The Math Circle

Robert Kaplan

Why are most children bored by math—or worse, frightened of it—when its beauties are so many and its pleasures so great? Probably because it is usually taught by people who fear it, and who therefore teach their fear.

To counter this (locally, but with hope of disseminating the ideas), three of us—my wife Ellen, Tomás Guillermo and I—have begun a Saturday morning Math Circle in space given us by Northeastern University in Boston (thanks to Andrei Zelevinsky), for interested 8-to-18 year-olds. Word of mouth alone brought us twenty-nine students for the first semester and forty for the second. Their giving up Saturday mornings of sleep, sports or music lessons is a sign of how much enthusiasm there is for math done with intensity and delight. We've now had to add a Thursday afternoon session at Harvard (thanks to Danny Goroff) for the overflow.

Our approach in The Math Circle is to pose questions and let congenial conversation take over. Conjectures emerge from a free-for-all, examples and counterexamples from the conjectures. Two steps forward are followed by a step back. What really is at issue here? How will we know when we've understood something? Is proving different from seeing? Where and with what should proofs begin, and how validate these beginnings? And if we get it, need we formalize it? Yes—following my old fencing-master's adage about holding the foil like a bird: tightly enough not to let it get away, not so tightly as to crush it. We don't want a '60s feel-good sense of math as expressive hand-waving. We explain that rigor without mortis consists in fluency at making a connected path back to foundations that will stand up to scrutiny. In our exchanges the students are developing the knack of pushing insight adventurously ahead while protecting the supply-lines that fuel it.

Here's an example of what happens, from the Thursday afternoon sessions with 8- to 11-year-olds. I began by asking them each to give me very large numbers, and a rather conservative list of integers developed on the board (Littlewood calculated that the gigantic stretches of time in Indian mythology "only" amounted to 10^{35} years—for example, for kids, 250 is, as it should be, way out there). Was there a last integer? Of course not, said 8-year-old Anna at once: if there were, just add one to it. I was struck by this instantaneous freeing of imagination with the passage to the general. They all took easily to one-one correspondences, via the need to show Martians who count 1, 2, 3, many; that a heap of thirteen pencils contains a larger "many" than a pile of eight coins (some sorted by threes, some paired up, all relished the Martian astonishment that there were two sizes of 'many'). By the end of the first hour each was pairing up the naturals with the evens or the odds or multiples of 7 ("Let's walk through the numbers in seven-league boots!" said one) or the negatives and positives, or the integers (a puzzling moment, followed by a cry of "Shift!"). The children left that first session in two minds: they saw and could not but assent to the results of their work, but they "knew" it couldn't be right—there were clearly only half as many even numbers as naturals.

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I was glad to see one boy return to the second session who had come to the first the way you go to a horror movie: frozen between fascination and flight. He had been brooding about the infinite, his mother told me, and found it terrifying, and had signed up for this class to overcome his fear. As you might expect, his were the most daring conjectures in what followed.

Could the positive rationals be counted? Passionate arguments raged about the lack of a starting place (the abyss of density opened, for the first time, for several there). I put the first few rows and columns on the board to claims ringing through the room that we could never get out of the row we once committed ourselves to—until a girl, quiet before this, asked if she could come up and began spinning a drunken spider's web from the upper left-hand corner. Another girl suggested how we might regularize the pattern and despair turning to glee, each now constructed a bus-route through these streets and avenues (to the accompaniment of popping insights and questions: "Look! Those diagonals are all repeats!" "But how can we tell ahead of time that the sixteenth stop will be 2/5?") The diagonal proof belongs now to each of them, for as Locke pointed out, "Property is that with which one has commingled one's labor."

We headed toward Cantor's second diagonal proof through the largely unfamiliar territory of decimals, browsing among repeating and nonrepeating growths (much ingenuity here, on their part, in concocting and explaining). Then a long digression on what it would mean not to have a one-one correspondence between two sets, and on proof by contradiction (this pushes the limits of how much can equably be held in mind), heralding the metaphorical entrance of a figure modeled on the late and unlamented Senator McCarthy, patting his jacket-pocket: "I have in my pocket a list of all the card-carrying decimals between 0 and 1", each card bearing its ordinal. They followed the proof complaining all the way, arguing with and explaining it to one another, ending up as flabbergasted as I am every time I think it through.

By the time I arrived for the next class, a girl with an uncannily precise mind was explaining Cantor's proof to a die-hard skeptic, countering his attempts to put "our decimal made up of 5s and 6s 'at the end of the list'". Now the pace accelerated as we came back to the visual: they called the proof they devised of the one-one correspondence between points on closed segments of different lengths "the circus tent". Giving names, as Adam first knew, is a sign of confidence. Since I'd told them about $\aleph_0$, they christened our new cardinality "zig-zag null".

And the open interval and the real line? This led to the greatest frustrations as well as ingenuity. At last, exhausted, with time running out and skepticism about everything creeping out of every corner, the boy who feared infinity had the saving insight. Tremendous relief, appreciation, exhilaration. But the last note was struck by the girl with the precise mind: "I see the proof and I accept it", she said, "and at the same time I don't. It doesn't tell us, does it, just which real number goes with the decimal we choose in (0, 1)? That makes me uneasy." A good note to end—and begin afresh—on.

The topics we chose—but more particularly the ways we worked on them—were designed to avoid two things which tend to go wrong with accelerated or "enrichment" math courses. One is children being taught the punch-lines without having worked their own way up to them ("I've already *had* Pascal's Triangle", says the child who knows neither its genesis nor implications). The second is teaching the magic that looks like math: Pavlovian training of ten-year-olds to push the right symbols to take derivatives of polynomials.

Why did we choose infinite sets for the youngest group? Because little kids love large numbers, the way they like elephants and dinosaurs: powerful friends in high places, but a liking tinged with titillating fear. Children have no reputations to protect—because they don't even know yet about reputation—so they are much better than older students at making and getting far-fetched ideas—not a card is held close to a chest. Unlike philosophy, math begins with awe and ends with wonder.

Ellen chose to work on polygon construction with the 8-to-11 year-olds. Polygon construction meant using actual straight-edges and compasses; and while the hands were busy, a casual conversation about constructability steadily moved from the context to the content of the course. One student to another: "What do you mean, you can't trisect all angles? If you can trisect a 90° angle by copying 30° angles, can't you copy some angle twice to get any angle?" The crux of the matter is to seize on such assertions so as to let the students find out for themselves what's at stake (coming to grips with, among other things, the mysteries of quantification).

A generation brought up with calculators has difficulty manipulating fractions. Instead of addressing this directly, Ellen led them to construct an equilateral triangle and a regular pentagon in the same circle, and to figure out how to construct a regular 15-gon—thus discovering that 1/3 minus 1/5 could be seen as the useful 2/15 rather than the unenlightening .13333…. The confidence that followed this self-won competence made them feel this world was their oyster: they could construct whole series of regular polygons. Then, why were the heptagon and the ennagon so resistant to their efforts? Fermat primes came up before the course ended; but as in all the best dramas, exeunt omnes in mysterium.

With the 11-to-14 year-olds Ellen worked on polyhedra. Conversation accompanying scissors-and-paste constructions led very quickly to the discovery of the Euler characteristic. They tested it with Schlegel diagrams, studied and were convinced by Cauchy's proof, then read Lakatos' dialogue *Proofs and Refutations* (about the Euler characteristic), taking parts, and stepping out of character again and again to argue with the protagonists. They were startled to recognize in themselves the traits of monster-barrer and monster-adjustor, skeptic and omni-ameliorator. As in the dialogue, the semester ended with everything up in the air. Should things be neatly tied up?

With the same group, I worked on number theory beginning with this peculiarity: why do the digits in half the period of the decimal expansion of 1/7 yield, when added to the other half, all 9s? (One 11-year-old immediately said: "You mean, one less than a power of 10"). And look, it's true of 1/13 too, and 1/101; but not of 1/2 or 1/5, much less 1/3 or 1/8. This took us on two long excursions: into geometric series and, through the idea of congruence, to Fermat's Little Theorem (which again they came up with themselves, just by messin' around). The far-shining goal of our initial puzzle got us through difficult stretches. There is a push-me-pull-you rhythm to the best of these classes: convictions put together the week before turn out to have been soldered, not welded, together, and come apart with flexing (how was it we got the sum of an infinite series?). We reconstruct them more solidly under the pressure of doubt.

The 14-to-18 year-olds worked on infinite sequences and series with Guillermo, then did projective geometry with me. These were the most hard-fought of all the classes: they wanted nothing told them, all was to be invented. They came up with convergence criteria of their own (named after their new inventors), approximating ever more closely to the curve of the topic's history. By judicious choice of examples and nudges at critical moments I moved them to where they could—and did—come up with Desargues' Theorem, followed by their vigorous, critical role as sous-chefs in cooking up its proof. Because they were very puzzled by the maneuver of having to pass out of the plane and back to it, some doubting the validity of the proof, others the universality of the theorem, we had to digress to the free projective plane on four points—which they found startling and disturbing. They took an inventor's pride in coming up with a proof of the uniqueness of the fourth harmonic point, and that left us, at the end (ten sessions are too few), able to conjecture the Fundamental Theorem and prove its existence part. A real advantage of projective geometry for students whose graphing calculators usually do their visualizing for them is that their spatial imagination is awaken and exercised.

Our Saturday format has been two one-hour classes (milk and cookies in between), followed by guest lecturers (for example, Mazur on the ABC conjecture, Diaconis on the card-shuffling that led him to become a mathematician). A good high school mathematics course brings a student up to the eighteenth century. Here they could see contemporary mathematicians working on the frontier in the same manner that they had been developing for the last two hours.

Because our clientele is growing for next year we'll be taking on another hand. We're thinking too of branching out to other cities. What may be hard to export is our style: we entertain all conjectures and questions with equal seriousness, letting them follow their conversational course and turning the current of that conversation into fruitful directions as unobtrusively as possible. If a line of inquiry hits a wall we tend to let it lie and strike off in another direction, rather than throwing our students a sophisticated assortment of scaling-ladders. What's left fallow one week tends to produce a flurry of ingenious growths by the next (and these dead ends are the material of the week's homework). We do our best to hold off introducing a symbol until its abbreviatory power is welcomed for packaging up what had become an unwieldy complex of relations. Best when the students come up with the symbol—and the need for it—themselves.

What have we learned from this? That the appetite for real math, done neither competitively nor scholastically but as the most exciting of the arts, is enormous. I see no limits to what children can learn, and am convinced that if you want to teach them A, and A implies B, work on B with them: A will be mastered en passant, painlessly, absorbed in the bones. I'm certain too that removing any question of time—or achievement—pressure lets understanding and technique blossom, as well as developing a delightfully collegial feeling in those involved and a sense of the enterprise as contained within larger frameworks of question and significance. The students come away certain that math is mysterious, equally certain that its mysteries are accessible; unsure whether we discover or invent it; confident in their growing competence, and with that heightened threshold of frustration, that odd combination of watchfulness and willfulness, that characterizes the practitioners of our craft.