A Few Results and Open Problems Regarding Incompressible Fluids

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Water in the oceans and air in the atmosphere are examples of incompressible fluids—fluids that do not change their volume while they flow. They display a truly amazing range of phenomena, from regular patterns to turbulence. Despite this complexity, incompressible fluids are described by only a few partial differential equations and two kinds of limits. In the first limit, time tends to infinity, and in the second limit, coefficients in the equation are varied. The two limits do not commute in general; they can also represent different physical and mathematical situations. I will try to illustrate these issues in an informal manner below.

Incompressible fluids: fluids that do not change their volume while they flow.

From Drops to Turbulence

The equations of motion of incompressible fluids are described as follows: Let $x \in \mathbf{R}^n$ denote a point in space, $n = 2$ or 3, and let $t \geq 0$ denote time. One associates to the velocity $u(x,t) \in \mathbf{R}^n$ a first-order differential operator, D_t:

$$D_t = \partial_t + u(x,t) \cdot \nabla.$$

D_t is the so-called material derivative; its characteristics, solutions of the ODE

$$\frac{dX}{dt} = u(X,t),$$

are called particle trajectories. The Navier-Stokes equations are

$$D_t u + \nabla p = \nu \Delta u + f,$$
$$\nabla \cdot u = 0.$$

The number $\nu > 0$ is the kinematic viscosity of the fluid. If one sets $\nu = 0$ in the equation above, one obtains the Euler equations of ideal fluids. The scalar function $p(x,t)$ represents pressure; its mathematical role is to maintain the constraint of incompressibility $\nabla \cdot u = 0$. The functions f represent body forces. The fluid occupies a region $G \subset \mathbf{R}^n$, and appropriate boundary conditions are prescribed. A great variety of physical situations are described by the nature of body forces and boundary conditions.

A drop falling from a faucet is an example of a familiar phenomenon that is described using the Navier-Stokes equations. This is a free boundary problem: the shape of the boundary between water and air changes in time under the pull of gravity. The Navier-Stokes equations are supplemented by additional equations for the boundary. These are given by the Gibbs–Thompson law that expresses the role played by the surface tension and links the pressure to the mean curvature of the free boundary. At the time when the drop pinches off there is a singularity in the

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interface [21]. This phenomenon has been the subject of recent experimental and theoretical efforts. The physical experiments [29] were matched to high-accuracy numerical simulations of simplified equations. The equations are approximations (so–called lubrication approximations) of the full Navier-Stokes free boundary problem, and they were obtained both from physical principles [17] and by systematic asymptotic expansion [1]. The singularity shape is believed to be universal [16] and stable; this is why the drastic (but not arbitrary) simplifications of the Navier-Stokes equations are able to capture it. A spectacular succession of self-similar events preceding the pinchoff has been observed experimentally and explained [3] using the simplified equations. These results are of high scientific quality: physical experiment, physical theory, and numerical experiments match each other. The ability of fluids to form drops in nonturbulent regimes is of obvious scientific and technological importance. There are some rigorous results, but to my knowledge the mathematical treatment of this problem is far from being complete.

I will turn now to questions related to turbulence. Two examples, Taylor-Couette and Rayleigh-Benard turbulence, can serve as a guide to posing questions regarding turbulence.

In the Taylor–Couette setting fluid is placed in the space between two concentric vertical cylinders; then one of the cylinders is rotated, entraining the fluid. In the Rayleigh–Benard setting fluid is placed in a closed container and heated from below. The external conditions are encoded in nondimensional parameters: Reynolds number, Rayleigh number. They represent a measure of the strength of the externally supplied energy (determined, for instance, by how fast the cylinders are rotated or how much the fluid is heated). The term *nondimensional* refers to scale invariance. The experimenters prepare an experiment at given values of the control parameters. The experiment [4, 25] is allowed to run for a sufficiently long time in order to make sure that one registers the time asymptotic regime. The information is recorded and processed, taking time averages. New values of the control parameters are selected and the process is repeated. This is possible because the experimentally controlled external forces are deterministic and the systems are assumed to be free of other external influences. In other words, the systems are assumed to be closed and controlled.

As the nondimensional control parameters are increased the behavior of the fluid changes, very roughly speaking, from simple to complex. At the high end of the parameter scale we find incompressible turbulence. The appropriate mathematical turbulence problem is thus: study the long-time behavior of solutions to Navier-Stokes equations at a fixed Reynolds number (fixed $\nu > 0$, boundary conditions and body forces f). Record it using appropriate measures or averages. Then increase the Reynolds number (for instance, decrease viscosity, keeping all other conditions the same) and study the large Reynolds number limit.

The order in which the limits are taken is crucial. In general the infinite Reynolds number and infinite time limits do not commute. The appropriate order for closed systems is taking the infinite time limit first, but taking the infinite Reynolds number limit before the infinite time limit might be appropriate for systems in which the forcing fluctuates rapidly in time.

Photo by Warren Bolster for Tony Stone Images

I will call the case in which the infinite time limit is taken first the "temporal limit" and the case in which one takes the control parameter to infinity first the "inviscid limit".

The Temporal Limit

The finite-dimensional dynamical system paradigm [28] has great merit for the study of transition to turbulence [18]. Mathematically and numerically motivated originally, it was later verified in a number of physical experiments. Consider for simplicity the case of two-dimensional, spatially periodic Navier-Stokes equations with time-independent forces. At a fixed Reynolds number the Navier-Stokes equations are solved by a nonlinear semigroup $S(t)$ in a Hilbert space H, the space of divergence-free square integrable velocities. The norm $|\cdot|_H$ is the square root of the total kinetic energy, and $S(t)u_0$ represents the velocity at time t. The semigroup is dissipative: there exists a compact set in H such that all trajectories $S(t)u_0$ belong to it for large enough t. The semigroup has a global attractor $\mathcal{A}$: a compact invariant ($S(t)(\mathcal{A}) = \mathcal{A}$) set that contains all omega limit sets. This set has finite fractal dimension [19]. The dimension is related to the Reynolds number [11], as predicted by the argument of Landau [24] concerning the number of degrees of freedom of turbulence. All Borel measures which are invariant under $S(t)$ are supported in $\mathcal{A}$. Consider the set of global solutions

$$\mathcal{G} = \{u_0 \in H;\ S(t)u_0 \text{ extends to } t \in \mathbf{R}\}.$$

The global attractor can be described by

$$\mathcal{A} = \left\{u_0 \in \mathcal{G};\ S(t)u_0 \text{ is bounded for } t \in \mathbf{R}\right\}.$$

Consider now the set of those global solutions that grow at most exponentially backward in time

$$\mathcal{M} = \left\{ u_0 \in \mathcal{G};\ \limsup_{t \to -\infty} \frac{\log |S(t)u_0|_H}{|t|} < \infty \right\}.$$

One can prove [12] that $\mathcal{M}$ is weakly dense in H. The dynamics on $\mathcal{M} \setminus \mathcal{A}$ can be related to the inviscid Eulerian dynamics: rescaling appropriately velocity and time, one obtains the Euler equations as the infinite negative time limit. Thus, for velocities selected from the rich invariant set $\mathcal{M} \setminus \mathcal{A}$, quasi-Eulerian dynamics is the past and finite dimensional dynamics on the Navier-Stokes attractor $\mathcal{A}$, the future.

One of the most common ways to analyze temporal data, such as the temperature $\theta(t)$ recorded at a given location in the Rayleigh-Benard convection experiment, is to compute the power spectrum,

$$P(\omega) = \lim_{T\to\infty} \frac{1}{T} \left| \int_0^T e^{-i\omega t}\theta(t)dt \right|^2.$$

This is a well-defined and even automated procedure; its mathematical justification stems from classic work on stochastic signals, going back to Wiener, Khintchin and Kolmogorov. At finite Rayleigh and Reynolds numbers, though, the signal is not a random process, rather it is the output of a finite dynamical system. Using the properties of solutions of Navier-Stokes equations, one can still make mathematical sense of the procedure. Moreover, one can prove that the power spectrum must decay at least exponentially at high frequencies [2]. Specifically, the power spectrum of the temporal data obtained by evaluating a solution at some fixed location is a well-defined positive Borel measure $P(d\omega)$ and satisfies

$$\int_{-\infty}^{\infty} e^{\tau|\omega|} P(d\omega) < \infty.$$

The positive constant $\tau = \tau(\mathcal{A})$ depends only on the attractor, in other words, only on the control parameters. Thus, slower than exponential decay—found by numerical treatment of experimental Rayleigh-Benard convection data—is not asymptotically correct in the high temporal frequency limit at fixed Rayleigh number. The reason why the numerical calculation led to an incorrect prediction is perhaps that numerical calculation may have used time series of experimental data sampled with too-long delays between successive measurements.

The temporal power spectrum of wind tunnel data provides strong experimental verification of one of the beacons in the subject of turbulence, the Kolmogorov theory [23]. Grounded in dimensional analysis, this theory proposes the existence of universal scaling behavior in fully developed turbulence. According to the Kolmogorov dissipation law, the energy dissipation rate

$$\epsilon = \nu \left\langle |\nabla u(x,t)|^2 \right\rangle$$

is a positive constant that is bounded independently of viscosity. The braces $< \cdots >$ represent ensemble average (functional integration). Kolmogorov assumed homogeneity and isotropy (invariance with respect to translations and rotations of the underlying probability distributions). In addition, the Kolmogorov theory assumes that there exists an interval, the so-called inertial range, which extends from a small length (the Kolmogorov dissipation scale η) to a large one (the integral scale ρ), such that, for $r \in [\eta, \rho]$, the variation

$$s(r) = \left\langle |u(x+y) - u(x)|^2 \right\rangle^{\frac{1}{2}}$$

of velocity across a distance $r = |y|$ can be determined from dimensional analysis:

$$(s(r))^2 \sim (\epsilon r)^{\frac{2}{3}}.$$

This is the Kolmogorov two-thirds law. It is one of the most important predictions of this theory. It implies that the energy spectrum behaves approximately like a $-\frac{5}{3}$ power in a range of wave numbers (Fourier variables).

Some modest progress on the mathematical aspects [5] of the issue of scaling can be summarized as follows. One considers a concrete averaging procedure in which the temporal limit has to be taken before ensemble average. There are no special assumptions regarding the ensemble average; in particular, no assumptions of homogeneity and isotropy are needed. One defines the structure function $s(y)$ (because of lack of isotropy s is not assumed to be radially symmetric) and dissipation ϵ as above. One considers ensembles $\mathcal{E}$ of solutions of the Navier-Stokes equations in $\mathbf{R}^3$ satisfying the following three assumptions: uniformly bounded velocities, uniformly bounded forces, and a mild technical assumption regarding s. Based on these three assumptions alone, one can prove that ϵ remains bounded uniformly as $\nu \to 0$. Moreover, the Kolmogorov two-thirds law holds as an upper bound:

$$(s(y))^2 \leq (\epsilon |y|)^{\frac{2}{3}}$$

for $|y|$ in an interval at the bottom of the inertial range. The Kolmogorov dissipation scale has a natural definition, so the statement is meaningful even if scaling has not been proved.

Proving a positive lower bound for ϵ that is uniform for arbitrarily large Reynolds numbers is a fundamental open problem.

There are few rigorous results concerning fully developed turbulence (i.e. the large Reynolds number asymptotics) in realistic models of closed systems driven at the boundary. The most reliable and reproducible quantities measured in high-quality experiments are bulk dissipation quantities. In the Taylor-Couette flow the bulk dissipation quantity is the total torque; in the Rayleigh-Benard case it is the Nusselt number, a measure of heat flux. These quantities obey empirical laws: they are given by certain functions of Reynolds or Rayleigh number. A method to estimate rigorously the asymptotic behavior of these functions for large Reynolds or Rayleigh numbers for systems driven at the boundary has been developed [6] in recent years. In addition to providing good agreement to experimental results, the method leads to new variational problems with spectral side conditions.

The Inviscid Limit

Here we consider the inviscid limit: that is, we fix time and let the control parameter tend to infinity. Let us focus on the Navier-Stokes equation. The first question is: what are the limiting equations? In realistic closed systems where the boundary effects are important, unstable boundary layers drive the system: the limit is not well understood. In the case of no boundaries (periodic solutions or solutions decaying at infinity) the issue becomes one of smoothness and rates of convergence. Indeed in $n = 2$, if the initial data are very smooth, then the limit is the Euler equation and the difference between Navier-Stokes solutions and corresponding Euler solutions is optimally small ($O(\nu)$). However, if the initial data are not that smooth, for instance in the case of vortex patches, then the situation changes. Vortex patches are solutions whose vorticity (antisymmetric part of the gradient) is a step function. They are the building blocks for the phase space of an important statistical theory [26, 27]. When one leaves the realm of smooth initial data, the inviscid limit becomes more complicated: internal transition layers form because the smoothing effect present in the Navier-Stokes solution is absent in the Eulerian solution. In the case of vortex patches with smooth boundaries, the inviscid limit is still the Euler equations, but there is a definite price to pay for rougher data: the difference between solutions (in L^2) is only $O(\sqrt{\nu})$ [13]. This drop in rate of convergence actually occurs—there exist exact solutions providing lower bounds. The question of the inviscid limit for the whole phase space of the statistical theory of [26] and [27] is open. If the initial data are more singular, then even the classic notion of weak solutions for the Euler equations might need revision [15], except when the vorticity is of one sign [14].

In the case of three spatial dimensions and smooth initial data, the inviscid limit is the Euler equation as long as the corresponding solution to the Euler equation is smooth [20, 7]. This might be a true limitation because of the possibility of finite time blowup. The blowup problem for the Euler ($\nu = 0$) equations is the following: do smooth data (for instance, $f = 0$, and smooth, rapidly decaying initial velocity) guarantee smooth solutions for all time? The answer is known to be yes only for $n = 2$, not known for $n = 3$. The main difference between the two-dimensional and the three-dimensional cases can be explained as follows. The Euler equations are equivalent to the requirement that a first-order differential operator Ω commute with D_t:

$$[D_t, \Omega] = 0.$$

Ω is associated to a divergence-free function $\omega = \omega(x,t) \in \mathbf{R}^n$, $\nabla \cdot \omega = 0$:

$$\Omega = \omega(x,t) \cdot \nabla.$$

The vanishing of the commutator can be expressed as an evolution equation for the coefficients ω. The equation is nonlinear because u and ω are coupled:

$$u = \mathcal{K} * \omega.$$

A difference between $n = 2$ and $n = 3$ lies in the nature of the coupling matrix $\mathcal{K}$. For $n = 2$

$$\mathcal{K}_{ij}^{(2E)}(y) = \delta_{ij} \log(|y|),$$

where δ_{ij} is the Kronecker delta. In three dimensions

$$\mathcal{K}_{ij}^{(3E)}(y) = \epsilon_{ijk}\frac{\hat{y}_k}{|y|^2},$$

where $\hat{y} = y/|y|$, ϵ_{ijk} is the signature of the permutation $(1,2,3) \mapsto (i,j,k)$ and repeated indices are summed. Note that the strength of the singularity of $\mathcal{K}^{(3E)}$ at the origin is of the order $2 = n-1$, whereas the strength of the singularity of $\mathcal{K}^{(2E)}$ is merely logarithmic. This is a crucial difference. Moreover, if $n = 2$, the equation $[D_t, \Omega] = 0$ is equivalent to an active scalar equation

$$D_t\theta = 0,$$

irrespective of the nature of $\mathcal{K}$. The natural analogue of the three-dimensional coupling in a two-dimensional model is

$$\mathcal{K}_{ij}^{(2QE)} = \delta_{ij}\frac{1}{|y|}.$$

This coupling has the correct singularity strength $1 = n-1$. It turns out that this is a physically significant model: the scalar θ represents temperature in a quasi-geostrophic (Coriolis forces balance pressure gradients) approximation of atmospheric flow. Recent numerical and analytic [8, 9, 10] results suggest that there are important geometric effects that can prevent blowup, both in the quasi-geostrophic active scalar and in the three-dimensional Euler equations. Namely, one can prove that in regions where the normalized direction field

$$\xi = \frac{\omega}{|\omega|}$$

is regularly directed there cannot be blowup. *Regularly directed* is a technical term: roughly speaking, it means that the direction $\pm\xi$ has Lipschitz extensions in regions of high $|\omega|$. A valid blowup scenario requires therefore the formation of a singular object in physical space. Such an object might be as simple as conical "elbows" in a pair of vortex tubes, and such structures might have been observed in numerical simulations [22].

Concluding Remarks

Among the most physically significant mathematical problems in the theory of incompressible fluids are problems related to singularities. The stable finite time singularities which develop in nonturbulent situations, such as the drops forming under the combined effects of gravity and surface tension, are an example. Another example arises in connection to turbulence theory. The correct limit for closed system turbulence is the infinite time limit. The important mathematical problem of finite time blowup for the Navier-Stokes equations belongs here. There is, however, no evidence that this problem has physical significance; blowup of solutions of the Navier-Stokes requires infinite momentum, that is, breakdown of the model. Rather, the physically important problem is posed by the Kolmogorov theory. A successful link of this theory to Navier-Stokes dynamics will have to produce a lower positive bound for the product of viscosity and average square of the gradients. This clearly requires large gradients in the limit of zero viscosity, suggesting finite time blowup in the Euler equations as the physically important problem. However, in general the temporal limit and the inviscid limit do not commute. The inviscid limit may be complicated, even in the absence of boundary effects, in the case of nonsmooth data. The question of blowup for the Euler equation is relevant in this context.

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