Why Are We Learning This?

Roman Kossak

The current mathematics education model probably will be remembered in the history of education as a spectacular failure. “Many of today’s entering college students do not know much about arithmetic, algebra, or functions; they know nothing about proofs or abstract systems; they cannot function on any symbolic abstract level; they think that mathematics equals algorithms; and they cannot relate mathematics to the world around them.”[1] Sadly, this statement also applies to many college graduates.

Obviously, there is an urgent need to change this situation, and reform is under way. The *Curriculum and Evaluation Standards for School Mathematics*, issued by the National Council of Teachers of Mathematics (NCTM), provides an excellent basis for a discussion of the role of mathematics education, as well as of specifics of the education process. I want to address one element in this large and complex picture—the college curriculum. I will not discuss the curriculum for future scientists and engineers, people who will be using mathematics as a professional tool. The subject of this article is mathematics for all. Mathematics for students of liberal arts, business and administration, and, what is very important, mathematics for future teachers. I am concerned with education of those who need mathematics in order to have some understanding of applications of mathematics and statistics in their professional fields, and also need it as an element of general education, where its role is similar to that of psychology, art, or history. To summarize my position briefly, I would like to see more elements of modern mathematics incorporated into college standards.

A good understanding of the mathematical profile of a typical student entering college is essential for the discussion of college curriculum. NCTM *Standards* promise an improvement in the quality of high school graduates’ mathematics education. I am, however, much less optimistic for reasons I will discuss in the first section of this article. In the second section I turn to the question of content of college and precollege mathematics. I focus there on one of the main methodological tenets of the NCTM *Standards*: learning mathematics by doing mathematics. In the last section I present the idea of a mathematics appreciation course. The idea is not entirely novel; similar courses are often offered to small groups of, usually weaker, students. The novelty of my proposal is the scale. I will present the idea of a course that should be more ambitious with respect to the content, and that should be taken by most students. I realize the difficulty this would involve, on both a theoretical and a practical level. Also, I am aware that to fully explain my views, I would need much more space; consequently, some of my remarks may seem more controversial than they are.

[1] *Judy Roitman, as quoted in “Looking for Gauss…Or Waiting for Godot”,* Notices *of the AMS, vol 41, no 9.*

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On Doing Mathematics in School

Mathematics is and always will be an important element of general education. I do not think this point of view can be seriously challenged. However, many questions still remain. First of all, what is this mathematics that we want everyone to know? A point of view that is quite often expressed is that it does not really matter. What is important is the special quality of mathematical reasoning, the power of mathematical problem solving, the ability to analyze complex problems with precision that is difficult to find in other disciplines. NCTM *Standards* make the general goals for all elementary and high-school students very clear. The goals are:

(1) That they learn to value mathematics.

(2) That they become confident in their ability to do mathematics.

(3) That they become mathematical problem solvers.

(4) That they learn to communicate mathematically.

(5) That they learn to reason mathematically.

If such curriculum reform is successful, then indeed it will be be a dramatic departure from the present situation. Clearly, achieving these goals has to be a long-range project, and the outline and examples presented in the text of the NCTM *Standards* confirm that what lies ahead will require a Herculean effort. The reform will be very difficult not only because of the effort it will have to involve, but there are also problems of a more fundamental nature. I will try to describe some of them.

The approach and the specific guidelines of the NCTM *Standards* rely on the explicitly mentioned assumption that "knowing mathematics" is "doing mathematics". Consequently, it is emphasized "… that learning should be guided by the search to answer questions—first at an intuitive, empirical level; then by generalizing; and finally by justifying (proving)" (page 10). "The standards specify that the instruction should be developed from problem situations" (page 11). While a lot can be said in favor of this approach, there are some troubling questions that I have not found addressed.

I think that those who "do mathematics" would agree that there are no simple, general patterns for creative mathematical thinking. This seems to be a very delicate process, very individualized, almost private. How does one become convinced that a certain argument really proves an alleged fact? At what moment does one see that a collection of examples supports a general claim? How do we know that a formal mathematical model we are considering really represents the real-world problem we want to solve? Don't we often need first an easy-to-analyze mathematical idealization of a real-world problem in order to realize what the problem is and how to formulate it correctly?

The tacit assumption seems to be that the student knows basic facts about the world around us and uses this knowledge in the development of mathematical concepts and techniques. In reality the situation is much more complex, and our understanding of the world develops together with our cognitive skills. Patterns of formal mathematical reasoning play an important role here. Individual experience and acquired knowledge and skills are entangled in this complicated process. How then will a teacher guide the unconvinced, and often unwilling, student through the intricate maze of his or her own thoughts? Will the teacher be tactful enough to allow different paces and different modes of mathematical reasoning developed by each student? Will he or she be knowledgeable enough not to suppress nonstandard but perfectly reasonable attempts to solve a problem? It is true, the teacher's task should be to create the environment in which the student will have a chance, through his or her own experimentation, to internalize mathematical concepts and to develop efficient patterns of creative thinking. But to what extent can this be controlled? It has always amazed me how hard it is to tell the difference between the obvious and the difficult. To appreciate the complexity of issues it is enough to recall one's own struggles with ideas that later became simple and perfectly clear. The methodology of the reformed teaching model has to provide answers to many specific questions, and I am sure reasonable answers are and will be provided. But how are we going to deal with all these problems in an average classroom? To successfully implement reforms, a new generation of teachers might be needed first.

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I was a first-hand witness to the educational reform in Poland about 20 years ago. A well-balanced curriculum, incorporating much of what is proposed in the NCTM *Standards*, was prepared by a panel of leading experts with the support of the Ministry of Education. Attractive textbooks, covering uniformly the material for 12 years of elementary and high school education, were prepared. An impressive nationwide campaign to inform and educate the teachers was carried out. For about a year, morning and late night television programs offered lectures and instruction concerning reforms (not only in mathematics). Special instruction was offered, much was written about reforms in professional journals and in the mass press. But despite all the effort, and despite the fact that mathematical education in Poland has traditionally been on an advanced level, these reforms were not a success. Not surprisingly, it turned out that, while so very much can be done by a good and well-prepared teacher, a dabbler often creates a disaster. And the reformers very soon found out that to teach the teacher is a daunting task.

The new style of instruction, promoted by the NCTM *Standards*, imposes new demands on the teacher. The teacher himself or herself must become a problem solver, and must be confident in his or her mathematical power. Future teachers will have to know and value mathematics much beyond the present curriculum. A lot of preparatory work will have to be done.

I do not want to engage in a discussion concerning the specifics of the NCTM *Standards*. The general direction of the proposed reforms is certainly worth supporting. The details, however, will not be easy, and, considering the dismal present state of affairs, we will have a long wait until the freshmen educated with the reformed model enter colleges. So, is there anything that can be done in the meantime to bring the practice of college mathematics closer to the ideals outlined in the NCTM *Standards*? I want to start with the issue that pertains to the content and method of teaching. The problem is, what should be the right balance between passive and active learning?

Mathematical Power vs. Power of Mathematics

My primary claim is that at the college level (and this also applies to precollege education) "knowing mathematics" is no longer "doing mathematics". First of all, there is too much to know. Second, much of what there is might not be suitable for the "genetic method", in which the material is developed through all stages starting from examples and intuitions. In areas like statistics or large parts of calculus, the volume and difficult nature of subject matter force us to limit the presentation to ready-to-use formulas and algorithms.

Now, can mathematics "be known" if it is not "done"? There is a tempting analogy between learning mathematics and learning arts. One does not have to practice art to learn a lot about it. Of course, we get a far superior insight into the world of organized sounds if we can read music and play an instrument. The same point applies to creativity of any kind. And indeed, there is time in the education process when students should experiment with painting, sculpting, writing, and performing. Time should be spent on development of artistic sensitivity and skills. This is the best way to learn and to become conscious of one's talents and interests. But then there is still art as a subject, as part of human history and experience, as a theoretical discipline, and one can learn quite a lot about all that without hands-on experience. The situation in mathematics is quite similar.

Already at the elementary level, some mathematical power can be gained just by knowing the methods that can be used to solve a particular problem. For example, if the problem is to solve a quadratic equation, I do not need to test my mathematical prowess and try various techniques of factoring. It is enough that I know what the quadratic formula is and how it should be applied. This is a simple example of a powerful tool—a piece of knowledge—that can be utilized in many contexts. The tool itself can be presented just as an algorithm. The full discussion and justification of the quadratic formula, although of much value, is not really necessary, unless done in the framework of a more general presentation of polynomial equations. There are many examples of facts, formulas, and theorems in algebra, geometry, and trigonometry that are worth knowing about but which do not seem to be particularly good subjects for active learning. The reason is quite often the level of difficulty. On the other hand, if we restrict ourselves only to what is tangible, to what can be accessed by students who actively reproduce fragments of algebra or geometry, and if we concentrate on the applied aspect, much of the material will remain out of reach.

I am afraid the new elementary and high school curriculum concentrating on techniques of problem solving and on creative mathematical reasoning, even if based on the right theoretical premises and with the support of the knowledge of psychology of learning, will produce a large number of frustrated and confused students. Perhaps paradoxically, this new type of instruction will put a larger burden on the student. On the other hand, when passive learning is permitted, then those who, for whatever reason, lack motivation to study still have a chance to perform satisfactorily by memorizing and reproducing parts of the material. This might not seem a great achievement if our aim is development of mathematical power, but I would like to dispute that. Passive learning has merit, as it forms a base of knowledge that can be used by the student when he or she becomes ready. Passive knowledge does not evaporate without a trace. It is true that most people, when asked about their school mathematics, do not remember much, if anything. But that does not mean that if they needed or wanted to go back and relearn parts of the material they used to know, they would have to go through the same learning process again. Our minds have fantastic storage capacity, and memories and associations do come back when needed.

If we concentrate on the methods of mathematical reasoning and if in the process of teaching this we neglect mathematical content, then we do not really have much control over the image of our subject that is created in the student's mind. If it becomes clear that what matters in class is not geometry, algebra, or arithmetic, because teachers test, rate, and compare thinking skills, then we might end up with even more students afflicted with math fear. There will be always those who think faster and always seem to know the right answer, and then there will be many others left behind, less and less willing to try their math talents in new challenges. If we pose a problem and the student's reaction is, "What should I do? How do I solve it? Do I know the answer? Do I remember how we did a similar problem the other day?" etc…, then I do not think we are doing a very good job. If our problem meets the response, "Hmm, that is interesting. Let me see. What does that mean? What happens if I take this element and move it over there?…" That is, if we can shift students' attention from their own involvement in the thinking process and let them concentrate on the subject, and if we let the power of mathematics do the job, that would be great progress.

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Mathematical structures *are* logic! Freed of the ambiguities of real-life phenomena, the world of mathematical concepts becomes a perfect field to practice logical thinking. Our cognitive skills develop from the moment of birth. We constantly learn how to recognize and analyze the signals we receive. Nature forces us to think rationally. Still our rational instinct is often overpowered by the complexity of issues and by the difficult process of separating our thoughts from our emotions. Mathematics allows one to see more clearly. Modern mathematics offers a remarkable range of structures at various levels of abstraction. Many geometric, algebraic, and combinatorial structures can be easily visualized, some with the use of software. Familiarity with these structures, the experience of performing mental and computer-aided experiments, should give the students a real chance to practice the art of reasoning in the pure, unobfuscated world of mathematical ideas.

There are two indispensable elements involved in teaching mathematics. First is clarity of the subject matter, which if used properly, mathematics provides free of charge. Second is students' attention—attention focused not on one's ability to reason, but on the content of what is discussed. Now, how do we get the student interested in the subject matter? No matter how hard we try, we cannot guarantee that the content of the real-world problems we choose as the base of our mathematical considerations will arouse interest and enthusiasm in young minds. Some will find what is offered intriguing and challenging; many will see it as superficial and boring. In the NCTM *Standards* we read, "The curriculum should be permeated with these goals [referring to the list of goals I quoted at the beginning of this article] so that they become commonplace in the lives of the students." It is very difficult to believe that this can be achieved. Contrary to what is claimed in the NCTM *Standards*, doing mathematics is not a common human activity, and that seems especially true about the MTV generation. Unfortunately, cultural stereotypes and popular images of people of science do not help in creating a conducive social climate. Why would a young person want to be good at math if this brings associations with some sort of oddity. After all, one of the greatest fears of adolescence is the fear of being perceived as a nerd. In the fight for mathematical literacy, we are facing very strong enemies.

In a precalculus class quite often we say, "Now, pay careful attention to what we are doing. We are learning techniques of equation solving. This will be very important later in calculus." Early in the calculus course, when we explain elementary notions, and then when we discuss limits and elements of curve sketching, we will say, "This is all rather theoretical now, but wait. Soon you will see how all of that can be applied." And the text will be accompanied by nice pictures of skiers speeding down a slope. When we finally get to the application stage, we will show the student how to compute the final velocity of a coin dropped from the top of the World Trade Center, or how to find dimensions of a soda can that minimize the amount of material used for its production, and then we will give examples of some rather simple business applications. And the student has every right to say: "So is this what all the fuss was about? Did I have to study so long just to learn that?"

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Applications of mathematics are important. The student has to learn about the methods, their applicability, and their limitations. But this is not enough. What makes calculus attractive is not only its computing power applied to real-life phenomena; calculus is interesting because it is difficult. The concept of continuity is far from clear. The notion of instantaneous rate of change is very dubious. It is very difficult to say what arc length is and what the area of an arbitrary two-dimensional figure should be. Most of the brief, and not so brief, calculus courses gloss over these topics. We can learn a lot by studying how particular methods of calculus can be applied in various disciplines, but I believe that we learn much more by understanding the development and general concepts of calculus. It is this kind of knowledge that gives insights into a mathematical approach toward solving difficult general problems, which might eventually turn out to be more useful in the everyday practice of the modern workplace. I believe that in our rapidly changing world what is most important is the knowledge of general principles and patterns that give one a good base for quickly learning the specifics needed in particular situations. Much more can be said about this, but I would like to turn to the questions, "Does anyone really need to learn about modern mathematics? Am I trying to sell a product that no one needs?"

Teaching About Mathematics

The role of undergraduate mathematical education is to equip students with basic mathematical knowledge to the extent that will allow them to read and understand common applications of mathematics and statistics. College graduates ought to know what makes a statistical argument valid and how to recognize simple cases of fraud and abuse using statistics. They must know why a small random sample can be used to accurately predict the outcome of a vote, and what makes the sample random. They should be aware of what is going on inside computer hardware, what tasks can be performed by a computer and with what accuracy, and also what, according to the present state of knowledge, a computer cannot and will not be able to do in the foreseeable future. Most applied knowledge of this kind is in the domain of statistics and computer science, where mathematics is used as a language and as a source of formulas and algorithms. Mathematics also provides necessary background and methods of precise analytic thinking that can be applied in any discipline. But all of that does not mean that mathematics in general college education should be given the role of a servant.

Mathematics is not just about algorithms and formulas. It is one of the most fascinating domains of science. It teaches us a lot about the world around us and about ourselves. The great mathematical power to explain what is difficult to grasp via direct experience and intuition is what makes it attractive to me and, I assume, to most of my colleagues. We should not be afraid to give students a chance to see that side of mathematics. Mathematics changes every day, theorems are proved, important problems are solved. Of course, most of it is difficult and technical, but still a lot can be made accessible in a popular presentation. We need to be able to show the position of mathematics among other sciences, and especially its role in disciplines whose primary goal is the explanation of the fundamental laws of nature. This, of course, is difficult with today's rapid expansion of knowledge. The standard calculus course supplemented with applications to mechanics or business problems is simply not enough.

When we teach algebra, why is so little time spent on its fundamental theorem? Why is the existence of formulas for solutions of cubic and quartic equations rarely mentioned? Why is the name Galois not mentioned at all? And what about number theory, with its unlimited supply of material? Why is so little done on this subject? I think we are making a mistake if we never give the student a chance to learn about the infinitude of prime numbers and at least about the Goldbach conjecture. Something is not right when a college graduate learns about the Fermat theorem only from the front page of The New York Times. I am not saying that we should teach all that. I am saying we should teach *about* it. The benefit of doing that first of all is motivation. The more I know about the subject, the more I am interested in it. Once I learn that something cannot be done, I value more the positive knowledge I have; I know more about what to expect later; I learn about the questions that can be asked, and the answers that can be obtained. All that passive knowledge facilitates my ability for further active study and enhances my mathematical power. There are many interesting fragments of mathematical knowledge that have this *welt-anshauung* value, and that can be discussed along with the study of college algebra, geometry, and calculus. A simple example is the statement that the graphs of any two distinct quadratic functions cannot meet at three distinct points. Students often find this intriguing, and it is an ideal point of departure to a general discussion of properties of conic sections, algebraic curves of higher degrees, and then of the analogous result concerning analytic functions.

Many topics related to the standard algebra-calculus-discrete mathematics curriculum might be used in a mathematics appreciation course. A course like that could be a part of the precollege curriculum. But there are also other important issues that surface only at a more advanced level. What is a number? What is a proof? What is mathematical knowledge, and what are its limits? What is space? How do abstract mathematical concepts relate to the modern understanding of physical space and time? These are important, deep, and difficult questions. Their value is not only in their subject matter, but also in the fact that they remain unanswered. While I realize very well what degree of difficulty is involved, I still maintain that foundational questions can and should be discussed at the undergraduate college level. I have had a chance to lecture on some of the above topics to students at Baruch College in New York. I found the students quite interested and responsive.

I discovered quite early in my student days that the subjects I find interesting are the subjects I know about. Some critical mass of information is necessary before I find a topic worthwhile. The vast majority of high school graduates lack that critical mass, and consequently they are not sufficiently prepared for mathematics at the college level. I suspect that every teacher of mathematics has at some point heard the question "Why are we learning this?" I have heard it many times, often expressed in a characteristic complaining tone. The answer to this question is never simple. We lecture on this or that topic very often only because it will be used later on in another topic which in turn happens to have important applications, but in an area that a typical student has yet to learn about. College math textbooks motivate almost every part of the material by introductory real-life examples. I have never found it very useful in the teaching process. The main reason is that the examples that are used are much more complex than the material we lecture on. The problem is not that the examples are not selected well. In most cases it is impossible to explain our purpose without going into complicated details of a technical or methodological nature. Students are forced, rather than convinced, to accept that what we do is applicable and useful (of course, I am not talking here about mathematics for scientists and engineers). I would like to eliminate the "Why are we learning this?" question by providing the answer far in advance. This should be done in the context of the mathematics appreciation course.

The mathematics appreciation course should contain a global overview of mathematics, including its history, and a discussion of some foundational questions. The overview should be supplemented with a more detailed discussion of selected topics. I would leave the choice of topics to the discretion of the teacher. Obviously, the choice should be directed toward arousing the interest of the students. Mathematics is an endless source of knowledge that can be made attractive for the general audience. I would not like to be misunderstood here. When I say that mathematics is attractive I am not referring to the attraction by the beauty of mathematics. The aesthetic value of mathematics is a separate issue. "The truth of the matter is that, though mathematics truth may be beauty, it can be only glimpsed after much hard thinking. Mathematics is difficult for many human minds to grasp because of its hierarchical structure: one thing builds on another and depends on it."[2]

Clearly, there are levels of mathematics where understanding and appreciation of beauty are reserved for the experts. But instead of beauty I would like to concentrate on relevance of mathematics. Moreover, I would like to see this relevance not only as the problem-solving power of mathematical techniques but also as the relevance of science that deals with the basic concepts present in our elementary human experience. The concepts of a number, a relation, a set, a structure, a space are the notions that we all create in the individual processes of the devel-

[2]*M. Holt and D.T.E. Marjoram,* Mathematics in a Changing World, *Walker, New York 1973.*

opment of cognitive thinking. It is exciting to see how the same notions are represented in mathematics and what mathematics has to say about them. The attraction here is not in the beauty of a particularly elegant mathematical argument but in the specific approach and analysis of fundamental notions. If this sounds like "new math" at this point, it should not. The difference is that I do not advocate teaching abstract concepts at early levels. What I want is to teach *about* these issues much later.

To learn about foundational issues, one does not need a great deal of technical knowledge. What is needed is some intellectual maturity. And this is the kind of maturity that we should expect of college graduates. Of course, there are various levels at which the foundational questions can be discussed, but this is very much like the arts. There is room for popular presentation of art, and many believe that a popular presentation is just a first step that might develop sensitivity and create further curiosity about the subject. I am convinced that we can do the same in mathematics.

I do not advocate teaching everyone foundations or philosophy of mathematics. A mathematics appreciation course should be designed along different lines, and there are many ways of doing it. Extensive literature exists that can be used for that purpose. "The Mathematical Experience" by Davis and Hersh is an excellent example of a text that could be used "as is". But there are countless other possibilities. "The Beauty of Doing Mathematics. Three Public Dialogues" by Serge Lang shows how much can be done in the field of popularizing mathematics. But essentially, it does not really matter much what and how we do it, as long as we give the student a chance to see beyond what his or her present perception of mathematics is. One could lecture about dynamical systems, topology, Galois theory (why not?), number theory, combinatorics, set theory, noneuclidean geometry, or, in fact, any favorite subject of the teacher. I am not saying that all of that is easy, or that all subjects are equally well-suited for popularization, but any well-motivated effort, any attractive, informal presentation of a fragment of modern mathematics, is worth a try.

There seems to be popular demand for this kind of knowledge. The evidence might be the surprising commercial success of books like "Brief History of Time" by Stephen Hawking, or Robert Penrose's "The Emperor's New Mind". Talking to my friends and to my wife (who is an artist), I have found that people are curious to find out what it is that mathematicians do, and they are not happy with the very narrow vision of mathematics that was offered to them at school and college. It seems shortsighted not to respond to this demand in a more organized way. I hope that the course I am proposing, presented in a stress-free, nonthreatening atmosphere uncommon in other mathematical courses, will allow many students to take a fresh look at the subject and will give their own mathematical experience and abilities a new perspective. This, in turn, may become a strong motivating influence that might positively affect the college mathematical experience of many.

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