1}$ which is possible due to (2.1). We then have the following system of $m+k$ linear equations:

$$(2.2) \quad \begin{aligned} \Psi_{-m} f_{-k} &= 0, \quad \Psi_{-m} f_{-k+1} + \Psi_{-m+1} f_{-k} = 0, \dots, \\ \Psi_{-m} f_{m-1} + \Psi_{-m+1} f_{m-2} + \dots + \Psi_{k-1} f_{-k} &= 0, \end{aligned}$$

expressing the fact that the terms containing z^{-m-k}, \dots, z^{-1} in the expression for Ψf vanish; i.e., s in (2.1) is holomorphic. This system of equations is homogeneous. Let F_i be the subspace in V comprised of the components f_i of all solutions $(f_{-k}, f_{-k+1}, \dots, f_{m-1})$ to the system.

The following lemma claims that the subspaces F_i , $i = -k, \dots, m-1$, constitute a flag denoted by F below. We were not able to find any reference for this fact. Similar arguments are used for the flag interpretation of opers [5].

Lemma 2.4. $F_{-k} \subseteq F_{-k+1} \subseteq \dots \subseteq F_{m-1} \subseteq V$.

Proof. Solutions to (2.2) can be found by an inductive procedure. The first equation is independent and homogeneous. The space of its solutions is exactly what we have denoted by F_{-k} . Solutions to the second equation can be found by plugging an arbitrary $f_{-k} \in F_{-k}$ and resolving the obtained system of equations. In particular, we can plug $f_{-k} = 0$. Then the second equation coincides with the first one, hence $F_{-k} \subseteq F_{-k+1}$.

The i th equation of (2.2) (at z^{-m-k+i}) is as follows:

$$\Psi_{-m} f_{-k+i} + \Psi_{-m+1} f_{-k+i-1} + \dots + \Psi_{-m+i} f_{-k} = 0.$$

The next equation has the form

$$\Psi_{-m} f_{-k+i+1} + \Psi_{-m+1} f_{-k+i} + \dots + \Psi_{-m+i} f_{-k+1} + \Psi_{-m+i+1} f_{-k} = 0.$$

If $f_{-k} = 0$, the $(i+1)$ th equation degenerates to the i th one. Hence every solution of the i th equation is a particular solution of the $(i+1)$ th equation, i.e., $F_{-m-k+i} \subseteq F_{-m-k+i+1}$. \square

The description of the sheaf $\Gamma_V(\Psi)$ is now as follows.

Lemma 2.5. $\Gamma_V(\Psi)$ is the sheaf of local meromorphic V -valued functions on Σ satisfying the following requirement for every $\gamma \in \Gamma$. Let f be such a function, and $f(z) = \sum f_i^\gamma z^i$ be its Laurent expansion at a $\gamma \in \Gamma$. Then it is required that $f_i^\gamma \in F_i^\gamma$ where $F_\gamma : \{0\} \subseteq \dots \subseteq F_i^\gamma \subseteq \dots \subseteq V$ is the flag corresponding to γ .

The proof immediately follows from Definition 2.3 and the definition of F . In Lemma 2.5 we consider the flag to be semi-infinite to the right, where F_i^γ becomes equal to V since a certain moment.

Definition 2.6. Given a matrix divisor Ψ we call the Lie algebra of meromorphic \mathfrak{g} -valued functions on Σ leaving invariant $\Gamma_V(\Psi)$ the *endomorphism algebra* of Ψ and denote it by $\text{End}(\Psi)$.

Next we give a description of the Lie algebra $\text{End}(\Psi)$ in terms of the flag configuration related to Ψ .

Let $\mathfrak{g} = \text{Lie}(G)$. Given a flag F consider the following filtration of \mathfrak{g} . Remember that V is a \mathfrak{g} -module. For every i consider a subspace $\tilde{\mathfrak{g}}_i \subseteq \mathfrak{g}$ such that $\tilde{\mathfrak{g}}_i F_j \subseteq F_{j+i}$ for every j . Then $\tilde{\mathfrak{g}}_i \subseteq \tilde{\mathfrak{g}}_{i+1}$ because $\tilde{\mathfrak{g}}_i F_j \subseteq F_{j+i} \subseteq F_{j+i+1}$, and $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$.

Lemma 2.7. $\text{End}(\Psi)$ is the subspace of the space of all \mathfrak{g} -valued meromorphic functions on Σ satisfying the following requirement for every $\gamma \in \Gamma$. Let L be such a function and let $L(z) = \sum L_i z^i$ be its Laurent expansion at a $\gamma \in \Gamma$. Then $L_i \in \tilde{\mathfrak{g}}_i$, $\forall i$.

It is instructive to keep in mind the homological interpretation of matrix divisors [27, 28] also. In this approach a matrix divisor is defined as a 0-cochain with coefficients in the sheaf $G(\mathcal{R})$ of rational G -valued functions whose coboundary is a 1-cocycle with coefficients in the sheaf $G(\mathcal{O})$ of regular G -valued functions. Thus an open covering $\{U_i\}$ of Σ is assigned with the system of local rational G -valued functions f_i such that its coboundary $f_{ij} = f_i f_j^{-1}$ is regular and regularly invertible on $U_i \cap U_j$, and $f_{ij} f_{jk} f_{ki} = 1$ on $U_i \cap U_j \cap U_k$ for every triple (i, j, k) . It is clear that the cocycle f_{ij} (the system of gluing functions in other terminology) is invariant with respect to the right action of any global G -valued function.

3. CANONICAL FORM OF A MATRIX DIVISOR. THE MODULI SPACE

Let \mathfrak{o} be a principal ideal ring (commutative, with the unit), let \mathfrak{k} be its quotient field, let G be a rank l Chevalley group over \mathfrak{k} , let $K = G_{\mathfrak{o}}$ be the same group over \mathfrak{o} , let H be a maximal torus in G , i.e., the subgroup generated by the 1-parameter subgroups $h'_i(t)$ where $h'_i(t)$ acts on every vector of a weight μ in V as a multiplication by $t^{\mu(h_i)}$, and let $h_i \in \mathfrak{h}$ ($i = 1, \dots, l$) form a base of the lattice L_V^* dual to L_V , the last being generated by the weights of the module V [16, Lemma 35, p. 58].

For an obvious reason $h'_i(t)$ is denoted also by t^{h_i} , and H consists of the elements of the form $t_1^{h_1} \cdot \dots \cdot t_l^{h_l}$ where $t_i \in \mathfrak{k} \setminus \{0\}$, $i = 1, \dots, l$. The following example shows how the module V affects the torus of the corresponding Chevalley group.

Example 3.1. Consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Up to the end of the example let h denote the canonical generator of the Cartan subalgebra. The set of weights of the standard \mathfrak{g} -module is equal to $\{\pm 1\}$ and to the adjoint module $\{\pm 2, 0\}$. Hence the dual lattice to the weight lattice L_V is generated by h in the first case and by $(1/2)h$ in the second case. The corresponding tori consist of elements of the form $t^{\rho(h)n}$, resp. $t^{\rho(h)n/2}$, where $n \in \mathbb{Z}$ and $\rho(h)$ denotes the corresponding representation operator (i.e., $\rho(h) = \text{diag}(1, -1)$ in the first case and $\rho(h) = \text{diag}(2, 0, -2)$ in the second case).

Let A^+ denote the chamber in H given by the products $t_1^{h_1} \dots t_l^{h_l}$ satisfying the condition $\prod_{i=1}^l t_i^{\alpha(h_i)} \in \mathfrak{o}$ for every positive root α .

Theorem 3.2 ([16], Theorem 21). *We have the following:*

- (a) $G = KA^+K$ (the Cartan decomposition).
- (b) For every element its A^+ -component in (a) is defined uniquely modulo $H \cap K$.

We need a particular case of Theorem 3.2 when $\mathfrak{o} = \mathbb{C}[[z]]$ is the ring of Taylor expansions in z and $\mathfrak{k} = \mathbb{C}((z))$ is the field of the Laurent expansions. The elements t_i up to units of the ring $\mathbb{C}[[z]]$ can be taken as $t_i = z^{d_i}$ where $d_i \in \mathbb{Z}$ ($i = 1, \dots, l$). Let $h = \sum_{i=1}^l d_i h_i$. Then $t_1^{h_1} \dots t_l^{h_l} = z^h$ (once again, z^h is defined on a module V such that $h \in L_V^*$ and operates as multiplication by $z^{\mu(h)}$ on the subspace of weight μ in V). The condition $\prod_{i=1}^l t_i^{\alpha(h_i)} \in \mathfrak{o}$ means that $z^{\alpha(h)}$ is holomorphic, i.e., $\alpha(h) \geq 0$, for every positive α . We conclude that $H = \{z^h | h \in L_V^*\}$ and $A^+ = \{z^h | h \in (L_V^*)_+\}$, where $(L_V^*)_+ = \{h \in L_V^* | \alpha(h) \geq 0 \ \forall \alpha \in R_+\}$. In this case we obtain the following Cartan decomposition for current groups out of Theorem 3.2. In certain particular cases it is also known as the factorization theorem in the theory of holomorphic vector bundles.

Theorem 3.3. *Let $G = G(\mathbb{C}((z)))$ be the Chevalley group over $\mathfrak{k} = \mathbb{C}((z))$ given by a semi-simple Lie algebra \mathfrak{g} over \mathbb{C} and its module V , let $K = G(\mathbb{C}[[z]])$ be the same group over the ring of Taylor series, and let H and A^+ be as introduced above. Then*

$$G = KA^+K,$$

and for any element its A^+ -component in the decomposition is determined uniquely up to $H_{\mathbb{C}}$.

By virtue of Theorem 3.3 the support of a divisor can be characterized as the set of those points in Σ for which $h \neq 0$.

Up to equivalence given by left multiplication by a distribution taking values in K we can assume that $\Psi = z^h k(z)$, $k(z) \in K$, where h and k depend on the point of $\text{supp } \Psi$. We call it the *reduced form* of the divisor.

Given a \mathfrak{g} -module V of highest weight χ and an $h \in (L_V^*)_+$ we introduce the following flag F in V (below $m = \chi(h)$, $m \in \mathbb{Z}_+$). First we define the grading of the module V :

$$(3.1) \quad V = \bigoplus_{i=-m}^m V_i \quad \text{where} \quad V_i = \{v \in V \mid hv = -iv\}.$$

Obviously

$$(3.2) \quad V_i = \bigoplus_{\mu(h)=-i} V_\mu,$$

where V_μ is the weight subspace of V of weight μ .

Next we define the flag $F : \{0\} \subseteq F_{-m} \subseteq \dots \subseteq F_m = V$ by setting

$$(3.3) \quad F_j = \bigoplus_{s=-m}^j V_s.$$

In particular, F_{-m} is generated by the highest weight vector. Let $Q = \mathbb{Z}(R)$ be the root lattice of \mathfrak{g} .

Lemma 3.4. *Let $h \in L_V^*$. Then F is nothing but the flag corresponding to the divisor $\Psi = z^h$ by virtue of Lemma 2.4.*

Proof. We resolve the equation (2.1) for $\Psi = z^h$: $f = z^{-h}s$ where $s(z)$ is holomorphic in the neighborhood of $z = 0$. Take

$$s(z) = \sum_{j=0}^{\infty} s_j z^j,$$

where $s_j \in V$ for every $j \geq 0$. Let $s_j = \sum_{i=-m}^m s_j^{(i)}$ be an expansion of s_j according to the grading of V , i.e., $s_j^{(i)} \in V_i$. Then

$$z^{-h}s_j = \sum_{i=-m}^m s_j^{(i)} z^i,$$

hence

$$f(z) = \sum_{j=0}^{\infty} \left(\sum_{i=-m}^m s_j^{(i)} z^i \right) z^j = \sum_{p=-m}^{\infty} \left(\sum_{i=-m}^p s_{p-i}^{(i)} \right) z^p.$$

Since $i \leq p$ in the internal sum, the last belongs to the subspace F_p . □

Remark 3.1. The flags close to those of the form (3.3) already occurred in [4] in the context of *infinitesimal parabolic structures*. In contrast to any parabolic structure our flag configurations appear as an intrinsic structure for a matrix divisor and the corresponding holomorphic vector bundle.

Let D be a non-negative divisor, $\Pi = \text{support } D$, $\Pi \cap \Gamma = \emptyset$, and $\Gamma_{\text{gl}}^D(\Psi) = \{f \in \Gamma_V(\Psi) \mid (f) + D + m\Gamma \geq 0, f \text{ is global}\}$.

Corollary 3.5. *Let $\dim \Gamma_{\text{gl}}^D(\Psi) = \dim V(\deg D - g + 1)$; in particular $\Gamma_{\text{gl}}^D(\Psi)$ is trivial unless $\deg D \geq g$.*

Proof. Let $\mathcal{F}^{D+m\Gamma}$ denote the space of global sections f satisfying the condition $(f) + D + m\Gamma \geq 0$, and let $l_{D+m\Gamma} = \dim \mathcal{F}^{D+m\Gamma}$. By the Riemann–Roch theorem $l_{D+m\Gamma} = (\dim V)(\deg D + m|\Gamma| - g + 1)$. However, the space of sections has a codimension in $\mathcal{F}^{D+m\Gamma}$ coming from the conditions $f_s \in \mathcal{F}_s^\gamma$ where F_s^γ , $s = -m, \dots, m$, are the flag subspaces at γ . The contribution of every $\gamma \in \Gamma$ to the codimension is equal to $\sum_{s=-m}^m \text{codim}_V F_s^\gamma$. By symmetry of the grading (3.1) the last is equal to $m \dim V$. The total codimension is equal to $(m \dim V)|\Gamma|$. Hence the rest of the dimension is equal to $(\dim V)(\deg D - g + 1)$, and it should be $\deg D - g + 1 > 0$ for non-triviality of the space of sections. □

The highest weight χ and the tuple $h = \{h_\gamma \in L_V^* \mid \gamma \in \Gamma\}$ are discrete invariants of a divisor. The matrix divisors also have moduli coming from the K -components of their canonical forms at the points in Γ and from elements $\gamma \in \Gamma$ themselves. Below we introduce two types of equivalence of matrix divisors and the corresponding moduli spaces.

Definition 3.6. Two matrix divisors are *equivalent* if they have the same sheaf of sections.

Remark 3.2. This equivalence is different from that given in [25, 26] for the purpose of classification of the holomorphic vector bundles. Following the last, two matrix divisors are equivalent if one of them can be taken to another by right multiplication by a (global) meromorphic G -valued function. In particular, the divisors equivalent in this sense may have different support.

Lemma 3.7. *Two matrix divisors are equivalent in the sense of Definition 3.6 if and only if they have the same flag configuration, in particular, the same support Γ .*

Proof. Immediately follows from Lemma 2.5. \square

We define a *flag set* in the same way as a flag configuration except that we relax the requirement that Γ is fixed. Then our second equivalence relation is as follows.

Definition 3.8. Two matrix divisors are equivalent if they have the same flag set.

Denote the moduli space of matrix divisors with given invariants χ , $h = \{h_\gamma \in L_V^* \mid \gamma \in \Gamma\}$ by \mathcal{M}_h^χ (it corresponds to the equivalence given by Definition 3.8), and with additionally fixed Γ by $\mathcal{M}_h^{\chi, \Gamma}$ (it corresponds to the equivalence given by Definition 3.6). Denote the part of the moduli space over a point $\gamma \in \Gamma$ by $\mathcal{M}_{h_\gamma}^\chi$.

It is our next step to represent $\mathcal{M}_h^{\chi, \Gamma}$ as a homogeneous space and describe the stationary group of a point of this space.

Observe that for a faithful \mathfrak{g} -module V one has $L_V^* \subseteq Q^*$ [16, Lemma 27]. Hence $\alpha(h_\gamma) \in \mathbb{Z}$ for every $\alpha \in R$, $\gamma \in \Gamma$. For every γ the h_γ defines a grading $\mathfrak{g} = \mathfrak{g}_{-k}^\gamma \oplus \dots \oplus \mathfrak{g}_k^\gamma$ and the corresponding increasing filtration $\tilde{\mathfrak{g}}_{-k}^\gamma \subset \dots \subset \tilde{\mathfrak{g}}_k^\gamma = \mathfrak{g}$ where

$$\mathfrak{g}_i^\gamma = \bigoplus_{\alpha(h_\gamma)=i} \mathfrak{g}_\alpha, \quad \tilde{\mathfrak{g}}_i^\gamma = \bigoplus_{\alpha(h_\gamma) \leq i} \mathfrak{g}_\alpha = \bigoplus_{s \leq i} \mathfrak{g}_s^\gamma.$$

In the next section we give equivalent definitions to these objects.

Let $\mathfrak{g}_\gamma = \{L(z) = \sum_{j=-k}^\infty L_j z^j \mid L_j \in \tilde{\mathfrak{g}}_j^\gamma, j = -k, \dots, \infty\}$. Let $G_\gamma \subset G(\mathbb{C}((z)))$ be the subgroup with the Lie algebra \mathfrak{g}_γ , $K_\gamma = K \cap G_\gamma$, $K_0 = \prod_{\gamma \in \Gamma} K_\gamma$.

Proposition 3.9. Let $\mathcal{M}_h^{\chi, \Gamma} = \underbrace{K \times \dots \times K}_{|\Gamma| \text{ times}} / K_0$.

Proof. By Lemma 2.7 $\prod_{\gamma \in \Gamma} G_\gamma$ is exactly the stationary subgroup of the given flag configuration in the group of all G -distributions. The proposition follows by

$$K_0 = \underbrace{K \times \dots \times K}_{|\Gamma| \text{ times}} \cap \prod_{\gamma \in \Gamma} G_\gamma.$$

\square

Theorem 3.10. The tangent space to $\mathcal{M}_h^{\chi, \Gamma}$ at the unit consists of elements of the form

$$\bigoplus_{\gamma \in \Gamma} \sum_{\alpha \in R^+ : \alpha(h_\gamma) > 0} a_\alpha^\gamma(z) x_\alpha, \quad a_\alpha^\gamma(z) \in \mathbb{C}[[z]] / z^{\alpha(h_\gamma)} \mathbb{C}[[z]],$$

where R is the root system of the Lie algebra \mathfrak{g} and x_α is the root vector of the root α .

Proof. Since $\mathcal{M}_{h_\gamma}^\chi = K / K_\gamma$, we have $T_e \mathcal{M}_{h_\gamma}^\chi = \mathfrak{k}(z) / \mathfrak{k}(z) \cap \mathfrak{g}_\gamma$, where \mathfrak{k} is the Lie algebra of the group K considered over \mathbb{C} .

Observe that \mathfrak{g}_γ can be characterized as the subalgebra in $\mathfrak{g}((z))$ consisting of elements of the form

$$L(z) = \sum_{\alpha \in R, i \geq \alpha(h_\gamma)} x_\alpha z^i.$$

Indeed, by definition $L_i \in \tilde{\mathfrak{g}}_i^\gamma$; hence $L_i = \sum_{s \leq i} L_i^s$ where $L_i^s \in \mathfrak{g}_s$, and $\mathfrak{g}_s = \bigoplus_{\alpha(h_\gamma)=s} \mathfrak{g}_\alpha$.

Hence the terms $x_\alpha z^i$, $i \geq \alpha(h_\gamma)$, are absent in the quotient $\mathfrak{k}(z) / \mathfrak{k}(z) \cap \mathfrak{g}_\gamma$. In particular, the whole lower triangle subalgebra $\tilde{\mathfrak{g}}_0^\gamma$ is filtered out because it is generated by x_α , $\alpha(h_\gamma) < 0$; hence only $x_\alpha z^i$ with $i < 0$ could be in the remainder ($i < \alpha(h_\gamma) \leq 0$), but there is no negative degrees in $\mathfrak{k}(z)$. Thus we are left with only upper nilpotents satisfying the condition $\alpha(h_\gamma) > 0$ and the corresponding powers z^i with $i < \alpha(h_\gamma)$. \square

Corollary 3.11. *We have the following:*

- (1) $\dim \mathcal{M}_{h_\gamma}^\chi = \sum_{s=1}^k s \dim \mathfrak{g}_s^\gamma,$
- (2) $\dim \mathcal{M}_h^{\chi, \Gamma} = \sum_{\gamma \in \Gamma} \sum_{s=1}^k s \dim \mathfrak{g}_s^\gamma \text{ } (\Gamma \text{ is fixed}),$
- (3) $\dim \mathcal{M}_h^\chi = \sum_{\gamma \in \Gamma} (1 + \sum_{s=1}^k s \dim \mathfrak{g}_s^\gamma) \text{ } (\Gamma \text{ is not fixed}),$

where k is defined above simultaneously with the grading.

Proof. Indeed, for every $\alpha \in R^+$ the number of moduli is equal to $\alpha(h_\gamma)$, hence

$$\dim \mathcal{M}_{h_\gamma}^\chi = \sum_{\alpha \in R^+} \alpha(h_\gamma).$$

Next,

$$\mathfrak{g}_s^\gamma = \bigoplus_{\alpha(h_\gamma)=s} \mathfrak{g}_\alpha,$$

i.e., $\dim \mathfrak{g}_s^\gamma = \#\{\alpha \mid \alpha(h_\gamma) = s\}$. Since $\dim \mathfrak{g}_\alpha = 1$ we obtain the first and the second claims. Claim (3) follows from (2) taking account of elements $\gamma \in \Gamma$ which are also counted as moduli. \square

We make one more step in the direction of weakening of the notion of equivalence of matrix divisors.

Definition 3.12. Two matrix divisors are equivalent if they have the same flag set up to a common left shift by a constant (in z and γ) element of the group G .

We denote the corresponding moduli space by $\widetilde{\mathcal{M}}_h^\chi$. A left shift of a flag by a constant element of G is a result of the conjugation of all components of the Cartan decomposition at the point; hence $\widetilde{\mathcal{M}}_h^\chi = \mathcal{M}_h^\chi / \text{Ad } G$.

Examples.

(1) For $G = GL(n)$, $V = \mathbb{C}^n$, $h_\gamma = \text{diag}(d_1, \dots, d_n)$ Theorem 3.10 gives $a_{ij} \in k(z)/z^{d_i-d_j}k(z)$ for the entry a_{ij} of an element of the tangent space. These are the conditions claimed in [25, 26] for the canonical form of any divisor. Here they have a different meaning.

Assume $h_\gamma = \text{diag}(1, 0, \dots, 0)$ for every γ . Then the corresponding grading is as follows: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\dim \mathfrak{g}_{\pm 1} = n-1$ [23]; i.e., according to Corollary 3.11(3) every point contributes $(n-1)+1 = n$ parameters to the dimension of the moduli space. If $|\Gamma| = ng$, then $\dim \mathcal{M}_h^\chi = n^2g$. Taking the quotient by $\text{Ad } G$ kills $n^2 - 1$ parameters, and we conclude that $\widetilde{\mathcal{M}}_h^\chi = n^2(g-1) + 1$, which coincides with the known dimension of the moduli space of holomorphic rank n vector bundles on Σ .

(2) Consider $G = SO(2n)$, $V = \mathbb{C}^{2n}$, where $h_\gamma = \text{diag}(1, 0, \dots, 0, -1, 0, \dots, 0)$ (-1 in the $(n+1)$ th position) for every γ . Then again $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\dim \mathfrak{g}_{\pm 1} = 2n-2$ [23]. Hence every $\gamma \in \Gamma$ contributes $2n-1$ to the dimension of the moduli space. If $|\Gamma| = ng$, then

$$\dim \mathcal{M} = (2n-1)ng - \dim G = (2n-1)n(g-1)$$

(the subtracting of $\dim G = (2n-1)n$ is due to the action of $\text{Ad } G$ as above).

(3) Let $G = Sp(2n)$, $V = \mathbb{C}^{2n}$, where $h_\gamma = \text{diag}(1, 0, \dots, 0, -1, 0, \dots, 0)$ (-1 in the $(n+1)$ th position) for every γ . Then $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\dim \mathfrak{g}_{\pm 1} = 2n-2$,

$\dim \mathfrak{g}_{\pm 2} = 1$ [23]. By Corollary 3.11(2) every point $\gamma \in \Gamma$ contributes $2 \cdot 1 + (2n - 2) + 1 = 2n + 1$ into the dimension of the moduli space. If $|\Gamma| = ng$, then

$$\dim \widetilde{\mathcal{M}}_h^\chi = (2n + 1)ng - \dim G = (2n + 1)n(g - 1)$$

(again the subtracting of $\dim G$ is due to the action of $\text{Ad } G$ as above, but this time $\dim G = (2n + 1)n$).

In all considered cases the dimension of the moduli space $\widetilde{\mathcal{M}}_h^\chi$ of matrix divisors coincides with the dimension of the corresponding moduli space of semi-stable G -bundles. To obtain the same result for $SL(n)$ and $G = SO(2n + 1)$ we would need to consider the matrix divisors with values in the conformal extensions of the corresponding groups (see the Introduction).

4. MODULI OF MATRIX DIVISORS AND LAX OPERATOR ALGEBRAS

Here we will establish an isomorphism of the tangent space at the unit to the moduli space of matrix divisors (with given discrete invariants) with the quotient of the space of M -operators by the Lax operator algebra (basically defined by the same invariants). The result goes back to [10]. For brevity, we consider only Chevalley groups with a trivial center here. We start with the definition of a Lax operator algebra.

Let \mathfrak{g} be a semi-simple Lie algebra over \mathbb{C} , let \mathfrak{h} be its Cartan subalgebra, and let $h \in \mathfrak{h}$ be such an element that $p_i = \alpha_i(h) \in \mathbb{Z}_+$ for every simple root α_i of \mathfrak{g} . If we denote the root lattice of \mathfrak{g} by Q , then h belongs to the positive chamber of the dual lattice Q^* .

For $p \in \mathbb{Z}$ let $\mathfrak{g}_p = \{X \in \mathfrak{g} \mid (\text{ad } h)X = pX\}$ and $k = \max\{p \mid \mathfrak{g}_p \neq 0\}$. Then the decomposition $\mathfrak{g} = \bigoplus_{i=-k}^k \mathfrak{g}_p$ gives a \mathbb{Z} -grading on \mathfrak{g} . For the theory and classification results on such kinds of gradings we refer to [29]. Call k a *depth* of the grading. Obviously,

$$\mathfrak{g}_p = \bigoplus_{\substack{\alpha \in R \\ \alpha(h)=p}} \mathfrak{g}_\alpha,$$

where R is the root system of \mathfrak{g} . Define also the following filtration on \mathfrak{g} : $\tilde{\mathfrak{g}}_p = \bigoplus_{q=-k}^p \mathfrak{g}_q$.

Then $\tilde{\mathfrak{g}}_p \subset \tilde{\mathfrak{g}}_{p+1}$ ($p \geq -k$), $\tilde{\mathfrak{g}}_{-k} = \mathfrak{g}_{-k}, \dots, \tilde{\mathfrak{g}}_k = \mathfrak{g}$, $\tilde{\mathfrak{g}}_p = \mathfrak{g}$, $p > k$.

As above, let Σ be a complex compact Riemann surface with two given non-intersecting finite sets of marked points: Π and Γ . Assume every $\gamma \in \Gamma$ to be assigned with an $h_\gamma \in Q_+^*$ and the corresponding grading and filtration. We equip the notation \mathfrak{g}_p , $\tilde{\mathfrak{g}}_p$ with the upper γ indicating that the grading (resp. filtration) subspace corresponds to γ . We stress that \mathfrak{g}_p^γ , $\tilde{\mathfrak{g}}_p^\gamma$ are the same as defined in the previous section. Let L be a meromorphic mapping $\Sigma \rightarrow \mathfrak{g}$ holomorphic outside the marked points which may have poles of an arbitrary order at the points in Π and has the expansion of the following form at every $\gamma \in \Gamma$:

$$(4.1) \quad L(z) = \sum_{p=-k}^{\infty} L_p z^p, \quad L_p \in \tilde{\mathfrak{g}}_p^\gamma,$$

where z is a local coordinate in the neighborhood of γ . For simplicity, we assume that the depth of grading k is the same all over Γ , though it would be no different otherwise.

We denote by \mathcal{L} a linear space of all such mappings. Since the relation (4.1) is preserved under commutator, \mathcal{L} is a Lie algebra called a *Lax operator algebra*.

Lax operator algebras have emerged in [11] due to the observation by I. Krichever [10] that the Lax operators of integrable systems with the spectral parameter on a Riemann

surface have very special Laurent expansions related to the Tyurin parameters of holomorphic vector bundles. In [18, 19] they were generalized to the form described here. For the current state of the theory of Lax operator algebras and their applications to integrable systems we refer to [23, 17, 18, 19, 20, 21, 22] and the references therein.

To give the Lax operator algebra description of the moduli space of matrix divisors we introduce the space of M -operators, the counterparts of Lax operators in the Lax pairs of integrable systems.

A meromorphic mapping $M : \Sigma \rightarrow \mathfrak{g}$, holomorphic outside Π and Γ , is called an M -operator if at any $\gamma \in \Gamma$ it has a Laurent expansion

$$(4.2) \quad M(z) = \frac{\nu_\gamma h_\gamma}{z} + \sum_{i=-k}^{\infty} M_i z^i,$$

where $M_i \in \tilde{\mathfrak{g}}_i^\gamma$ for $i < 0$, $M_i \in \mathfrak{g}$ for $i \geq 0$, and $\nu_\gamma \in \mathbb{C}$. We denote by $\mathcal{M}_{\Pi, \Gamma, h}$ the space of M -operators corresponding to given data Π, Γ, h (as above, $h = \{h_\gamma \mid \gamma \in \Gamma\}$). Obviously, $\mathcal{L}_{\Pi, \Gamma, h} \subset \mathcal{M}_{\Pi, \Gamma, h}$.

A mapping taking a meromorphic function supported at $\Pi \cup \Gamma$ to the set of its Laurent expansions at the points $\gamma \in \Gamma$ is called a *localization map*. For a Lax operator algebra the localization map is of the form $\mathcal{L}_{\Pi, \Gamma, h} \rightarrow \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$.

Theorem 4.1. *For the tangent space at the unit to the moduli space of matrix divisors we have*

$$(4.3) \quad T_e \mathcal{M}_h^{\chi, \Gamma} \cong \mathcal{M}_{\Pi, \Gamma, h} / \mathcal{L}_{\Pi, \Gamma, h},$$

independently of Π . The isomorphism is given by the localization map.

Proof. At any point $\gamma \in \Gamma$ the main parts of the Laurent expansions for L - and M -operators satisfy the same conditions, hence vanish in the quotient $\mathcal{M}_{\Pi, \Gamma, h} / \mathcal{L}_{\Pi, \Gamma, h}$. As for the Tailor parts, the coefficients at z^i in the quotient run over $\mathfrak{g} / \tilde{\mathfrak{g}}_i = \mathfrak{g}_{i+1} \oplus \dots \oplus \mathfrak{g}_k$ for $i = 0, \dots, k-1$ and vanish for $i \geq k$. Hence

$$\mathcal{M}_{\Pi, \Gamma, h} / \mathcal{L}_{\Pi, \Gamma, h} = \bigoplus_{\gamma \in \Gamma} \left\{ \sum_{i=1}^k L_i z^i \mid L_i \in \bigoplus_{i < s} \mathfrak{g}_s^\gamma \right\},$$

where it is exactly the quotient of localizations on the right-hand side of the relation. It is similar to the proof of Corollary 3.11 that

$$\bigoplus_{i < s} \mathfrak{g}_s^\gamma = \bigoplus_{i < \alpha(h_\gamma)} \mathfrak{g}_\alpha,$$

hence

$$\mathcal{M}_{\Pi, \Gamma, h} / \mathcal{L}_{\Pi, \Gamma, h} = \bigoplus_{\gamma \in \Gamma} \left\{ \sum_{i=1}^k L_i z^i \mid L_i \in \bigoplus_{i < \alpha(h_\gamma)} \mathfrak{g}_\alpha \right\}.$$

The last space coincides with $T_e \mathcal{M}_h^{\chi, \Gamma}$ by Theorem 3.10. \square

Remark 4.1. The right-hand side of the relation (4.3) depends on χ as far as χ determines the choice of h (remember that $h = \{h_\gamma \in L_V^* \mid \gamma \in \Gamma\}$, where V is the module of the highest weight χ and L_V is its weight lattice). Since the dependence of the right-hand side of (4.3) on h is explicit in the notation the author thinks fit not to point out explicitly its dependence on χ .²

²The author is grateful to the referee, who pointed out a necessity of clarifying this remark.

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REFERENCES

- [1] Atiyah, M.F. *Vector bundles over an elliptic curve*, Proc. London Math. Soc., V. 7 (1957), 414-452. MR0131423
- [2] Biswas, I., Dhillon, A., Hurtubise, J., Wentworth, R. A. *A symplectic analog of the Quot scheme*. C. R. Math. Acad. Sci. Paris 353 (2015), no. 11, 995-999. MR3419849
- [3] Grothendieck, A. *Sur la classification des fibrés holomorphes sur la sphère de Riemann*, Amer. J. Math. 79 (1957), 121-138. MR0087176
- [4] Faltings, G. *Stable G-bundles and projective connections*, J. Algebraic Geometry, V. 2 (1993), 507-568. MR1211997
- [5] Feigin, B.L. *Integrable systems, shaffle algebras and the Bethe equations*. Trans. Moscow Math. Soc. 2016, 203-246. MR3643971
- [6] Hitchin, N. *Stable bundles and integrable systems*, Duke Math. J. **54**:1 (1987), 91-114. MR885778
- [7] Hitchin, N. *The self-duality equations on a Riemann surface*, Proc. Lond. Math. Soc. (3), **55**(3):1, 1987, 59-126. MR887284
- [8] Krichever, I.M., Novikov, S.P. *Holomorphic bundles over Riemann surfaces and the Kadomtsev-Petviashvili (KP) equation. I*. Funct. Anal. and Appl. , 4, **12** (1978), 276-286. MR0515628
- [9] Krichever, I.M., Novikov, S.P. *Holomorphic bundles over algebraic curves and non-linear equations*. Russ. Math. Surv., 6, **35** (1980), 53-79. MR601756
- [10] Krichever, I.M. *Vector bundles and Lax equations on algebraic curves*. Comm. Math. Phys. 229, 229-269 (2002). arXiv:Hep-th/0108110. MR1923174
- [11] Krichever, I.M., Sheinman, O.K. *Lax operator algebras*. Funct. Anal. i Prilozhen., **41** (2007), no. 4, p. 46-59. arXiv:math.RT/0701648. MR2411605
- [12] Levin, A., Olshanetsky, M., Smirnov, A., Zotov, A. *Characteristic classes and integrable systems. General construction*, arXiv:1006.0702.
- [13] Mehta, V.B., Seshadri, C.S. *Moduli of vector bundles on curves with parabolic structures*. Math. Ann., V. 248, p. 205-239 (1980). MR575939
- [14] Narasimhan, M.S., Seshadri, C.S. *Stable and unitary vector bundles on a compact Riemann surface*. Ann. Math., V. 82, 540-564, 1965. MR0184252
- [15] Seshadri, C.S. *Moduli of vector bundles on curves with parabolic structures*, Bull. Amer. Math. Soc. 83:1 (1977), 124-126. MR0570987
- [16] Steinberg, R. *Lectures on Chevalley groups*. Yale University, 1967. MR0466335
- [17] Sheinman, O.K. *Current algebras on Riemann surfaces*, De Gruyter Expositions in Mathematics, 58, Walter de Gruyter GmbH & Co. KG, Berlin-Boston, 2012, ISBN: 978-3-11-026452-4, 150 pp. MR2985911
- [18] Sheinman, O.K. *Lax operator algebras and gradings on semi-simple Lie algebras*. Doklady Mathematics, V. 461, no. 2, 2015, 143-145. MR3442778
- [19] Sheinman, O.K. *Lax operator algebras and gradings on semi-simple Lie algebras*. Transform. Groups, Vol. 21, No. 1, 2016, 181-196. DOI 10.1007/s00031-015-9340-y. MR3459709
- [20] Sheinman, O.K. *Hierarchies of finite-dimensional Lax equations with a spectral parameter on a Riemann surface, and semi-simple Lie algebras*. Theoret. and Math. Phys., 185:3 (2015), 1816-1831. MR3438634
- [21] Sheinman, O.K. *Semi-simple Lie algebras and Hamiltonian theory of finite-dimensional Lax equations with the spectral parameter on a Riemann surface*, Contemp. Problems of Math. Mech. and Math. Phys. Proc. Steklov Inst. Math., M:MAIK, 2015, V.290, P.178-188. MR3488791
- [22] Sheinman, O.K. *Global current algebras and localization on Riemann surfaces*, Moscow Math. Journ., V. 15, n. 4, (2015), p. 833-846. MR3438837

- [23] Sheinman, O.K. *Lax operator algebras and integrable systems*. Russian Math. Surveys, 71:1 (2016), 109–156. MR3507465
- [24] Tyurin, A.N. *On the classification of rank 2 vector bundles over an algebraic curve of arbitrary genus*, Izv. AN SSSR. Ser. Math. (1964) 28:1 p. 21–52. Also published in: Tyurin, A.N., Vector Bundles, Collected Works, V.1, F. Bogomolov, A. Gorodentsev, V. Pidstrigach, M. Reid, and N. Tyurin eds., Göttingen, 2008. MR2742585
- [25] Tyurin, A.N. *Classification of vector bundles over an arbitrary genus algebraic curve*. Amer. Math. Soc. Transl. Ser. 2, 63 (1967), 245–279.
- [26] Tyurin, A.N. *On classification of rank n vector bundles over an arbitrary genus algebraic curve*. Amer. Math. Soc. Transl. Ser. 2, 73 (1968), 196–211.
- [27] Tyurin, A.N. *On the classifications of rank 2 vector bundles on an algebraic curve of an arbitrary genus*. Izv. AN SSSR, ser. math. 28:1 (1964), 21-52 (see also Collected papers).
- [28] Tyurin, A.N. *Geometry of modules of vector bundles*, Russ Math. Surv., 29:6(180) (1974), 59-88 (see also Collected papers). MR0404261
- [29] Vinberg, E.B., Gorbatshevich, V.V., Onischik A.L. *Structure of Lie groups and Lie algebras*, Itogi nauki i tekhniki, Ser.: Contemp. problems of math. Fundamental directions. M: VINITI, 1990. V. 41, P. 5-258. MR1056486
- [30] Weil, A. *Généralisation des fonctions abeliennes*, J. Math. Pures et Appl., v.17 (1938), 47-87.

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