The Geometry of *p*-adic Symmetric Spaces

Jeremy Teitelbaum

Many of the geometric objects of interest to number theorists arise as quotients of classical symmetric spaces by discrete subgroups of Lie groups. For example, the Riemann surfaces known as "modular curves", which play a central role in Wiles's proof of Fermat's Last Theorem, are the quotients of the upper half plane by certain arithmetically defined discrete subgroups of $SL_2(\mathbf{R})$.

A typical symmetric space $X = G/K$ arises as the quotient of a Lie group G by a maximal compact subgroup. The deep theory of Shimura varieties teaches us that, if Γ is a discrete subgroup of G and the correct technical conditions are satisfied, the double coset space $\Gamma\backslash G/K$ can be realized as the zero set of a system of polynomial equations. In fact, these equations will have coefficients in a field of algebraic numbers L which is finite dimensional over $\mathbf{Q}$. In other words, the double coset space is an "algebraic variety defined over L". Therefore, it makes sense to consider Diophantine problems such as the number and distribution of points on $\Gamma\backslash X$ which have coordinates in L or its extensions. The theory known as the Langlands program then relates such Diophantine questions about $\Gamma\backslash X$ to the representation theory of G.

Under certain circumstances, work of Tate, Mumford, and Drinfeld has provided us with an alternative construction of the important spaces $\Gamma\backslash G/K$ based on p-adic rather than complex analysis. (For those unfamiliar with p-adic fields, I will give a lightning introduction below.) The idea of uniformizing algebraic varieties by p-adic analytic spaces is due to Tate, who showed how an elliptic curve E over a p-adic field F can be constructed as the quotient of the multiplicative group of F by the infinite cyclic subgroup generated by a single element (which, by tradition, is called q.) Tate's construction works only if the elliptic curve E over F acquires a certain type of singularity over the residue field of F; for example, if E is defined by a Weierstrass equation $y^2 = x^3 + Ax + B$, with A and B in the p-adic field $\mathbf{Q}_p$, then the cubic $x^3 + Ax + B$ has a double root mod p. This condition means that E lies at the boundary of the parameter space for nonsingular elliptic curves. Mumford generalized Tate's work and applied it to understand the boundary of the parameter spaces of curves of arbitrary genus. Mumford's construction of curves is the one-dimensional version of the theory of p-adic uniformization.

In general, p-adic uniformization replaces complex symmetric spaces X with p-adic analytic spaces $\mathcal{X}$ (one in each dimension). The p-adic general linear group acts on $\mathcal{X}$, and, as with complex symmetric spaces, the quotients of $\mathcal{X}$ by discrete groups sometimes turn out to be algebraic varieties defined over number fields. Amazingly, the spaces constructed in this way

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Jeremy Teitelbaum is associate professor, Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago. His e-mail address is `jeremy@math.uic.edu`. His home page is `http://zariski.math.uic.edu/~jeremy`.

The author is supported by the National Science Foundation. This paper is the written version of a talk presented at the March 1995 meeting of the Central Section of the AMS in Chicago.

are often the same Shimura varieties which arise out of complex symmetric spaces. (This is illustrated in Figure 1.) For an explicit example of the situation depicted in Figure 1, consider the subgroup $\Gamma = \Gamma_0(14) \subset SL_2(\mathbf{Z})$, where $SL_2(\mathbf{Z})$ is the group of two-by-two integer matrices with determinant one and $\Gamma_0(14)$ is the subgroup of such matrices with lower left entry divisible by 14. The group Γ acts on the upper half plane $\mathcal{H}$, and the quotient space $\mathcal{H}/\Gamma$ is a Riemann surface which is almost compact—it is missing a few points. The compact surface obtained by inserting these points is called $X_0(14)$, and one can show by complex function theory that this Riemann surface has genus one.

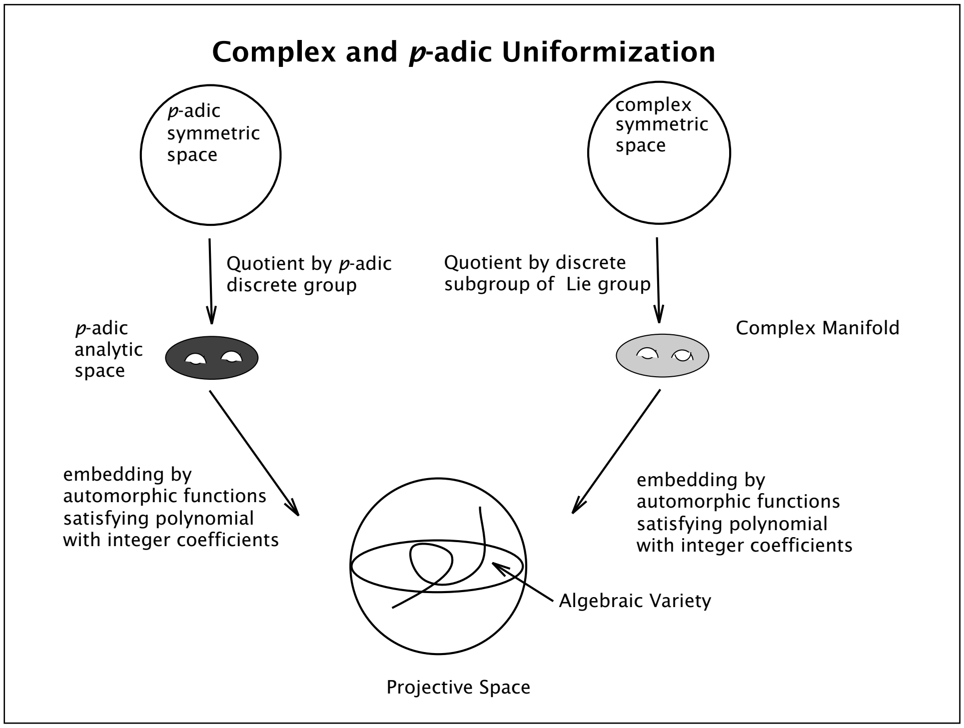

Figure 1

I have described an example of the right-hand column of Figure 1. By using functions on the upper half plane with good transformation properties under Γ (i.e., modular forms and modular functions), one can show that the Riemann surface $X_0(14)$ is isomorphic to the set of solutions to the homogeneous equation

$$(*) \qquad y^2z + xyz + yz^2 = x^3 + 4xz^2 - 6z^3$$

in the complex projective plane. Notice that this equation has integer coefficients.

This Riemann surface has an alternative, p-adic construction, along the lines of the left-hand column of Figure 1. For the purposes of this explanation, let $\mathcal{X}$ denote the one-dimensional p-adic symmetric space (which is often called the "p-adic upper half plane"). Let $\mathbf{H}$ be the division algebra of Hamilton quaternions and let $p = 7$. It is not hard to construct an embedding

$$\iota : \mathbf{H} \to M_2(\mathbf{Q}_7)$$

where $M_2(\mathbf{Q}_7)$ is the ring of two-by-two matrices with 7-adic entries. Let $R = \mathbf{Z}[i, j, k, (1 + i + j + k)/2]$, which is Hamilton's ring of integral quaternions, and let

$$\Gamma = \{x \in R[\frac{1}{7}] : x\overline{x} = 7^{2n}, n \in \mathbf{Z}\}.$$

This group of quaternions embeds via ι into $GL_2(\mathbf{Q}_7)$, and the quotient $\mathcal{X}/\Gamma$ is the same curve defined in equation (*).

The construction I have outlined here is a single example which follows from a general theory of p–adic uniformization of one–dimensional Shimura varieties (Shimura curves) due to Cerednik and Drinfeld [7]. Recently, work of Berkovich, Gross–Hopkins, Rapaport–Zink, and others has greatly generalized the range of situations in which a theory of this type can be applied; see their works cited in the references.

In this paper, I will describe Drinfeld's n-dimensional p-adic symmetric space $\mathcal{X}$, which is the starting point for this theory. This space has an intricate geometry with interesting connections to hyperplane arrangements, the theory of buildings, and representation theory. I will assume from now on that $n > 1$; the one-dimensional space, or p-adic upper half plane, which I mentioned in the example above has a somewhat different feel and is treated in [19]. After a lightning introduction to the p-adic numbers, I will begin by discussing the Bruhat–Tits building for $PGL_{n+1}(\mathbf{Q}_p)$, which provides a "skeleton" for $\mathcal{X}$; then I will describe $\mathcal{X}$ itself and finally I will discuss some recent work with Peter Schneider which emphasizes the connections between the analytic properties of $\mathcal{X}$ and combinatorial geometry. By the end, I hope to convey some feeling for the geometry of $\mathcal{X}$.

The p-adic Numbers

For each prime number p there is a field $\mathbf{Q}_p$ containing the rational numbers $\mathbf{Q}$ as a dense subfield. The p-adic field $\mathbf{Q}_p$ is constructed as the topological completion of $\mathbf{Q}$ in the topology defined by the norm

$$|a/b|_p = p^{(\omega_p(b) - \omega_p(a))}$$

where $\omega_p(a)$ is the exact power of p dividing an integer a.

In $\mathbf{Q}_p$, the usual notion of closeness is replaced by the notion "two numbers are close if they are congruent modulo a high power of p". The closure $\mathbf{Z}_p$ of the integers in $\mathbf{Q}_p$ is a com-

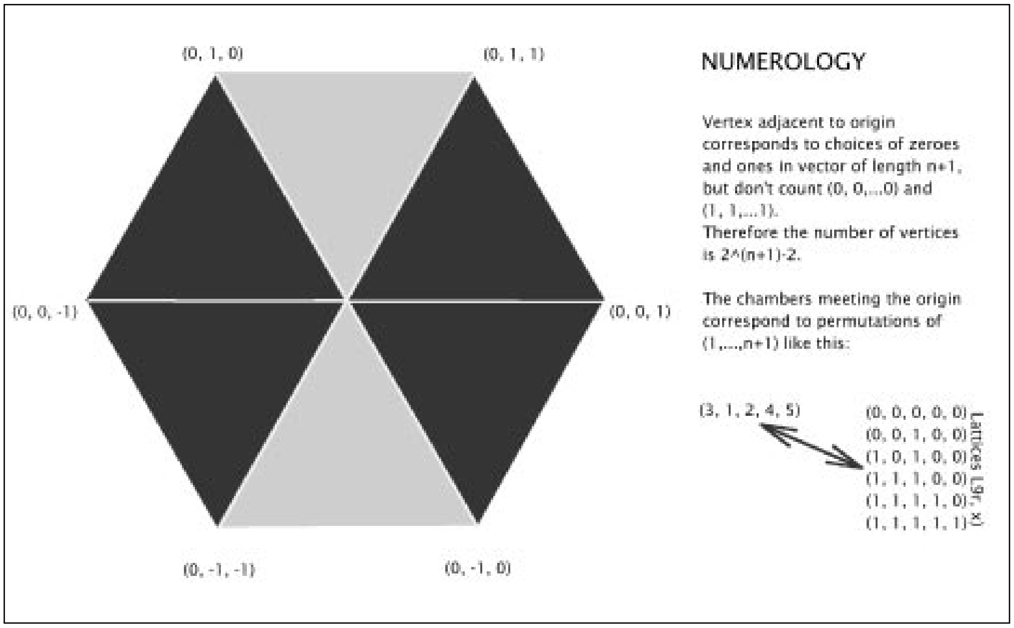

Figure 2

basis $\{y_0, \ldots, y_n\}$ of V. The group K is the stabilizer of the standard lattice $\mathbf{Z}_p^{n+1} \subset V$, and G permutes the lattices transitively, so G/K is precisely the set of lattices up to isomorphism. Two lattices L and L' are equivalent if there is an $a \in \mathbf{Q}_p^*$ such that $L = aL'$.

The Bruhat–Tits building B is a representation of the incidence relations among lattices. More precisely, let B^0, the vertices of B, be the set of equivalence classes $[L]$ of lattices (so that B^0 is $\mathrm{PGL}_{n+1}(\mathbf{Q}_p)/K$.) By definition, the edges B^1 of B correspond to pairs $[L], [L']$ of classes where $L \supset L' \supset pL$ (strict inclusions). The 2-faces of B correspond to triples $[L], [L'], [L'']$ where $L \supset L' \supset L'' \supset pL$, and so on. Since L/pL is an $n+1$ dimensional vector space over $\mathbf{Z}_p/p\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$, the maximum length of a chain is $n+1$, so all simplices have dimension at most n. Also, since G acts on the lattices, it acts on the building B.

Defined in this abstract way, the Bruhat–Tits building B is difficult to visualize. To help, I will describe a particular subset of B called *an apartment.* Let L be the standard lattice, which is the $\mathbf{Z}_p$ span of the standard basis $\{x_0, \ldots, x_n\}$ of V. One way to construct other lattices is to take a vector of integers $r_0, \ldots, r_n$ and let

$$L(r, x) = \mathbf{Z}_p\text{—span of } \{p^{r_0}x_0, \ldots, p^{r_n}x_n\}.$$

Notice that two lattices, $L(r, x)$ and $L(r', x)$, are equivalent if and only if $r_i - r_i'$ is constant, independent of i, and $L(r, x) \supset L(r', x) \supset pL(r, x)$ if and only if $r_i \leq r_i' \leq r_i + 1$. The standard apartment A is the subset of the building B consisting of the lattice classes $[L(r, x)]$.

The best way to visualize A is as an n-dimensional Euclidean space tiled by regular n-simplices. For example, if $n = 2$, A is the plane tiled by equilateral triangles. There are a number of ways to see this. For example, let W be an $n+1$-dimensional Euclidean space coordinatized by the r-vectors. The points with integer coordinates in W correspond to the $L(r, x)$, with adjacent points in the integer lattice in W adjacent in the sense of the building. The Euclidean symmetry group generated by translation in r and permutation of coordinates acts transitively on the integral points in W. To move from lattices to lattice classes, project W into the subspace perpendicular to the vector $(1, \ldots, 1)$; the image of the integral r-points is then a lattice (in the usual sense) which is symmetric

pact local ring with prime ideal generated by p; any p-adic number may be written $p^m u$ where u is an invertible element of $\mathbf{Z}_p$.

In some sense, $\mathbf{Q}_p$ is a version of the real numbers (like the reals, it contains $\mathbf{Q}$ as a dense subfield). To obtain a version of the complex numbers, one must take the topological completion of an algebraic closure of $\mathbf{Q}_p$. The resulting field, $\mathbf{C}_p$, is of infinite transcendence degree over $\mathbf{Q}_p$ (in sharp contrast to the situation with the usual complex numbers, of degree two over the reals).

The Bruhat–Tits Building

Since classical symmetric spaces arise as quotients G/K of a Lie group by a maximal compact subgroup, a first approach to building a p-adic space with similar properties is to consider coset spaces. Let G be the group $\mathrm{GL}_{n+1}(\mathbf{Q}_p)$ of invertible $(n+1) \times (n+1)$ matrices with p-adic entries. Then the subgroup $K = \mathrm{GL}_{n+1}(\mathbf{Z}_p)$ consisting of matrices with p-adic integral entries and unit determinant is a maximal compact subgroup. The differences between the p-adic and classical situation become apparent immediately when one observes that K is both open and closed in G. This means that the natural topology on G/K is discrete, which makes it ill-suited for any sort of analytic theory.

This situation can be improved by following the construction of Bruhat and Tits (which is beautifully presented in Brown's book [1]). The set G/K should be properly viewed as the set of vertices of an n-dimensional simplicial complex called a "building".

First, let us reinterpret G/K in terms of lattices. Let $V = \mathbf{Q}_p^{n+1}$ be an $(n+1)$ dimensional vector space over $\mathbf{Q}_p$, on which G acts in the standard way. A *lattice* in V is the $\mathbf{Z}_p$ span of a under the induced action of the Euclidean symmetry group, which is the group of the tiling by simplices.

It is fun to work out some of the numbers involved here; for example, consider the standard lattice $L = L(0, \dots, 0)$. A lattice class $[L(r, x)]$ is adjacent to $[L]$ provided that r consists of only zeros and ones, there are 2^{n+1} such choices, though $(0, \dots, 0)$ and $(1, \dots, 1)$ both correspond to L. Thus there are $2^{n+1} - 2$ vertices in A adjacent to L. Also, a top dimensional simplex in A corresponds to a *flag*, or descending chain $L \supset L' \supset \dots \supset pL$, of lattices of length n; such a flag can be specified by giving a sequence of r–vectors, each consisting only of zeros and ones; starting from $(0, \dots, 0)$, ending at $(1, \dots, 1)$, and each differing from the preceeding in only one place. It is not hard to see that there are $n!$ such sequences. (See Figure 2 for a picture in the case n=2.)

The complex B as a whole is made up of a union of apartments A (each of which we think of as a Euclidean space tiled by simplices). These apartments meet and branch along the faces of the tiling. While the whole construction is too complicated to describe in detail here, remember that the original apartment A was based on the choice of basis $\{x_0, \dots, x_n\}$ for L. Had we chosen a different basis, say, $\{y_0, \dots, y_n\}$ for L, we could have repeated our entire construction and built an apartment $A(y)$. With some thought one can show that $A(y)$ is different from A, though both A and $A(y)$ contain the vertex $[L]$.

Figure 3 illustrates this branching phenomenon when n=2. It shows one hexagon in A and then the many faces which meet along a given edge. Each of these faces may be used as the starting point for the continuation of the tiling into another apartment. I should also remark that I have not drawn the additional branching along the six radial lines leaving the central vertex.

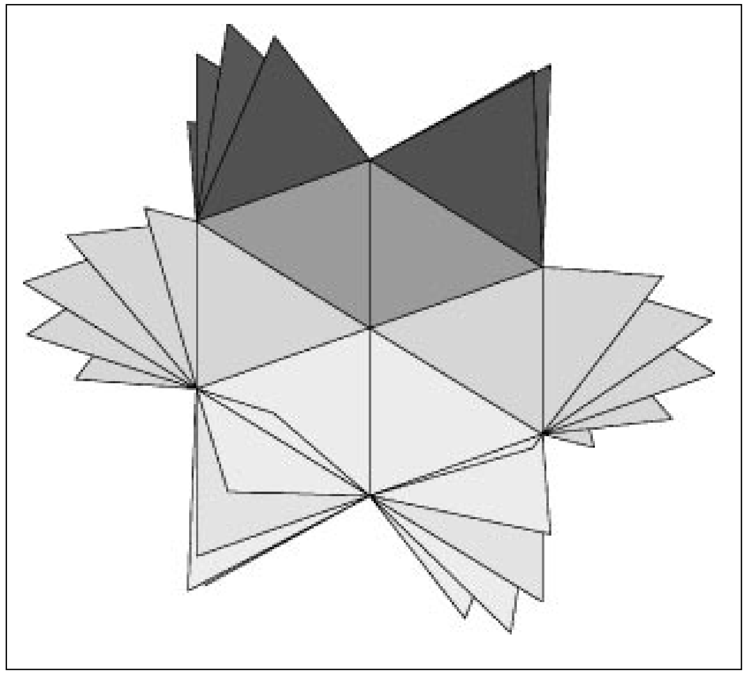

Figure 3

Drinfeld's *p*-adic Symmetric Space

The Bruhat–Tits building, which is an important and beautiful object, is nevertheless useless for constructing algebraic varieties. For that, one needs an object which supports a theory of analytic functions. A space which meets this requirement was defined by Drinfeld in his papers [6] and [7].

Let H be the set of all linear forms in $n+1$ variables $x_0, \dots, x_n$ with coefficients in $\mathbf{Q}_p$. Any linear form ℓ in H defines a hyperplane in projective n-space over $\mathbf{C}_p$; Drinfeld's symmetric space $\mathcal{X}$ is the set of points which *do not* lie on any such hyperplane.

Put another way, a point $x = [a_0, \dots, a_n]$ in $\mathbf{P}^n(\mathbf{C}_p)$ lies in $\mathcal{X}$ if the a_i are linearly independent over $\mathbf{Q}_p$. (Remember that $\mathbf{C}_p$ is of uncountable dimension over $\mathbf{Q}_p$, so there are lots of such points!) The matrices in G have coefficients in $\mathbf{Q}_p$, so the linear forms in H are permuted by G; consequently, the space $\mathcal{X}$ is preserved by G.

To put this space in some perspective, let me point out that a great deal of study has been devoted to the topology of *hyperplane complements* over $\mathbf{C}$; these are complex manifolds obtained by deleting finitely many hyperplanes from complex projective space. (See [14], for example.) Drinfeld's space $\mathcal{X}$ is a p-adic version of such a hyperplane complement, with the further twist that the set of missing hyperplanes is infinite.

Drinfeld's space supports a theory of analytic functions, differential forms, deRham cohomology, and so forth. To see how such analytic objects might be constructed, choose a set of representatives $\ell_1, \dots, \ell_m$ for the linear forms in H, mod p^k, for some k. Assume that the coefficients of the ℓ_i are p-adic integers and that at least one of these coefficients is a p-adic unit. Let $\mathcal{X}_k$ be the set of points x in $\mathcal{X}$ such that $|\ell_i(x)|_p > p^{-k}$. In terms of hyperplane arrangements, $\mathcal{X}_k$ is the complement to a p-adic neighborhood of a finite hyperplane arrangement. Any rational function on projective space whose denominator involves only the linear forms ℓ_i will be defined on all of $\mathcal{X}_k$. The holomorphic functions on $\mathcal{X}_k$ are those functions which are uniformly approximable by such rational functions. The $\mathcal{X}_k$ cover $\mathcal{X}$ as $k \to \infty$, and a function on $\mathcal{X}$ is holomorphic if it restricts to a holomorphic function on each $\mathcal{X}_k$.

Drinfeld's space is both like and unlike classical symmetric spaces. Like the classical symmetric spaces which arise in the theory of Shimura varieties, it can be interpreted as a pa-

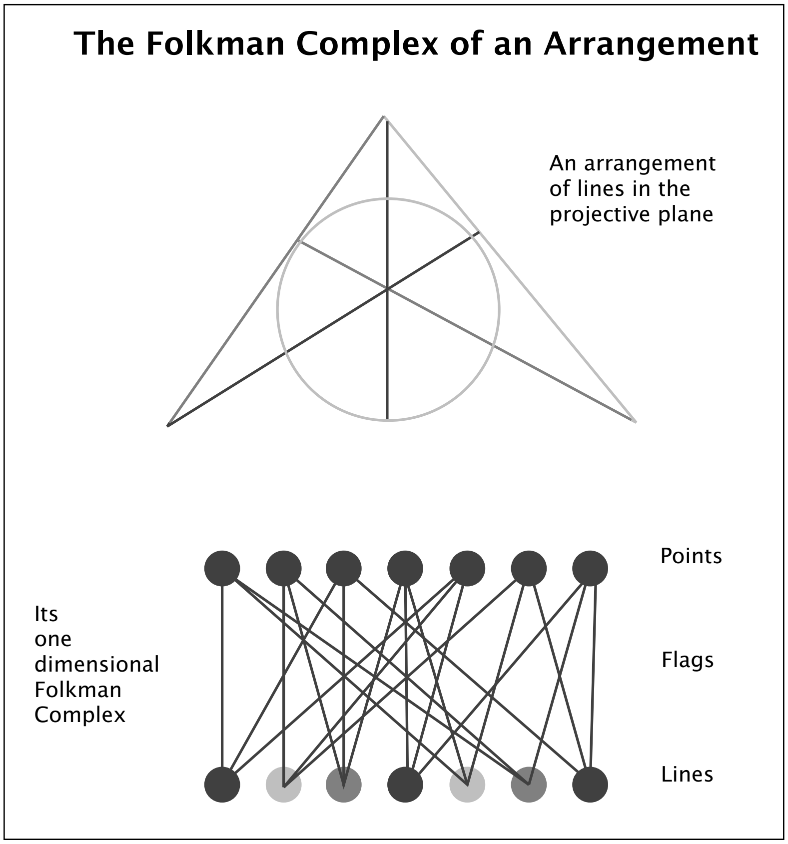

Figure 4

rameter space for something interesting—a certain class of formal groups—and, as hoped for in Figure 1, its quotients by appropriate arithmetic groups are projective algebraic varieties. However, $\mathcal{X}$ is not a coset space nor is it simply connected—indeed, it has interesting deRham and etale cohomology.

Analysis Meets Combinatorics

Having introduced both the Bruhat–Tits building and Drinfeld's space $\mathcal{X}$, we are in a position to explore the relationship between these two objects.

The most fundamental relationship is the existence of a map $r : \mathcal{X} \to B$ called the *reduction map.* This map commutes with the G action and enables us to think of Drinfeld's space as a p-adic neighborhood of the Bruhat–Tits building.

To describe r, we must relate points in $\mathcal{X}$ to lattices. The simplest way to do this is to adopt the point of view of Goldman and Iwahori [9]. Suppose that $x = [x_0, \ldots, x_n]$ is a point in $\mathcal{X}$, and recall that V is the standard $(n+1)$ dimensional vector space over $\mathbf{Q}_p$. Let

$$L(x,t) = \{(a_0, \ldots, a_n) : |\sum a_i x_i|_p \leq p^{-t}\}.$$

Since the a_i are in $\mathbf{Q}_p$ and the x_i are linearly independent over $\mathbf{Q}_p$, it follows that $L(x,t)$ is a lattice. In addition, it is clear that $L(x,1) = pL(x,0)$. Varying t from 0 to 1, one obtains finitely many different lattices $L(x,t)$ between $L(x,0)$ and $L(x,1)$; these finitely many lattices determine a face of the building B. The point $x \in \mathcal{X}$ then maps under r to the face determined by these lattices.

The existence of the reduction map implies many connections between $\mathcal{X}$ and B. One of the most striking of these connections is provided by the theorem of Schneider and Stuhler on the cohomology of $\mathcal{X}$. For simplicity, we will state this theorem only for $H^n_{DR}(\mathcal{X}, \mathbf{Q}_p)$; to do this we need to introduce the notion of harmonic function on B.

The n-simplices of B are called *chambers.* If F is a codimension one-face of a chamber Δ, then Δ meets $p+1$ other chambers along the face F. A ($\mathbf{Q}_p$-valued) *harmonic function* f on B is a function which assigns an element of $\mathbf{Q}_p$ to each chamber Δ in such a way that, first, changing the orientation of Δ reverses the sign, and, second,

$$\sum f(\Delta) = 0$$

where the sum is over all chambers with a common codimension one- face F.

Theorem (Schneider–Stuhler). *The deRham cohomology group $H^n_{DR}(\mathcal{X}, \mathbf{Q}_p)$ is isomorphic to the space of harmonic functions on B.*

Boundaries

To correctly interpret the Schneider–Stuhler result, we need one more piece to our picture. This is the introduction of a "boundary". This will enable us to understand the Schneider–Stuhler result as the assertion that the deRham class of an n form on $\mathcal{X}$ is determined by its "boundary values".

To provide some motivation for the construction of boundaries, recall that Drinfeld's space $\mathcal{X}$ is a hyperplane complement. Turning for the moment to the classical situation, suppose we consider the complex manifold obtained by deleting a finite set H of hyperplanes from $\mathbf{P}^n(\mathbf{C})$. A fundamental theorem in this subject relates the deRham cohomology of this space to the incidence relations among the hyperplanes in H.

More precisely, build a finite simplicial complex F whose vertices F^0 correspond to intersections $H_1 \cap \ldots \cap H_r$ of arbitrary numbers of elements of H. The i-simplices of this complex then correspond to flags of elements of F^0. (A simple example is shown in Figure 4.) This finite simplicial complex is called the Folkman complex, and the theory of hyperplane arrangements tells us that the deRham cohomology $H^n_{DR}(\mathbf{P}^n(\mathbf{C}) - H, \mathbf{C})$ is isomorphic to $\tilde{H}_{n-1}(F, \mathbf{C})$. (See [14], Chapter 4.) In the p-adic situation, the finite set of complex hyperplanes H is replaced by the infinite set of all hyperplanes defined by linear forms with coefficients in $\mathbf{Q}_p$. Nevertheless, it makes sense to speak of the Folkman complex in this situation. Since we have every hyperplane to work with, the vertices of this Folkman complex turn out to correspond to every linear subspace of $\mathbf{P}^n(\mathbf{Q}_p)$, and the simplices correspond to partial flags of such subspaces. The simplicial complex constructed in this way has another name—it is the spherical, or Tits, building associated to GL_{n+1}.

A top dimensional simplex of the Tits building is determined by a flag of subspaces of $\mathbf{P}^n$. (Note: Not a flag of lattices, but of subspaces!) The stabilizer of such a flag is a Borel subgroup P of G—for example, the lower triangular group—and since G permutes the flags transitively, the set of top dimensional simplices of the building can be identified with G/P. In addition, using the p-adic topology on the space of hyperplanes, one can show that the Tits building carries a topology making this identification a homeomorphism.

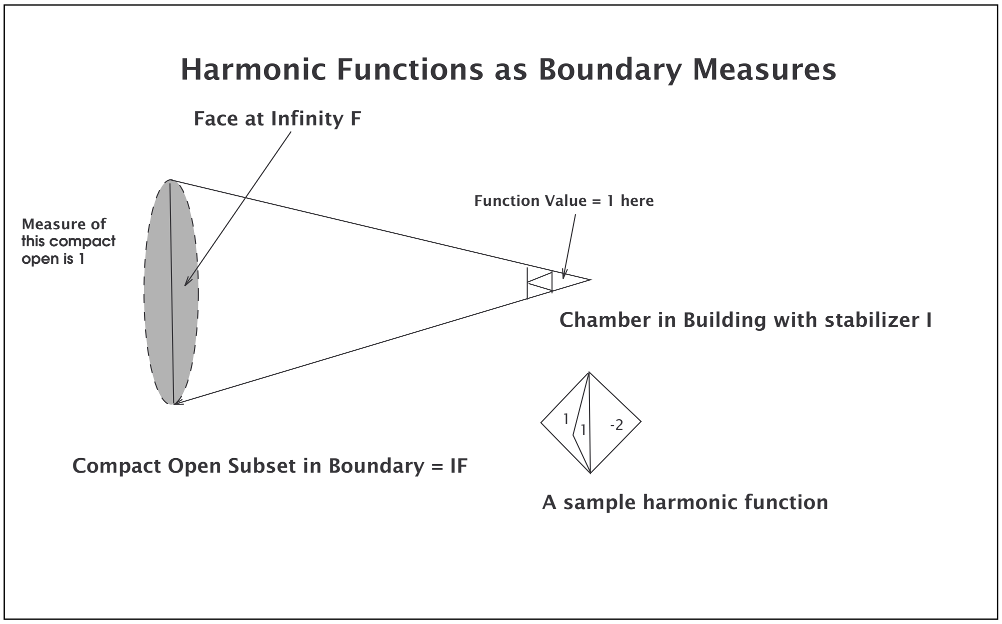

Figure 5

The spherical Tits building captures the incidence relations among hyperplanes we have deleted to construct $\mathcal{X}$, but it is a simplicial complex and therefore does not seem to be a suitable candidate for a "boundary" to $\mathcal{X}$. Remarkably, however, Borel and Serre [4] showed that the Tits building *is* the topological boundary of the Bruhat–Tits building B in a very natural way. One very imprecise way to think about this result is to view subspaces as limiting cases of lattices; since the building B captures incidence relations among lattices, its boundary should capture incidence relations among subspaces. Although this would take too long to explain in detail, let us at least outline Schneider and Stuhler's result which relates harmonic functions on B to measures on G/P.

First, suppose we have a chamber Δ in B. The chamber Δ belongs to an apartment A, and this apartment determines an unordered basis $\{y_0, \dots, y_n\}$ of our vector space V. We could show that choosing one vertex of Δ amounts to fixing an ordering of the y_i. Believing this, notice that an ordered basis is just a flag of subspaces; call this flag F. Now let I be the subgroup of G which stabilizes Δ with our fixed choice of vertex. Then the orbit IF of F is an open set of flags in G/P. Our harmonic function f on B assigns a number to Δ and thus to the open set IF. (See Figure 5.) Schneider and Stuhler show that sets of the form IF form a basis for the topology on G/P and that the harmonicity of the function f means that the corresponding function on open sets in G/P is finitely additive. This sets up the correspondence between harmonic functions on B and boundary measures.

Remark: It is worth pointing out to the experts that the space of harmonic functions on B is naturally isomorphic to the $\mathbf{Q}_p$-linear dual of the Steinberg representation of G.

An Integral Transform

I will conclude our discussion of geometry and analysis on $\mathcal{X}$ by stating a recent theorem of Peter Schneider and the author which complements the Schneider–Stuhler result. Since the Schneider–Stuhler theorem interprets the deRham cohomology of $\mathcal{X}$ in terms of measures on the "boundary" G/P, one might ask to what extent one can recover a differential form from its boundary measure. As in complex analysis, this reconstruction problem is solved by an integral kernel.

Theorem. *Let λ be a $\mathbf{Z}_p$-valued harmonic function on B, viewed as a measure on G/P. There is a function $k(x,g)$ on $\mathcal{X} \times G/P$, analytic in x and continous in g, such that*

$$F_\lambda(x) = \int_{G/P} k(x,g)\,d\lambda(g)$$

is an analytic function on $\mathcal{X}$. Let

$$\omega_\lambda = F_\lambda dz_1 \wedge \cdots \wedge dz_{n-1}$$

where $z_0, \dots, z_{n-1}$ are specific coordinates on $\mathcal{X}$ which we will not define here. Then ω_λ corresponds to λ under the Schneider–Stuhler isomorphism, and $\omega_{g\lambda} = g_* \omega_\lambda$.

Perhaps the most interesting aspect of this theorem is the G-equivariance of the integral. It asserts that a deRham class corresponding to a p-adically bounded measure has a canonical representative (namely, the one obtained by integration against the kernel). Thus the combinatorially defined representation of G on the space of harmonic functions occurs in the infinite-dimensional p-adic vector space of n-forms on $\mathcal{X}$.

Conclusion

The theory of p-adic symmetric spaces blends the theory of buildings, representation theory, geometry of hyperplane complements, and number theory into an intricate and beautiful picture.

To learn more about the theory of p-adic symmetric spaces and p-adic uniformization, I recommend beginning with the theory of the Tate elliptic curve, which is described in Silverman's second volume [18]. Mumford describes the theory of uniformization of curves in [12]; an easier place to look first is the book by Gerritzen and van der Put [11], which adopts the point of view of p-adic analytic geometry. The book by Gekeler [8] describes the theory of uniformization over p-adic fields of characteristic p, which grows out of Drinfeld's theory of elliptic modules [6]. These works are accessible to second-year graduate students.

For deeper investigation, one of the most important papers in this subject is [7], in which Drinfeld explains how the p-adic symmetric spaces function as parameter spaces for a kind of formal group, and uses this to relate quotients of the one-dimensional symmetric space to classical Shimura curves. (This result was also obtained by Cerednik, using different methods.) Drinfeld's paper is notoriously condensed, and it has been amplified and explained by Boutot and Carayol [3] in a paper more than ten times as long. Both Drinfeld and Boutot–Carayol use very sophisticated algebraic geometry.

To learn more about buildings, I recommend the pleasure of reading Kenneth Brown's book *Buildings* [1], which clearly explains the Bruhat–Tits building which I have talked about above. Orlik and Terao's comprehensive work on hyperplane arrangements [14] provides an excellent overview of this related theory.

For the general theory of the higher-dimensional p-adic symmetric space, one should see the work of Drinfeld on elliptic modules [6] and Mustafin's paper [13]. Schneider and Stuhler compute the cohomology of Drinfeld's space $\mathcal{X}$ and its quotients in [16], using techniques from representation theory as well as the theory of buildings and some complicated combinatorics. The integral transform theorem I have stated will appear in [17], although I have described earlier work on the one-dimensional case in the expository paper [19].

Finally, Rapaport and Zink [15] have introduced a much wider class of p-adic symmetric spaces and have greatly expanded the range of examples of Shimura varieties with p-adic uniformizations. Their work is a vast simultaneous generalization of the kind of p-adic uniformization I have talked about here and of an apparently different kind of set-up than that discussed in the paper of Gross and Hopkins [10].

References

[1] K. Brown, *Buildings*, Springer, Berlin, 1989.

[2] V. Berkovich, *Spectral theory and analytic geometry over non-archimedean fields*, translated by N. Koblitz, Amer. Math. Soc., Providence, RI, 1990.

[3] J. F. Boutot and H. Carayol, *Uniformisation p-adique des courbes de Shimura: les théorèmes de Cerednik et de Drinfeld*, Courbes Modulaires et Courbes de Shimura, Asterisque **196–197** (1991).

[4] A. Borel and J. P. Serre, *Cohomologie d'immeubles et de groupes S–arithmetiques*, Topology **15** (1976), 211–232.

[5] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local* I, *Donnees radicielles valuees*, Publ. Math. I.H.E.S. **41** (1972).

[6] V. G. Drinfeld, *Elliptic modules* (Russian), Mat. Sb. **94** (1974), 594–627; English translation Math. USSR-Sb. **23** (1976), 561–592.

[7] V. G. Drinfeld, *Coverings of p-adic symmetric regions*, Funct. Anal. Appl. **10** (1976), 107–115.

[8] E. U. Gekeler, *Drinfeld modular curves*, Lecture Notes in Math., vol. 1231, Springer, New York, 1986.

[9] O. Goldman and N. Iwahori, *The space of p-adic norms*, Acta Math.**109** (1963), 137–177.

[10] B. Gross and M. Hopkins, *Equivariant vector bundles on the Lubin–Tate moduli space*, Topology and Representation Theory (Evanston, IL 1992), Contemporary Mathematics, vol. 158, Amer. Math. Soc., Providence, RI, 1994.

[11] L. Gerritzen and M. van der Put, *Schottky groups and Mumford curves*, Lecture Notes in Mathematics, vol. 817, Springer, New York, 980.

[12] D. Mumford, *An analytic construction of degenerating curves over complete local rings* (1972), 129–174.

[13] G. A. Mustafin, *Nonarchimedean uniformization*, Math. USSR Sb. **34** (1978), 187–214.

[14] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Springer, Berlin, 1991.

[15] M. Rapaport and T. Zink, *Non-archimedian period domains*, preprint.

[16] P. Schneider and U. Stuhler, *The cohomology of p-adic symmetric spaces*, Inventiones Mathematicae **105** (1991), 47–122.

[17] P. Schneider and J. Teitelbaum, *An integral transform for p- adic symmetric spaces*, Duke Math. J. (to appear).

[18] J. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Math. vol. 151, Springer, New York, 1995.

[19] J. Teitelbaum, *Rigid analytic modular forms: An integral transform approach*, The Arithmetic of Function Fields (D. Goss, D. R. Hayes, and M. I. Rosen, eds.), deGruyter, Berlin, 1992.