Geometric Tomography

R. J. Gardner

1. Introduction

Tomography, from the Greek $\tau\acute{o}\mu o\varsigma$, a slice, is by now an established and active area of mathematics. The word is usually associated with computerized tomography, the dramatic applications of which include the CAT scanner in medicine. In such applications, information is collected about sections of a density distribution. By various techniques, this information is synthesized to yield a reconstruction of the density distribution itself. For example, the CAT scanner produces images of two-dimensional sections of human patients from X rays taken in a finite number of directions. Such a reconstruction is always merely approximate, the accuracy depending on the number of X rays, since no finite set of X rays determines a density distribution uniquely.

. . . from the Greek $\tau\acute{o}\mu o\varsigma$, a slice.

At the Conference on Tomography at Oberwolfach in 1990, the author introduced the term *geometric tomography*. In the author's book [21], the following definition is offered: "*Geometric tomography* is the area of mathematics dealing with the retrieval of information about a geometric object from data about its sections, or projections, or both." The use of the word *geometric* here is deliberately vague. A special case would be the study of sections or projections of a convex body or polytope, but it is sometimes more appropriate to consider star-shaped bodies, compact sets, or even Borel sets. The word *projection* is used in the sense of a shadow, that is, the usual orthogonal projection on a line or plane.

By considering only a strict subclass of density distributions, one can sometimes obtain uniqueness in the inverse problem of determining a set from partial knowledge of its sections. For example, the author and McMullen [23] proved that there are certain prescribed sets of four directions in n-dimensional Euclidean space E^n, such that the X rays of a convex body in these directions distinguish it from all other convex bodies. (In this context, an X ray gives the lengths of all chords of the body parallel to the direction of the X ray; see Figure 1.) In such situations one might formulate and implement an algorithm providing an arbitrarily accurate reconstruction from a fixed finite set of X rays.

Even when the collected data do not force a unique solution, interesting questions of practical importance can be raised. For instance, what is the best way to estimate the volume of a three-dimensional convex body, given the areas of its projections on planes? Lutwak [34] obtained a striking upper bound by applying the Petty projection inequality, one of a slew of deep affine isoperimetric inequalities surveyed in [37]. Lutwak's bound, which extends to E^n, is indeed affine invariant—it yields the exact volume for ellipsoids!

R. J. Gardner is professor of mathematics at Western Washington University in Bellingham, WA. His e-mail address is gardner@baker.math.wwu.edu.

Much of geometric tomography, as expounded in [21], consists of inverse problems concerning data of one of three types: X rays, measurements of projections, and measurements of concurrent sections. We shall explain these terms and provide just a sampler of the many results in each category, focusing on uniqueness and ignoring other types of data and intriguing topics such as stability, reconstruction, and estimates of volume. We stress that while the current state of the art often makes convexity a convenient assumption, this is usually unnecessary, except where projections are involved, and sometimes even inappropriate.

2. X Rays

In 1917, J. Radon showed that a density distribution in the plane is determined by its X rays taken in every direction. To prove this, Radon found an inversion formula for an integral transform now called the Radon transform; the formula is a key ingredient in computerized tomography, though much more is required in practice (see [42]).

Geometric tomography is concerned with sets, that is, densities taking only the values 0 or 1. Suppose that C is a compact subset of E^n and u is a *direction*, a unit vector in the unit n-sphere S^{n-1}. Denote by $u^\perp$ the $(n-1)$-dimensional subspace orthogonal to u. The *parallel X ray of C in the direction u* gives for each $x \in u^\perp$ the linear measure of the intersection of C with the line through x parallel to u. A slight strengthening of Radon's theorem implies that if U is an infinite set of directions in S^1, then the parallel X rays of a compact set in E^2 in the directions in U distinguish it from any other compact set.

In 1963, P. C. Hammer posed several problems concerning the determination of planar convex bodies by finite sets of X rays. Here there are simple examples of nonuniqueness: A regular n-gon P and its rotation Q by π/n about its center have equal parallel X rays in each of the directions of the edges of the regular $2n$-gon R formed by the convex hull of P and Q. Moreover, since affine maps preserve the ratios of lengths of parallel line segments, the images ϕP and ϕQ of these n-gons under an affine map ϕ have equal parallel X rays in each of the directions of the edges of the affinely regular polygon ϕR. The author and McMullen [23] proved that these are the only sets of directions to be avoided in order to obtain uniqueness. To be precise: If U is a finite set of directions in S^1, then the corresponding parallel X rays of a planar convex body distinguish it from any other such body if and only if U is not a subset of the directions of the edges of an affinely regular polygon. In contrast to Radon's theorem, the proof does not use integral transforms, though these also represent a major tool in the geometric branch of tomography.

To obtain the result mentioned in Section 1, one observes that the cross ratio of the slopes of any four edges of a regular polygon is an algebraic number. This remains true of an affinely regular polygon, since affine maps preserve cross ratio. Therefore a set of four directions whose slopes have a transcendental cross ratio will ensure that the corresponding parallel X rays determine each planar convex body. Convex bodies in E^n can also be determined by parallel X rays in such a set of four directions lying in the same 2-dimensional plane, since the X rays then determine each 2-dimensional slice parallel to this plane.

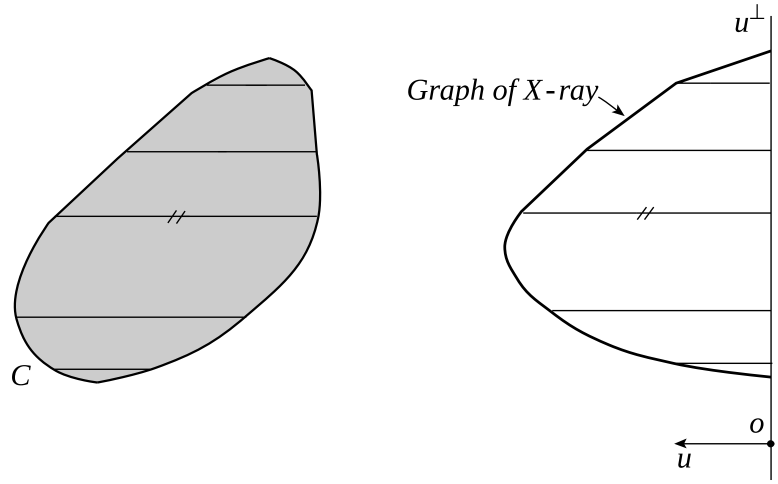

Figure 1
A parallel X ray of a convex body C.

One can also imagine an X ray in which the beams are not parallel but rather emanate from a single point; in fact, such "fan-beam X rays" are employed in modern CAT scanners. Mathematically, the *point X ray of C at a point p* in E^n gives for each $u \in S^{n-1}$ the linear measure of the intersection of C with the line through p parallel to u.

The problem of how best to determine convex bodies by point X rays has not been completely solved. One point X ray is clearly insufficient; see Figure 2. There are results concerning X rays at two points—Falconer [15] found a clever way of applying a version of the stable manifold theorem—but these require advance knowledge of the position of the body relative to the two points. The best unrestricted uniqueness theorem was established by Volčič [49]: Every set of four noncollinear points in the plane has the property that X rays of a planar convex body at these points distinguish it from any other such body.

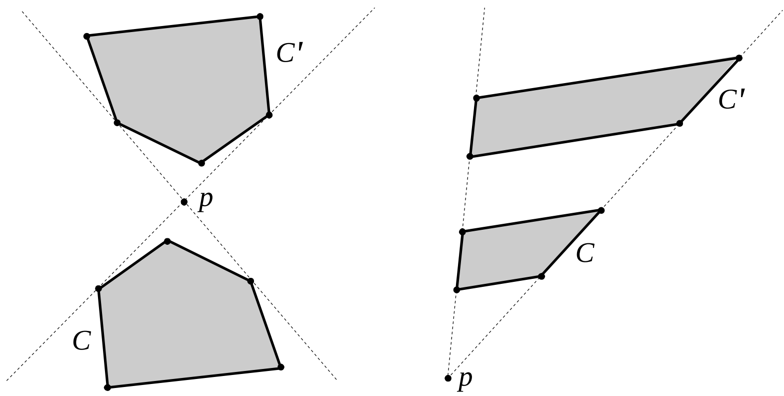

Figure 2
Polygons with equal point X rays at p.

Many well-known problems fit squarely into the framework of geometric tomography. For example, the notorious equichordal problem can be phrased as follows: Is there a planar body, bounded by a simple closed curve and star shaped with respect to two interior points p and q, whose point X rays at p and q are both constant? Posed in 1917, the equichordal problem generated deep studies by E. Wirsing, R. Schäfke, H. Volkmer and others. Now the problem has apparently been laid to rest: a tour de force by Rychlik [44] offers a negative solution.

3. Projections

If the compact set C is a nonempty *body*—that is, C is the closure of its interior—then the support of the X ray of C in a direction u is just the projection $C|u^\perp$ of C on $u^\perp$, so the latter yields strictly weaker information. This link is one reason for allowing projections as well as sections in geometric tomography. (Polar duality provides another; see below.)

Holes and dents can be invisible in shadows, so the class of convex bodies is the largest convenient class for this type of data. Let us consider two basic measurements of their projections. The *width function* of a convex body K gives for each $u \in S^{n-1}$ the length of the projection (of K) on the line through the origin parallel to u. The *brightness function* of K gives for each $u \in S^{n-1}$ the volume of its projection on $u^\perp$. (The *volume* of a k-dimensional body always means its k-dimensional volume.) In E^3, then, the brightness function of a convex body gives the areas of its shadows cast on planes.

Mixed volumes furnish an ideal vehicle for problems involving projections as well as a unified treatment of metric quantities such as volume, surface area, and mean width. They play a central role in the Brunn-Minkowski theory, founded by H. Minkowski in the last decade of the nineteenth century after groundwork of J. Steiner and H. Brunn. (Schneider's book [46] is a lucid and comprehensive guide to this labyrinthine palace, and references to everything in this section are supplied there.) Minkowski's discoveries included generalizations of the classic isoperimetric inequality and employed basic tools of analysis such as measure, integral, and spherical harmonics. The Brunn-Minkowski theory quickly produced dozens of theorems on the important sets of constant width or brightness (the front cover picture depicts one of the latter) and characterizations of spheres or ellipsoids in terms of various measurements of their projections.

Perhaps the single most important development from the point of view of geometric tomography came with a uniqueness theorem of A. D. Aleksandrov. In 1937, he (and, independently, W. Fenchel and B. Jessen) introduced the surface area measure of a convex body, a concept allowing many results in convex geometry to be relieved of unnecessary differentiability assumptions. This innovation is employed in Aleksandrov's uniqueness theorem for projections: A convex body in E^n that is *centered*, centrally symmetric with center at the origin, is determined, among all such bodies, by its brightness function. The proof hinges on two facts: An integral transform called the cosine transform is injective on the even continuous functions on S^{n-1}, and a convex body is determined up to translation by its surface area measure. The latter is a consequence of the Aleksandrov-Fenchel inequality, a profound generalization of the isoperimetric inequality that has unexpected connections with other areas, for example, algebraic geometry.

In the later development of the Brunn-Minkowski theory, the concept of a *projection body* is of special significance. This is a centered convex body whose support function—giving for each $u \in S^{n-1}$ the distance from the origin to a supporting hyperplane orthogonal to u—is the brightness function of another convex body. Figure 3 depicts the projection bodies of a regular tetrahedron and double cone. To illustrate the role of projection bodies, suppose measurements of a centered convex body in E^n, $n \geq 3$, yield the comparative information that its brightness function is smaller than that of another such body. Can one conclude that its volume is also smaller? Sometimes called Shephard's problem, this was solved in 1967 by C. M. Petty and R. Schneider; the answer is yes if the body with the larger brightness function is a projection body, but no in general for each n.

4. Concurrent Sections

In geometric tomography there is a remarkable and mysterious correspondence between projections and concurrent sections, sections through a fixed point that can be taken as the origin. Some connection between the two can be expected from the polar duality familiar to all geometers. If K is a convex body containing the origin in its interior, denote the polar body of K by K^*; (see [46], p. 33). (If K is centered, its support function, extended from S^{n-1} to a positively homogeneous function on E^n, is a norm on E^n. Then K^* is the unit ball in the corresponding normed space (E^n, K^*). This is the source of much interplay between convexity and functional analysis.) If S is a subspace, then (see [38], p. 70) it is easily shown that $K^* \cap S = (K|S)^*$, where the polar operation on the right is taken in S. Apart from a few special situations, however, this equation sheds little light on the puzzle.

If, in the definitions above, we replace the words "projection on" with "section by", the width function of a convex body becomes its point X ray at the origin and the brightness function becomes a new function called the *section function* that gives for each $u \in S^{n-1}$ the volume of the intersection with $u^\perp$. When the measured data concern planar sections through the origin, however, it is more appropriate to work with star bodies than with convex bodies. A *star body* is a body, containing the origin and star shaped with respect to the origin, whose radial function—giving for each $u \in S^{n-1}$ the distance from the origin to the boundary in the direction u—is continuous. (For another, more general definition, see [24].) The definitions of point X ray at the origin and section function extend naturally to star bodies.

We have mentioned one of the founders of the analytical basis of tomography, J. Radon. Another was P. Funk, whose paper [17] proves: A centered star body in E^3 is determined, among all such bodies, by its section function. Reformulated, Funk's uniqueness theorem says that an integral transform called the spherical Radon transform is injective on even continuous functions on S^{n-1}. The latter was actually proved earlier by Minkowski for $n = 3$; subsequently, proofs for general n were found by H. Helgason, C. M. Petty, and R. Schneider.

Though perfect for projections, the Brunn-Minkowski theory has comparatively little relevance for sections. In 1975, Lutwak [33] initiated a "dual Brunn-Minkowski theory", in which projections of convex bodies are replaced by sections of star bodies. He observed that the integrals over S^{n-1} of certain expressions involving powers of radial functions behave in many ways just like mixed volumes, and called them dual mixed volumes. Moreover Hölder's inequality, when reformulated in terms of dual mixed volumes, looks just like the Aleksandrov-Fenchel inequality, with the inequality reversed!

In these early ingredients, the dual theory is more Bauhaus than Byzantine, though no less effective. But Lutwak's dual theory includes much more than this. In [35], for example, he introduced the notion of an *intersection body*, a centered star body whose radial function is the section function of another star body. The intersection body of a convex body need not be convex, but a theorem of H. Busemann, fundamental for his theory of area in Finsler spaces, can be reinterpreted as saying that the intersection body of a centered convex body is convex. Some examples of convex intersection bodies are illustrated in Figure 4. On the left, a centered cube C of side length $1/2$ includes its intersection body, created from a program written by Fred Pickle; on the right, a centered cylinder K of radius 1 and height 2 is shown with its (translated) intersection body.

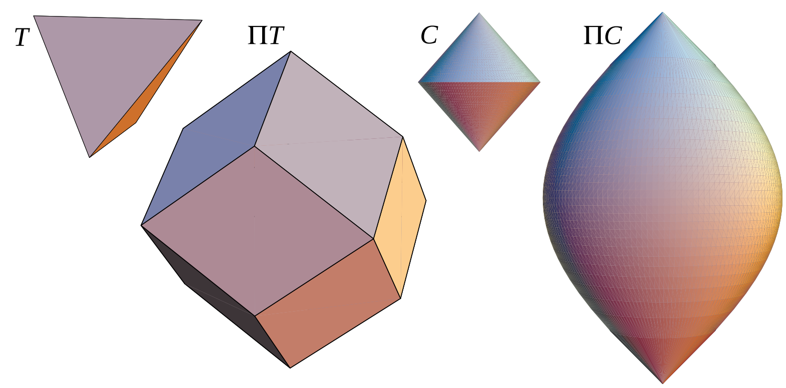

Figure 3 Projection bodies.

The Brunn-Minkowski theory and Lutwak's dual theory feed from each other, with great benefit for both and for geometric tomography. This process occurs via a sort of dictionary that allows transport from one theory to the other. In this dictionary, convex bodies, projections, the support function, the brightness function, mixed volumes, the cosine transform, and projection bodies in the Brunn-Minkowski theory correspond to star bodies, sections by planes through the origin, the radial function, the section function, dual mixed volumes, the spherical Radon transform, and intersection bodies, respectively, in the dual theory. In this way, the uniqueness theorems of Aleksandrov and Funk, for example, translate into each other. Moreover, a proof in one theory can sometimes be translated into a proof in the other.

When we translate Shephard's problem according to Lutwak's dictionary, we obtain the following problem: If the section function of a centered star body in E^n, $n \geq 3$, is smaller than that of another such body, is its volume also smaller? The answer is yes if the body with the *smaller* section function is an intersection body, but is in general no for each n. (The word "smaller" is due to the inequality reversal mentioned above.)

The first part of this result is a theorem of Lutwak [35]. Lutwak was actually motivated by a variant of this problem, obtained by replacing the word "star" in the previous paragraph by "convex". This is precisely the well-known Busemann-Petty problem, one of a list of ten problems in [10] inspired by investigations in Minkowskian geometry. Lutwak's theorem still applies, of course, and intersection bodies—and, by the way, nonconvex star bodies—also play a crucial role in the solution to the Busemann-Petty problem. In fact, the answer to the Busemann-Petty problem is yes for a given n if and only if each sufficiently smooth centered convex body in E^n is an intersection body. This discovery led the author [18,19,20] and Zhang [52,53] to the complete solution; it turns out that the answer is no for each $n \geq 4$ but an unqualified yes if $n = 3$. Along the way, Ball [1] established the precise upper bound of $\sqrt{2}$ for the section function of a centered cube of unit side length. While expected, the result is tricky to prove, and, incidentally, yields a negative answer to the Busemann-Petty problem for $n \geq 10$.

Recent work of Goodey, Lutwak, and Weil [25] suggests that a profound synthesis of the Brunn-Minkowski theory and its dual may one day be revealed, and the mystery explained at last.

5. Public Relations Department

Computerized tomography generally deals with sections of density distributions. Despite this, it is worth noting that geometric objects do occur in the medical literature. For example, the authors of [11] approximate the human heart by a convex body and attempt to use two X rays to obtain a reconstructed image. However, applications of geometric tomography are perhaps more likely to occur in several other areas.

Skiena [48] envisages that what he calls "geometric probing" will be of use in robotics. A robot may need to be equipped with a sensing device to help identify the shape and position of geometric objects, for example, in picking machine parts off a conveyor belt or in moving around an environment in which objects are approximated by polyhedra. Skiena investigates the possibility of using X rays, and this leads him to consider X rays of (convex or nonconvex) polytopes. The fresh "interactive" viewpoint of computer science brought the natural and practical concept of *successive determination*, in which X rays already taken can be consulted in deciding the direction for the next X ray; see [14] and [22]. X rays of geometric objects also occur in Horn's book [29] on robot vision.

On September 19, 1994, a minisymposium with the title "Discrete Tomography", organized by Larry Shepp of AT&T Bell Labs, was held at DIMACS. Some time earlier, Peter Schwander, a physicist at AT&T Bell Labs in Holmdel, had asked Shepp for help in obtaining three-dimensional information at the atomic level from two-dimensional images taken by an electron microscope. A new technique, based on high-resolution transmission electron microscopy, can effectively measure the number of atoms lying on each line in certain directions. (At present, this can be achieved only for some crystals and in a constrained set of crystallographic directions—lattice directions for the crystal lattice.) The aim is to determine the three-dimensional crystal from information of this sort obtained from a number of different directions. This leads to the problem of determining a finite set from its projections (counted with multiplicity) on a finite number of planes. While it formally belongs to geometric tomography, the problem is given only very brief mention in [21]; partial answers can be found in [7] and [16].

Suppose that C is a compact set in E^n and S is a k-dimensional subspace, where $1 \leq k \leq n - 1$. The *k-dimensional X ray of C parallel to S* gives for each $x \in S^\perp$ the volume of the intersection of C with the translate of S containing x. The ordinary X ray corresponds to $k = 1$. In [12], it is observed that the problem of radar target estimation is related to the determination of a three-dimensional body from its two-dimensional X rays. The k-dimensional X ray of a convex polytope P, parallel to a subspace in general position with respect to the vertices of P, is a piecewise polynomial of degree at most k, at least $(k - 1)$-times continuously differentiable. In short, it is a *spline*. Of particular consequence in spline theory are the cases when P is a simplex or a cube, the latter giving rise to *box splines*. Box splines are of considerable interest in computer-aided design; see [13].

If C is a body and u is a direction, the maximum volume of sections of C by hyperplanes orthogonal to u is sometimes called an *HA measurement*. This strange term derives from the study of the Fermi surface of a metal—the surface of a body formed, in velocity space, by velocity states occupied at absolute zero by valence electrons of the metal. Knowledge of the Fermi surface gives valuable information about various physical properties such as conductivity. The maximal cross-sectional areas of the body bounded by a Fermi surface can be measured by means of the de Haas-van Alphen effect, magnetism induced in the metal by a strong magnetic field at a low temperature, hence the name HA measurement. More details and references are provided by Klee [32].

Projection bodies appear in quite surprising guises. A strong form of Liapounov's theorem says that they are precisely the ranges of vector-valued measures. A centered convex polyhedron is equidissectable with a cube if and only if it is a projection body. If K is a centered convex body in E^n, the normed space (E^n, K^*) defined above is isometric to a subspace of $L_1(0,1)$ if and only if the body is a projection body. Projection bodies have also found application in stochastic geometry, random determinants, Hilbert's fourth problem, mathematical economics, and other areas; see [26,36], and the references given there.

Santaló ([45], p. 282) quotes a definition of *stereology*, due to H. Elias, as the exploration of three-dimensional space from two-dimensional sections or projections of solid bodies. The relatively new term was coined at a meeting, principally of biologists and metallurgists, on the Feldberg, Germany, in 1961. Stereology is closely related to geometric tomography, but focuses on statistical estimates from random sections. Many pertinent references can be found in [45] and in Weil's survey [50]. Mathematical morphology, image analysis, and pattern recognition also overlap with geometric tomography.

6. Try Your Hand?

Space permits only one open problem from each of the three categories referred to at the end of Section 1. They all happen to concern convex bodies, but many of the seventy or so open questions listed in [21] do not.

Question 1. *In E^3, is there a prescribed finite set of directions in general position such that the X rays of any convex body in these directions will distinguish it from any other such body?*

Here, general position means that no three of the directions lie in a plane. The four directions in the theorem of the author and McMullen [23] (see Sections 1 and 2) all lie in the same plane. Unfortunately, an arbitrarily small perturbation of a suitable set of four directions can destroy the uniqueness property, so that the theorem lacks stability. Question 1 is especially interesting because it seems possible that *any* set of seven (mutually nonparallel) directions in general position will do; an example found by the author, Volčič, and Wills ([21], Theorem 2.2.2) shows that not every set of six directions in general position has the required property.

Question 2. *Is there a nonspherical convex body in E^n, $n \geq 3$, of constant width and constant brightness?*

This tough old nut goes back to Nakajima [41], who showed that the answer is no for $n = 3$ under an additional smoothness assumption. Nonspherical convex bodies of constant width have been around since Euler named them orbiforms, and Blaschke ([8], pp. 151–4) constructed nonspherical convex bodies whose brightness functions are constant. Centered balls are the only star bodies whose point X rays at the origin and section functions are both constant; this answer to the dual form of Question 2 is given in [24] (see also [27]).

Question 3. *Is there a constant c, independent of n, such that if H and K are centered convex bodies in E^n with the section function of H smaller than that of K, then the volume of H is smaller than c times the volume of K?*

A complete answer to a problem on Busemann and Petty's list ([10], Problem 2) would also answer Question 3; but, as formulated here, with the emphasis on the universal constant c, the question is a quite recent one. Bourgain [9] proved that c can be replaced by $c_n = O(n^{1/4})$. The solution to the Busemann-Petty problem

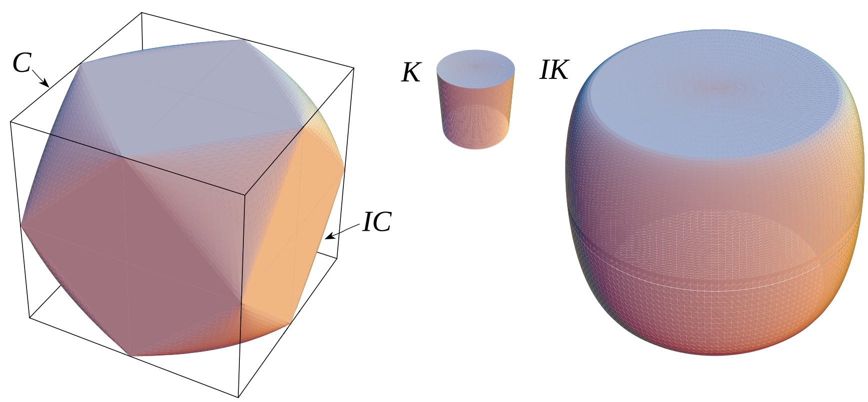

Figure 4 Intersection bodies.

(see Section 4) shows that $c_3 = 1$ is possible, but $c_n > 1$ for $n \geq 4$. Sometimes called the hyperplane problem or the slicing problem, Question 3 is also known as the maximal slice problem, in view of one of several equivalent forms examined at length by Ball [2] and Milman and Pajor [39]. Contributions to the slicing problem, which is of great significance in the local theory of Banach spaces, have been made by Ball [3,4], Junge [30,31], and Zhang [54].

If we replace the section function in Question 3 with the brightness function, the answer is no. Ball [5,6] shows that in this case c can be replaced by $c_n = O(n^{1/2})$ and that this order is the correct one, even for arbitrary convex bodies.

7. Postscript

There must be thousands of mathematical results involving sections or projections of compact sets. Inevitably, several worthy topics were omitted. For example, Dvoretzky's theorem says that given $k \in \mathbb{N}$, each centered convex body of sufficiently high dimension has an "almost spherical" k-dimensional central section. This seeded a whole branch of Banach space theory concentrating on the properties of high-dimensional convex bodies, expounded, for example, in the books of Milman and Schechtman [40] and Pisier [43]. Another example is Crofton's intersection formula and others from integral geometry, for which the texts of Schneider ([46], Chapter 4) or Schneider and Weil [47] can be recommended.

The scope can also be widened in other ways. There are already some results on X rays in projective space, and several of the concepts discussed above carry over to the more general homogeneous spaces, as in the work of Helgason [28] and others. In fact, readers should interpret or even broaden the definition of geometric tomography above to suit their tastes. Fascinating new possibilities arise, for example, in inverse problems involving intersections with circles and spheres, treated in the eloquent article by Zalcman [51].

A definition should not sever new shoots of mathematics, nor shade its beauty.

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