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ON SOME MODULES OF COVARIANTS FOR A REFLECTION GROUP

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To Ernest Vinberg on the occasion of his 80th birthday

Abstract. Let $\mathfrak{g}$ be a simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Weyl group W. We build up a graded isomorphism $(\bigwedge \mathfrak{h} \otimes \mathcal{H} \otimes \mathfrak{h})^W \to (\bigwedge \mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$ of $(\bigwedge \mathfrak{g})^{\mathfrak{g}} \cong S(\mathfrak{h})^W$-modules, where $\mathcal{H}$ is the space of W-harmonics. In this way we prove an enhanced form of a conjecture of Reeder for the adjoint representation.

1. Introduction

Let W be a finite irreducible real reflection group, and let S be a set of Coxeter generators. Let V be the euclidean space affording a reflection representation of W. Consider the ring A of complex valued polynomial functions on V. Let $2 \leq d_1 \leq d_2 \leq \cdots \leq d_r$, $r = \dim V$, be the degrees of any set of homogeneous generators $\psi_1, \dots, \psi_r$ of the polynomial ring A^W. Now consider the ideal J of A generated by $\psi_1, \dots, \psi_r$ and set $\mathcal{H} = A/J$.

Let $\mathcal{W} = \bigwedge V \otimes A$ be the Weil algebra, which we regard as graded by

$$\deg(q \otimes k) = \deg(q) + 2\deg(k)$$

for $q \in \bigwedge V$ and $k \in A$ homogeneous elements. Consider now the graded ring $\mathcal{B} = \bigwedge V \otimes \mathcal{H} = \bigoplus_q \mathcal{B}_q$ and its special elements

$$p_i = \pi(d(1 \otimes \psi_i)) \in \mathcal{B}^W, \tag{1.1}$$

d being the de Rham differential on $\mathcal{W}$ (cf. (3.1)) and $\pi\colon \mathcal{W} \to \mathcal{B}$ the quotient map. A classical theorem of Solomon states that $\mathcal{B}^W = \bigwedge(p_1, \dots, p_r)$ (cf. Proposition 4.2). Let $\mathcal{D} = \hom_W(V, \mathcal{B})$. Fix a W-invariant nondegenerate symmetric bilinear $(-,-)$ form on V (unique up to multiplication by a nonzero constant). There is a natural $\mathcal{W}$-valued bilinear form E on $\mathcal{W} \otimes V$ defined by

$$E(w_1 \otimes v_1, w_2 \otimes v_2) = (v_1, v_2)w_1w_2 \tag{1.2}$$

for $v_1, v_2 \in V$, $w_1, w_2 \in \mathcal{W}$. Since J is an ideal, the form pushes down to $\mathcal{B} \otimes V \cong \hom(V, \mathcal{B})$, where we identify V with V^* using the bilinear form $(-,-)$. Passing to the invariants, we obtain a $\mathcal{B}^W$-valued bilinear form, still denoted by E, on the $\mathcal{B}^W$-module $\mathcal{D} = \hom_W(V, \mathcal{B})$.

Our main result is the following theorem, a more precise version of which is given in Theorem 5.1 (see also Proposition 4.5).

Theorem 1.1.

(1) *$\mathcal{D}$ is a free module, with explicit generators $f_i, u_i, i = 1, \dots, r$, over the exterior algebra $\bigwedge(p_1, \dots, p_{r-1})$.*

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(2) *There are nonzero constants* $k_i \in \mathbb{Q}$ *such that* $E(f_i, u_{r-i+1}) = k_i p_r$ *for each* $i = 1, \dots, r$. *The multiplication by* p_r *is selfadjoint for the form* E. *It is given by the formulas*

$$p_r f_i = -\sum_{j=1,\, j\neq i}^{r} k_j^{-1} E(f_i, u_{r-j+1}) f_j, \quad i = 1, \dots, r, \tag{1.3}$$

$$p_r u_i = -\sum_{j=1,\, j\neq i}^{r} k_j^{-1} E(f_i, u_{r-j+1}) u_j, \quad i = 1, \dots, r. \tag{1.4}$$

Statement (1) has been proven, for well-generated complex reflection groups, in [RS].

As a consequence of Theorem 1.1 we give a positive answer to a special case of a conjecture of Reeder, in an "enhanced" formulation due to Reiner and Shepler; see § 2 for details.

2. Preliminaries, motivations and outline of proof of Theorem 1.1

The framework of Reeder's conjecture is Lie-theoretic, so let us revert to this context and fix notation.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra (over $\mathbb{C}$) of rank r. Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$. Let Δ be the corresponding root system, W the Weyl group, Δ^+ a positive system and ρ the Weyl vector. Observe that as a W-module, $\mathfrak{h}$ is the reflection representation. We will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form which restricts to a W-invariant bilinear form on $\mathfrak{h}$ which we choose as our form $(-,-)$. Let Q, P denote the root and weight lattices, P^+ the cone of dominant integral weights.

The exterior algebra $\bigwedge \mathfrak{g}$ has been extensively studied as representation of $\mathfrak{g}$ (see e. g. [Kos65,Kos97]). We are concerned with Reeder's paper [Ree97], where the author studies the isotopic components in $\bigwedge \mathfrak{g}$ of representations whose highest weight is "near" 2ρ or "near" 0 w.r.t. the usual partial order on dominant weights. The nearness condition about 0 is made precise in the following.

Definition 2.1. A irreducible finite dimensional highest weight module V_λ with highest weight $\lambda \in Q \cap P^+$ is said to be *small* if twice a root of $\mathfrak{g}$ is not a weight of V_λ.

Given $\lambda \in Q \cap P^+$, the zero-weight space $0 \neq V_\lambda^0 \subset V_\lambda$ is a W-module. We introduce the following generating functions:

$$P(V_\lambda, \bigwedge \mathfrak{g}, u) = \sum_{n\geq 0} \dim \hom_{\mathfrak{g}}(V_\lambda, \bigwedge^n \mathfrak{g}) u^n,$$
$$P_W(V_\lambda^0, \mathcal{B}, u) = \sum_{q\geq 0} \dim \hom_W(V_\lambda^0, \mathcal{B}_q) u^q.$$

In [Ree97, Conjecture 7.1] Reeder proposed the following relation between these generating series when V_λ is small,

$$P(V_\lambda, \bigwedge \mathfrak{g}, u) = P_W(V_\lambda^0, \mathcal{B}, u), \tag{2.1}$$

and verified it in rank less than or equal to 3. The conjecture has two different motivations. Let G be a compact Lie group with complexified Lie algebra $\mathfrak{g}$ and let $T \subset G$ be a maximal torus. Consider the W-action on both factors of the manifold $T \times G/T$. The Weyl map $T \times_W G/T \to G$ induces an isomorphism in cohomology, which in terms of invariants reads as an isomorphism of graded vector spaces

$$(\bigwedge \mathfrak{g})^{\mathfrak{g}} \cong H^*(G) \cong H^*(T \times G/T)^W \cong (\bigwedge \mathfrak{h}^* \otimes \mathcal{H})^W = \mathcal{B}^W.$$

Conjecture (2.1) is the natural extension of this graded isomorphism to covariants of small representations. On the other hand, Broer [Bro95] has shown that, exactly for small representations, Chevalley restriction can be generalized to covariants. Let $S(\mathfrak{g})$ (resp. $S(\mathfrak{h})$) denote the symmetric algebra of $\mathfrak{g}$ (resp. of $\mathfrak{h}$). Chevalley restriction theorem gives an isomorphism $S(\mathfrak{g})^{\mathfrak{g}} \simeq S(\mathfrak{h})^W$. Broer proves that restriction also induces an isomorphism of graded $S(\mathfrak{g})^{\mathfrak{g}} \simeq S(\mathfrak{h})^W$-modules between $\hom_{\mathfrak{g}}(V_\lambda, S(\mathfrak{g}))$ and $\hom_W(V_\lambda^0, S(\mathfrak{h}))$.

Curiously enough, conjecture (2.1) in type A was implicitly proven in literature before [Ree97] appeared: the left hand side was computed by Stembridge [Ste87], whereas the right hand side appears in [KP90, Mol92] (in a more general context). Further related work appears in [Ste05], where Stembridge provides methods which can be reasonably applied for a case by case proof of the conjecture (see the discussion at the end of Section 3 in [RS]).

Set $\Gamma = \left(\bigwedge \mathfrak{g}\right)^{\mathfrak{g}} \cong \mathcal{B}^W$. In Corollary 5.2 we prove that Theorem 1.1 implies the following.

Theorem 2.2. *There is a degree preserving isomorphism of* Γ*-modules*

$$(\textstyle\bigwedge \mathfrak{h} \otimes \mathcal{H} \otimes \mathfrak{h})^W \to (\textstyle\bigwedge \mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}. \tag{2.2}$$

We are also able to build up a module isomorphism like (2.2) for the little adjoint representation, i.e., the highest weight module $\mathfrak{g}_s$ with highest weight the highest short root of Δ (provided two different root lengths exist); see Corollary 6.6. Indeed, in § 6, we prove an analogue of Theorem 1.1 for the "Weyl group side" of the little adjoint representation; see Theorem 6.5.

The statement of Theorem 2.2 cannot be extended from the adjoint representation to a general small representation: the small module $S^3(\mathbb{C}^3)$ for $\mathfrak{g} = sl(3,\mathbb{C})$ admits as zero-weight space the sign representation *sign* of the symmetric group S_3, but an easy analysis shows that a graded isomorphism of Γ-modules

$$\hom_{S_3}(sign, \textstyle\bigwedge \mathfrak{h} \otimes \mathcal{H}) \cong \hom_{sl(3,\mathbb{C})}(S^3(\mathbb{C}^3), \textstyle\bigwedge sl(3,\mathbb{C}))$$

cannot exist. Nevertheless, in § 7 we provide a speculative approach to a possible extension of Reeder's conjecture. We build up, for any $\mathfrak{g}$-module V, a map Φ^V from covariants of type V in $\bigwedge \mathfrak{g}$ to covariants of type V^0 in $\bigwedge \mathfrak{h} \otimes \mathcal{H}$ (see (7.2)). We conjecture that Φ^V is injective for any V. A result of Reeder would then imply that Φ^V is an *isomorphism of graded vector spaces* when V is small, hence implying Reeder's conjecture.

Our approach to Theorem 1.1 is motivated by our previous work with Procesi [DCPP15] on covariants of the adjoint representation in $\bigwedge \mathfrak{g}$. It is a classical fact that the invariant algebra Γ is an exterior algebra $\bigwedge(P_1, \dots, P_r)$ over primitive generators P_i of degree $2d_i - 1$. The main subject of [DCPP15] is the study of the module of covariants $\mathcal{A} = \hom_{\mathfrak{g}}\left(\mathfrak{g}, \bigwedge \mathfrak{g}\right)$; we prove the following three facts (assume for simplicity of exposition that all exponents $1 = m_1 \leq \dots \leq m_r$ of $\mathfrak{g}$ are distinct).

(1) $\mathcal{A}$ is a free module over $\bigwedge(P_1, \dots, P_{r-1})$ of rank $2r$. A set of free generators is given by the $\mathfrak{g}$-equivariant maps

$$f_i^\wedge(x) = \frac{1}{\deg(P_i)}\iota(x)P_i, \quad u_i^\wedge(x) = \frac{2}{\deg(P_i)}\iota(\mathbf{d}(x))P_i, \quad i = 1, \dots, r,$$

where $x \in \mathfrak{g}$, ι denotes interior multiplication in the exterior algebra and $\mathbf{d}$ is the usual Chevalley–Eilenberg coboundary operator for Lie algebra cohomology.

(2) The Killing form on $\mathfrak{g}$ induces an invariant graded symmetric bilinear $\bigwedge \mathfrak{g}$-valued form on $\bigwedge \mathfrak{g} \otimes \mathfrak{g}$ given, for $a, b \in \bigwedge \mathfrak{g}$, $x, y \in \mathfrak{g}$, by

$$e(a \otimes x, b \otimes y) = (x, y) a \wedge b,$$

which restricts to a Γ-valued form on $\mathcal{A}$. Then, for each pair i, j there exists a nonzero rational constant $c_{i,j}$ such that

$$e(f_i^\wedge, f_j^\wedge) = e(u_i^\wedge, u_j^\wedge) = 0, \tag{2.3}$$

$$e(f_i^\wedge, u_j^\wedge) = e(f_j^\wedge, u_i^\wedge) = \begin{cases} c_{i,j} P_k & \text{if } m_i + m_j - 1 = m_k \text{ is an exponent,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

(3) Set $c_i := c_{i,r-i+1}$. The Γ-module structure of $\mathcal{A}$ is expressed by the following relations:

$$P_r f_i^\wedge = -\sum_{j=1,\, j\neq i}^{r} c_j^{-1} e(f_i^\wedge, u_{r-j+1}^\wedge) f_j^\wedge, \quad i = 1, \dots, r, \tag{2.5}$$

$$P_r u_i^\wedge = -\sum_{j=1,\, j\neq i}^{r} c_j^{-1} e(f_i^\wedge, u_{r-j+1}^\wedge) u_j, \quad i = 1, \dots, r. \tag{2.6}$$

Similar results are obtained in [DCMFPP14] for covariants of the little adjoint representation.

In § 3 we define, in the context of finite reflection groups, equivariant maps $u_i, f_i \in \hom_W\big(\mathfrak{h}, \mathcal{H} \otimes \bigwedge \mathfrak{h}\big), i = 1, \dots, r$, of suitable degrees for which statements (1), (2), (3) hold upon replacing $P_i, e, u_i^\wedge, f_i^\wedge$ with p_i, E, u_i, f_i, respectively.

The definition (4.1) of the f_i is natural after definition (1.1). The key technical point is getting the analog of relations (2.3), (2.4). For that purpose it is necessary to introduce carefully chosen elements u_i, whose definition (4.3) involves a variation of Dunkl's operators. We are then able to prove Proposition 4.5 in the adjoint setting and Proposition 6.4 in the little adjoint setting, which are the "symmetric" analogs of statement (2).

3. Symmetric picture

As in the Introduction, let W be a finite irreducible real reflection group, and let S be a set of Coxeter generators. Let V be the euclidean space affording a reflection representation of W (which we assume to be irreducible) and let $(\cdot, \cdot)$ be the positive definite W-invariant symmetric bilinear on V. We will identify V and V^* when convenient via the invariant bilinear form. Let $T \subset W$ be the set of reflections. It is well-known that T is the union of at most two conjugacy classes T_ℓ and T_p. Let choose for every $s \in T$ a nonzero vector α_s orthogonal to the reflection hyperplane $\mathrm{Fix}(s)$ (so that $s(\alpha_s) = -\alpha_s$), and let Δ^+ be the set of such vectors; then $\Delta^+ \cup -\Delta^+$ is a root system in the sense of [Hum90, 1.2].

Consider the ring A of complex valued polynomial functions on V (which is also $S(V)$, under the identification $V \cong V^*$). One knows that A^W is a polynomial ring on $\dim V = r$ homogeneous generators $\psi_1, \dots, \psi_r$ of degrees $2 \le d_1 \le d_2 \le \cdots \le d_r$. Now consider the ideal J of A generated by $\psi_1, \dots, \psi_r$ and set $\mathcal{H} = A/J$. This is a graded representation of W whose ungraded character is the regular character. It is a well-known fact that $A \simeq A^W \otimes \mathcal{H}$ as a A^W-module.

Let $\mathcal{W} = \bigwedge V \otimes A$ be the Weil algebra, which we regard as graded by

$$\deg(q \otimes k) = \deg(q) + 2\deg(k)$$

for $q \in \bigwedge V$ and $k \in A$ homogeneous elements. Using the duality between V and V^* we think of $\mathcal{W}$ as the algebra of differential forms on V with polynomial coefficients.

So, $\mathcal{W}$ is equipped with the usual de Rham differential d given by

$$d(q \otimes k) = \sum_{i=1}^{r} (x_i \wedge q) \otimes \frac{\partial k}{\partial x_i}, \tag{3.1}$$

where $\{x_1, \dots, x_r\}$ is an orthonormal basis of V. Under our grading, d has clearly degree -1.

Consider now the graded ring

$$\mathcal{B} = \bigwedge V \otimes \mathcal{H} = \mathcal{W}/\mathcal{W}J.$$

We denote by $\pi \colon \mathcal{W} \to \mathcal{B}$ the quotient homomorphism; sometimes, abusing notation, we also denote by π the quotient map $A \to \mathcal{H}$. It is clear that $\mathcal{B}$ inherits a grading from $\mathcal{W}$.

Together with d we also have the Koszul differential δ given by the derivation

$$\delta(x_i \otimes 1) = 1 \otimes x_i, \quad \delta(1 \otimes f) = 0, \ f \in A, \tag{3.2}$$

which has degree 1 under our grading.

Since δ is W-equivariant and the ideal $\mathcal{W}J$ is preserved by δ, δ induces a differential on $\mathcal{B}$. On the other hand, $\mathcal{W}J$ is clearly not preserved by d and we need to introduce a further differential on $\mathcal{W}$.

For $s \in T$ consider the operator

$$\nabla_s = (d \log \alpha)(1 - s) = (\alpha \otimes 1)\frac{1-s}{\alpha},$$

where $\alpha = \alpha_s \in \Delta^+$. Remark that ∇_s does not depend on the choice of α and acts on the Weil algebra $\mathcal{W}$. The following properties of ∇_s are clear from its definition.

Lemma 3.1.
(1) *If* $\omega \in \mathcal{W}^W$, $\nabla_s(\omega) = 0$.
(2) $\nabla_s(\omega\nu) = \nabla_s(\omega)\nu + (s\omega)\nabla_s(\nu)$, $\omega, \nu \in \mathcal{W}$.

Lemma 3.1 implies that the ideal $\mathcal{W}J$ is preserved by ∇_s, so we get an operator on the algebra $\mathcal{B}$.

We now remark that if $\omega = a \otimes b$, $a \in \bigwedge V$, $b \in A$,

$$a \otimes b - s(a \otimes b) = (a - s(a)) \otimes b + s(a) \otimes (b - s(b))$$

and we have the next lemma.

Lemma 3.2. *If* $a \in \bigwedge V$, *then* $\alpha_s \wedge (a - s(a)) = 0$.

Proof. If $x \in V$, then $x - s(x)$ is a multiple of α_s and we are done. Let $a = a' \wedge x'$ with a' of degree t and $x \in V$. Then, by induction,

$$\begin{aligned}\alpha_s \wedge (a - s(a)) &= \alpha_s \wedge (a' - s(a')) \wedge x' + s(a') \wedge (x' - s(x')) \\ &= \alpha_s \wedge (a' - s(a')) \wedge x' + (-1)^t s(a') \wedge \alpha_s \wedge (x' - s(x')) = 0. \qquad \square\end{aligned}$$

It follows that

$$\nabla_s(a \otimes b) = (\alpha_s \wedge s(a)) \otimes \frac{(1-s)(b)}{\alpha_s}. \tag{3.3}$$

We now choose a function $c \colon T \to \mathbb{C}$ constant on conjugacy classes and set

$$D_c := \sum_{s \in T} c(s)\nabla_s, \tag{3.4}$$

and consider it as an operator both on $\mathcal{W}$ and on $\mathcal{B}$. Notice that, since clearly $D_c(\mathcal{W}^W) = 0$ and any element of $\mathcal{B}^W$ can be lifted to a W-invariant element of $\mathcal{W}$, we get the following.

Proposition 3.3. *If* $u \in \mathcal{B}^W$, *then* $D_c(u) = 0$.

Lemma 3.4. *If $w \in W$, then $w^{-1} D_c w = D_c$.*

Proof. We have

$$w^{-1} D_c w(\omega) = w^{-1}\Big(\sum_{s\in T} c(s)\nabla_s(w\omega)\Big) = \sum_{s\in T} c(s)\nabla_{w^{-1}sw}(\omega) = D_c(\omega),$$

since the function c is constant on conjugacy classes. □

Proposition 3.5. *Let U be an irreducible W-module and $x \in \mathcal{H}$ or $x \in A$ be such that it generates a copy of U. Fix $s_\ell \in T_\ell$, $s_p \in T_p$. Then*

$$\delta D_c(x) = \left(c(s_\ell)|T_\ell|\left(1 - \frac{\chi_U(s_\ell)}{\chi_U(1)}\right) + c(s_p)|T_p|\left(1 - \frac{\chi_U(s_p)}{\chi_U(1)}\right)\right)x. \tag{3.5}$$

Proof. By the definitions

$$\delta D_c(x) = \sum_{s\in T} c(s)(x - s(x)),$$

so that $\delta D_c(x) \in U$. Since U is irreducible and δD_c commutes with the W-action, we get that $\delta D_c(x) = \gamma x$, γ a constant. Computing traces we get

$$\gamma\chi_U(1) = (c(s_\ell)|T_\ell| + c(s_p)|T_p|)\chi_U(1) - c(s_\ell)|T_\ell|\chi_U(s_\ell) - c(s_p)|T_p|\chi_U(s_p),$$

from which (3.5) is clear. □

Finally we see that D_c gives a differential both on $\mathcal{W}$ and on $\mathcal{B}$. Indeed we have a proposition.

Proposition 3.6. $D_c^2 = 0$.

Proof. We have

$$D_c^2 = \sum_{(s,t)\in T\times T} c(s)c(t)\nabla_s\nabla_t.$$

Now

$$\nabla_s\nabla_t(a\otimes b) = (\alpha_s \wedge s(\alpha_t) \wedge st(a)) \otimes \left(\frac{b - t(b)}{\alpha_s\alpha_t} - \frac{s(b) - st(b)}{\alpha_s s(\alpha_t)}\right).$$

If $s = t$, clearly $\alpha_s \wedge s(\alpha_s) = -\alpha_s \wedge \alpha_s = 0$, so we can assume $s \neq t$.

We now consider the space $V_{s,t}$ spanned by α_s and α_t and the dihedral subgroup $W_{s,t}$ generated by s, t. If we set $U = \alpha_t^\perp \cap \alpha_s^\perp$, we clearly get that $V = V_{s,t} \oplus U$ and we can write as a linear combination of elements of the form $a = a' \otimes u$ with $a' \in \bigwedge V_{s,t}$ and $u \in \bigwedge U$, each homogeneous. Then if a' is of positive degree we get $\alpha_s \wedge s(\alpha_s)a' = 0$ so that we get possibly nonzero contributions to $\nabla_s\nabla_t(a\otimes b)$ only when $a' = 1$. By linearity we can assume that $a \in \bigwedge U$ so that $st(a) = a$ and we get

$$\nabla_s\nabla_t(a\otimes b) = (a\otimes 1)\left(\alpha_s \wedge s(\alpha_t) \otimes \left(\frac{b - t(b)}{\alpha_s\alpha_t} - \frac{s(b) - st(b)}{\alpha_s s(\alpha_t)}\right)\right).$$

We can even assume that $a = 1$ and look at

$$\nabla_s\nabla_t(1\otimes b) = (\alpha_s \wedge s(\alpha_t)) \otimes \left(\frac{b - t(b)}{\alpha_s\alpha_t} - \frac{s(b) - st(b)}{\alpha_s s(\alpha_t)}\right).$$

Furthermore notice that all the contributions to the right hand side come from either multiplying or dividing by vectors in $V_{s,t}$ or applying elements in $W_{s,t}$. From these considerations we deduce that we can really assume that $W = W_{s,t}$ and the claim follows from Lemma 3.7 below. □

Consider a dihedral group D generated by the reflections s, t subject to the relation $(st)^n = 1$, so that its set of reflections is formed by the n-elements $s_1 = s$, $s_2 = sts$, $s_3 = ststs$, $\dots$, $s_n = (st)^{n-1}s = t$.

Let $\mathfrak{h}$ be a reflection representation of D (meaning that $\mathfrak{h}$ is the direct sum $\mathfrak{h}_{s,t} \oplus U$, with $\mathfrak{h}_{s,t}$ the irreducible 2-dimensional reflection representation of D as above). We choose as usual $\alpha_i = \alpha_{s_i}$, $i = 1, \dots, n$, and consider the ring

$$R = \bigwedge \mathfrak{h}_{s,t} \otimes S(\mathfrak{h})\left[\prod \alpha_i^{-1}\right]$$

and the twisted group algebra $R[D]$.

Lemma 3.7. *The element*

$$\sum_{r=1}^{n} c(s_r) d\log\alpha_r (1 - s_r)$$

has zero square.

Proof. As we have seen, each summand of

$$\left(\sum_{i=1}^{n} c(s_r)\alpha_r \otimes \frac{1}{\alpha_r}(1 - s_r)\right)^2$$

comes from a pair of reflections (s_i, s_j) and is of the form

$$q_i q_j (\alpha_i \wedge s_i(\alpha_j)) \otimes \left(\frac{1 - s_j}{\alpha_i \alpha_j} - \frac{s_i(1 - s_j)}{\alpha_i s_i(\alpha_j)}\right),$$

where we set $c(s_h) = q_h$ for each h. So the pairs (s_i, s_j) give to s_j the contribution

$$q_j \sum_{i=1}^{n} q_i (\alpha_i \wedge s_i(\alpha_j)) \otimes \frac{1}{\alpha_i \alpha_j}. \tag{3.6}$$

On the other hand, the pairs (s_j, s_i) give to s_j the contribution

$$q_j \sum_{i=1}^{n} q_i (\alpha_j \wedge s_j(\alpha_i)) \otimes \frac{1}{\alpha_j s_j(\alpha_i)}. \tag{3.7}$$

We have already seen that we can assume that $j \neq i$. Then setting $\alpha_h = s_j(\alpha_i)$, and observing that $q_h = q_i$, we get that (3.7) becomes

$$q_j \sum_{h=1}^{n} q_h (\alpha_j \wedge \alpha_h) \otimes \frac{1}{\alpha_j \alpha_h}.$$

On the other hand, $\alpha_i \wedge s_i(\alpha_j) = \alpha_i \wedge \alpha_j$, so that (3.6) equals

$$q_j \sum_{i=1}^{n} q_i (\alpha_i \wedge \alpha_j) \otimes \frac{1}{\alpha_i \alpha_j}.$$

Thus the coefficient of s_j is clearly 0.

We now pass to the coefficient of $s_i s_j$. This is equal to

$$q_i q_j (\alpha_i \wedge s_i(\alpha_j)) \otimes \frac{1}{\alpha_i s_i(\alpha_j)} = q_i q_j (\alpha_i \wedge \alpha_j) \otimes \frac{1}{\alpha_i s_i(\alpha_j)}.$$

For each $h = 1, \dots, n$, $s_i s_j = s_h s_{h+j-i}$ and $q_i q_j = q_h q_{h+j-i}$. So one needs to verify that

$$\sum_{h=1}^{n} d\log\alpha_h \wedge d\log s_h(\alpha_{h+j-i}) = \sum_{h=1}^{n} d\log\alpha_h \wedge d\log\alpha_{h+i-j} = 0.$$

If we take a cycle $c = (u_1, u_2, \dots, u_d)$, we claim that, setting $u_{d+1} = u_1$,

$$\sum_{r=1}^{d} d\log\alpha_{u_i} \wedge d\log(\alpha_{u_{i+1}}) = 0.$$

If the cycle has length 2 this is obvious. If it has length 3, a simple computation shows that

$$d\log\alpha_{u_1} \wedge d\log\alpha_{u_2} + d\log\alpha_{u_2} \wedge d\log\alpha_{u_3} = d\log\alpha_{u_1} \wedge d\log\alpha_{u_3},$$

which is our relation.

We proceed now by induction and, using the above relation, we substitute and get the relation using the cycle $(u_1, u_3, \dots, u_d)$. Let us fix $m = j-i$ and consider the permutation $\sigma(h) = m + h \pmod n$ (choosing as remainders $1, \dots, n$). Now decompose it into cycles and apply the previous claim to get the result. □

Remark 3.8. We are going to call D_c a Dunkl differential. Operators of this kind on differential forms already appear in the paper [DdJO94].

4. The bilinear form

If W is crystallographic, hence it is the Weyl group associated to a simple Lie algebra $\mathfrak{g}$, we recall that, by Chevalley theorem, restriction gives an isomorphism between $S(\mathfrak{g})^{\mathfrak{g}}$ and A^W, the polynomial ring of W invariant functions on the Cartan subalgebra. We then fix homogenous generators $\psi_1, \dots, \psi_r$ of the polynomial ring $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \simeq A^W$ in such a way that they induce by transgression the generators $P_1, \dots, P_r$ of $\left(\bigwedge \mathfrak{g}\right)^{\mathfrak{g}}$ considered in § 1. On the other hand, considering $\psi_1, \dots, \psi_r$ in A^W, we can introduce the elements p_i (cf. (1.1)).

In the case W is not crystallographic, we choose the homogenous generators $\psi_1, \dots, \psi_r$ of the polynomial ring A^W arbitrarily and proceed to define the elements p_i, $i = 1, \dots, r$, as above.

Remark 4.1. A priori the definition of the elements p_i depends on the choice of the generators $\psi_1, \dots, \psi_r$ of the polynomial ring A^W. However, if $J \subset A^W$ denotes as above the ideal of elements of positive degree it is immediate to see that the p_i depend only on the induced basis $\bar\psi_1, \dots, \bar\psi_r$ of J/J^2.

Indeed if $z = \psi_i - \psi_i' \in J^2$ and $z = \sum_j x_j y_j, x_i, y_j \in J$, then

$$\pi(d(1 \otimes z)) = \pi\left(\sum_j d(x_j) y_j + \sum_j x_j d(y_j)\right) = 0,$$

proving the claim.

We have the following theorem of Solomon, which is reproved here for the reader's convenience.

Proposition 4.2. *$\mathcal{B}^W$ is the graded exterior algebra generated by the elements p_i defined in* (1.1)*.*

Proof. First notice that $\dim \mathcal{B}^W = \dim \bigwedge V = 2^r$. Second, notice that for each $i = 1, \dots, r$ the element p_i is of degree $(1, 2d_i - 2)$, that is of total degree $2d_i - 1$. It is clear that $p_i p_j = -p_j p_i$, so it suffices to show that $\prod_{j=1}^r p_j \neq 0$. Now let us remark that the element

$$\Delta = \det(\frac{\partial_i \psi_j}{\partial x_i})$$

spans the copy of the sign representation of W of lowest possible degree. Thus $\Delta \notin J$. Furthermore

$$\prod_{j=1}^{r} p_j = x_1 \wedge x_2 \wedge \cdots \wedge x_r \otimes \pi(\Delta) \neq 0$$

and the claim follows. □

Remark 4.3. In the crystallographic case, we get a natural isomorphism between $\left(\bigwedge \mathfrak{g}\right)^{\mathfrak{g}}$ and $\mathcal{B}^W$.

We now consider $\mathcal{D} = \hom_W(V, \mathcal{B})$, and the following special element of $\mathcal{D}$,

$$f_i(v) = \pi(1 \otimes \partial_v \psi_i), \tag{4.1}$$

where ∂_v denotes the directional derivative in the direction $v \in V$ and $i = 1, \ldots, r$.

Notice that with respect to a orthonormal basis $\{x_i\}$ of V, we have

$$f_i = \sum_{j=1}^{r} \pi(1 \otimes \frac{\partial \psi_i}{\partial x_j}) \otimes x_j.$$

Moreover, by (3.2), for every $v \in V$,

$$\delta(f_i(v)) = 0. \tag{4.2}$$

Fix a function $c\colon T \to \mathbb{C}$ constant on conjugacy classes as in the previous section. Set $|T|_c = c(s_\ell)|T|_\ell + c(s_p)|T|_p$ and define

$$u_i(v) = \frac{r}{2|T|_c} D_c f_i(v). \tag{4.3}$$

Proposition 4.4. *For every $v \in V$, $\delta(u_i(v)) = f_i(v)$.*

Proof. Apply Proposition 3.5 taking $U = V$ (notice that we have assumed that V is irreducible) and $x = f_i$. Since $\chi_U(s_p) = \chi_U(s_\ell) = r - 2$, the claim follows. □

From now on, we will use the constant function $c = 1$ on T and set $D = D_1$. Recall the natural $\mathcal{W}$-valued bilinear form E on $\mathcal{W} \otimes V$ defined by (1.2) and its restriction to a $\mathcal{B}^W$-valued bilinear form on the $\mathcal{B}^W$-module $\mathcal{D}$.

Proposition 4.5.
(1) *$E(f_i, f_j) = 0$.*
(2) *Assume that $d_i \neq d_j$ for $i \neq j$. Then*

$$E(u_i, f_j) = E(u_j, f_i) = \begin{cases} k_{i,j} p_s & \text{if there exists } s \text{ such that } d_i + d_j - 2 = d_s, \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

with $k_{i,j} \neq 0$. Furthermore, if W is crystallographic, then

$$k_{i,j} = c_{i,j}, \tag{4.5}$$

where $c_{i,j}$ is the constant introduced in (2.4).

Proof. Let us choose a orthonormal basis $\{x_i\}$ for V. Then, since $E(f_i, f_j) \in \mathcal{B}^W$ and

$$E(f_i, f_j) = \sum_{s=1}^{r} \pi(1 \otimes \frac{\partial \psi_i}{\partial x_s} \frac{\partial \psi_j}{\partial x_s}),$$

we have that

$$\sum_{s=1}^{r} (\frac{\partial \psi_i}{\partial x_s})(\frac{\partial \psi_j}{\partial x_s}) \in J,$$

hence (1) follows.

To see part (2), notice that $E(u_i, f_j) \in \left(\bigwedge^1 V \otimes \mathcal{H}\right)^W$ so that if there is no s for which $d_i + d_j - 2 = d_s$, then by Proposition 4.2 we have $E(u_i, f_j) = 0$.

Assume $d_i + d_j - 2 = d_s$. Then we have that necessarily $E(u_i, f_j) = k_{i,\,j} p_k$, $k_{i,\,j} \in \mathbb{C}$ again by Proposition 4.2.

We have to prove that $k_{i,\,j} \neq 0$. Lifting to $\mathcal{W}$ and applying δ we obtain

$$E(d\psi_i, d\psi_j) = k_{i,\,j}\psi_k + b, \quad b \in J^2.$$

If $k = r = 2$ this statement is obvious. If $k = r$, so that the indices i, j are complementary, we can then apply the argument of Proposition 2.9 from [DCPP15] and deduce $k_{i,\,j} \neq 0$.

This completes the proof in the noncrystallographic case, since the only pairs d_i, d_j with $d_i + d_j - 2 = d_s$ either have $d_i = 2$ or $d_j = 2$ or i, j are complementary; this is clear for dihedral groups: for H_3 the degrees are $2, 6, 10$ and for H_4 they are $2, 12, 20, 30$ so everything is readily verified.

It remains to treat the crystallographic case. But this follows from [DCPP15, 2.7.2], from which also the equality $k_{i,\,j} = c_{i,\,j}$ is easily deduced. □

Remark 4.6. Type D_{2n}, where a basic degree of multiplicity 2 appears, is handled as in [DCPP15, Proposition 1.3].

Proposition 4.7.

$$E(u_j, u_i) = 0. \tag{4.6}$$

Proof. Consider $u, v \in \mathcal{B}$. We have

$$D(uv) = (Du)v + \sum_{s \in S} s(u)\nabla_s(v) = (Du)v - uDv + \sum_{s \in S}(u + s(u))\nabla_s(v).$$

But

$$(u + s(u))\nabla_s(v) = (-1)^{\deg(u)}(1 - s)(d\log\alpha_s(u + s(u))v).$$

Since $d\log\alpha_s(u + s(u))$ is fixed by s, we have

$$s((u + s(u))\nabla_s(v)) = -(u + s(u))\nabla_s(v).$$

Thus, since the usual scalar product is W-invariant, we deduce that

$$s((u + s(u))\nabla_s(v))$$

is orthogonal to the W-invariants. So, also

$$(uv) - (Du)v + uDv$$

is orthogonal to the W-invariants. From this, reasoning as in [DCPP15, Lemma 2.15], we get that

$$DE(f_j, u_i) - E((1 \otimes D)f_j, u_i) + E(f_j, (1 \otimes D)u_i) = 0.$$

However, by Proposition 3.3, $DE(f_j, u_i) = 0$, by Proposition 3.6 $E(f_j, (1 \otimes D)u_i) = 0$, so that (4.6) follows. □

5. Main theorem

Theorem 5.1.

(1) *$\mathcal{D}$ is a free module, with basis the elements $f_i, u_i, i = 1, \ldots, r$, over the exterior algebra $\bigwedge(p_1, \ldots, p_{r-1})$.*

(2) *Let $k_i = k_{i,r-i+1}$ with $k_{i,\,j}$ defined as in (4.4). Then for each $i = 1, \ldots, r$*

$$E(f_i, u_{r-i+1}) = k_i p_r.$$

The multiplication by p_r is selfadjoint for the form E and it is given by the formulas:

$$p_r f_i = -\sum_{j=1,\, j\neq i}^{r} k_j^{-1} E(f_i, u_{r-j+1}) f_j, \quad i = 1, \dots, r, \tag{5.1}$$

$$p_r u_i = -\sum_{j=1,\, j\neq i}^{r} k_j^{-1} E(f_i, u_{r-j+1}) u_j, \quad i = 1, \dots, r. \tag{5.2}$$

Proof. (1) Suppose that we have a relation

$$\sum_{i=1}^{r} \lambda_i u_i + \sum_{j=1}^{r} \mu_j f_j = 0.$$

Then apply $1 \otimes \delta$ and by (4.2) and Proposition 4.4 get

$$\sum_{i=1}^{r} \lambda_i f_i = 0.$$

So if we prove that the f_i are linearly independent, we get $\lambda_i = 0$ for all i and in turn that also all the μ_j are 0.

Remark that, if there is a nontrivial relation

$$\sum_{j=1}^{r} \mu_j f_j = 0,$$

we may assume that it is homogeneous. Moreover, given an index j, multiplying by a suitable element of $\bigwedge(p_1, \dots, p_{r-1})$ we can reduce ourselves to the case in which $\mu_j = p_1 \wedge p_2 \wedge \dots \wedge p_{r-1}$.

Notice now that the coefficient μ_h of the terms $\mu_h f_h$ for which $d_h < d_j$ has degree higher than the maximum allowed degree, hence it is zero. Thus, if we choose for j the maximum for which $\mu_j \neq 0$, we are reduced to prove that

$$p_1 \wedge p_2 \wedge \dots \wedge p_{r-1} f_j \neq 0. \tag{5.3}$$

By part (2) of Proposition 4.5 we have $E(f_j, u_{r-j+1}) = k_r p_r$, hence

$$E(p_1 \wedge p_2 \wedge \dots \wedge p_{r-1} f_j, u_{r-j+1}) = k_r\, p_1 \wedge p_2 \wedge \dots \wedge p_{r-1} \wedge p_r \neq 0.$$

(2) Using Propositions 4.5, 4.7, one can mimic the proof of [DCPP15, Theorem 1.4]. We briefly explain how to proceed, omitting for simplicity the case D_{2n}.

Consider the relation for u_i. We have

$$p_r u_i = \sum_{j=1}^{r} H_j u_j + \sum_{j=1}^{r} K_j f_j, \tag{5.4}$$

where the $H_j, K_j \in \bigwedge(p_1, \dots, p_{r-1})$. Applying the differential $1 \otimes \delta$ we get

$$p_r f_i = \sum_{j=1}^{r} H_j f_j. \tag{5.5}$$

Thus the relation for f_i involves only the f_j's. Also we have that the relation is homogeneous.

For each j, taking the scalar product with u_{r-j+1}, we have

$$\begin{aligned} p_r E(f_i, u_{r-j+1}) &= H_j E(f_j, u_{r-j+1}) + \sum_{h\neq j} H_h E(f_h, u_{r-j+1}) \\ &= H_j k_j p_r + \sum_{h\neq j} H_h E(f_h, u_{r-j+1}). \end{aligned}$$

Since the terms $\sum_{h\neq j} H_h E(f_h, u_{r-j+1})$ do not involve p_r, we must have

$$\sum_{h\neq j} H_h E(f_h, u_{r-j+1}) = 0, \quad -E(f_i, u_{r-j+1})p_r = H_j k_j p_r. \tag{5.6}$$

If $i \neq j$ we have that $E(f_i, u_{r-j+1})$ is not a multiple of p_r and we deduce that

$$E(f_i, u_{r-j+1}) = -k_j H_j.$$

If $i = j$ we deduce $H_j = 0$, so finally (5.5) becomes

$$p_r f_i + \sum_{i\neq j} k_j^{-1} E(f_i, u_{r-j+1}) f_j = 0. \tag{5.7}$$

Since $E(f_i, u_{r-i+1}) = k_i p_i$, formula (5.7) is indeed formula (5.1), as required. We go back to formula (5.4), which we now write as

$$p_r u_i = -\sum_{j=1}^{r} k_j^{-1} E(f_i, u_{r-j+1}) u_j + \sum_{j=1}^{r} K_j f_j. \tag{5.8}$$

Take the scalar product of both sides of (5.8) with u_{r-j+1}. We get

$$p_r E(u_i, u_{r-j+1}) = -\sum_{j=1}^{r} k_j^{-1} E(f_i, u_{r-j+1}) E(u_j, u_{r-j+1}) + \sum_{j=1}^{r} K_j E(f_j, u_{r-j+1}).$$

Since $E(u_h, u_k) = 0$, we deduce that

$$k_j K_j p_r + \sum_{i,\, i\neq j} K_i E(f_i, u_{r-j+1}) = 0.$$

We claim that all K_j are zero. Indeed the only product containing p_r is $k_j K_j p_r$. Since each element of Γ can be written in a unique way in the form $a + bp_r$ with $a, b \in \bigwedge(p_1, \dots, p_{r-1})$, we deduce $K_j = 0$ as desired. $\square$

Using (4.5) one gets the following corollary, which obviously implies Reeder's conjecture (2.1) for $\mathfrak{g}$.

Cororllary 5.2. *The map*

$$p_i \mapsto P_i, \quad u_i \mapsto u_i^{\wedge}, \quad f_i \mapsto f_i^{\wedge}, \quad 1 \leq i \leq r,$$

extends to an isomorphism of graded $\mathcal{B}^W$-modules $\left(\bigwedge \mathfrak{h} \otimes \mathcal{H} \otimes \mathfrak{h}\right)^W \to \left(\bigwedge \mathfrak{g} \otimes \mathfrak{g}\right)^{\mathfrak{g}}$.

6. The Weyl group side of the little adjoint representation

Suppose that W contains two distinct conjugacy classes of reflections T_ℓ, T_p. Set $r_\ell = |T_\ell \cap S|$, $r_p = |T_p \cap S|$. Denote by H_{T_ℓ} the subgroup of W generated by the reflections $s \in T_\ell$, and by W_{T_p} the reflection subgroup of W generated by the reflections $s \in T_p \cap S$. The following fact is proven in [Pan12, Proposition 2.1].

Lemma 6.1. *$W = W_{T_p} \ltimes H_{T_\ell}$ so W/H_{T_ℓ} is canonically isomorphic to W_{T_p}. Symmetrically $W = W_{T_\ell} \ltimes H_{T_p}$, so W/H_{T_p} is canonically isomorphic to W_{T_ℓ}.*

Let us now consider the reflection representation U of W_{T_p}. Since W_{T_p} is a quotient of W, we may consider U as a W-module.

Consider now V as a H_{T_ℓ}-module. Since H_{T_ℓ} is generated by reflections, the ring $A^{H_{T_\ell}}$ is a polynomial ring generated by homogeneous generators $\bar\psi_1, \dots, \bar\psi_n$. Let $J_{H_{T_\ell}}$ be the ideal in $A^{H_{T_\ell}}$ generated by $\bar\psi_1, \dots, \bar\psi_n$. Clearly W acts on $\bar V = J_{H_{T_\ell}}/J^2_{H_{T_\ell}}$, and we have the following.

Proposition 6.2. *For the W-module $\bar V$ one has:*
(1) $\bar V \simeq U \oplus \bar V^W$;
(2) $\dim \bar V^W = |T_\ell \cap S|$;
(3) *the submodule $U \subset \bar V$ is homogeneous of degree $d_n/2 - (r_p - 1)r_\ell$.*

Proof. The proof is a case by case check. Let us start recalling that we have two distinct conjugacy classes of reflections precisely in the following cases: $B_n = C_n$, $I_2(2m)$, F_4.

Type B_n. Let us choose an orthonormal basis $e_1, \dots, e_n$ of V in such a way that the conjugacy class T_ℓ is given by the n reflections with respect to the coordinate hyperplanes, the other class T_p by the reflections with respect to the hyperplanes of equation $x_i \pm x_j$, $i < j$.

The group H_{T_ℓ} is clearly isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$, and identifying A with $K[x_1, \dots, x_n]$ using the coordinates associated to our basis, it turns out that

$$A^{H_{T_\ell}} = K[x_1^2, \dots, x_n^2].$$

Moreover, W_{T_p} is the symmetric group S_n, acting on $\bar V = \langle x_1^2, \dots, x_n^2\rangle$ by the permutation representation. It clearly follows that $\bar V = U \oplus \bar V^W$. So $\bar V$ and hence U is contained in the homogeneous component of degree 2 and the rest is clear since $d_n = 2n$, $r_\ell = 1$ $r_p = n - 1$, so that $q = d_n/2 - (r_p - 1)r_\ell = 2$.

Let us now exchange the roles of T_ℓ and T_p. In this case H_{T_p} is a Weyl group of type D_n, so H_{T_p} has index 2 in W and $W_{T_\ell} \simeq \mathbb{Z}/2\mathbb{Z}$. We have that

$$A^{H_{T_p}} = K[\psi_0, \psi_1, \dots, \psi_{n-1}],$$

where $\psi_i = \sum_{h=1}^n x^{2i}$ is a basic invariant for B_n of degree $2i$ for $i = 1, 2, \dots, n - 1$, while $\psi_0 = x_1 \dots x_n$. It is now clear that $\bar V = \langle \psi_0, \dots, \psi_1, \dots, \psi_n\rangle$ and $U = K\psi_0$, while $\langle \psi_1, \dots, \psi_n\rangle = \bar V^W$. The remaining statement is clear.

Type $I_2(2m)$. In this case the roles of T_ℓ and T_p are completely symmetric, so we shall treat only one case. We have

$$H_{T_\ell} = I_2(m), \quad W_{T_p} \simeq \mathbb{Z}/2\mathbb{Z}. \quad A^{H_{T_\ell}} = K[\psi_1, \psi_2],$$

while $A^W = K[\psi_1, \psi_2^2]$ with $\deg \psi_1 = 2$, $\deg \psi_2 = m$. From this, everything follows.

Type F_4. Also in this case the roles of T_ℓ and T_p are completely symmetric, so we shall treat only one case. The group H_{T_ℓ} is of type D_4 and $W_{T_p} = S_3$. Let $\psi_1, \psi_2, \psi_3, \psi_4$ be basic invariants for H_{T_ℓ} of degrees $2, 4, 4, 6$, respectively. The basic invariants for W occur in degrees $2, 6, 8, 12$. We can choose ψ_1, ψ_4 to be basic invariants for W. We claim that the action of W_{T_p} on $\langle \psi_2, \psi_3\rangle$ is given by its reflection representation. Indeed, since $\langle \psi_2, \psi_3\rangle$ cannot contain invariants for W_{T_p}, the only other possibility is that W_{T_p} acts on $\langle \psi_2, \psi_3\rangle$ by two copies of the sign representation. If this were the case we would have that the degree 8 component of A^W would have dimension at least 5 while we know that it has dimension 3. Finally, $d_n = 12$, $r_\ell = r_2 = 2$ so $d_n/2 - (r_p - 1)r_\ell = 4$. □

Now take a W-invariant complement to $J^2_{H_{T_\ell}}$ in $J_{H_{T_\ell}}$ which we can clearly identify with $\bar V$. Then

$$A^{H_{T_\ell}} = K[\bar V] = K[U] \otimes K[\bar V^W].$$

Set $\widetilde A = K[U]$. Let $\phi_1, \dots, \phi_{r_p}$ be homogeneous polynomial generators of $\widetilde A^W$. Consider the ideal J kernel of the quotient $\pi\colon A \to \mathcal{H}$. Then

Lemma 6.3. *$\widetilde J := J \cap \widetilde A$ is the ideal generated by $\phi_1, \dots, \phi_{r_p}$.*

Proof. Take a homogeneous basis $\phi_{r_p+1},\dots,\phi_n$ of $\bar V^W$. Then $J=(\phi_1,\dots,\phi_n)$. Take $a\in J\cap A^{H_{T_\ell}}$, and write $a=\sum_i b_i\phi_i$, with $b_i\in A$. Applying to a the operator

$$R=\frac{1}{|H_{T_\ell}|}\sum_{g\in H_{T_\ell}} g$$

we get

$$a=\sum_i R(b_i)\phi_i,$$

so that, since $R(b_i)\in A^{H_{T_\ell}}$, $J\cap A^{H_{T_\ell}}$ is generated by $\phi_1,\dots,\phi_n$. But then $\widetilde J$ is clearly generated by $\phi_1,\dots,\phi_{r_p}$. □

Let us now double all degrees. The inclusion $\widetilde A\subset A$ multiplies the degrees by $q=d_n-2(r_p-1)r_\ell$. Furthermore Lemma 6.3 clearly implies that we have an inclusion of $\widetilde{\mathcal H}=\widetilde A/\widetilde J$ into $\mathcal H$, which also multiplies the degrees by $q=d_n-2(r_p-1)r_\ell$.

In each case W_{T_p} is the symmetric group S_{r_p+1}, so that $\deg\phi_i=2(i+1)q$, each $j=1,\dots,r_p$. In particular ϕ_{r_p} has degree $(d_n-2(r_p-1)r_\ell)(r_p+1)$, which one checks easily to equal $2d_n$. We deduce that ϕ_{r_p} is a highest degree generator of both $\widetilde A^W$ and A^W.

We define for each $i=1,\dots,r_p$, the W-equivariant map $g_i\colon U\to\bigwedge V\otimes\mathcal H$, given, for $u\in U$, by

$$g_i(u)=1\otimes\pi'(\partial_u\phi_i)=\sum_{j=1}^{s}(u,y_j)(1\otimes\pi'(\frac{\partial\phi_i}{\partial y_j})).$$

By the above discussion g_i is homogeneous of degree $2iq$. Let us now take the operator D introduced in (3.4) (with $c=1$) and notice that clearly its restriction to $A^{H_{T_\ell}}$ equals

$$D_{(p)}:=\sum_{s\in T_p}\nabla_s.$$

Define

$$v_i(u)=\frac{s}{2|T_p|}D_{(p)}g_i(v). \tag{6.1}$$

By repeating the proof of Proposition 4.4, we then get $(1\otimes\delta)(v_i)=g_i$. Furthermore, using a W-invariant bilinear form on U and reasoning exactly as in (1.2), we obtain a bilinear for on the module $\mathcal E=\hom_W(U,\mathcal B)$ with values in $\mathcal B^W$ which we still denote by E. We have

Proposition 6.4. *Let $1\le i,j\le r_p$. Then:*
(1) $E(g_i,g_j)=0$;
(2) $E(v_i,g_j)=E(v_j,g_i)$

$$=\begin{cases} m_{i,j}p_k & \text{if there exists } k \text{ such that } d_i+d_j-2=d_k,\\ 0 & \text{otherwise}\end{cases} \tag{6.2}$$

with $m_{i,j}\neq 0$.

Proof. We have seen that any set $\phi_1,\dots,\phi_{r_p}$ of homogeneous polynomial generators of $\widetilde A^W$ is part of a set of polynomial generators for A^W and that ϕ_{r_p} is the highest degree generator for both $\widetilde A^W$ and A^W. Furthermore, by Proposition 3.5, we have that $(1\otimes\delta)(v_i)=g_i$ for all $i=1,\dots,r_p$. At this point, everything follows right away from Propositions 6.4 and 4.7 applied to the group W_{T_p}. □

Let us now consider the $\mathcal B^W$-module $\mathcal D_p:=\hom_W(U,\mathcal B)$. We get, repeating word by word, the proof of Theorem 5.1,

Theorem 6.5.
(1) $\mathcal{D}_p$ *is a free module, with basis the elements* $g_i, v_i, i = 1, \ldots, r_p$, *over the exterior algebra* $\bigwedge(p_1, \ldots, p_{r_p-1})$.
(2) *The multiplication by* p_{r_p} *is selfadjoint for the form* E. *Setting* $m_i = m_{i,r+1-i}$, *it is given by the formulas*

$$p_r g_i = - \sum_{j=1,\, j\neq i}^{r_p} m_j^{-1} E(g_i, v_{r_p-j+1}) g_j, \quad i = 1, \ldots, r_p, \tag{6.3}$$

$$p_r v_i = - \sum_{j=1,\, j\neq i}^{r_p} m_j^{-1} E(g_i, v_{r_p-j+1}) v_j, \quad i = 1, \ldots, r_p. \tag{6.4}$$

In the case in which W is the Weyl group of a simple Lie algebra $\mathfrak{g}$, which is of course nonsimply laced, our representation U is the zero weight space of the irreducible $\mathfrak{g}$-module $\mathfrak{g}_s$ whose highest weight is the dominant short root, which in fact is small. Using Theorem 6.5, and [DCMFPP14], one can then easily deduce the following corollary which obviously implies Reeder's conjecture (2.1) for $\mathfrak{g}_s$.

Cororllary 6.6. *The map*

$$p_i \mapsto P_i, \quad v_i \mapsto u_i^{\wedge}, \quad g_i \mapsto f_i^{\wedge}, \quad 1 \le i \le r$$

extends to an isomorphism of graded $\mathcal{B}^W$*-modules* $\left(\bigwedge \mathfrak{h} \otimes \mathcal{H} \otimes U\right)^W \to \left(\bigwedge \mathfrak{g} \otimes \mathfrak{g}_s\right)^{\mathfrak{g}}$.

7. A possible extension of Reeder's conjecture

Consider the bracket map $[-,-] \colon \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$. Dualizing and using the isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ given by the Killing form, we get a linear map $\mathfrak{g} \to \bigwedge^2 \mathfrak{g}$. Since $\bigwedge^{\text{even}} \mathfrak{g}$ is a commutative algebra, this linear map extends to homomorphism of algebras $s \colon S(\mathfrak{g}) \to \bigwedge^{\text{even}} \mathfrak{g}$.

The inclusion $\mathfrak{h} \subset \mathfrak{g}$ also gives an inclusion of rings $j \colon S(\mathfrak{h}) \to S(\mathfrak{g})$. Composing with s we get the homomorphism, $\tau \colon S(\mathfrak{h}) \to \bigwedge^{\text{even}} \mathfrak{g}$.

Let, as in § 1, J be the ideal in $S(\mathfrak{h})$ generated by the W-invariants vanishing in 0. Recall that the ideal J has a canonical complement $\mathcal{A}$, the so-called harmonic polynomials, i.e. those elements in $S(\mathfrak{h})$ killed by all constant coefficients W-invariant differential operators without constant term.

We have the following.

Proposition 7.1. *The restriction of the homomorphism* $\tau \colon S(\mathfrak{h}) \to \bigwedge^{\text{even}} \mathfrak{g}$ *to* $\mathcal{A}$ *is injective.*

Proof. Let $\Delta^+ \subset \mathfrak{h}^* \simeq \mathfrak{h}$ denote the set of positive roots. Take the Weyl denominator polynomial

$$P = \prod_{\alpha \in \Delta^+} \alpha = \prod_{\alpha \in \Delta^+} t_\alpha$$

(where $t_\alpha \in \mathfrak{h}$ is defined by $\lambda(t_\alpha) = (\alpha, \lambda)$, $\lambda \in \mathfrak{h}^*$). We know that W acts on P by the sign representation and that in degree $N = |\Delta^+|$ the homogeneous component $\mathcal{A}_N$ of $\mathcal{A}$ is spanned by P.

Recall from [Kos97, (89)] that, if $\{x_i\}, \{x^i\}$ are dual basis of $\mathfrak{g}$ w.r.t. the chosen invariant form, then

$$s(x) = \frac{1}{2} \sum_i x_i \wedge [x^i, x], \quad x \in \mathfrak{g}.$$

Now fix root vectors e_β, $\beta \in \Delta^+$ and choose $e_{-\beta}$ such that $(e_\beta, e_{-\beta}) = 1$. A simple computation using the above formula for s shows that for any $\alpha \in \Delta^+$ we have

$$\tau(t_\alpha) = \sum_{\beta\in\Delta^+} (\beta, \alpha) e_\beta \wedge e_{-\beta}.$$

It follows that

$$\tau(P) = per(A) \prod_{\beta\in\Delta^+} (e_\beta \wedge e_{-\beta}),$$

where $per(A)$ is the permanent of the matrix $A = ((\beta, \alpha))$.

Now A is a positive semidefinite matrix. It follows that its permanent is nonzero. Indeed by [MM65], one has

$$per(A) \geq \frac{N!}{(\rho,\rho)^N} P(\rho)^2 = \frac{N!}{(\rho,\rho)^N} \prod_{\alpha\in\Delta^+} (\alpha, \rho)^2 > 0,$$

where ρ is the half sum of positive roots, which is well-known to be regular. This proves our claim in degree N.

Let us now consider $\mathcal{A}_m$. We have $m \leq N$ otherwise $\mathcal{A}_m = \{0\}$ and there is nothing to prove. So we can assume $m < N$. Take $0 \neq a \in \mathcal{A}_m$. We then know that there is an element $b \in \mathcal{A}_{N-m}$ such that $ab = P + r$ with $r \in J_N$. Assume $\tau(a) = 0$. Then $\tau(r) = -\tau(P)$. Consider the W-module U spanned by r. Then $U \subset J_N$ and τ gives a surjective W-equivariant homomorphism $U \to \mathbb{C}\tau(P)$. We deduce that U and hence J_N contains a copy of the sign representation of W, contrary to the fact that P spans the only copy of the sign representation in degree N. It follows that $\tau(a) \neq 0$, proving our claim. $\square$

Recall that there is a W-equivariant degree preserving isomorphism between $\mathcal{A}^*$ and $\mathcal{H}$. Since $\bigwedge^{\text{even}} \mathfrak{g}$ is selfdual, dualizing τ we obtain a surjective degree preserving map

$$\phi\colon \bigwedge^{\text{even}} \mathfrak{g} \to \mathcal{H}.$$

Let p be the projection to $p\colon \bigwedge \mathfrak{g} \to \bigwedge \mathfrak{h}$ and $\pi\colon \bigwedge \mathfrak{g} \to \bigwedge^{\text{even}} \mathfrak{g}$ the projection on the even part. Using these, we can build up the map

$$\Phi\colon \bigwedge \mathfrak{g} \xrightarrow{\wedge^*} \bigwedge \mathfrak{g} \otimes \bigwedge \mathfrak{g} \xrightarrow{\mathrm{Id} \otimes \pi} \bigwedge \mathfrak{g} \otimes \bigwedge^{\text{even}} \mathfrak{g} \xrightarrow{p\otimes\phi} \bigwedge \mathfrak{h} \otimes \mathcal{H}. \tag{7.1}$$

Let V be any finite dimensional irreducible $\mathfrak{g}$-module. Denote by V^0 its zero weight space and by $i\colon V^0 \hookrightarrow V$ the natural inclusion. If $f \in \hom(V, \bigwedge \mathfrak{g})$, we may consider

$$\Phi_f^V := \Phi \circ f \circ i \in \hom(V_0, \bigwedge \mathfrak{h} \otimes \mathcal{H}).$$

Clearly, by equivariance, if $f \in \hom_{\mathfrak{g}}(V, \bigwedge \mathfrak{g})$, then $\Phi_f^V \in \hom_W(V^0, \bigwedge \mathfrak{h} \otimes \mathcal{H})$.

Hence we have a graded map

$$\Phi^V\colon \hom_{\mathfrak{g}}(V, \bigwedge \mathfrak{g}) \to \hom_W(V^0, \bigwedge \mathfrak{h} \otimes \mathcal{H}), \quad \Phi^V(f) = \Phi_f^V. \tag{7.2}$$

Conjecture. *For any finite dimensional irreducible $\mathfrak{g}$-module V, the map Φ^V is injective.*

Remark 7.2. Since $\dim \hom_{\mathfrak{g}}\big(V, \bigwedge \mathfrak{g}\big) = \dim \hom_W\big(V^0, \bigwedge \mathfrak{h} \otimes \mathcal{H}\big)$ if (and only if) V is small (cf. [Ree97, Corollary 4.2]), the above conjecture implies Reeder's conjecture.

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