The Graph of the Truncated Icosahedron and the Last Letter of Galois

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The use of group theory by chemists to determine certain properties of suitable molecules is a well-established procedure and there is a vast literature on the subject. For a single molecule the group involved is the molecule's symmetry group which, up to conjugacy, can be considered as a subgroup of $O(3)$ (necessarily finite, assuming the molecule is non-trivial). The most significant part of the symmetry group is its intersection, G, with $SO(3)$ and

$$G \subset SO(3)$$

will be referred to as the molecule's proper symmetry group. This use of the word proper in connection with subgroups of $O(3)$ will be maintained throughout the paper.

From the point of view of mathematics the groups G in question—with exactly one exception (up to conjugacy)—are not very interesting since they are easily constructed solvable groups. The one exception is a group which is isomorphic to the alternating group A_5. As a subgroup of $SO(3)$ it is the proper symmetry group of the icosahedron (and the dodecahedron). It is the unique finite subgroup of $SO(3)$ which equals its own commutator subgroup—in fact, it is the unique non-abelian simple finite subgroup of $SO(3)$. It can be thought of as the beginning or smallest member of the family of non-abelian finite simple groups. From the point of view of the McKay correspondence it (or more exactly its "double cover") corresponds to the exceptional Lie group E_8.

A group isomorphic to A_5 will be referred to as an icosahedral group and any structure admitting such a group as a symmetry group is said to have icosahedral symmetry. Given the unique role of the icosahedral group in group theory, any natural structure having icosahedral symmetry surely deserves special attention. In this paper we will be concerned with one such structure—a structure which seems to be appearing with increasing frequency in the scientific literature.

Prior to the discovery of the Fullerenes, around ten years ago, the only known form of pure solid carbon was graphite and diamonds. These two forms are crystalline materials where the bonds between the carbon atoms exhibit hexagonal and tetrahedral structures, respectively. In neither of these two substances, however, are there isolated molecules of pure carbon. On the other hand, in

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Fullerene one finds for the first time a pure carbon crystalline solid with well-defined carbon molecules. See e.g. the top of p. 58 in [6]. Mathematically these molecules can be described as convex polyhedrons where the faces are either hexagons or pentagons and each vertex (carbon atom) is the endpoint of three edges (carbon bonds). The fact that the Euler characteristic of the 2-sphere is 2 easily implies that the number of pentagonal faces is necessarily 12. Hint: Express the number of vertices, edges and faces in terms of the number of pentagons and the number of hexagons. See e.g. [8], p. 16.

Among the many Fullerene molecules the most prominent and the most studied is C_{60}

Fullerenes exhibit remarkable chemical and physical properties (e.g. superconductivity, ferromagnetism, tremendous stability) and have been the objects of a vast amount of research throughout the world. The shape of the molecules is such that they can behave as cages, encapsulating other atoms or molecules. For an up-to-date account of Fullerenes, see [7]. A catalogue of possible Fullerene applications is given in Chapter 20 of [7] and in part III of [11].

Among the many Fullerene molecules the most prominent and the most studied is C_{60}. (The term Fullerene is sometimes used to refer specifically to this molecule.) In the case of C_{60} the corresponding polyhedron is the truncated icosahedron (see chapter 8, §4, in [5]). This polyhedron is seen on the surface of a soccer ball. It has thirty-two faces. That is, besides the twelve pentagons there are twenty hexagons. It is of significance that the twelve pentagons are isolated from one another. Chemists believe that the isolation of the pentagons in a Fullerene molecule is a requirement for stability. See e.g. bottom of p. 4 in [8]. C_{60} is the smallest Fullerene molecule in which this isolation occurs.

The truncated icosahedron is also found as the polyhedral coat of a number of viruses. For a survey of the polyhedral structure of viruses, including electron microscope pictures, see [9]. Another important biological occurrence of a truncated icosahedral structure is in a substance—called a clathrin—which is concerned with the release of neurotransmitters into the synapses of neural networks. See p. 365 in [15] (I thank S. Sternberg for this latter reference) and, in more detail, in Chapter 5 of [11].

There are ninety edges in the truncated icosahedron, sixty of which bound the twelve pentagons and separate hexagons from pentagons. We refer to these sixty edges as pentagonal edges. The remaining thirty edges separate hexagons from hexagons and are referred to here as hexagonal edges. According to pp. 46–47 in [8] the pentagonal edges in C_{60} are single carbon bonds and the hexagonal edges are double carbon bonds. Thus each hexagon, reminiscent of a benzine ring, has alternating single and double bonds. However, unlike benzine, single and double bonds in these hexagons of C_{60} remain fixed.

The truncated icosahedron has sixty vertices. Because of the isolation of the pentagonal faces each vertex lies on a unique pentagon. In this way the pentagons define a natural equivalence relation on the set of vertices—partitioning the set of vertices into twelve equivalence classes where each class is the 5-element set of vertices of one of the pentagons. By abuse of terminology we will generally refer to these 5-element sets themselves as pentagons.

One also notes that at each vertex there are three edges, two pentagonal and one hexagonal. The structure of a truncated icosahedron is completely determined by the graph Γ of its vertices and edges. The proper symmetry group G of C_{60}, or of the truncated icosahedron, is an (60-element) icosahedral group. The group G operates in a simple transitive way on the set V of vertices. Thus, given a pair of ordered vertices there is a unique proper symmetry which carries the first to the second. That is, the action of G on V is equivalent to the action of G on itself by left translations. In particular the action of G on V does not "see" the edge structure in Γ. This, in our opinion, points to an inadequacy of G in dealing with many questions about the nature of C_{60}. In the expectation that a group-based harmonic analysis will eventually lead to a deep understanding of the remarkable molecule C_{60}, it seems to be highly desirable to be able to express the full structure of Γ group theoretically. The edge structure in Γ determines a 60×60 adjacency matrix H and consequently H would be expressed group theoretically. In this connection it is useful to point out that the eigenvalues of H, via what is called a Huckel approximation, enter into the determination of the molecular energy levels of C_{60}. See e.g. [8], p. 44 and §9 in [3].

If p is a prime number let F_p denote the finite field of p elements. The group $Sl(2,p)$ is the group of all 2×2 matrices with entries in F_p having determinant 1 and $PSl(2,p)$ is $Sl(2,p)$ modulo its (2-element if p is odd) center. The group $PSl(2,p)$ is simple if $p\geq 5$ and $PSl(2,5)$ is an icosahedral group. The icosahedral group $PSl(2,5)$ admits an embedding into $PSl(2,11)$ and the relationship (see section headed "The Embedding of $PSl(2,5)$ into $PSl(2,11)$) and Galois' Letter to Chevalier", p. 964) between these two groups is quite remarkable. This relationship has much to do with a statement (see p. 964) made by Galois in his famous letter to Chevalier written on the night before his life-ending duel. The set of all elements of order 11 in $PSl(2,11)$ decomposes into two conjugacy classes, each of which has sixty elements. The choice of the embedding (there are two such inequivalent embeddings) of $PSl(2,5)$ in $PSl(2,11)$ favors one of the conjugacy classes, say M. It will be proved that the conjugacy class M has a natural structure of the graph of a truncated icosahedron. In effect, the model we are proposing for C_{60} is such that each carbon atom can be labeled by an element of order 11 in $PSl(2,11)$ in such a fashion that the carbon bonds can be expressed in terms of the group structure of $PSl(2,11)$. It will be seen that the twelve pentagons are exactly the intersections of M with the twelve Borel subgroups of $PSl(2,11)$. (A Borel subgroup is any subgroup which is conjugate to the group $PSl(2,11)$ defined in (2).) In particular the pentagons are the maximal sets of commuting elements in M. The most subtle point is the natural existence of the hexagonal bonds. This will arise from a group theoretic linkage of any element of order 11 in one Borel subgroup with a uniquely defined element of order 11 in another Borel subgroup.

The Graph of the Icosahedron and a Conjugacy Class in A_5

Let $\tau=\frac{1+\sqrt{5}}{2}$ be the golden number. A rectangle is referred to as a golden rectangle if the ratio of the larger side to the smaller side is the golden number. In R^2 the rectangle with the four vertices $\{(\pm 1,\pm\tau)\}$ is a golden rectangle. Let V be the set of 12 points in R^3 obtained from the vertices of the 3 mutually perpendicular golden rectangles

$$V=\{(\pm 1,\pm\tau,0),(0,\pm 1,\pm\tau),(\pm\tau,0,\pm 1)\} \tag{1}$$

Then V is the set of vertices of an icosahedron P where $\{c,d\}\subset V$ define an edge of P if the scalar product $<c|d>=\tau$. See $3\cdot 75$ in [5]. Waxing poetic, one is strongly tempted to describe the icosahedron P as a symphony in the golden number.

Let $A\subset SO(3)$ be the group of all rotations which stabilize the 12-element set V. Then A is an icosahedral group. For any integer $j>1$, let $A(j)\subset A$ denote the set of all elements in A of order j. The $A(j)\neq\emptyset$ only if $j=2,3,or\ 5$ and where $|\,|$ denotes set cardinality

$$|A(2)|=15$$
$$|A(3)|=20$$
$$|A(5)|=24$$

For any prime p (or in fact power of a prime) the group $PSl(2,p)$ naturally operates (transitively) on $p+1$ points. More specifically, it operates on the projective line $\mathsf{F}_p\cup\{\infty\}$ over F_p as the group of fractional transformations $x\mapsto\frac{ax+b}{cx+d}$. The isotropy subgroup is the Borel subgroup

$$B=\left\{\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix}|a\in\mathsf{F}_p^*,\,b\in\mathsf{F}_p\right\} \quad \textit{modulo the center} \tag{2}$$

where F_p^* is the multiplicative group of invertible elements in F_p. Thus for the case at hand the projective line is the "flag manifold" for $PSl(2,p)$—the set of all conjugates of the Borel subgroup B. Since $A\simeq PSl(2,5)$ this means that A naturally operates on six points. Geometrically this is readily seen on the icosahedron P where the six points may be taken to be six pairs of antipodal vertices. But, of course, $A\simeq A_5$ so that A operates on a set $\mathcal{F}$ of five "objects". Looking at the icosahedron P this is less obvious. Each edge and its antipodal edge are the edges of a golden rectangle defining in this way fifteen golden rectangles on P. These fifteen golden rectangles break up uniquely into five sets of three mutually orthogonal golden rectangles (one set of which is given by (1)). This is a geometric way of seeing the five objects. Algebraically it is simpler. Define a relation on the 15-element set $A(2)$ where if $\sigma,\rho\in A(2)$ then $\sigma\sim\rho$ if σ commutes with ρ. Marvelously in this case the relation is an equivalence relation and there are five equivalence classes each of which has three elements. Thus algebraically we can take

$$\mathcal{F}=\textit{the set of maximal commuting subsets of } A(2) \tag{3}$$

Of course there is a natural bijective correspondence between $A(2)$ and the set of fifteen golden rectangles. Each element in $A(2)$ defines a 180-degree rotation in the plane of the corresponding golden rectangle.

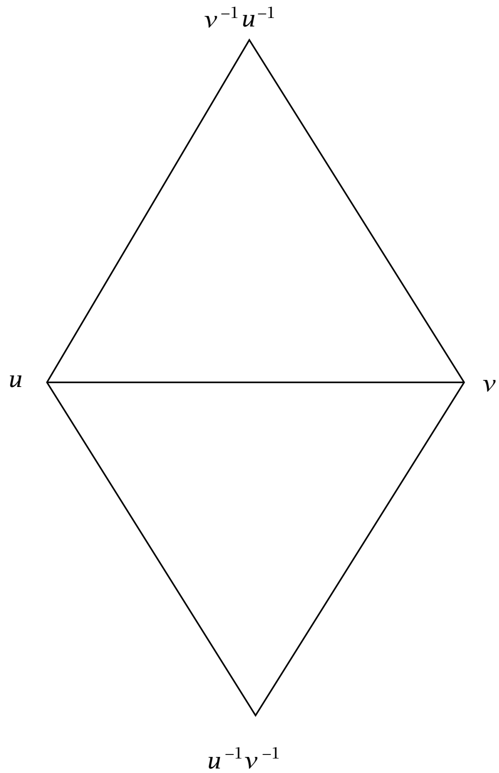

Figure 1

To what extent does A "see" the icosahedron itself? By this we mean: where in A can we find the graph of the vertices and edges of an icosahedron? It was our solution to this question, Theorem 1, given next, which led to the main result of the paper (given in the section entitled "The Graph of the Truncated Icosahedron and the Conjugacy Class M in $PSl(2,11)$", p. 965)—the solution of a similar question when the icosahedron is replaced by the truncated icosahedron. The sets $A(2)$ and $A(3)$ are conjugacy classes in A. On the other hand, $A(5)$ decomposes into a union of two conjugacy classes, say C and C' each with twelve elements. Being conjugacy classes, they are, of course, A-sets with respect to the action of conjugation and both are closed under inversion. The map $C \to C'$, $\sigma \mapsto \sigma^2$ is an A-bijection.

Consider one of the two, say $C \subset A(5)$, conjugacy classes of elements of order 5 in A. We will define a graph Δ whose set of vertices is C. Normally it is not a common practice in group theory to consider whether or not the product of two elements in a conjugacy class is again an element in that conjugacy class. However such a consideration here turns out to be quite productive. For $u, v \in C$ we define

$$\{u, v\} \text{ to be an edge of } \Delta, \text{ if } uv \in C \tag{4}$$

Note that this is well defined since, being conjugate elements, $uv \in C$ if and only if $vu \in C$. Note also that whatever graph has been defined, it is necessarily invariant under the icosahedral group A. But in fact that there are exactly thirty edges and indeed one has the icosahedron. The following theorem is proved in [13].

Theorem 1. *The graph Δ is isomorphic to the graph of vertices and edges of an icosahedron. With respect to such an isomorphism for any $u \in C$ the vertex corresponding to u^{-1} is the antipode of the vertex correponding to u.*

If c and d are vertices of an icosahedron and $\{c, d\}$ is an edge, we will call d a neighbor of c. Each vertex c, of course, has five neighbors. The five neighbors of the antipode of c will be referred to as coneighbors of c. Any vertex d not equal to c or its antipode is either a neighbor of c or a coneighbor of c, but not both. Accordingly we refer to the pair $\{c, d\}$ as neighbors or coneighbors. The same terminology of neighbor and coneighbor will be used for the conjugacy class C in A.

With an example, we illustrate Theorem 1 for the icosahedral group A_5. Let C be the conjugacy class of the permutation cycle $u = (1,2,3,4,5) \in A_5$. If $w = (1,5,2,4,3)$ then one readily has $w \in C$. However

$$(1,2,3,4,5)(1,5,2,4,3) = (1,4,2)$$

so that necessarily $\{u, w\}$ are coneighbors. But then u and v must be neighbors where $v = w^{-1}$. Indeed $v = (1,3,4,2,5)$ and

$$\begin{aligned} uv &= (1,2,3,4,5)(1,3,4,2,5) \\ &= (1,5,3,2,4) \end{aligned}$$

and $(1,5,3,2,4) \in C$.

Remark 2. It follows easily from the example above that if $u, w \in C$ are arbitrary then u and u are coneighbors if and only if uw has order 3.

Being a polyhedron, any edge of the icosahedron is the boundary of two faces. For the icosahedron the faces are triangles. Hence if $\{u, v\} \subset C$ are neighbors, there are exactly two other elements $w \in C$ which are neighbors of both u and v. Since $(vu)^{-1} = u^{-1}v^{-1}$ and $(uv)^{-1} = v^{-1}u^{-1}$ satisfy these conditions they must be two other elements as indicated in Figure 1.

Starting with the edge defined by u and v, and choosing the orientation of the icosahedron so that the edge $\{u, v^{-1}u^{-1}\}$ is obtained from $\{u, v\}$ by counterclockwise rotation, the five neighbors of u can be expressed in terms of u and v, as indicated in Figure 2, exhibiting the five faces of the icosahedron which have u as a vertex.

Of course the five coneighbors of u are the inverses of the five neighbors.

Remark 3. Note that a comparison of Figure 1 and Figure 2 yields the equation

$$u^2vu^{-2} = u^{-1}v^{-1} \tag{5}$$

Since u has order 5 the equation (5) is equivalent to the condition that $x = u^{-2}v$ has order 2. By Remark 2 the element $ux = u^{-1}v$ has order 3. These three relations involving u and x are a presentation (referred to in (12) as a standard presentation) of the icosahedral group and such a presentation will later be seen (Theorem 7) to define, as a Cayley graph, the graph of the truncated icosahedron.

For non-solvable groups it is difficult to keep track of commutators of pairs of elements. The icosahedral group is the group of smallest order which equals its own commutator subgroup so that it is not without interest to see how commutators in A distribute themselves. For the conjugacy class C, we will now see that this distribution is neatly expressed in terms of the three orthogonal golden rectangles. Assume $u, v \in C$ are neighbors. Then $\{u, v, u^{-1}, v^{-1}\}$ are the vertices of one of the fifteen golden rectangles. On the other hand, given two elements in any group, there are eight ways of forming a commutator and these eight expressions decompose naturally into two sets with four expressions in each. For u and v these are set forth in the second and third columns of the array

$$\begin{array}{ccc} u & uvu^{-1}v^{-1} & uv^{-1}u^{-1}v \\ v & v^{-1}u^{-1}vu & vu^{-1}v^{-1}u \\ u^{-1} & vuv^{-1}u^{-1} & v^{-1}uvu^{-1} \\ v^{-1} & u^{-1}v^{-1}uv & u^{-1}vuv^{-1} \end{array} \tag{6}$$

Theorem 2. *The elements in A defined by the twelve expressions in (6) are distinct and lie in C —and hence in fact exhaust C. Furthermore the elements in any one of the 3-columns are the vertices of a golden rectangle and the three golden rectangles are orthogonal (i.e. correspond to orthogonal golden rectangles in the icosahedron $P \subset \mathbb{R}^3$). In particular the second and third golden rectangles are the unique (among the fifteen golden rectangles) golden rectangles which are orthogonal to the first.*

This theorem is proved as Theorem 1.18 in [13].

The following diagram (Figure 3) illustrates Theorem 2 and gives additional expressions (see §1 in [13]) for some of the vertices of the icosahedron in the terms of a pair $\{u, v\} \subset C$ of neighbors.

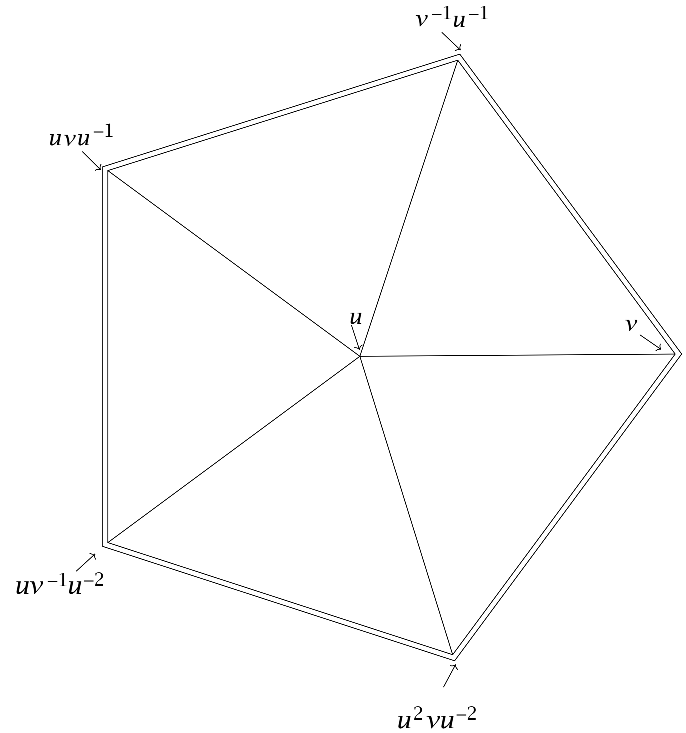

Figure 2

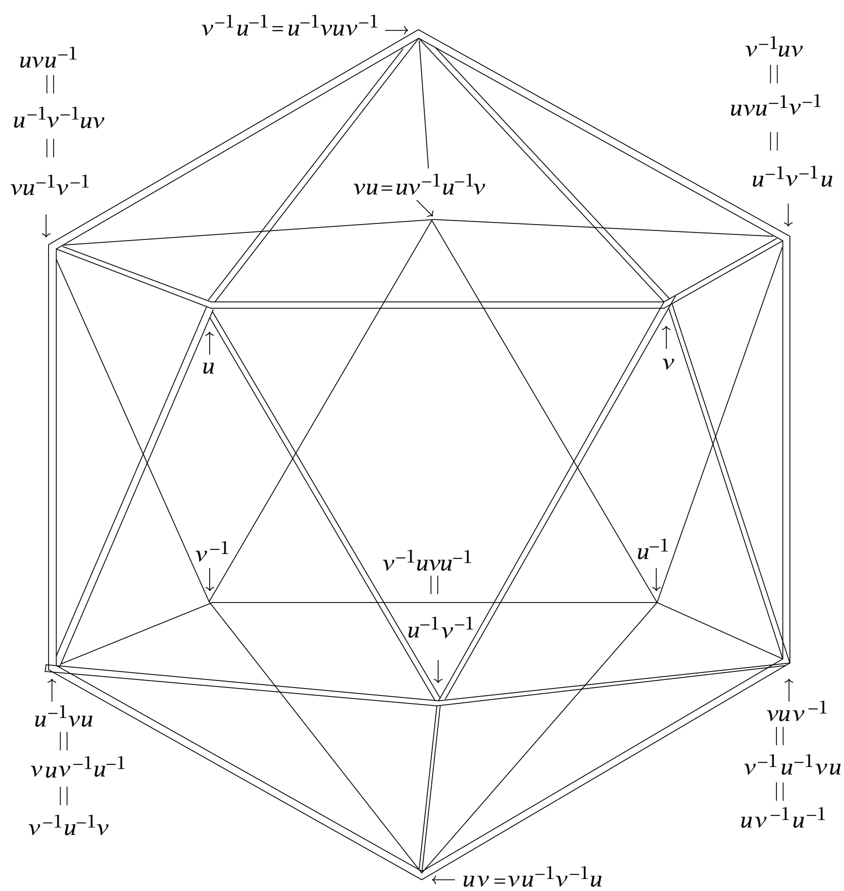

Figure 3

The Embedding of $PSl(2,5)$ into $PSl(2,11)$ and Galois' Letter to Chevalier

A major theme in what follows is the relationship between the 60-element icosahedral group $PSl(2,5)$ and the 660-element group $PSl(2,11)$. Up to conjugacy the group $PSl(2,5)$ embeds into $PSl(2,11)$ as a subgroup in two different ways (which, however, are interchanged by an outer automorphism of $PSl(2,11)$). That the relationship between the group $PSl(2,11)$ and its subgroup $PSl(2,5)$ is distinguished and extraordinary goes back to Galois. It will be discussed shortly. That the relationship has to do with the icosahedron is discussed in [3] and is further developed in [13].

As we noted before the 6-element projective line $Proj_1(\mathsf{F}_5)$ over the field F_5, as a $PSl(2,5)$-set, may be identified with the six pairs of antipodal vertices on the icosahedron. The point is that the 12-element projective line $Proj_1(\mathsf{F}_{11})$ over the field F_{11}, as a $PSl(2,11)$-set, may be identified with the full set of vertices V of the icosahedron in such a fashion that there exists a partition of $Proj_1(\mathsf{F}_{11})$ into six pairs of elements whose stabilizing subgroup of $PSl(2,11)$ is $PSl(2,5)$, recovering the action of $PSl(2,5)$ on $Proj_1(\mathsf{F}_5)$. Actually there exists two inequivalent such partitions corresponding to the two embeddings of $PSl(2,5)$ in $PSl(2,11)$. As evidence that the bijective correspondence between $Proj_1(\mathsf{F}_{11})$ and V is more than a coincidence it was shown in [3] that the graph of vertices and edges of the icosahedron may be neatly expressed in terms of the cross-ratio in $Proj_1(\mathsf{F}_{11})$. See Theorem 1 in [3].

We are concerned here not with the graph of the icosahedron but with the more sophisticated graph associated with the truncated icosahedron. A step in this direction will be a consequence of an idea inherent in Galois' result. The three groups A_4, S_4, and A_5 are, up to conjugacy, the only finite subgroups of $SO(3)$ which operate irreducibly on R^3. They are, of course, the proper symmetry groups of the five Platonic solids. For the tetrahedron there is A_4. For the octahedron and the cube, there is S_4, and for the icosahedron and dodecahedron there is A_5. From the point of view of the McKay correspondence the groups A_4, S_4, and A_5 correspond, respectively, to the simple Lie groups E_6, E_7, and E_8.

Now if p is a prime number then the group $PSl(2,p)$ is simple whenever $p \geq 5$ and it of course operates on the $p+1$-element projective line over the field F_p. Galois' result, reported in his letter to Chevalier (see p. 268 in [4] and p. 214 in [10]) is that if $p > 11$ then $PSl(2,p)$ cannot operate, non-trivially, on a set with fewer than $p+1$-elements. In particular it cannot operate

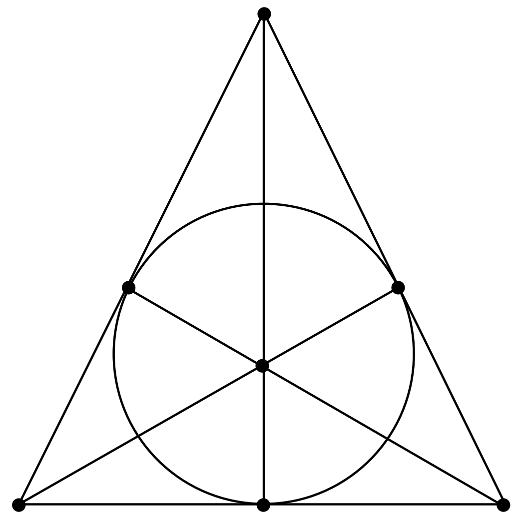

Figure 4

non-trivially on a set with p elements when $p > 11$. This may be expressed as follows. The cyclic group Z_p of order p embeds uniquely in $PSl(2,p)$, up to conjugacy, as the Sylow p-subgroup. It is the unipotent radical of a Borel subgroup and pulled up to $Sl(2,p)$ may be taken to be

$$\mathsf{Z}_p = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} | x \in \mathsf{F}_p \right\} \tag{7}$$

The result of Galois is that if $p > 11$,

> *There exists no subgroup of $PSl(2,p)$ which is complementary to Z_p.*

By this we mean that there exists no subgroup F such that set theoretically $PSl(2,p) = F \times \mathsf{Z}_p$. The notation here, of course, does not mean direct product of groups. It means that every element g can be uniquely written $g = fz$ where $f \in F$ and $z \in \mathsf{Z}_p$. Implicit in Galois' statement is certainly the knowledge that his statement is not true for the three cases of a simple $PSl(2,p)$ where $p \leq 11$—namely $p = 5, 7, 11$. It is surely a marvelous fact, that for the three exceptional cases, the groups F which run counter to Galois' statement are precisely the symmetry groups of the Platonic solids. Namely one has

$$\begin{aligned} PSl(2,5) &= A_4 \times \mathsf{Z}_5 \\ PSl(2,7) &= S_4 \times \mathsf{Z}_7 \\ PSl(2,11) &= A_5 \times \mathsf{Z}_{11} \end{aligned} \tag{8}$$

These exceptional cases merit elaboration. The fact that $PSl(2,5)$ operates on five points was discussed on page 961 (see (3)) in connection with the commutativity equivalence relation on the set of elements of order 2 in $PSl(2,5)$. This action sets up the isomorphism of $PSl(2,5)$ with A_5. The group $PSl(2,7)$ is isomomorphic to $PSl(3,2)$. The 3 here implies that $PSl(3,2)$ operates on a projective plane and the 2 implies that the plane is over the field of two elements. This plane has $1+2+2^2=7$ lines and 7 points, exhibiting seven objects on which $PSl(2,7)$ operates. The plane is classically represented by the diagram in Figure 4.

Of course our main interest is in the final and most sophisticated case, $PSl(2,11)$. Its action on eleven points arises from its symmetry of a special and remarkable geometry. The eleven points will be the field F_{11} itself—as represented by the integers from 1 to 11 where of course $11 = 0$. The set of non-zero squares in F_{11} is the set $\{1,3,4,5,9\}$. Now using the additive structure in F_{11} translate this five-element set by each of the elements in F_{11}. One then obtains the following eleven five-element sets

$$
\begin{array}{c}
1,3,4,5,9\\
2,4,5,6,10\\
3,5,6,7,11\\
1,4,6,7,8\\
2,5,7,8,9\\
3,6,8,9,10\\
4,7,9,10,11\\
1,5,8,10,11\\
1,2,6,9,11\\
1,2,3,7,10\\
2,3,4,8,11
\end{array}
\tag{9}
$$

If we regard these sets as lines then it may be noted that any two distinct lines intersect in exactly two points and any two distinct points lie in exactly two lines. That is, intersection sets up a bijection between the 55-element set of all pairs of distinct points and the 55-element set of all pairs of distinct lines. This is referred to as a biplane geometry on p. 7 in [1]. If we identify the symmetric group S_{11} with the permutation group of the set F_{11} then the subgroup of S_{11} which stabilizes the set of these eleven lines is isomorphic to $PSl(2,11)$. This then yields an embedding of $PSl(2,11)$ into S_{11} or an action of $PSl(2,11)$ on eleven objects. The isotropy subgroup of $PSl(2,11)$ at a point in F_{11} is isomorphic to $PSl(2,5)$. The second embedding of $PSl(2,5)$ in $PSl(2,11)$ is obtained by taking the isotropy subgroup of a line instead of a point.

The biplane geometry on eleven elements is directly related to the existence of 12×12 Hadamard matrices. A Hadamard matrix is a square matrix all of whose elements are either 1 or -1 and which has orthogonal columns (and hence has orthogonal rows). The set $\mathcal{H}_n$ of all $n\times n$ Hadamard matrices is clearly stable under the action of the group G_n of all row and column permutations and sign changes. If $R\in\mathcal{H}_{12}$ then, first of all using sign changes, we may make the last row and the last column have only 1's. If R' is the complementary 11×11 principal minor of R then, since the first 11 columns of R are orthogonal to the last column, each column of R' must have exactly five 1's. Taking the row indices of these five 1's defines a 5-element subset of $\{1,\dots,11\}$. The eleven 5-element subsets obtained in this way must make a biplane geometry in order that the columns of R be orthogonal—and conversely. In particular (9) yields the following 12×12 Hadamard matrix.

$$
R=\begin{pmatrix}
1&-1&-1&1&-1&-1&-1&1&1&1&-1&1\\
-1&1&-1&-1&1&-1&-1&-1&1&1&1&1\\
1&-1&1&-1&-1&1&-1&-1&-1&1&1&1\\
1&1&-1&1&-1&-1&1&-1&-1&-1&1&1\\
1&1&1&-1&1&-1&-1&1&-1&-1&-1&1\\
-1&1&1&1&-1&1&-1&-1&1&-1&-1&1\\
-1&-1&1&1&1&-1&1&-1&-1&1&-1&1\\
-1&-1&-1&1&1&1&-1&1&-1&-1&1&1\\
1&-1&-1&-1&1&1&1&-1&1&-1&-1&1\\
-1&1&-1&-1&-1&1&1&1&-1&-1&1&1\\
-1&-1&1&-1&-1&-1&1&1&1&-1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1
\end{pmatrix}
\tag{10}
$$

Remark 5. Andrew Gleason informed me that it is easy to show that $\mathcal{H}_{12}$ is a single G_{12} orbit. This, of course, implies that one can obtain all elements in $\mathcal{H}_{12}$ by applying G_{12} to (10). He also conjectures that $n=12$ is maximum such that $\mathcal{H}_n$ is a single G_n-orbit, assuming of course that n is such that $\mathcal{H}_n$ is not empty.

The Graph of the Truncated Icosahedron and the Conjugacy Class *M* in *PSl*(2, 11)

Now returning to (8), instead of considering, as above, the 11-element $PSl(2,11)$-set $PSl(2,11)/H$ where H is the subgroup A_5, consider the 60-element $PSl(2,11)$-set $PSl(2,11)/H$ where the subgroup H is chosen to be Z_{11}. We now raise the question of the possibility that $X = PSl(2,11)/\mathsf{Z}_{11}$ has a natural structure of the graph of the truncated icosahedron. Evidence for this is that one immediately sees a canonical $PSl(2,11)$ invariant decomposition of X into a union of twelve "pentagons". Namely if B is the normalizer of Z_{11} in $PSl(2,11)$ then from general principles the group of B/Z_{11} operates (on the right) faithfully on X and the action not only commutes with the action of $PSl(2,11)$ but is its full centralizer in the full group of permutations of X. But B is just that Borel subgroup of $PSl(2,11)$ whose unipotent radical is Z_{11}. Thus

$$B/\mathsf{Z}_{11} \simeq \mathsf{Z}_5 \tag{11}$$

The orbits of B/Z_{11} are the twelve pentagons. In addition, from (8), X is a principal homogeneous space for A_5 which is exactly how A_5 operates on the vertices of a truncated icosahedron. What is missing, of course, is the most sophisticated part of the graph of the truncated icosahedron—namely the hexagonal edges—the edges which tie together the various pentagons (the double carbon bonds in C_{60}).

A graph, namely a Cayley graph, is defined on a group by choosing generators. In the case of an icosahedral group there is a classical choice of a pair of generators. Actually, there are three such (which are easily related to one another) depending on whether one wants the generators to have orders 2 and 5, or 2 and 3, or 3 and 5. We make the first choice. A presentation of the icosahedral group—which we refer to as a standard presentation—is given by a pair of non-trivial group elements ϕ, τ satisfying the relations

$$\phi^5 = 1, \quad \tau^2 = 1, \quad (\phi\tau)^3 = 1 \tag{12}$$

See e.g. Lemma 13.6, p. 120 in [14]. Non-trivial elements ϕ, τ in an icosahedral group A' define a Cayley graph on A'. The following theorem says that this Cayley graph is the graph of a truncated icosahedron. Actually it is more convenient for us to express the result in terms of a principal homogeneous space (i.e. trivial isotropy groups) rather than the group itself.

Theorem 7 is just a more detailed version of Theorem 1 in [3]. It is proved in [3] and as Theorem 2.10 in [13].

Theorem 7. *Let M be any 60-element set and let S_{60} be the full group of permutations of M. Assume that $\phi, \tau \in S_{60}$ satisfy the relations (12) and that ϕ, τ and $\phi\tau$ have no fixed points in M. Then the subgroup $A' \subset S_{60}$ generated by ϕ and τ is an icosahedral group and M is a principal homogeneous space for A'. Let A be the centralizer of A' in S_{60} so that A is also an icosahedral group (A and A' are each other's centralizer) and M is also a principal homogeneous space for A.*

Let Γ be the graph on M so that for any $x \in M$ the edges (three of them) containing x are $\{x, \phi x\}$, $\{x, \phi^{-1}x\}$ and $\{x, \tau x\}$. Then Γ is isomorphic to the graph of a truncated icosahedron where the two pentagonal edges containing x are $\{x, \phi x\}$ and $\{x, \phi^{-1}x\}$ and the unique hexagonal edge containing x is $\{x, \tau x\}$.

Finally, A is the proper symmetry group of Γ.

If we apply Theorem 7 to the coset space $X = PSl(2,11)/\mathsf{Z}_{11}$ we note that the element ϕ has been essentially given to us. It is a generator of the "torus" B/Z_{11} operating from the right on X. Explicitly what is missing is the permutation τ which with ϕ defines a standard presentation of the centralizer of A_5 in the permutation group of X. To find τ we seek a more natural setting for X, where there is more unlying structure, and we will find it as a conjugacy class M of elements of order 11 in $PSl(2,11)$.

Let A be a fixed choice of an icosahedral subgroup of $PSl(2,11)$. (Recall there are two such subgroups, up to conjugacy.) Then if Z_{11} is the unipotent radical of any Borel subgroup one has, using multiplication, the set (non-group) direct product

$$PSl(2,11) = A \times \mathsf{Z}_{11} \tag{13}$$

If Z_{11}^* is the set of elements of order 11 in Z_{11} then, of course, $|\mathsf{Z}_{11}^*| = 10$. Since there are twelve Borel subgroups it follows that there are 120 elements of order 11 in $PSl(2,11)$. Since B has two 5-element orbits on Z_{11}^* it then follows that there are 2 conjugacy classes M, M' of elements of order 11 and each has sixty elements. Let M be either one of these two classes.

Remark 9. Actually, as stated in Theorem 11 below, the choice of A will be seen to favor one of these classes over the other. Any non-trivial outer automorphism of $PSl(2,11)$ interchanges the two choices of A and also interchanges the two choices of M.

Since the centralizer of any element in M is the unipotent radical of the unique Borel subgroup B which contains it, one has an isomorphism

$$M \simeq X = PSl(2,11)/\mathsf{Z}_{11} \tag{14}$$

of $PSl(2,11)$-sets. The twelve "pentagons" $P \subset M$ then turn out to be the subsets of the form

$$P = B' \cap M \tag{15}$$

where B' is a Borel subgroup of $PSl(2,11)$.

Remark 10. In group theoretic terms it follows from (15) that the pentagons can be characterized as the maximal commutative subsets of M. Implicit in this statement is the fact that—as in $A(2)$ (see p. 961)—commutativity in M is an equivalence relation and the pentagons are the equivalence classes.

If p is a prime number and M_p is a conjugacy class of elements of order p in $PSl(2,p)$ then note that Galois' result can be interpreted as saying that M_p cannot be a principal homogeneous space for the conjugation action of a subgroup of $PSl(2,p)$, in case $p > 11$. In particular then, the Cayley graph structure on M which will be now be constructed can have no analogous generalization for M_p when $p > 11$.

The use of the term pentagon implies that there is an implicit graph structure on the orbits of the torus B/Z_{11}. Up to now this has not been made clear. But in fact there are two polygonal Cayley graphs on the cyclic group $B/\mathsf{Z}_{11} = \mathsf{Z}_5$—each defining a pentagonal structure on this group and consequently on its orbits. These are illustrated as the single and double edge structure in Figure 5. Note that $n = 5$ is the minimal value of n for which Z_n has more than one polygonal Cayley graph.

The action of this Z_5 can be expressed in terms of group multiplication in $PSl(2,11)$. For

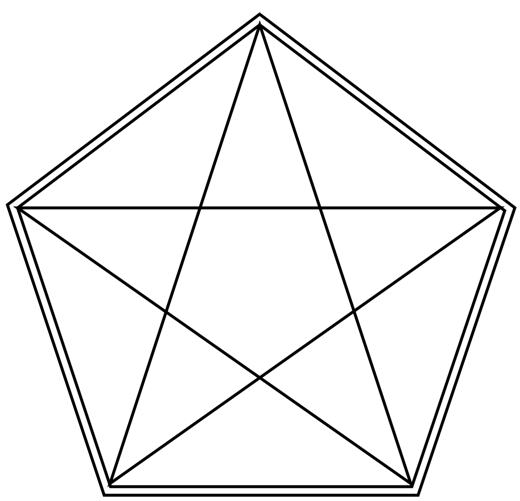

Figure 5

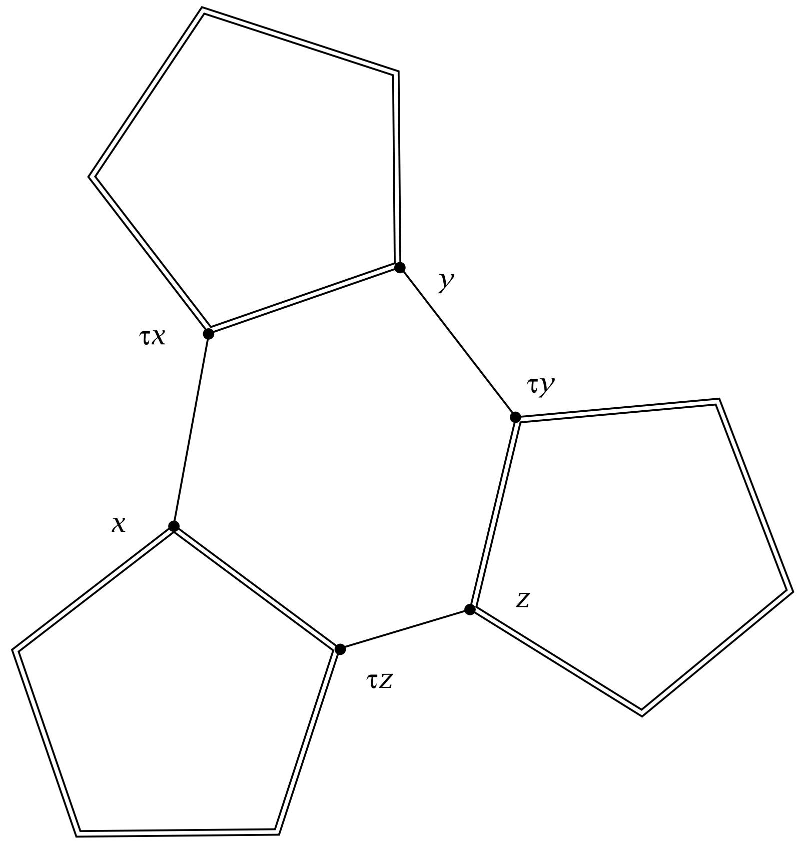

Figure 6

one pentagonal structure the two pentagonal edges containing any $x \in M$ are $\{x, x^3\}$ and $\{x, x^4\}$. For the other structure the edges are $\{x, x^9\}$ and $\{x, x^5\}$—noting that $x^4 = x^{-3}$ and $x^5 = x^{-9}$. A choice of one these two determines ϕ (see Theorem 11) up to inversion.

Finding the graph of the truncated icosahedron in the conjugacy class M, group theoretically, is clearly analogous, and was motivated by, finding, as in Theorem 1, the graph of the icosahedron in a conjugacy class C of elements of order 5 in the icosahedral group A. There is another heuristic reason for taking M to be the vertices of the truncated icosahedron. It goes something like this: The twelve vertices of the icosahedron have been identified here, and previously in [3], with the points of the "flag manifold" of $PSl(2,11)$—the twelve Borel subgroups of $PSl(2,11)$. Truncating a polyhedron can be viewed as a finite analogue of blowing up points in algebraic varieties. In the operation of blowing up points one replaces the point by objects in the tangent space at the point. Changing tangent space to cotangent space, it is a well known and standard fact that an orbit of the principal unipotent elements of a semisimple Lie group embeds naturally in the cotangent bundle of its flag manifold. Thus regarding truncation of an icosahedron as the operation of replacing a Borel subgroup B', regarded as an element (of the "flag manifold"), by the elements of the pentagon $B' \cap M$ is not entirely without motivation.

The subtle point in the construction of the graph of the truncated icosahedron in the conjugacy class M is the determination of the permutation τ of M of order 2 which with ϕ defines a standard presentation of the centralizer of the conjugation action of A on M. Given any $x \in M$ the pair $\{x, \tau x\}$ would define the hexagonal edge containing x. In effect, then what τ is to do is to assign to any $x \in M$, which, of course, is a unipotent element lying in a unique Borel subgroup, not only another Borel subgroup

B' but a distinguished unipotent element τx in that subgroup. I know of no operation for general groups which does that sort of thing. Geometrically this linking together of the pentagons is represented locally as dotted lines in the diagram (Figure 6).

(Of course, to represent bonds in C_{60} the single and double lines in Figure 6 should be reversed).

In the notation of the following theorem the map τ is given by putting $\tau x = \rho_x x$. Theorem 11 is proved as the main theorem, Theorem 3.30, in [13]. Although its proof makes no use of computers, it was, however, discovered after a long series of Maple computations involving $PSl(2,11)$.

Theorem 11. *Let A be any icosahedral subgroup of $PSl(2,11)$. Let $A(2)$ be the set of all elements of order 2 in A. Then for any $x \in PSl(2,11)$ of order 11 there exists a unique element $\sigma_x \in A(2)$ such that the commutator $\rho_x = x^{-1}\sigma_x x \sigma_x$ is again in $A(2)$. Moreover, there exists a unique choice C_A of a (12-element) conjugacy class of elements of order 5 in A and a unique choice of a (60-element) conjugacy class, M, of elements of order 11 in $PSl(2,11)$ such that if $x \in M$ there exists $u \in C_A$ which normalizes the cyclic groups generated by x and $\sigma_x x \sigma_x$ and is such that the pair u, ρ_x defines a standard presentation of A. (That is, $u\rho_x$ has order 3). Furthermore if $x \in M$ then $\rho_x x \in M$ and a truncated icosahedral structure Γ is defined on M where the two pentagonal edges containing x are $\{x, x^3\}$ and $\{x, x^{-3}\}$ and the unique hexagonal edge containing x is $\{x, \rho_x x\}$.*

Next given $x \in M$ the element $u \in C_A$, satisfying the condition above, is unique up to inversion and may be fixed by the condition that $uxu^{-1} = x^5$. As such write $u = u(x)$. The space $\mathcal{P}$ of twelve pentagons is then parameterized by C_A so that, for $u \in C_A$, one has $P_u \in \mathcal{P}$ where $P_u = \{x \in M \mid u = u(x)\}$. In addition the icosahedral graph structure Δ induced on $\mathcal{P}$ by Γ is such that $\{P_u, P_v\}$ is an edge if and only if $uv \in C_A$.

Tables, which among other things, label the vertices of the truncated icosahedron and their neighbors by elements in M are given in [12] as well as in [13].

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